Branching Processes with Immigration in a Random Environment – the Grincevičius–Grey setup

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Abstract

We determine the tail asymptotics of the stationary distribution of a branching process with immigration in a random environment, when the immigration distribution dominates the offspring distribution. The assumptions are the same as in the Grincevičius–Grey theorem for the stochastic recurrence equation.

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1. Introduction and the main result

A branching process with immigration in a random environment is a usual Galton–Watson process with immigration, where the offspring and immigration distribution in each generation is governed by an independent identically distributed (iid) sequence of probability measures. Let Δ denote the set of probability measures on $\mathbb{N} = \{0, 1, \ldots\}$, and consider the Borel-σ-algebra on it induced by the total variation distance. Let $\xi, \xi_0, \xi_1, \ldots$ be an iid sequence in Δ^2, the components $\xi_n = (\nu_{\xi_n}, \nu^\circ_{\xi_n})$ represent the offspring and immigration distribution in the consecutive generations. Let $X_0 = x \in \mathbb{N}$, and

$$X_{n+1} = \sum_{i=1}^{X_n} A_i^{(n+1)} + B_{n+1} =: \theta_{n+1} \circ X_n + B_{n+1}, \quad n \geq 0, \tag{1}$$

where conditioned on the environment $\mathcal{E} = \sigma(\xi_0, \xi_1, \ldots)$, the variables $\{A_i^{(n)}, B_n : n \in \mathbb{N}, i \geq 1\}$ are independent, and for n fixed, $(A_i^{(n)})_{i \geq 1}$ are iid with distribution ν_{ξ_n}, and B_n has distribution $\nu^\circ_{\xi_n}$. Note that given the environment the random variables are independent however, ν_ξ and ν°_ξ may depend. The variable $A_i^{(n)}$ is the number of offsprings of the ith element in the $(n-1)$st generation, and B_n is the number of immigrants in the nth generation. To ease notation we write $\theta_n \circ x = \sum_{i=1}^{x} A_i^{(n)}$.

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For a random measure $\xi = (\nu_\xi, \nu_\xi^\circ)$ in Δ^2 let A has distribution ν_ξ conditionally on ξ, and denote

$$m(\xi) = \sum_{i=1}^{\infty} i \nu_\xi(\{i\}) = \mathbb{E}[A|\xi],$$

the conditioned expectation of its offspring distribution. Our standing assumption is that for some $\kappa > 0$

$$\mathbb{E}(m(\xi)^{1\vee\kappa}) < 1, \quad \mathbb{E}(A^{(1\vee\kappa)+\delta}) < \infty, \quad \mathbb{E}(B^\delta) < \infty, \quad \text{for some } \delta > 0, \tag{2}$$

where $a \vee b = \max\{a, b\}$. Then, by the Jensen inequality the process is subcritical, i.e. $\mathbb{E}(\log m(\xi)) < 0$, which implies that the corresponding branching process in a random environment without immigration dies out almost surely, see e.g. the recent monograph by Kersting and Vatutin [12, Section 2.2]. Furthermore, by the conditional Jensen's inequality $m(\xi)^{(1\vee\kappa)+\delta} \leq \mathbb{E}[A^{(1\vee\kappa)+\delta}|\xi]$, thus (2) also implies that $\mathbb{E}(m(\xi)^{\kappa+\delta}) < \infty$, which is used implicitly later.

By Lemma 3 below under condition (2) the Markov chain (1) has a unique stationary distribution X_∞, which using backward iteration can be represented as

$$X_\infty \stackrel{\mathcal{D}}{=} \sum_{i=0}^{\infty} \theta_0 \circ \theta_1 \circ \ldots \circ \theta_{i-1} \circ B_i, \tag{3}$$

where $\stackrel{\mathcal{D}}{=}$ stands for equality in distribution. The stationary distribution X_∞ satisfies the distributional fixed point equation

$$X \stackrel{\mathcal{D}}{=} \sum_{i=1}^{X} A_i + B,$$

where $(\xi, B, A_1, A_2, \ldots)$ and X on the right-hand side are independent, and conditionally on ξ the variables $B, A_1, A_2, \ldots$ are independent, and $A_1, A_2, \ldots$ are iid ν_ξ, and B has distribution ν_ξ°.

The main result of the paper characterizes the regular variation of the stationary distribution.

Theorem. *Assume that there is a $\kappa > 0$ such that (2) holds. Let ℓ be a slowly varying function. Then*

$$\mathbb{P}(B > x) \sim \frac{\ell(x)}{x^\kappa}, \quad \text{as} \quad x \to \infty, \tag{4}$$

if and only if

$$\mathbb{P}(X_\infty > x) \sim \frac{\ell(x)}{x^\kappa} \frac{1}{1 - \mathbb{E}(m(\xi)^\kappa)}, \quad \text{as} \quad x \to \infty. \tag{5}$$

The implication (4) $\Rightarrow$ (5) was investigated in deterministic environment, in which case $m(\xi) \equiv \mathbb{E}(A) \in (0, 1)$ is deterministic. Then we obtain a generalization of Theorem 2.1.1 in Basrak et al. [4], where it is assumed that $\kappa \in (0, 2)$, and if $\kappa \geq 1$ then $\mathbb{E}(A^2) < \infty$. (In fact for $\kappa < 1$ we need that $\mathbb{E}(A^{1+\delta}) < \infty$ for some $\delta > 0$, whereas in [4] only $\mathbb{E}(A) < 1$ is needed.) Theorems 2.1 and 2.2 by Foss and Miyazawa [9] covers more general tail behavior, not only regular variation. If $\mathbb{E}(B) < \infty$ then their Theorem 2.1 implies our result, while if $\mathbb{E}(B) = \infty$ it is assumed in [9, Theorem 2.2] that either $\mathbb{E}(A^2) < \infty$, or $1 - \int_x^\infty \mathbb{P}(A > u) \mathrm{d}u$ is subexponential. In the latter case, we only require that $\mathbb{E}(A^{1+\delta}) < \infty$ for some δ, which does not seem to imply the subexponential condition (on a related question, on the subexponential property of the integrated tail distribution, see Section 1.4 in Embrechts et al. [7]). Therefore, in some cases our results are new even in the deterministic setup. To the best of our knowledge, the converse implication (5) $\Rightarrow$ (4) was not treated earlier.

In the random environment setup the tail behavior was studied in [3] under a different assumption. In [3], the main assumptions are Cramér's condition

$$\mathbb{E}(m(\xi)^\kappa) = 1, \quad \text{for some} \quad \kappa > 0, \tag{6}$$

and $\mathbb{E}(A^\kappa) < \infty$, $\mathbb{E}(B^\kappa) < \infty$. If further weak assumptions are satisfied, then the tail of X_∞ is regularly varying with index $-\kappa$, that is $\mathbb{P}(X_\infty > \cdot) \in \mathcal{RV}_{-\kappa}$. Therefore, in [3] the tail behavior is governed by the offspring distribution, while in the present paper the tail is determined by the immigration.

The main idea of the proof is very simple. If a random sum $\sum_{i=1}^B A_i$ is large, where the summands are small, then, by the law of large numbers, it is asymptotically $B\mathbb{E}(A)$, see [9, Remark 2.4]. More precisely, we rely on the similarity between the asymptotic behavior of the branching process (X_n) in (1) and the stochastic recurrence equation defined by $Y_0 = y \geq 0$,

$$Y_{n+1} = C_{n+1} Y_n + D_{n+1}, \quad n \geq 0, \tag{7}$$

where $(C, D), (C_1, D_1), \ldots$ are iid random vectors, with nonnegative components. For recent monographs on the stochastic recurrence equation we refer to Buraczewski et al. [5], and to Iksanov [11]. The tail behavior of the stationary distribution, or, which is the same the solution to the corresponding stochastic fixed point equation, is well-understood, see e.g. [5, Section 2.4]: If $\mathbb{E}(C^\kappa) = 1$, and $\mathbb{E}(D^\kappa) < \infty$ then by the Kesten–Grincevičius–Goldie theorem (called Kesten–Goldie theorem in [5]) the tail is asymptotically $kx^{-\kappa}$, for some $k > 0$. For an extension of this result see Kevei [13]. While, if $\mathbb{E}(C^\kappa) < 1$ and $\mathbb{E}(C^{\kappa+\delta}) < \infty$ for some $\delta > 0$, then by the Grincevičius–Grey theorem $\mathbb{P}(D > \cdot) \in \mathcal{RV}_{-\kappa}$ if and only if $\mathbb{P}(Y_\infty > \cdot) \in \mathcal{RV}_{-\kappa}$, where Y_∞ is the stationary distribution of (7).

The regular variation of the stationary distribution of a Markov chain has a lot of consequences. Then, the well-known theory of regularly varying time series apply, see e.g. the recent monograph by Kulik and Soulier [16]. The conditions to apply the general theory, that is ergodicity, anticlustering, and vanishing small values were all established in [3]. In particular, if (2) and (5) hold then all the results in Section 3 in [3] hold true in the current setup, because in the proofs only the regular variation of the stationary distribution was used, and that $\mathbb{E}(m(\xi)^\alpha) < 1$, $\mathbb{E}(B^\alpha) < \infty$, for some $\alpha > 0$. In particular, the tail process of (X_n), the convergence of the point process, and the central limit theorems are given in Theorems 3–5 in [3].

2. Proof

For the analysis of the stationary distribution we need the tail behavior of randomly stopped sums of identically distributed, conditionally independent summands, where the number of terms dominates the summands. Such results for independent summands were subject of intensive investigations, see e.g. Barczy et al. [2, Proposition D.3], Aleškevičienė et al. [1, Theorem 1.2], Faÿ et al. [8, Proposition 4.3], Robert and Segers [17]. In the iid case the next statement coincides with Proposition D.3 in [2] for $\kappa \geq 1$, while for $\kappa < 1$ we need $\mathbb{E}(A^{1+\delta}) < \infty$ for some $\delta > 0$, whereas in [2] only $\mathbb{E}(A) < 1$ is assumed.

Lemma 1. *Let $A, A_1, \ldots$ be identically distributed random variables, independent given the random element ζ, and put $m(\zeta) = \mathbb{E}[A|\zeta]$. Let B be an nonnegative integer-valued random variable, independent of the A's and ζ. Assume that $\mathbb{P}(B > x) = \frac{\ell(x)}{x^\kappa}$, $\mathbb{E}(A^{(1\vee\kappa)+\delta}) < \infty$, for some $\kappa > 0$, $\delta > 0$, and a slowly varying function ℓ. Then, as $x \to \infty$*

$$\mathbb{P}\left(\theta \circ B\right) = \mathbb{P}\left(\sum_{i=1}^{B} A_i > x\right) \sim \mathbb{P}(B > x)\, \mathbb{E}(m(\zeta)^\kappa).$$

The reason for the change in the terminology from ξ to ζ is that we want to use the result for more generations when $\zeta = (\xi_0, \xi_1, \ldots, \xi_{i-1})$.

Proof. To ease notation write $\widetilde{A}_i = A_i - m(\zeta)$.

First we prove that as $x \to \infty$ for some $\alpha > \kappa$

$$\mathbb{P}\left(\sum_{i=1}^{B} \widetilde{A}_i > x\right) = o(x^{-\alpha}). \tag{8}$$

In what follows, nonspecified limits are meant as $x \to \infty$, and c is finite positive constant, whose value is irrelevant, and may change from line to line.

By the assumption on B, for any $\beta \in (0, \kappa)$ we have $\mathbb{E}(B^\beta) < \infty$. Assume that $\kappa > 1$. Then by [3, Lemma 2 (ii)], for $\kappa < \alpha < \kappa + \delta < 2\kappa$ there exists $c = c(\alpha)$ depending only on α, such that for any $n \geq 1$

$$\mathbb{E}\left(\left|\sum_{i=1}^{n} \widetilde{A}_i\right|^\alpha\right) \leq c n^{1\vee\frac{\alpha}{2}}\, \mathbb{E}\left(\mathbb{E}[|\widetilde{A}|^\alpha|\zeta]\right) \leq c n^{1\vee\frac{\alpha}{2}}\, \mathbb{E}(A^\alpha).$$

Thus

$$\mathbb{E}\left(\left|\sum_{i=1}^{B} \widetilde{A}_i\right|^\alpha\right) \leq c\, \mathbb{E}(B^{1\vee\frac{\alpha}{2}})\, \mathbb{E}(A^\alpha) < \infty. \tag{9}$$

For $\kappa \leq 1$, choose α and η such that $\alpha - \eta < \kappa < \alpha < \kappa + \delta$, $2\eta \leq \alpha < 1 + \eta$, and $\frac{\alpha}{\alpha - \eta} < 1 + \delta$. This is clearly possible, if α is close enough to κ and η is small. Then, by [3, Lemma 2 (i)] (the exponent $\alpha - \eta$ is missing in the statement, but it appears in the proof)

$$\mathbb{E}\left(\left|\sum_{i=1}^{n} \widetilde{A}_i\right|^{\alpha}\right) \leq c n^{\alpha-\eta}\, \mathbb{E}\left((\mathbb{E}[|\widetilde{A}|^{\frac{\alpha}{\alpha-\eta}}|\zeta])^{\alpha-\eta}\right) \leq c n^{\alpha-\eta}\, (\mathbb{E}(A^{\frac{\alpha}{\alpha-\eta}}))^{\alpha-\eta},$$

implying that

$$\mathbb{E}\left(\left|\sum_{i=1}^{B} \widetilde{A}_i\right|^{\alpha}\right) \leq c\, \mathbb{E}(B^{\alpha-\eta})\, (\mathbb{E}(A^{\frac{\alpha}{\alpha-\eta}}))^{\alpha-\eta} < \infty. \tag{10}$$

In both cases (8) follows.

Next

$$\begin{aligned} \mathbb{P}\left(\sum_{i=1}^{B} A_i > x\right) &= \mathbb{P}\left(\sum_{i=1}^{B} \widetilde{A}_i + Bm(\zeta) > x\right) \\ &\leq \mathbb{P}\left(\sum_{i=1}^{B} \widetilde{A}_i > \varepsilon x\right) + \mathbb{P}(Bm(\zeta) > (1-\varepsilon)x) \\ &\sim \mathbb{E}(m(\zeta)^{\kappa})\, (1-\varepsilon)^{-\kappa}\, \mathbb{P}(B > x), \end{aligned} \tag{11}$$

by (8), a version of Breiman's lemma (see e.g. [5, Lemma B.5.1]) and the regular variation of $\mathbb{P}(B > x)$. Breiman's lemma is indeed applicable, as $\mathbb{E}(m(\zeta)^{(1\vee\kappa)+\delta}) \leq \mathbb{E}(A^{(1\vee\kappa)+\delta}) < \infty$. Similarly, for the lower bound

$$\begin{aligned} \mathbb{P}\left(\sum_{i=1}^{B} A_i > x\right) &\geq \mathbb{P}\left(Bm(\zeta) > (1+\varepsilon)x, \sum_{i=1}^{B} \widetilde{A}_i > -\varepsilon x\right) \\ &\geq \mathbb{P}(Bm(\zeta) > (1+\varepsilon)x) - \mathbb{P}\left(\left|\sum_{i=1}^{B} \widetilde{A}_i\right| \geq \varepsilon x\right) \\ &\sim \mathbb{E}(m(\zeta)^{\kappa})\, (1+\varepsilon)^{-\kappa}\, \mathbb{P}(B > x). \end{aligned} \tag{12}$$

Letting $\varepsilon \to 0$ in (11) and (12), the result follows. □

As an immediate consequence, we obtain the following.

Corollary. *For any $i \geq 0$ as $x \to \infty$*

$$\mathbb{P}(\theta_0 \circ \ldots \circ \theta_{i-1} \circ B_i > x) \sim \mathbb{P}(B > x)(\mathbb{E}(m(\xi)^{\kappa}))^{i}. \tag{13}$$

Next we need the existence of the moments of a branching process in a random environment and a bound for their decay. To ease notation write $\theta_0 \circ \ldots \circ \theta_{i-1} = \Theta_{i-1}$, and $m(\xi_0) \ldots m(\xi_{i-1}) = \Pi_{i-1}$, $i = 0, 1, 2, \ldots$, with the convention $\Theta_{-1} \circ B_0 = B_0$ and $\Pi_{-1} = 1$. Let $Z_0 = 1$ and $Z_n = \Theta_{n-1} \circ 1$, thus Z_n is a branching process in a random environment without immigration.

Lemma 2. *If* $\mathbb{E}(A^\alpha) < \infty$ *and* $\mathbb{E}(m(\xi)^\alpha) < \infty$ *then* $\mathbb{E}(Z_n^\alpha) < \infty$ *for any* $n \geq 0$. *Furthermore, if* $\mathbb{E}(m(\xi)^\alpha) < 1$, *then there exist* $\rho \in (0,1)$, *and* $c > 0$, *such that for any* $n \geq 0$

$$\mathbb{E}(Z_n^\alpha) \leq c\rho^n. \tag{14}$$

If $\alpha \leq 1$ or $\alpha > 1$ and $\mathbb{E}(m(\xi)^\alpha) > \mathbb{E}(m(\xi))$, then under further assumptions Lemma 3.1 by Buraczewski and Dyszewski [6] gives precise asymptotics. Furthermore, Proposition 3.1 in the first arXiv version of [6] states the necessary bound. For completeness, we sketch the proof as given in the first arXiv version of [6].

Proof. If $\alpha \leq 1$, then (14) with $c = 1$ and $\rho = \mathbb{E}(m(\xi)^\alpha)$ follows from the conditional Jensen inequality, proving both statements.

Let $\alpha > 1$. For any $\varepsilon > 0$ there exists $C = C(\alpha, \varepsilon) > 0$ such that $(x+y)^\alpha \leq (1+\varepsilon)x^\alpha + Cy^\alpha$ for $x > 0, y > 0$. Therefore, we have by (9)

$$\begin{aligned} \mathbb{E}(Z_n^\alpha) &= \mathbb{E}\left[\left(Z_{n-1} m(\xi_{n-1}) + \sum_{i=1}^{Z_{n-1}} (A_i^{(n-1)} - m(\xi_{n-1}))\right)^\alpha\right] \\ &\leq (1+\varepsilon)\,\mathbb{E}(m(\xi)^\alpha)\,\mathbb{E}(Z_{n-1}^\alpha) + c\,\mathbb{E}\left(Z_{n-1}^{1\vee\frac{\alpha}{2}}\right)\mathbb{E}(A^\alpha), \end{aligned} \tag{15}$$

proving the first statement.

Assume now that $\mathbb{E}(m(\xi)^\alpha) < 1$. Choose $\varepsilon > 0$ small enough so that $\widetilde{\rho} := (1+\varepsilon)\mathbb{E}(m(\xi)^\alpha) < 1$. Then by (15)

$$\mathbb{E}(Z_n^\alpha) \leq \widetilde{\rho}\,\mathbb{E}(Z_{n-1}^\alpha) + c\,\mathbb{E}(Z_{n-1}^{1\vee\frac{\alpha}{2}}). \tag{16}$$

For $\alpha \leq 2$, we obtain

$$\mathbb{E}(Z_n^\alpha) \leq \widetilde{\rho}\,\mathbb{E}(Z_{n-1}^\alpha) + c\,(\mathbb{E}(m(\xi)))^n,$$

which after iteration implies (14) with $\rho > \max\{\widetilde{\rho}, \mathbb{E}(m(\xi))\}$. Once we have the exponential decrease for $\alpha \in (1,2]$, by (16) we have it for $\alpha \in (2,4]$, and similarly (14) follows for any α by induction. Here, we used that the function $\lambda(\alpha) = \mathbb{E}(m(\xi)^\alpha)$ is convex, $\lambda(0) = 1$, and $\lambda'(0) < 0$ (by subcriticality). □

The following statement on the existence and uniqueness of the stationary distribution appeared implicitly in the proof of [3, Lemma 3]. In the multitype setting Theorem 3.3 in Key [15] gives conditions for the existence of a limiting distribution.

Lemma 3. *Assume that* $\mathbb{E}(A^\alpha) < \infty$, $\mathbb{E}(m(\xi)^\alpha) < 1$, *and* $\mathbb{E}(B^\alpha) < \infty$ *for some* $\alpha > 0$. *Then the Markov chain in* (1) *has a unique stationary distribution* X_∞ *defined in* (3).

Proof. We first show that $\mathbb{E}(X_\infty^\alpha) < \infty$, implying the existence of a stationary distribution. For $\alpha \leq 1$ by the conditional Jensen inequality

$$\begin{aligned} \mathbb{E}[(\Theta_{i-1} \circ B_i)^\alpha] &= \mathbb{E}(\mathbb{E}[(\Theta_{i-1} \circ B_i)^\alpha | \xi_0, \ldots, \xi_{i-1}, B_i]) \\ &\leq \mathbb{E}((B_i m(\xi_0) \ldots m(\xi_{i-1}))^\alpha) = \mathbb{E}(B^\alpha)(\mathbb{E}(m(\xi)^\alpha))^i, \end{aligned}$$

while for $\alpha > 1$ by Minkowski's inequality and Lemma 2

$$\mathbb{E}[(\Theta_{i-1} \circ B_i)^\alpha] \leq c\mathbb{E}(B^\alpha)\rho^i$$

for some $\rho < 1$. In both case $\mathbb{E}(X_\infty^\alpha) < \infty$ follows, for $\alpha \leq 1$ by subadditivity, for $\alpha > 1$ by Minkowski's inequality.

The uniqueness follows from the same argument as in the proof of [3, Lemma 3], showing the existence of an accessible atom. □

For the implication (5) ⇒ (4) we need the random sum version of Lemma 2 in Grey [10].

Lemma 4. *Let N be a nonnegative integer valued random variable for which $\mathbb{P}(N > x) \sim C\ell(x)x^{-\kappa}$, with some $C > 0$, and a slowly varying function ℓ. Let $(\xi, B, A_1, A_2, \ldots)$ be independent of N, and conditionally on ξ the variables $(B, A_1, A_2, \ldots)$ are independent, $A_1, A_2, \ldots$ are iid ν_ξ, and B has distribution ν_ξ°. Assume condition* (2)*. Then as $x \to \infty$*

$$\mathbb{P}(B > x) \sim \frac{\ell(x)}{x^\kappa} \iff \mathbb{P}\left(B + \sum_{i=1}^N A_i > x\right) \sim (1 + C\,\mathbb{E}(m(\xi)^\kappa))\frac{\ell(x)}{x^\kappa}.$$

Proof. The statement follows from Lemma 2 in [10], since the right-hand side above is equivalent to

$$\mathbb{P}(B + Nm(\xi) > x) \sim (1 + C\,\mathbb{E}(m(\xi)^\kappa))\ell(x)x^{-\kappa}.$$

Indeed,

$$B + \sum_{i=1}^N A_i = B + Nm(\xi) + \sum_{i=1}^N \widetilde{A}_i,$$

where the sum on the right-hand side, by (9) or (10) has finite moment of order $\alpha > \kappa$, implying that its tail is $o(x^{-\alpha})$. □

We are ready to prove the main result.

Proof of the Theorem. As in [10], implication (5) ⇒ (4) follows from Lemma 4 with the choice $N = X_\infty$ and $C = 1/(1 - \mathbb{E}(m(\xi)^\kappa))$.

We turn to (4) ⇒ (5). We follow the proof of the Grincevičius–Grey theorem in [5, Sect. 2.4.3].

For $K > 1$ we use the decomposition

$$X_\infty \overset{\mathcal{D}}{=} \left(\sum_{i=0}^K + \sum_{i=K+1}^\infty\right)\Theta_{i-1} \circ B_i =: \widetilde{X}_K + \widetilde{X}^K.$$

For any $i > j \geq 0$

$$\begin{aligned}
&\mathbb{P}(\Theta_{i-1} \circ B_i > x, \Theta_{j-1} \circ B_j > x) \\
&\leq \mathbb{P}(\Pi_{i-1}B_i > (1-\varepsilon)x, \Pi_{j-1}B_j > (1-\varepsilon)x) \\
&\quad + \mathbb{P}(\Theta_{i-1} \circ B_i - \Pi_{i-1}B_i > \varepsilon x) + \mathbb{P}(\Theta_{j-1} \circ B_j - \Pi_{j-1}B_j > \varepsilon x).
\end{aligned} \tag{17}$$

We have by (2.4.52) in [5] that

$$\mathbb{P}(\Pi_{i-1}B_i > (1-\varepsilon)x, \Pi_{j-1}B_j > (1-\varepsilon)x) = o(\mathbb{P}(B > x)).$$

For the last two terms in (17) we apply (8). Indeed, the assumptions of Lemma 1 are satisfied with the random element ζ being i iid random environments, $\zeta = (\xi_0, \ldots, \xi_{i-1})$, and A having distribution $\Theta_{i-1} \circ 1 = Z_i$, with the notation introduced before Lemma 2. By Lemma 2 we have $\mathbb{E}(Z_i^{(1\vee\kappa)+\delta}) < \infty$, thus (8) holds with some $\alpha > \kappa$. Substituting back into (17) we obtain as $x \to \infty$

$$\mathbb{P}(\Theta_{i-1} \circ B_i > x, \Theta_{j-1} \circ B_j > x) = o(\mathbb{P}(B > x)). \tag{18}$$

Thus by [5, Lemma B.6.1] and (13)

$$\mathbb{P}(\widetilde{X}_K > x) \sim \mathbb{P}(B > x) \sum_{i=0}^{K} (\mathbb{E}(m(\xi)^\kappa))^i. \tag{19}$$

The result follows from (19) with $K \to \infty$, provided we show that

$$\lim_{K\to\infty} \limsup_{x\to\infty} \frac{\mathbb{P}(\widetilde{X}^K > x)}{\mathbb{P}(B > x)} = 0. \tag{20}$$

We have

$$\begin{aligned} \mathbb{P}(\widetilde{X}^K > x) \le \mathbb{P}\left(\sum_{i=K+1}^{\infty} (\Theta_{i-1} \circ B_i - \Pi_{i-1}B_i) > \frac{x}{2} \right) \\ + \mathbb{P}\left(\sum_{i=K+1}^{\infty} \Pi_{i-1}B_i > \frac{x}{2} \right). \end{aligned} \tag{21}$$

For the second term above, by the proof in the usual Grincevičius–Grey theorem, see [5, (2.4.53)]

$$\lim_{K\to\infty} \limsup_{x\to\infty} \frac{\mathbb{P}\left(\sum_{i=K+1}^{\infty} \Pi_{i-1}B_i > \frac{x}{2}\right)}{\mathbb{P}(B > x)} = 0.$$

For the first term in (21), by Markov's inequality for $\alpha > 0$ to be specified later

$$\mathbb{P}\left(\sum_{i=K+1}^{\infty} (\Theta_{i-1} \circ B_i - \Pi_{i-1}B_i) > \frac{x}{2} \right) \le \frac{2^\alpha}{x^\alpha} \mathbb{E} \left| \sum_{i=K+1}^{\infty} (\Theta_{i-1} \circ B_i - \Pi_{i-1}B_i) \right|^\alpha.$$

For $\kappa > 1$ choose α such that $\kappa < \alpha < \kappa + \delta < 2\kappa$ and $\mathbb{E}(m(\xi)^\alpha) < 1$. Then by (9)

$$\begin{aligned} \mathbb{E}\left|\Theta_{i-1} \circ B_i - \Pi_{i-1}B_i\right|^\alpha &\le c\,\mathbb{E}(B^{1\vee\frac{\alpha}{2}})\,\mathbb{E}[(\Theta_{i-1} \circ 1)^\alpha)] \\ &\le c\,\mathbb{E}(B^{1\vee\frac{\alpha}{2}})\rho^i, \end{aligned}$$

for some $\rho < 1$, by (14). Therefore, using the Minkowski inequality

$$\mathbb{E}\left(\left|\sum_{i=K+1}^{\infty}(\Theta_{i-1}\circ B_i - \Pi_{i-1}B_i)\right|^{\alpha}\right) \leq \left(c[\mathbb{E}(B^{1\vee\frac{\alpha}{2}})]^{\frac{1}{\alpha}}\sum_{i=K+1}^{\infty}\rho^{\frac{i}{\alpha}}\right)^{\alpha} \leq c\rho^K.$$

For $\kappa = 1$ choose $\alpha > 0$ and $\eta > 0$ such that $\alpha - \eta < 1 < \alpha < 1 + \delta$, $2\eta \leq \alpha < 1 + \eta$, $\frac{\alpha}{\alpha-\eta} < 1 + \delta$, and $\mathbb{E}(m(\xi)^{\alpha/(\alpha-\eta)}) < 1$. This is possible by choosing α close enough to 1 and $\eta > 0$ small enough. Using the Minkowski inequality together with (10) and (14),

$$\begin{aligned}
&\mathbb{E}\left(\left|\sum_{i=K+1}^{\infty}(\Theta_{i-1}\circ B_i - \Pi_{i-1}B_i)\right|^{\alpha}\right) \\
&\leq \left(\sum_{i=K+1}^{\infty}\left[\mathbb{E}\left(|\Theta_{i-1}\circ B_i - \Pi_{i-1}B_i|^{\alpha}\right)\right]^{\frac{1}{\alpha}}\right)^{\alpha} \\
&\leq \left(c[\mathbb{E}(B^{\alpha-\eta})]^{\frac{1}{\alpha}}\sum_{i=K+1}^{\infty}\left(\mathbb{E}((\Theta_{i-1}\circ 1)^{\frac{\alpha}{\alpha-\eta}})\right)^{\frac{\alpha-\eta}{\alpha}}\right)^{\alpha} \\
&\leq c\,\mathbb{E}(B^{\alpha-\eta})\rho^{K(\alpha-\eta)}.
\end{aligned}$$

Finally, for $\kappa < 1$ choose $\alpha > 0$ and $\eta > 0$ such that $\alpha - \eta < \kappa < \alpha < (\kappa + \delta) \wedge 1$, $2\eta \leq \alpha < 1 + \eta$, $\frac{\alpha}{\alpha-\eta} < 1 + \delta$, and $\mathbb{E}(m(\xi)^{\frac{\alpha}{\alpha-\eta}}) < 1$. This is possible by choosing α close enough to κ and $\eta > 0$ small enough. First by subadditivity, next by (10) and (14), for some $\rho < 1$

$$\begin{aligned}
&\mathbb{E}\left(\left|\sum_{i=K+1}^{\infty}(\Theta_{i-1}\circ B_i - \Pi_{i-1}B_i)\right|^{\alpha}\right) \leq \sum_{i=K+1}^{\infty}\mathbb{E}\left(|\Theta_{i-1}\circ B_i - \Pi_{i-1}B_i|^{\alpha}\right) \\
&\leq c\,\mathbb{E}(B^{\alpha-\eta})\sum_{i=K+1}^{\infty}(\mathbb{E}((\Theta_{i-1}\circ 1)^{\frac{\alpha}{\alpha-\eta}}))^{\alpha-\eta} \\
&\leq c\mathbb{E}(B^{\alpha-\eta})\,\rho^K.
\end{aligned}$$

Therefore, (20) holds in each case, and thus the proof is complete. □

Remark. The proof works in the deterministic environment case, however then it is much simpler. Indeed, $\Pi_{i-1} = \mu^i$, where $\mu = \mathbb{E}(A)$, and (17) becomes trivial because of the independence of B_i and B_j. Furthermore, for the second term in (21) instead of referring to the proof of the Grincevičius–Grey theorem in [5], by the Potter bounds we have for $\gamma \in (\mu, 1)$, $\varepsilon = \frac{\kappa}{2} \wedge 1$, for x large enough

$$\begin{aligned}
\mathbb{P}\left(\sum_{i=K+1}^{\infty}\mu^i B_i > \frac{x}{2}\right) &\leq \sum_{j=0}^{\infty}\mathbb{P}\left(\mu^{j+K+1}B_j > \frac{x}{2}(1-\gamma)\gamma^j\right) \\
&\leq 2\sum_{j=0}^{\infty}\mathbb{P}(B > x)\left(\frac{(1-\gamma)\gamma^j}{2\mu^{j+K+1}}\right)^{-\kappa+\varepsilon},
\end{aligned}$$

implying the necessary bound. The first term in (21) can be handled the same way, since (14) holds for $\alpha \geq 1$ by bound (11) in Kevei and Wiandt [14], while for $\alpha < 1$ it follows from Jensen's inequality.

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References

[1] A. Aleškevičienė, R. Leipus, and J. Šiaulys. Tail behavior of random sums under consistent variation with applications to the compound renewal risk model. *Extremes*, 11(3):261–279, 2008.

[2] M. Barczy, Z. Bősze, and G. Pap. Regularly varying non-stationary Galton-Watson processes with immigration. *Statist. Probab. Lett.*, 140:106–114, 2018.

[3] B. Basrak and P. Kevei. Limit theorems for branching processes with immigration in a random environment. *Extremes*, 25(4):623–654, 2022.

[4] B. Basrak, R. Kulik, and Z. Palmowski. Heavy-tailed branching process with immigration. *Stoch. Models*, 29(4):413–434, 2013.

[5] D. Buraczewski, E. Damek, and T. Mikosch. *Stochastic Models with Power-Law Tails. The Equation X=AX+B.* Springer Series in Operations Research and Financial Engineering. Springer, 2016.

[6] D. Buraczewski and P. Dyszewski. Precise large deviation estimates for branching process in random environment. *Ann. Inst. Henri Poincaré Probab. Stat.*, 58(3):1669–1700, 2022.

[7] P. Embrechts, C. Klüppelberg, and T. Mikosch. *Modelling extremal events*, volume 33 of *Applications of Mathematics (New York).* Springer-Verlag, Berlin, 1997.

[8] G. Faÿ, B. González-Arévalo, T. Mikosch, and G. Samorodnitsky. Modeling teletraffic arrivals by a Poisson cluster process. *Queueing Syst.*, 54(2):121–140, 2006.

[9] S. Foss and M. Miyazawa. Tails in a fixed-point problem for a branching process with state-independent immigration. *Markov Process. Related Fields*, 26(4):613–635, 2020.

[10] D. R. Grey. Regular variation in the tail behaviour of solutions of random difference equations. *Ann. Appl. Probab.*, 4(1):169–183, 1994.

[11] A. Iksanov. *Renewal Theory for Perturbed Random Walks and Similar Processes.* Probability and Its Applications. Springer, 2016.

[12] G. Kersting and V. Vatutin. *Discrete Time Branching Processes in Random Environment.* ISTE Ltd and John Wiley & Sons, Inc., 2017.

[13] P. Kevei. A note on the Kesten–Grincevičius–Goldie theorem. *Electron. Commun. Probab.*, 21(51):1–12, 2016.

[14] P. Kevei and P. Wiandt. Moments of the stationary distribution of subcritical multitype Galton-Watson processes with immigration. *Statist. Probab. Lett.*, 173:Paper No. 109067, 6, 2021.

[15] E. S. Key. Limiting distributions and regeneration times for multitype branching processes with immigration in a random environment. *Ann. Probab.*, 15(1):344–353, 1987.

[16] R. Kulik and P. Soulier. *Heavy-tailed time series.* Springer-Verlag New York, 2020.

[17] C. Y. Robert and J. Segers. Tails of random sums of a heavy-tailed number of light-tailed terms. *Insurance Math. Econom.*, 43(1):85–92, 2008.