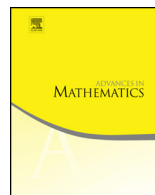




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Lipschitz images and dimensions ☆

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ABSTRACT

We consider the question which compact metric spaces can be obtained as a Lipschitz image of the middle third Cantor set, or more generally, as a Lipschitz image of a subset of a given compact metric space.

In the general case we prove that if A and B are compact metric spaces and the Hausdorff dimension of A is bigger than the upper box dimension of B , then there exist a compact set $A' \subset A$ and a Lipschitz onto map $f: A' \rightarrow B$.

As a corollary we prove that any ‘natural’ dimension in \mathbb{R}^n must be between the Hausdorff and upper box dimensions.

We show that if A and B are self-similar sets with the strong separation condition with equal Hausdorff dimension and A is homogeneous, then A can be mapped onto B by a Lipschitz map if and only if A and B are bilipschitz equivalent.

For given $\alpha > 0$ we also give a characterization of those compact metric spaces that can be obtained as an α -Hölder image of a compact subset of \mathbb{R} . The quantity we introduce for this turns out to be closely related to the upper box dimension.

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1. Introduction

The characterization of compact sets in \mathbb{R}^2 which can be covered by a Lipschitz image of $[0, 1]$ was given in the celebrated paper of P. W. Jones [14], which was generalized to \mathbb{R}^d by K. Okikiolu [27]. This is called the analyst's traveling salesman theorem. Later R. Schul [30] and I. Hahlomaa [12] considered the question in Hilbert spaces and general metric spaces, respectively. Motivated by these results, we formulated the following question.

Question 1.1. (a) *What compact metric spaces can be obtained as a Lipschitz image of a given compact metric space X or at least as a Lipschitz image of a subset of X ?*

(b) *For example, what compact metric spaces can be obtained as a Lipschitz image of the middle third Cantor set?*

1.1. Lipschitz images of general compact metric spaces

M. Laczkovich [19] posed the following problem.

Problem 1.2 (Laczkovich, 1991). Let $A \subset \mathbb{R}^d$ be a measurable set with positive d -dimensional Lebesgue measure. Is there a Lipschitz onto map $f: A \rightarrow [0, 1]^d$?

For $d = 1$ the positive answer is only an easy exercise. For $d = 2$ the affirmative answer was given first by D. Preiss [29]. By modifying partly the argument of Preiss, J. Matoušek [22] provided a simpler proof based on a well-known combinatorial lemma due to Erdős and Szekeres. Meanwhile, P. Jones noticed that the $d = 2$ case easily follows from an earlier result of N. X. Uy [31], see the argument of Jones in [1] after Theorem 2.6. Problem 1.2 is still open for dimensions $d \geq 3$.

The following result of A. G. Vituškin, L. D. Ivanov, and M. S. Melnikov [32] shows (see also [16] for a less concise proof) that Laczkovich's problem does not remain true if we only assume that E has positive d -dimensional Hausdorff measure in a larger dimensional space.

Theorem 1.3 (Vituškin–Ivanov–Melnikov, 1963). *There exists a compact set K on the plane such that K has positive 1-dimensional Hausdorff measure but there exists no Lipschitz onto map $f: K \rightarrow [0, 1]$.*

Let \dim_H and $\overline{\dim}_B$ denote the Hausdorff and upper box dimensions, respectively. Keleti, Máthé, and Zindulka [17] proved the following positive result under the stronger assumption $\dim_H A > d$.

Theorem 1.4 (Keleti–Máthé–Zindulka, 2014). *Let A be a compact metric space with $\dim_{\text{H}} A > d$. Then there exists a Lipschitz onto map $f: A \rightarrow [0, 1]^d$.*

Our first main result is the following generalization of Theorem 1.4. Here we state it only for Lipschitz maps but in fact we prove a more general statement for Hölder maps.

Theorem 1.5. *Let A and B be compact metric spaces such that $\dim_{\text{H}} A > \overline{\dim}_{\text{B}} B$. Then there exist a compact set $A' \subset A$ and a Lipschitz onto map $f: A' \rightarrow B$.*

In order to get a Lipschitz onto map defined on the whole A we need to extend the Lipschitz function obtained in Theorem 1.5 to A . To this end, we prove the following lemma, which shows that this is always possible if A is an ultrametric space (see the definition in Section 2).

Lemma 1.6. *Let X be a compact ultrametric space and Y be any complete metric space. Then for any $A \subset X$ any Lipschitz function $f: A \rightarrow Y$ can be extended to X as a Lipschitz function with the same Lipschitz constant.*

Since it is well known and easy to show (see Lemma 2.2) that any self-similar set with the strong separation condition (SSC) is bilipschitz equivalent to an ultrametric space (see the definitions in Section 2), Theorem 1.5 and Lemma 1.6 immediately give the following.

Corollary 1.7. *Let A be a complete ultrametric space or a self-similar set with the strong separation condition (SSC) and let B be a compact metric space such that $\dim_{\text{H}} A > \overline{\dim}_{\text{B}} B$. Then A can be mapped onto B by a Lipschitz map.*

In Section 4 we also prove a result (Proposition 4.1 (2)), which is slightly stronger than Corollary 1.7: it can be also applied for some cases when $\dim_{\text{H}} A = \overline{\dim}_{\text{B}} B$. To give a partial answer to Question 1.1 (b) we give a more explicit sufficient condition (Corollary 4.5) for the case when A is the middle third Cantor set.

In many cases, for example if A is connected and B is non-connected, we clearly cannot have a Lipschitz map from A onto B but we might hope for an extension to a space that contains B , obtaining a cover of B by a Lipschitz image of A , similarly as in the classical analyst's traveling salesman theorem. It is easy to prove that any real valued Lipschitz function defined on a subset of a metric space X has a Lipschitz extension to the whole space X , see e.g. [11, p. 202]. Applying this extension in each coordinate, we obtain that every Lipschitz function from a subset of a metric space X to \mathbb{R}^n has a Lipschitz extension to the whole X . Using this and Kirszbraun's extension theorem [18], Theorem 1.5 yields the next corollary.

Corollary 1.8. *Suppose that X and Y are Hilbert spaces or X is an arbitrary metric space and $Y = \mathbb{R}^n$ for some n . Let A be a compact subset of X and B be a compact subset of Y such that $\dim_{\text{H}} A > \overline{\dim}_{\text{B}} B$. Then B can be covered by a Lipschitz image of A , that is, there is a Lipschitz map $f: A \rightarrow Y$ with $B \subset f(A)$.*

From Theorem 1.5 we also deduce the following result, which might be interesting in itself. It states that any ‘natural’ concept of dimension must be in between the Hausdorff and the upper box dimensions if we do not assume σ -stability, and in between the Hausdorff and the packing dimensions (\dim_{P} , see the definition in Section 2), if we require σ -stability. Here we state it only for dimensions defined on compact sets; in Section 6 we prove a stronger statement for more general domains.

Corollary 1.9. *Let n be a positive integer and let D be a function from the family of compact subsets of \mathbb{R}^n to $[0, n]$. Suppose that*

- (i) *Lipschitz functions cannot increase D , that is, for any compact $K \subset \mathbb{R}^n$ and Lipschitz function $f: K \rightarrow \mathbb{R}^n$ we have $D(f(K)) \leq D(K)$,*
- (ii) *D is monotone, that is, $A \subset B$ implies that $D(A) \leq D(B)$, and*
- (iii) *if $S \subset \mathbb{R}^n$ is a homogeneous self-similar set with the SSC then $D(S) = \dim_{\text{H}} S$.*

Then for every compact set $K \subset \mathbb{R}^n$ we have

$$\dim_{\text{H}}(K) \leq D(K) \leq \overline{\dim}_{\text{B}}(K).$$

Furthermore, if we also require that D is σ -stable, that is,

- (iv) *$D(\cup_{i=1}^{\infty} K_i) = \sup_i D(K_i)$ whenever $\cup_{i=1}^{\infty} K_i$ and all K_i are compact,*

then for every compact set $K \subset \mathbb{R}^n$ we have

$$\dim_{\text{H}}(K) \leq D(K) \leq \dim_{\text{P}}(K).$$

The above result sheds some light on why the intermediate dimensions introduced by Falconer, Fraser and Kempton [9] and the generalized intermediate dimensions introduced by Banaji [4] span between the Hausdorff and upper box dimensions, and their σ -stable versions [7] are between the Hausdorff and packing dimensions.

Note that there are other fractal dimensions, such as the Assouad dimension, which do not lie between the Hausdorff and the upper box dimensions. However, this does not contradict Corollary 1.9, since these dimensions do not satisfy all the assumptions; for example, Lipschitz maps can increase the Assouad dimension.

1.2. Hölder images of compact metric spaces

In order to prove Theorem 1.5 we need to consider Hölder images of $[0, 1]$.

Question 1.10. For given $\alpha > 0$ which compact metric spaces B can be obtained as an α -Hölder image of a compact subset of $[0, 1]$?

This is basically the Hölder version of the analyst's traveling salesman problem, for which M. Badger, L. Naples, and V. Vellis [2] found a deep sufficient condition if $\alpha < 1$ and B is a subset of a Euclidean or Hilbert space. We provide a different type of characterization in Theorem 1.12.

Definition 1.11. Let (X, d) be a metric space and let n be a positive integer. By $\text{Sym}(n)$ we denote the set of $\{1, \dots, n\} \rightarrow \{1, \dots, n\}$ bijections. For $x_1, \dots, x_n \in X$ and $s > 0$ define

$$Z^s(x_1, \dots, x_n) = \max \left\{ \sum_{j=1}^{k-1} (d(x_{i_j}, x_{i_{j+1}}))^s : 1 = i_1 < \dots < i_k = n \right\},$$

$$\delta^s(X) = \sup \left\{ \min_{\pi \in \text{Sym}(n)} Z^s(x_{\pi(1)}, \dots, x_{\pi(n)}) : x_1, \dots, x_n \in X, n \geq 1 \right\}.$$

Theorem 1.12. Let X be a compact metric space and $s > 0$. Then there exist a compact set $D \subset [0, 1]$ and a $1/s$ -Hölder onto map $g: D \rightarrow X$ if and only if $\delta^s(X) < \infty$.

Remark 1.13. It is well known that if $\alpha > 1$ then every α -Hölder function $f: [0, 1] \rightarrow \mathbb{R}^n$ is constant. Indeed, on one hand $f([0, 1])$ is connected, on the other hand its Hausdorff dimension is at most $1/\alpha < 1$. This happens only because $[0, 1]$ is connected, which is illustrated by the following example. Let $0 < \beta < \gamma < 1$ be arbitrary and let C_β and C_γ be the homogeneous self-similar Cantor sets of dimensions β and γ , respectively. Then it is easy to see that the natural bijection $\varphi: C_\gamma \rightarrow C_\beta$ is γ/β -Hölder.

The next theorem connects δ^s to the upper box dimension.

Theorem 1.14. If X is a compact metric space with $\overline{\dim}_B X < s$, then $\delta^s(X) < \infty$.
Moreover, for any compact metric space X we have

$$\overline{\dim}_B X = \inf \{s > 0 : \delta^s(X) < \infty\}.$$

In fact, we prove both Theorems 1.12 and 1.14 in a slightly stronger, quantitative form in Section 3. Combining Theorems 1.12 and 1.14 immediately gives the following corollary.

Corollary 1.15. *If $\alpha > 0$ and B is a compact metric space with $\overline{\dim}_B B < 1/\alpha$ then B can be obtained as the α -Hölder image of a compact subset of $[0, 1]$.*

For the special case when $B \subset \mathbb{R}^m$ and $\alpha \leq 1$, Corollary 1.15 is essentially known: it follows easily from [3, Theorem 2.3] or the more general [2, Theorem 1.1].

By an extension theorem of Minty [26, Theorem 1 (ii)], if $\alpha \leq 1$, $D \subset [0, 1]$ and H is a Hilbert space then any α -Hölder map $g: D \rightarrow H$ can be extended to $[0, 1]$ as an α -Hölder map. Thus Corollary 1.15 has the following direct consequence.

Corollary 1.16. *If B is a compact subset of a Hilbert space and $\overline{\dim}_B B < 1/\alpha$ and $\alpha \leq 1$ then B can be covered by an α -Hölder image of $[0, 1]$.*

1.3. Self-similar sets with the strong separation condition

Let $A \subset \mathbb{R}^n$ and $B \subset \mathbb{R}^m$ be self-similar sets with the strong separation condition (SSC). It is well known (see e.g. in [8]) that for such sets all dimensions agree, including Hausdorff dimension and upper box dimension. Since Lipschitz maps cannot increase Hausdorff dimension, A cannot be mapped onto B by a Lipschitz map if $\dim A < \dim B$. On the other hand, by a special case of Corollary 1.7, A can be mapped onto B by a Lipschitz map if $\dim A > \dim B$. For this special case Deng, Weng, Xiong and Xi [6] proved a stronger result by showing that in this case $\dim A > \dim B$ implies that there is a bilipschitz embedding $f: B \rightarrow A$. (Note that by Lemma 1.6, the inverse of f can be extended to A , so the existence of a bilipschitz embedding $f: B \rightarrow A$ indeed implies that A can be mapped onto B by a Lipschitz map.)

So it remains to study the $\dim A = \dim B$ case. In [6] Deng, Weng, Xiong and Xi also proved that if A and B are self-similar sets with the SSC and $\dim A = \dim B$ then B can be bilipschitz embedded to A if and only if A and B are bilipschitz equivalent.

There is a vast literature establishing conditions under which two metric spaces, or more specifically two self-similar sets are bilipschitz equivalent, see e.g. [10,21,33,34] and the references therein. In 1992 Falconer and Marsh [10] found necessary algebraic conditions for two self-similar sets with the SSC to be bilipschitz equivalent. In 2010 Xi [33] gave a necessary and sufficient condition. For the case when one of the sets is the middle third Cantor set C , the Falconer-Marsh necessary condition is also sufficient and it is very simple: a self-similar set B with the SSC and with $\dim B = \dim C$ is bilipschitz equivalent to C if and only if all similarity ratios of B are integer powers of 3.

We prove that this condition is necessary even to have a Lipschitz onto map from C onto B . Therefore we can answer a special case of Question 1.1: a self-similar set B with the SSC and $\dim B = \dim C$ can be obtained as a Lipschitz image of the Cantor set C if and only if B and C are bilipschitz equivalent. In Section 7 we prove this not only for the Cantor set but also for any homogeneous self-similar set with the SSC:

Theorem 1.17. *Let A and B be self-similar sets with the SSC such that A is homogeneous. Suppose that $\dim_H A = \dim_H B$. Then A can be mapped onto B by a Lipschitz map if and only if A and B are bilipschitz equivalent.*

1.4. Structure of the paper

The paper is organized as follows. Section 2 contains some basic definitions and results we use later. We prove Theorems 1.12 and 1.14 in somewhat stronger forms in Section 3, Theorem 1.5 and some related results in Section 4, Lemma 1.6 in Section 5, Corollary 1.9 in a stronger form in Section 6 and Theorem 1.17 in a stronger form in Section 7. Finally, in Section 8 we pose some questions.

2. Preliminaries

Let (X, d) be a metric space. For $A, B \subset X$ let $\text{dist}(A, B) = \inf\{d(x, y) : x \in A, y \in B\}$. For $x \in X$ and $r > 0$ we denote by $\overline{B}(x, r)$ the closed ball of radius r centered at x , and by $N(X, r)$ the minimal number of closed balls of radius r that cover X . The *upper box dimension* of a bounded set $X \subset \mathbb{R}^n$ is defined as

$$\overline{\dim}_B X = \limsup_{r \rightarrow 0+} \frac{\log N(X, r)}{\log(1/r)}.$$

The *packing dimension* of a set $X \subset \mathbb{R}^n$ can be defined as the σ -stable modification of upper box dimension:

$$\dim_P X = \inf \left\{ \sup_i \overline{\dim}_B X_i : X = \cup_{i=1}^{\infty} X_i, X_i \text{ is bounded} \right\}. \quad (2.1)$$

For more on these dimensions and for the concepts of the *Hausdorff dimension* \dim_H and *s-dimensional Hausdorff measure* \mathcal{H}^s see e.g. the books [8] and [23].

A metric space (X, d) is called *ultrametric* if the triangle inequality is replaced with the stronger inequality

$$d(x, y) \leq \max\{d(x, z), d(y, z)\} \quad \text{for all } x, y, z \in X.$$

This is equivalent to the property that if $a \leq b \leq c$ are sides of a triangle in X , then $b = c$. The following useful fact follows easily from the definition.

Fact 2.1. *Let (X, d) be an ultrametric space. For all $x, y \in X$ and $r > 0$ either $\overline{B}(x, r) \cap \overline{B}(y, r) = \emptyset$ or $\overline{B}(x, r) = \overline{B}(y, r)$.*

Suppose that $K \subset \mathbb{R}^n$ is a self-similar set with contracting similarity maps $\{f_i\}_{1 \leq i \leq m}$, that is, $K = \cup_{i=1}^m f_i(K)$. We say that K satisfies the *strong separation condition* (SSC) if

$f_i(K) \cap f_j(K) = \emptyset$ for all $1 \leq i < j \leq m$. We say that K is *homogeneous* if the similarity ratios of all the similarity maps f_i are the same.

Let (X, d_X) and (Y, d_Y) be metric spaces and $f: X \rightarrow Y$. We say that f is α -Hölder if there exists a finite constant C such that for all $x, z \in X$ we have

$$d_Y(f(x), f(z)) \leq C d_X(x, z)^\alpha.$$

If we can take $C = 1$ in the above equation, then f is called α -1-Hölder. The 1-Hölder functions are also called *Lipschitz* functions, and the 1-1-Hölder functions are called *Lipschitz-1* functions.

The metric spaces X and Y are said to be *bilipschitz equivalent* if there exists a bijection $f: X \rightarrow Y$ such that both f and its inverse are Lipschitz.

A subset A of a separable complete metric space X is called *analytic* if it can be obtained as a continuous image of a complete separable metric space Y . It is well known that all Borel sets are analytic, see e.g. [15].

For completeness we provide a proof for the following useful fact.

Lemma 2.2. *Let $K \subset \mathbb{R}^n$ be a self-similar set obtained from the similarity maps f_1, \dots, f_m with similarity ratios r_1, \dots, r_m , respectively. If K satisfies the strong separation condition then it is bilipschitz equivalent to an ultrametric space (Y, d) , where (Y, d) depends only on the similarity ratios r_1, \dots, r_m .*

Proof. Let $Y = \{1, \dots, m\}^{\mathbb{N}}$ with the metric $d((i_1, i_2, \dots), (j_1, j_2, \dots)) = r_{i_1} \cdot \dots \cdot r_{i_k}$ whenever $i_1 = j_1, \dots, i_k = j_k$ but $i_{k+1} \neq j_{k+1}$. It is easy to check that (Y, d) is an ultrametric space and it clearly depends only on r_1, \dots, r_m . It is also easy to check that the function $f((i_1, i_2, \dots)) = \bigcap_{k=1}^{\infty} f_{i_1} \circ \dots \circ f_{i_k}(K)$ is well defined and gives a bilipschitz equivalence between (Y, d) and K . \square

3. Hölder images of compact subsets of the real line

The goal of this section is to prove Theorems 1.12 and 1.14. First we need two lemmas. The proof of the first one is a standard compactness argument.

Lemma 3.1. *Let (X, d) be a compact metric space and $\alpha, \ell > 0$. Then there exist a compact set $D \subset [0, \ell]$ and an α -1-Hölder onto map $g: D \rightarrow X$ if and only if for any finite subset $V \subset X$ there exists a finite set $W \subset [0, \ell]$ and an α -1-Hölder onto map $h: W \rightarrow V$.*

Proof. One implication is clear, we prove the other one. Let $B = \{b_1, b_2, \dots\}$ be a dense subset of X . By the assumption of the lemma for every n there exist $W_n = \{t_{n,1}, \dots, t_{n,n}\} \subset [0, \ell]$ and an α -1-Hölder map $h_n: W_n \rightarrow X$ such that $h_n(t_{n,i}) = b_i$ for every $i = 1, \dots, n$. By taking convergent subsequences, we can get a nested sequence of infinite index sets $\mathbb{N} \supset I_1 \supset I_2 \supset \dots$ such that for every k the subsequence $(t_{n,k})_{n \in I_k}$ converges. Let a_k be the limit and let $g(a_k) = b_k$.

We claim that g is α -1-Hölder on $A = \{a_1, a_2, \dots\}$. As h_n is α -1-Hölder, we have $d(b_i, b_j) \leq |t_{n,i} - t_{n,j}|^\alpha$ for all n . Fix $i < j$. Since $I_j \subset I_i$ we get that $(t_{n,i})_{n \in I_j} \rightarrow a_i$ and $(t_{n,j})_{n \in I_j} \rightarrow a_j$. Therefore, $d(g(a_i), g(a_j)) = d(b_i, b_j) \leq |a_i - a_j|^\alpha$, so g is indeed α -1-Hölder on A .

Let D be the closure of $\{a_1, a_2, \dots\}$. Then $D \subset [0, \ell]$ is compact and g clearly extends to D as an α -1-Hölder function. Since the compact set $g(D)$ contains the dense set B , we have $g(D) = X$, that is, $g: D \rightarrow X$ is onto. \square

Lemma 3.2. *For any metric space (X, d) , $1 \leq i < j$, $x_1, \dots, x_j \in X$ and $s > 0$ we have the following inequalities (recall Definition 1.11):*

$$Z^s(x_1, \dots, x_i) + Z^s(x_i, \dots, x_j) \leq Z^s(x_1, \dots, x_j), \quad (3.1)$$

$$Z^s(x_1, \dots, x_j) - Z^s(x_1, \dots, x_i) \geq Z^s(x_i, \dots, x_j) \geq (d(x_i, x_j))^s. \quad (3.2)$$

Proof. The inequality (3.1) and the second part of (3.2) follows from the definition, the first part of (3.2) clearly follows from (3.1). \square

Now we can prove Theorem 1.12. In fact, we prove a bit more.

Theorem 3.3. *Let X be a compact metric space and $s > 0$. Then there exist a compact set $D \subset [0, 1]$ and a $1/s$ -Hölder onto map $g: D \rightarrow X$ if and only if $\delta^s(X) < \infty$.*

Moreover, if $\delta^s(X) < \infty$, then

$$\delta^s(X) = \min\{\ell > 0 : \exists D \subset [0, \ell] \text{ compact and } g: D \rightarrow X \text{ } (1/s)\text{-Hölder onto}\}.$$

Proof. First we prove that the existence of a compact set $D \subset [0, \ell]$ and a $(1/s)$ -1-Hölder onto map $g: D \rightarrow X$ implies that $\delta^s(X) \leq \ell$. Let $x_1, \dots, x_n \in X$ be arbitrary and pick $t_i \in g^{-1}(\{x_i\})$. There exists a permutation $\pi \in \text{Sym}(n)$ such that $t_{\pi(1)} \leq \dots \leq t_{\pi(n)}$. Since g is $(1/s)$ -1-Hölder, we obtain

$$d(x_{\pi(i)}, x_{\pi(i')}) \leq |t_{\pi(i)} - t_{\pi(i')}|^{1/s}$$

for any i, i' . Thus for any $1 = i_1 < \dots < i_k = n$ we have

$$\sum_{j=1}^{k-1} (d(x_{\pi(i_j)}, x_{\pi(i_{j+1})}))^s \leq \sum_{j=1}^{k-1} |t_{\pi(i_j)} - t_{\pi(i_{j+1})}| \leq \ell.$$

Therefore $Z^s(x_{\pi(1)}, \dots, x_{\pi(n)}) \leq \ell$, which implies that $\delta^s(X) \leq \ell$.

Finally, we show that if $\delta^s(X) < \infty$ then there exist a compact set $D \subset [0, \delta^s(X)]$ and a $(1/s)$ -1-Hölder onto map $g: D \rightarrow X$. Let $V = \{x_1, \dots, x_n\} \subset X$ be an arbitrary finite subset of X enumerated such that $Z^s(x_1, \dots, x_n) \leq \delta^s(X)$. For each $i = 1, \dots, n$ define $a_i = Z^s(x_1, \dots, x_i)$, $h(a_i) = x_i$ and $W = \{a_1, \dots, a_n\}$. Clearly, $W \subset [0, \delta^s(X)]$ is finite,

and by (3.2) of Lemma 3.2 we obtain that $h: W \rightarrow V$ is a $(1/s)$ -1-Hölder onto map. Since $V \subset X$ was an arbitrary finite subset, by Lemma 3.1 the proof is complete. \square

The following theorem clearly contains Theorem 1.14. For the notation $N(X, r)$ recall Section 2.

Theorem 3.4. *For any compact metric space (X, d) the following statements hold:*

$$(1) \delta^s(X) \leq \left(\frac{2 \operatorname{diam} X}{u(1-u)} \right)^s \sum_{n=1}^{\infty} N(X, u^n) \cdot u^{ns} \quad \text{for all } s > 0 \text{ and } 0 < u < 1;$$

$$(2) \text{ If } s > \overline{\dim}_B X \text{ then } \delta^s(X) < \infty;$$

$$(3) \overline{\dim}_B X = \inf\{s > 0: \delta^s(X) < \infty\};$$

$$(4) \overline{\dim}_B X = \inf\{s > 0: \exists D \subset [0, 1] \text{ and } 1/s\text{-Hölder onto } g: D \rightarrow X\}.$$

Proof. It is easy to see that, by the definition of the upper box dimension, (1) implies (2). It is clear that (2) implies ‘ \geq ’ in (3). The other inequality in (3) follows from the observation that $\overline{\dim}_B X > s$ implies that for any $\varepsilon > 0$ there exists an ε -net of large cardinality in X , which clearly implies that $\delta^s(X)$ is large. By Theorem 3.3, statements (3) and (4) are clearly equivalent. Therefore it remains to prove (1).

By rescaling X , we can suppose that $\operatorname{diam} X = 1$. Let $a_n = N(X, u^n)$ and let H_n be the set of centers of a collection of a_n balls of radius u^n that covers X . Since $\operatorname{diam} X = 1$, we have $a_0 = 1$.

We define an infinite tree such that for each integer $n \geq 0$ the vertices of the n th level are the points of H_n , and let us endow H_n with an arbitrary ordering $(H_n, <_n)$. Since $\bigcup_{h \in H_n} \overline{B}(h, u^n) = X$, for every $h' \in H_{n+1}$ there exists an $h \in H_n$ with $d(h, h') \leq u^n$. Join h' by an edge to exactly one such $h \in H_n$. Let T be the tree we obtained. Consider the infinite branches of T . By the compactness of X and since $u < 1$ the vertices of every infinite branch $(h_n)_{n \geq 0}$ of T converge to a point of X .

We claim that the converse also holds: for any $x \in X$ there exists an infinite branch $(h_n(x))_{n \geq 0}$ of T that converges to x . Indeed, let $W_x = \bigcup_{n=0}^{\infty} \{h \in H_n: x \in \overline{B}(h, u^n)\}$ and let V_x consists of W_x and all the ancestors of all points of W_x . Then the points of V_x form a subtree of T with arbitrarily long branches. Since every degree of T is finite, König’s lemma implies that the subtree V_x has an infinite branch $(h_n(x))_{n \geq 0}$. Let y be the limit of $h_n(x)$. Since by definition $h_n(x) \in W_x$ for infinitely many n , this implies that $y = x$, which completes the proof of the claim.

Using the orderings $(H_n, <_n)$, the lexicographic ordering gives an order \prec on X : let $x \prec y$ if for some $n \geq 1$ we have $h_0(x) = h_0(y), \dots, h_{n-1}(x) = h_{n-1}(y)$ and $h_n(x) <_n h_n(y)$. Let $y_1, \dots, y_k \in X$ be arbitrarily fixed. To prove (1), and so also the theorem, it is enough to show that

$$y_1 \prec \dots \prec y_k \implies \sum_{j=1}^{k-1} (d(y_j, y_{j+1}))^s \leq \left(\frac{2}{u(1-u)} \right)^s \sum_{n=1}^{\infty} a_n \cdot u^{ns}. \quad (3.3)$$

Assume $y_1 \prec \dots \prec y_k$. We partition $\{y_1, \dots, y_{k-1}\}$ as follows: for all $n \geq 1$ define

$$Y_n = \{y_j : j \leq k-1, h_i(y_j) = h_i(y_{j+1}) \text{ for all } 0 \leq i < n \text{ and } h_n(y_j) < h_n(y_{j+1})\}.$$

Note that $h_n : (Y_n, \prec) \rightarrow (H_n, <_n)$ is a strictly increasing map, so $|Y_n| \leq |H_n| = a_n$.

For each $x \in X$ we have $d(h_i(x), h_{i+1}(x)) \leq u^i$ and $h_i(x) \rightarrow x$ as $i \rightarrow \infty$, so for all $n \geq 1$ we obtain

$$d(h_{n-1}(x), x) \leq \sum_{i=n-1}^{\infty} u^i = \frac{u^{n-1}}{1-u}.$$

Suppose that $y_j \in Y_n$. Then $h_{n-1}(y_j) = h_{n-1}(y_{j+1})$ and we obtain

$$d(y_j, y_{j+1}) \leq d(y_j, h_{n-1}(y_j)) + d(y_{j+1}, h_{n-1}(y_{j+1})) \leq \frac{2u^{n-1}}{1-u}.$$

Therefore

$$\begin{aligned} \sum_{j=1}^{k-1} (d(y_j, y_{j+1}))^s &= \sum_{n=1}^{\infty} \sum_{y_j \in Y_n} (d(y_j, y_{j+1}))^s \\ &\leq \sum_{n=1}^{\infty} |Y_n| \cdot \left(\frac{2u^{n-1}}{1-u} \right)^s \\ &= \left(\frac{2}{u(1-u)} \right)^s \sum_{n=1}^{\infty} a_n u^{ns}, \end{aligned}$$

so (3.3) holds. This completes the proof of the theorem. \square

Remark 3.5. For $X \subset \mathbb{R}$ it is easier to calculate $\delta^s(X)$ since in this case in Definition 1.11 it is clear which permutation $\pi \in \text{Sym}(n)$ gives the minimum. For example, consider the middle third Cantor set C and let $s = \log 2 / \log 3$, then one can easily see that $\delta^s(C) = \infty$. Thus, by Theorem 3.3, C cannot be obtained as a $1/s$ -Hölder image of a compact subset of \mathbb{R} . On the other hand, by Theorem 3.4 and Corollary 1.15, for any $t > \overline{\dim}_{\text{B}} C = s$ we have $\delta^t(C) < \infty$ and C can be obtained as a $1/t$ -Hölder image of a compact subset of \mathbb{R} .

4. Hölder images of compact metric spaces, Lipschitz images of the Cantor set

The goal of this section is to prove Theorem 1.5 (for Hölder maps) and some related results. We combine results from Section 3 with arguments and results from [17]. First we prove the following statement, which is in fact a stronger version of Corollary 1.7.

Proposition 4.1. *Let A be a compact ultrametric space with positive t -dimensional Hausdorff measure and B be a compact metric space with $\delta^s(B) < \infty$.*

- (1) *There exists a compact set $A' \subset A$ and a t/s -Hölder onto map $f: A' \rightarrow B$.*
 (2) *If $t = s$ then A can be mapped onto B by a Lipschitz map.*

Proof. (1): By [17, Theorem 2.1, Lemma 2.3] any compact ultrametric space with positive t -dimensional Hausdorff measure can be mapped onto $[0, 1]$ by a t -Hölder function. Hence there exists a t -Hölder onto map $h: A \rightarrow [0, 1]$. By Theorem 1.12, there exist a compact $D \subset [0, 1]$ and a $1/s$ -Hölder onto map $g: D \rightarrow B$. Let $A' = h^{-1}(D)$. Then $A' \subset A$ is clearly compact and $g \circ h: A' \rightarrow B$ is a t/s -Hölder onto map.

(2): By (1) there exists an $A' \subset A$ and a Lipschitz onto map $f: A' \rightarrow B$. Lemma 1.6 implies that f can be extended to A as a Lipschitz map. \square

Remark 4.2. Recall that the middle third Cantor set C is bilipschitz equivalent to an ultrametric space. Let $s = \log 2 / \log 3$. Proposition 4.1(2) gives a sufficient condition for Question 1.1(b): a compact metric space B can be covered by a Lipschitz image of C if $\delta^s(B) < \infty$. On the other hand, as we saw in Remark 3.5, $\delta^s(C) = \infty$, so the $B = C$ example shows that $\delta^s(B) < \infty$ is not a necessary condition for Question 1.1 (b).

Theorem 1.5 is clearly the $\alpha = 1$ special case of the following theorem.

Theorem 4.3. *Let A and B be compact metric spaces such that $\dim_H A > \alpha \overline{\dim}_B B$ for some $\alpha > 0$. Then there exists a compact set $A' \subset A$ and an α -Hölder onto map $f: A' \rightarrow B$.*

Proof. Let $t \in (\alpha \overline{\dim}_B B, \dim_H A)$. By a deep theorem of Mendel and Naor [25], $\dim_H A > t$ implies that there exists a compact set $A' \subset A$ such that $\dim_H A' > t$ and A' is bilipschitz equivalent to an ultrametric space. Since $\overline{\dim}_B B < t/\alpha$, Theorem 1.14 implies that $\delta^{t/\alpha}(B) < \infty$. Then Proposition 4.1 (1) implies that there is an α -Hölder onto map $f: A' \rightarrow B$. \square

Remark 4.4. By Howroyd's theorem [13], if A is an analytic subset of a separable complete metric space X with $\dim_H A > s$ for some s then there exists a compact set $A' \subset A$ with $\dim_H A' > s$. Thus in Theorem 4.3 (and so also in Theorem 1.5) A does not have to be compact, it is enough to assume that A is an analytic subset of a separable complete metric space.

The following result gives partial answer to Question 1.1 (b). We state the essentially trivial necessary condition (2) just to show that the obtained necessary and sufficient conditions are not very far from each other. Note that the sufficient condition in (1) is weaker than the condition $\overline{\dim}_B B < \log 2 / \log 3$ in Corollary 1.7: it also allows some compact metric spaces with $\overline{\dim}_B B = \log 2 / \log 3$.

Corollary 4.5. *Let C be the middle third Cantor set and let B be an arbitrary compact metric space. Let $b_n = N(B, 3^{-n})$.*

- (1) *If $\sum_{k=1}^{\infty} b_n/2^n$ converges then C can be mapped onto B by a Lipschitz map.*
 (2) *If C can be mapped onto B by a Lipschitz map then the sequence $b_n/2^n$ must be bounded.*

Proof. Let $s = \log 2 / \log 3$. Applying Theorem 3.4 (1) for $u = 1/3$ implies that $\delta^s(B) < \infty$. Since C is bilipschitz equivalent to an ultrametric space and its s -dimensional Hausdorff measure is positive, Proposition 4.1(2) completes the proof of (1). The necessary condition (2) easily follows from $N(C, 3^{-n}) \leq 2^n$. \square

Remark 4.6. Neither the condition of (1), nor the condition of (2) can be necessary and sufficient. If $B = C$ then $\sum b_n/2^n$ clearly diverges but the identity map is a trivial Lipschitz onto map. On the other hand, we show a compact set $B \subset \mathbb{R}^2$ such that $b_n/2^n$ is bounded but for $s = \log_3 2$ the s -dimensional packing measure of B is infinite. Since C clearly has finite s -dimensional packing measure, and the packing measure of a Lipschitz image is at most a finite multiple of the packing measure of the original set, this indeed implies that there is no Lipschitz onto map $f: C \rightarrow B$. Consider a Bedford–McMullen carpet $B \subset \mathbb{R}^2$ (see [5] or [24]) with $m = 3^2$ rows and $n = 3^4$ columns, such that the pattern $D \subset \{0, \dots, n-1\} \times \{0, \dots, m-1\}$ has 8 elements and 3 are in one row and 5 are in another one. Then the projection to the second coordinate, $\pi(D)$, has 2 elements. Then by [24] we have the formula

$$\overline{\dim}_B B = \log_m |\pi(D)| + \log_n \frac{|D|}{|\pi(D)|} = \log_3 2,$$

and an easy calculation also yields that $b_n/2^n$ is bounded. On the other hand, as the 2 non-empty rows of D have different cardinalities, the s -dimensional packing measure of B is infinite by [28, Theorem 1.1].

5. Extensions, proof of Lemma 1.6

In this section we prove Lemma 1.6. A usual way to prove an extension result like Lemma 1.6 is to prove a statement about retracts. In this case we need the following.

Lemma 5.1. *Let (X, d) be a compact ultrametric space and let $A \subset X$ be a compact subset. Then there exists a Lipschitz-1 retraction $g: X \rightarrow A$, that is, a function such that $g(x) = x$ for any $x \in A$.*

Proof. By a *sphere* we mean a set of the form $S(x, r) = \{y \in X : d(x, y) = r\}$, where $x \in X$ and $r > 0$. From each sphere S such that $S \cap A \neq \emptyset$ we choose a point $p(S) \in S \cap A$. For $x \in A$ let $g(x) = x$, for $x \in X \setminus A$ let $g(x) = p(S(x, \text{dist}(x, A)))$.

It remains to check that g is Lipschitz-1. Let $x, y \in X$ be given, we need to prove that $d(g(x), g(y)) \leq d(x, y)$. This is clear if $x, y \in A$.

Now assume $x \in A$, $y \notin A$. By definition $d(y, g(y)) = \text{dist}(y, A) \leq d(y, x)$. Hence the ultrametric property for the triangle with vertices $x, y, g(y)$ yields $d(g(y), g(x)) = d(g(y), x) \leq d(y, x)$.

Finally, suppose $x, y \notin A$. Assume to the contrary that $d(x, y) < d(g(x), g(y))$. The definition of g implies that

$$d(x, g(x)) = \text{dist}(x, A) \leq d(x, g(y)) \quad \text{and} \quad d(y, g(y)) = \text{dist}(y, A) \leq d(y, g(x)). \quad (5.1)$$

Applying (5.1) and the ultrametric property for the triangles with vertices $x, g(x), g(y)$, and $y, g(y), g(x)$, respectively, we obtain

$$d(g(x), g(y)) \leq d(x, g(y)) \quad \text{and} \quad d(g(x), g(y)) \leq d(y, g(x)). \quad (5.2)$$

Then $d(x, y) < d(g(x), g(y))$ and inequalities (5.2) imply $d(x, y) < d(x, g(y))$ and $d(x, y) < d(y, g(x))$. Hence the ultrametric property for the triangles with vertices $x, y, g(y)$, and $x, y, g(x)$, respectively yields

$$d(y, g(y)) = d(x, g(y)) \quad \text{and} \quad d(x, g(x)) = d(y, g(x)). \quad (5.3)$$

Then inequalities (5.1) and (5.3) imply

$$d(x, g(x)) \leq d(x, g(y)) = d(y, g(y)) \leq d(y, g(x)) = d(x, g(x)),$$

so

$$d(x, g(x)) = d(x, g(y)) = d(y, g(y)) = d(y, g(x)). \quad (5.4)$$

Let $r = \text{dist}(x, A) = d(x, g(x))$, then (5.4) yields $r = d(y, g(y)) = \text{dist}(y, A)$. Then (5.2) and (5.4) imply $d(x, y) < d(g(x), g(y)) \leq d(x, g(y)) = r$. As $d(x, y) < r$, Fact 2.1 yields $S(x, r) = S(y, r)$. Hence by the definition of g we have $g(x) = g(y)$, so $d(x, y) < d(g(x), g(y)) = 0$, which is a contradiction. The proof is complete. \square

Proof of Lemma 1.6. Since Y is complete one can easily extend f to the closure of A as a Lipschitz function with the same Lipschitz constant, hence we can assume that A is closed and so also compact. Let $g: X \rightarrow A$ be the Lipschitz-1 retract provided by Lemma 5.1. Then $f \circ g: X \rightarrow Y$ is clearly a Lipschitz extension of f with the same Lipschitz constant. \square

6. Dimensions - proof of Corollary 1.9

In this section we prove the following stronger form of Corollary 1.9. Note that the technical assumptions about the domain \mathcal{F} of D (see the first sentence and (v)) hold

for any of the reasonable domains like all subsets, all Borel sets, all closed sets or all compact sets, so Corollary 1.9 is indeed a special case of Theorem 6.1. Recall that a set is called F_σ if it can be written as a countable union of closed sets.

Theorem 6.1. *For a fixed positive integer n let \mathcal{F} be any family of subsets of \mathbb{R}^n that contains all compact subsets of \mathbb{R}^n . Suppose that a function $D: \mathcal{F} \rightarrow [0, n]$ has the following properties.*

- (i) *Lipschitz functions cannot increase D for compact sets, that is, for any compact $K \subset \mathbb{R}^n$ and Lipschitz function $f: K \rightarrow \mathbb{R}^n$ we have $D(f(K)) \leq D(K)$,*
- (ii) *D is monotone, that is, $A, B \in \mathcal{F}$, $A \subset B$ implies that $D(A) \leq D(B)$, and*
- (iii) *if $S \subset \mathbb{R}^n$ is a homogeneous self-similar set with the SSC then $D(S) = \dim_H S$.*

Then the following statements hold.

- a) *For every analytic $A \in \mathcal{F}$ (in particular for any Borel $A \in \mathcal{F}$) we have*

$$\dim_H A \leq D(A).$$

- b) *For any bounded $A \in \mathcal{F}$ we have*

$$D(A) \leq \overline{\dim}_B A.$$

- c) *If we also require that*

- (iv) *D is σ -stable for compact sets, that is, $D(\cup_{i=1}^\infty K_i) = \sup_i D(K_i)$ whenever $\cup_{i=1}^\infty K_i \in \mathcal{F}$ and every K_i is compact, and*
- (v) *\mathcal{F} contains all F_σ sets or $A \cap F \in \mathcal{F}$ for any $A \in \mathcal{F}$ and closed $F \subset \mathbb{R}^n$,*

then for every $A \in \mathcal{F}$ we have

$$D(A) \leq \dim_P A.$$

Proof. a) Suppose that for some analytic $A \in \mathcal{F}$ we have $\dim_H A > D(A)$. Then by Howroyd's theorem [13] there exists a compact set $K \subset A$ with $\dim_H K > D(A)$. By the monotonicity of D we have $D(K) \leq D(A)$, so we obtain that $D(K) < \dim_H K$. Let $S \subset \mathbb{R}^n$ be a homogeneous self-similar set with the SSC such that

$$D(K) < \overline{\dim}_B S < \dim_H K.$$

By Theorem 1.5 there exists a compact set $K' \subset K$ and a Lipschitz onto map $f: K' \rightarrow S$. Then, using our assumptions we obtain

$$D(S) = D(f(K')) \leq D(K') \leq D(K) < \overline{\dim}_B S = D(S),$$

which is a contradiction.

b) Suppose that for some bounded $A \in \mathcal{F}$ we have $\overline{\dim}_B A < D(A)$. Let K be the closure of A . Then K is compact and by the monotonicity of D we get $\overline{\dim}_B K = \overline{\dim}_B A < D(A) \leq D(K)$. Let $S \subset \mathbb{R}^n$ be a homogeneous self-similar set with the SSC such that

$$\overline{\dim}_B K < \dim_H S < D(K).$$

By Corollary 1.7 there exists a Lipschitz onto map $f: S \rightarrow K$, and our assumptions imply

$$D(S) = \dim_H S < D(K) = D(f(S)) \leq D(S),$$

which is a contradiction.

c) Let $A \in \mathcal{F}$. Using (2.1) and the fact that taking closure does not change the upper box dimension we get that

$$\dim_P A = \inf \left\{ \sup_i \overline{\dim}_B \overline{A_i} : A = \cup_{i=1}^{\infty} A_i, A_i \text{ is bounded} \right\}.$$

So let $A = \cup_{i=1}^{\infty} A_i$ such that each A_i is bounded. Since each $\overline{A_i}$ is compact, part (b) implies $\sup_i \overline{\dim}_B \overline{A_i} \geq \sup_i D(\overline{A_i})$. Therefore, to complete the proof it is enough to show that $\sup_i D(\overline{A_i}) \geq D(A)$.

If \mathcal{F} contains all F_σ sets then we can apply (iv) and (ii) to obtain

$$\sup_i D(\overline{A_i}) = D(\cup_{i=1}^{\infty} \overline{A_i}) \geq D(A).$$

If $A \cap F \in \mathcal{F}$ for any $A \in \mathcal{F}$ and closed $F \subset \mathbb{R}^n$ then we apply first (ii) then (iv) for the partition $A = \cup_{i=1}^{\infty} (A \cap \overline{A_i})$ to obtain

$$\sup_i D(\overline{A_i}) \geq \sup_i D(A \cap \overline{A_i}) = D(A).$$

We considered the alternative conditions of (v), which completes the proof. \square

Remarks 6.2. (1) As one can see from the proof, we used only three facts about the family \mathcal{H} of homogeneous self-similar sets with the SSC:

- (A) every $S \in \mathcal{H}$ is compact,
- (B) $\dim_H S = \overline{\dim}_B S$ for any $S \in \mathcal{H}$ and
- (C) for any $0 \leq a < b \leq n$ there exists a set $S \in \mathcal{H}$ such that $a < \dim_H S < b$.

Thus in condition (iii) of Theorem 6.1 the class of self-similar sets with the SSC can be replaced by any other family \mathcal{H} of compact subsets of \mathbb{R}^n with the above properties. For example, we could also use sets of the form $A \times \dots \times A$, where A is a middle- c Cantor set for some rational c .

(2) For a fixed \mathcal{F} let

$$D_1(A) = \sup\{\dim_{\mathbb{H}} K : K \subset A \text{ compact}\} \quad (A \in \mathcal{F}).$$

Since part (a) of Theorem 6.1 can be applied to any compact subset of \mathbb{R}^n , using also the monotonicity condition (ii), we obtain that (a) can be replaced by the following, more general statement:

a') For any $A \in \mathcal{F}$ we have $D_1(A) \leq D(A)$.

Since the Hausdorff dimension clearly satisfies (i), (ii) and (iii), by (a) of Theorem 6.1 it is the smallest function on \mathcal{F} that satisfies (i), (ii) and (iii), provided \mathcal{F} contains only analytic sets. Note that D_1 also satisfies conditions (i), (ii) and (iii), so on a general \mathcal{F} this is the smallest function that satisfies (i), (ii) and (iii).

Recall that a set $B \subset \mathbb{R}^n$ is called a Bernstein set if neither B , nor $\mathbb{R}^n \setminus B$ contains any uncountable compact subset. It is well known that such sets exist, see e.g. [15]. Then clearly $D_1(B) = D_1(\mathbb{R}^n \setminus B) = 0$ but B or $\mathbb{R}^n \setminus B$ has Hausdorff dimension n . This shows that the condition that A is analytic cannot be removed from Theorem 6.1 (a).

(3) The following example shows that in Theorem 6.1 one cannot remove assumption (v) about the domain \mathcal{F} of D , not even if we require σ -stability for all sets of \mathcal{F} , that is, if we replace (iv) by the following stronger assumption:

(iv') $D(\cup_{i=1}^{\infty} A_i) = \sup_i D(A_i)$ whenever $\cup_{i=1}^{\infty} A_i \in \mathcal{F}$ and $A_i \in \mathcal{F}$ for all i .

Since there are only continuum many F_{σ} sets and every set of cardinality continuum has more than continuum many subsets, there exists an unbounded non- F_{σ} set $E \subset \mathbb{R}^n$ such that $\dim_{\mathbb{P}} E < n$. Let \mathcal{F} consist of E and all compact subsets of \mathbb{R}^n and let D_2 be the packing dimension for the compact sets and let $D_2(E) = n$. This \mathcal{F} and D_2 have all the required properties but (v), still $D_2(A) \leq \dim_{\mathbb{P}}(A)$ does not hold for $A = E$.

On the other hand, we claim that if we replace (iv) by the even stronger assumption that

(iv'') $D(A) \leq \sup_i D(K_i)$ whenever $A \subset \cup_{i=1}^{\infty} K_i$, $A \in \mathcal{F}$ and every K_i is compact

then assumption (v) about \mathcal{F} can be dropped. Indeed, in this case in the proof of (c) the needed inequality $\sup_i D(\overline{A_i}) \geq D(A)$ directly follows from (iv'').

7. Lipschitz functions between self-similar sets

In this section we prove Theorem 1.17. We need some preparation. First we prove the following algebraic lemma, which might be known but for completeness we present a proof.

Lemma 7.1. *Let $q > 1$ be an integer, which is not of the form n^k for any integers $n, k > 1$ and let r_1, \dots, r_m be rational numbers. Then*

$$q^{r_1} + \dots + q^{r_m} = 1 \quad (7.1)$$

implies that every r_i is an integer.

Proof. Assume to the contrary that (7.1) holds and not every r_i is an integer. Let n be the smallest common divisor of r_1, \dots, r_m . Then each r_i can be written as $r_i = k_i + \frac{p_i}{n}$, where $k_i \in \mathbb{Z}$ and $p_i \in \{0, 1, \dots, n-1\}$. Define the algebraic number $z = q^{1/n}$ and the rational polynomial

$$R(x) = -1 + \sum_{i=1}^m q^{k_i} x^{p_i}.$$

We obtain that $R(z) = 0$ by (7.1), and clearly $\deg(R) < n$. Since not every r_i is an integer, $\max_i p_i > 0$, so $\deg(R) > 0$. Hence the minimal polynomial of z over \mathbb{Q} has degree less than n .

Finally, we prove that $P(x) = x^n - q$ is the minimal polynomial of z over \mathbb{Q} , which will be a contradiction. Clearly $P(z) = 0$. By [20, Theorem 9.1, Chapter 6] the polynomial $Q(x) = x^u - v$ for positive integers u, v is irreducible over \mathbb{Q} if and only if for every prime $p|u$ we have $v^{1/p} \notin \mathbb{Q}$, which is equivalent to $v^{1/p} \notin \mathbb{N}^+$. As $q > 0$ is not a perfect power, this implies that P is irreducible over \mathbb{Q} . Therefore, P is indeed the minimal polynomial of z , and the proof is complete. \square

Notation 7.2. For metric spaces (or subsets of metric spaces) A and B we denote by $F(A, B)$ the minimal number of Lipschitz-1 images of A that can cover B , that is,

$$F(A, B) = \min \left\{ k : \exists f_1, \dots, f_k : A \rightarrow B \text{ Lipschitz-1 s.t. } \cup_{i=1}^k f_i(A) = B \right\}.$$

For a metric space A and an $r > 0$ we denote by rA the scaled copy of A in which every distance is the r -multiple of the corresponding distance in A .

Fact 7.3. *For any metric spaces A and B and $r > 0$ we have $F(rA, rB) = F(A, B)$.*

Lemma 7.4. *Let A and B be metric spaces and $r > 0$. Suppose that $A = \cup_{i=1}^N A_i$ such that every A_i is isometric to rA . Then*

$$F(A, B) \geq \left\lceil \frac{F(rA, B)}{N} \right\rceil.$$

Furthermore, if $\text{dist}(A_i, A_j) \geq \text{diam } B$ for every $i \neq j$, then

$$F(A, B) = \left\lceil \frac{F(rA, B)}{N} \right\rceil.$$

Proof. Let $m = F(rA, B)$. If B can be covered by less than m/N Lipschitz-1 images of A then we get a cover of B by less than m Lipschitz-1 images of rA , which would contradict $m = F(rA, B)$. Therefore, $F(A, B) \geq \left\lceil \frac{F(rA, B)}{N} \right\rceil$.

If for every $i \neq j$ we have $\text{dist}(A_i, A_j) \geq \text{diam } B$, then from any N Lipschitz-1 onto maps $f_i: rA \rightarrow B$ we can get a Lipschitz-1 onto map $f: A \rightarrow B$. Thus $F(A, B) \leq \lceil m/N \rceil = \left\lceil \frac{F(rA, B)}{N} \right\rceil$. \square

Lemma 7.5. Let A and B be metric spaces. Suppose that $B = \cup_{j=1}^n B_j$ and for every $i \neq j$ we have $\text{dist}(B_i, B_j) > \text{diam } A$. Then

$$F(A, B) = \sum_{j=1}^n F(A, B_j).$$

Proof. Since $\text{dist}(B_i, B_j) > \text{diam } A$, no Lipschitz-1 image of A can intersect more than one B_i . This implies the claim. \square

Recall that a real valued function f defined on a metric space X is called *lower semicontinuous* if whenever $x, x_1, x_2, \dots \in X$, $f(x_n) \leq y$ for every n and $x_n \rightarrow x$ then $f(x) \leq y$. It is well known that such a function has a minimum on any compact set.

Lemma 7.6. For any compact metric spaces A and B , the function $f(x) = F(xA, B)$ is lower semicontinuous on $(0, \infty)$.

Proof. By definition, it is enough to prove that if $z_n \rightarrow z$ and $F(z_n A, B) \leq k$ for all n , then $F(zA, B) \leq k$. Let $f_{i,n}: z_n A \rightarrow B$ be Lipschitz-1 functions such that $\cup_{i=1}^k f_{i,n}(A) = B$ for all n . For each n let $h_n: zA \rightarrow z_n A$ be the natural bijection between zA and $z_n A$. As the sequences $\{f_{i,n} \circ h_n\}_{n \geq 1}$ are equicontinuous for all $1 \leq i \leq k$, using the Arzelà-Ascoli theorem k -times and passing to a subsequence we may assume that there are functions $f_i: zA \rightarrow B$ such that $f_{i,n} \circ h_n \rightarrow f_i$ uniformly for all $1 \leq i \leq k$. Then clearly every f_i is a Lipschitz-1 function. Finally, it is enough to see that $\cup_{i=1}^k f_i(zA) = B$, which easily follows from the uniform convergence and that $\cup_{i=1}^k (f_{i,n} \circ h_n)(zA) = B$ for all n . \square

Now we are ready to prove Theorem 1.17. We prove it in the following stronger form.

Theorem 7.7. *Let A be a homogeneous self-similar set with the strong separation condition, that is, A is the disjoint union of q similar r -scaled copies of A for some $r > 0$ real and $q > 1$ integer. Let k be the maximal positive integer such that $\sqrt[k]{q}$ is integer. Let B be any self-similar set with the SSC such that $\dim_{\text{H}} A = \dim_{\text{H}} B$. Then the following statements are equivalent:*

- (i) A can be mapped onto B by a Lipschitz map;
- (ii) All similarity ratios of B are positive integer powers of $\sqrt[k]{r}$;
- (iii) A and B are bilipschitz equivalent.

Proof. If $q = n^k$ for some integers $n, k > 1$ then it is easy to see that A is bilipschitz equivalent to a homogeneous self-similar set $A' = \sqcup_{i=1}^n f_i(A')$ with the SSC such that each f_i has similarity ratio $\sqrt[k]{r}$. Therefore we can suppose that $k = 1$.

Let $s = \dim_{\text{H}} A = \dim_{\text{H}} B$. Then $qr^s = 1$. Let the similarity ratios of B be β_1, \dots, β_m and so $\beta_1^s + \dots + \beta_m^s = 1$. For every i let $\alpha_i = -\log_q \beta_i^s > 0$. Hence $\beta_i^s = q^{-\alpha_i} = (r^{\alpha_i})^s$, so $\beta_i = r^{\alpha_i}$ and

$$q^{-\alpha_1} + \dots + q^{-\alpha_m} = 1. \quad (7.2)$$

First we prove that (ii) implies (iii). Since $k = 1$, (ii) implies that $\alpha_1, \dots, \alpha_m$ are positive integers. We claim that in this case (7.2) implies that A can be also written as a self-similar set using similarity ratios $r^{\alpha_1}, \dots, r^{\alpha_m}$. Indeed, we can clearly suppose that $\alpha_1 \leq \dots \leq \alpha_m$. Let $A_1 \subset A$ be an r^{α_1} -scaled copy of A , let $A_2 \subset A \setminus A_1$ be an r^{α_2} -scaled copy of A , and so on. One can easily check that $\alpha_1 \leq \dots \leq \alpha_m$ and (7.2) imply that we never get stuck and finally we obtain a partition $A = A_1 \cup \dots \cup A_m$ such that each A_i is an r^{α_i} -scaled copy of A , which completes the proof of the claim. Since A and B are self-similar sets with the SSC and they can be obtained using the same similarity ratios, Lemma 2.2 implies that A and B are bilipschitz equivalent.

Since (iii) clearly implies (i), it remains to prove that (i) implies (ii). So we suppose that A can be mapped onto B by a Lipschitz map and we need to show that every α_i is a positive integer. Using (7.2) and that q is not of the form $q = n^k$ for any integers $n, k > 1$, by Lemma 7.1 it is enough to prove that every α_i is rational.

Let g_1, \dots, g_m denote the similarity maps of ratios β_1, \dots, β_m , respectively, such that $B = \sqcup_{j=1}^m g_j(B)$. We can clearly suppose that

$$\min_{i \neq j} \text{dist}(g_i(B), g_j(B)) > \frac{\text{diam } A}{r}. \quad (7.3)$$

Let

$$z(t) = F(r^{-\{t\}}A, B)q^{\{t\}},$$

where $\{\cdot\}$ denotes fractional part. Then $z(t)$ is clearly a 1-periodic function.

Now we claim that for any $t \in \mathbb{R}$ we have

$$z(t) \geq \sum_{j=1}^m q^{-\alpha_j} z(t + \alpha_j). \quad (7.4)$$

Indeed, noting that $r^{\lfloor t \rfloor} A$ is the union of $N = q^{\lfloor t + \alpha_j \rfloor - \lfloor t \rfloor}$ copies of $r^{\lfloor t + \alpha_j \rfloor} A$, we can apply Lemma 7.4 and Fact 7.3 to get

$$\begin{aligned} F(r^{\lfloor t \rfloor} A, r^t g_j(B)) q^{\{t\}} &\geq q^{\lfloor t \rfloor - \lfloor t + \alpha_j \rfloor} F(r^{\lfloor t + \alpha_j \rfloor} A, r^t g_j(B)) q^{\{t\}} \\ &= F(r^{-\{t + \alpha_j\}} A, B) q^{\{t + \alpha_j\} - \alpha_j} \\ &= q^{-\alpha_j} z(t + \alpha_j). \end{aligned} \quad (7.5)$$

By (7.3) we can apply Lemma 7.5 to $r^{\lfloor t \rfloor} A$ and $r^t B = \cup_{j=1}^m r^t g_j(B)$. Using Fact 7.3, Lemma 7.5 and (7.5) we obtain

$$\begin{aligned} z(t) &= F(r^{-\{t\}} A, B) q^{\{t\}} = F(r^{\lfloor t \rfloor} A, r^t B) q^{\{t\}} \\ &= \sum_{j=1}^m F(r^{\lfloor t \rfloor} A, r^t g_j(B)) q^{\{t\}} \geq \sum_{j=1}^m q^{-\alpha_j} z(t + \alpha_j), \end{aligned}$$

which completes the proof of (7.4).

Choose $\delta \in (0, 1)$ such that for every $j \in \{1, \dots, m\}$ we have

$$\alpha_j \notin \mathbb{Z} \implies \text{dist}(\alpha_j, \mathbb{Z}) > \delta. \quad (7.6)$$

By Lemma 7.6, z is lower semicontinuous on $[0, 1)$. This implies that z has a minimum w on $[0, 1 - \delta]$ obtained at a point $u \in [0, 1 - \delta]$.

First we consider the case when w is the minimum of z on \mathbb{R} . Then (7.4), the minimality of w , and (7.2) imply that

$$w = z(u) \geq \sum_{j=1}^m q^{-\alpha_j} z(u + \alpha_j) \geq \sum_{j=1}^m q^{-\alpha_j} w = w,$$

which implies that $z(u + \alpha_j) = w$ for every j . If not every α_j is rational then, since z is 1-periodic, we obtain that z is constant on a dense set. But this is impossible, since $z(t)$ is the product of a nonzero integer valued function and $q^{\{t\}}$.

It remains to consider the case when w is not the minimum of z on \mathbb{R} . Since w is the minimum of the 1-periodic z on $[0, 1 - \delta]$, this implies that there exists a $v \in (1 - \delta, 1)$ such that $z(v) < w$. Then (7.6) implies that for any non-integer α_j we have $\{v + \alpha_j\} \in [0, 1 - \delta]$ and so $z(v + \alpha_j) \geq w > z(v)$. Since for integer α_j clearly $z(v + \alpha_j) = z(v)$, by (7.2) we get that if not every α_j is integer then

$$z(v) < \sum_{j=1}^m q^{-\alpha_j} z(v + \alpha_j),$$

which contradicts (7.4). The proof is complete. \square

8. Open questions

In Theorem 7.7 the assumption that A is homogeneous is essential since for general self-similar sets with the SSC (ii) and (iii) are not equivalent, the characterization by Xi [33] of bilipschitz equivalent self-similar sets with the SSC is much more complicated. But (i) and (iii) might be equivalent without assuming homogeneity; that is, we do not know if Theorem 1.17 holds for any self-similar sets with the SSC.

Another possible and natural way to improve Theorem 1.17 is to assume only that A can be mapped onto a subset of B of positive measure. We do not know if this stronger result holds. If the answer to both of the above questions is positive then one might ask if both improvements can be made at the same time.

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