Strong renewal theorem and local limit theorem in the absence of regular variation

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Abstract

We obtain a strong renewal theorem with infinite mean beyond regular variation, when the underlying distribution belongs to the domain of geometric partial attraction a semistable law with index $\alpha \in (1/2, 1]$. In the process we obtain local limit theorems for both finite and infinite mean, that is for the whole range $\alpha \in (0, 2)$. We also derive the asymptotics of the renewal function for $\alpha \in (0, 1]$.

1 Introduction

Strong renewal theorems (SRT) with infinite mean that have regularly varying (with parameter $\alpha \in [0, 1]$) underlying renewal distributions are nowadays completely understood. The SRT in the one-sided lattice case with $\alpha \in (1/2, 1)$ has been obtained by Garsia and Lamperti [10] and it was later generalized to the nonarithmetic case by Erickson [9]. The latter also treats the case $\alpha = 1$. As noted in [10], the mere regular variation is insufficient in the range $\alpha \in (0, 1/2)$. The problem of finding necessary and sufficient conditions has recently been solved by Caravenna and Doney [6] directly in the two-sided case. For more information on improved sufficient conditions for this problematic range we refer to [6]. For a complete treatment of the two-sided $\alpha = 1$ case we refer to Berger [3]. Very recently, Uchiyama [22] obtained asymptotic results for the renewal function for relatively stable variables with infinite mean. A nonnegative random variable is relatively stable if and only if its truncated mean is slowly varying. This roughly corresponds to the case $\alpha = 1$, but the tail is not necessarily regularly varying. To the best of our knowledge, Uchiyama's paper is the only one where the infinite mean case in the absence of regular variation is treated. We also remark that renewal theory with no moments (roughly, the $\alpha = 0$ case) has been dealt with in [2].

In this paper we are interested in SRT with infinite mean beyond regular variation. More precisely, we focus on distributions in the domain of geometric partial attraction of a semistable law. The class of semistable laws, introduced by Paul Lévy, is a natural extension of stable laws. They are the limits of appropriately centered and normed sums of iid random variables along geometrically increasing subsequences. Analytically, the tail of

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the Lévy measure of the non-Gaussian stable laws are $x^{-\alpha}$, for some $\alpha \in (0,2)$, while for semistable laws an additional logarithmically periodic factor appears. The same logarithmically periodic function appears in the characterization of the domain of geometric partial attraction. A brief background on semistable laws is provided in Section 3. For definitions, properties, and history of semistable laws we refer to Sato [19, Chapter 13], Megyesi [16], Csörgő and Megyesi [8], and the references therein.

Our main results on SRT for the case of one-sided $\alpha \in (1/2,1)$ semistable renewal distributions are Theorem 4 (arithmetic case) and Theorem 5 (nonarithmetic and nonlattice cases). Unlike in [10] and [9], we cannot use the precise asymptotic of the characteristic function. Although the characteristic function asymptotic in Theorem 1 is an important ingredient of our proofs, the strategy is the systematic use of local limit theorems (LLT). The LLTs for semistable laws that we obtain here for both finite and infinite mean, that is for the whole range $\alpha \in (0,2)$, are new. These are Theorem 2 (lattice case) and Theorem 3 (nonlattice case).

Note that concerning LLT lattice and nonlattice distributions have to be treated separately, while concerning renewal theorems arithmetic and nonarithmetic distributions are different. Our proof of the SRT relies on the LLT. For arithmetic distributions we use the lattice LLT, while for nonlattice distributions we use the nonlattice LLT. In the proof of the remaining case for nonarithmetic lattice distributions, we use the lattice LLT together with the fact that the irrational rotation is uniquely ergodic, therefore it smooths out the mass at infinity. In particular, our proof in the nonarithmetic case is different from Erickson's [9] method.

As clarified in [6, Section 4.1] via probabilistic arguments, local limit results (namely, LLT and Local Large Deviation) are sufficient to prove SRT for the regularly varying case in range $\alpha \in (1/2,1)$. An analytic proof of this fact is absent in the literature. Our proof of Theorem 4 does precisely this while answering the current question on SRT in the semistable setting. In the process we show that the proofs in [10] and [9] can be written using just the LLT together with a 'rough' asymptotics of the characteristic function.

While the characteristic function asymptotics for $\alpha = 1$ in Theorem 1, are considerably more difficult than for the range $\alpha \in (0,1)$, the proof of the SRT (Theorem 6) is in fact simpler, and was obtained in a more general setup in [22].

In Theorem 7 we obtain the asymptotics of the renewal function for $\alpha \in (0,1]$ semistable renewal distributions. Previous similar, partial results are obtained in Kevei [13, Theorem 2.1] and in the authors' previous paper [15, Theorem 2], which provide a Karamata type theorem in the absence of strict regular variation. The basic observation used in the proof of Theorem 7 is that the semistable limit theorem obtained in [8] in terms of characteristic functions (but not LLT) together with an inversion formula can be used to obtain the asymptotics of the renewal function. This type of argument is not needed (although it makes sense) in the regular variation setting because the Karamata Tauberian theorem gives the desired result.

All the proofs are gathered together in Section 7.

2 Characteristic function asymptotics

Let X be a random variable with distribution function $F(x) = \mathbb{P}(X \leq x)$. Put $\overline{F}(x) = 1 - F(x)$. For $r > 1$ introduce the set of logarithmically periodic functions

$$\begin{aligned}\mathcal{P}_r = \Big\{ p : (0,\infty) \to (0,\infty) : \inf_{x \in [1,r]} p(x) > 0,\ p \text{ is bounded,} \\ \text{right-continuous, and } p(xr) = p(x),\ \forall x > 0 \Big\}.\end{aligned}$$

Assume that for some $r > 1$, $\alpha \in (0,1)$, and a slowly varying function ℓ

$$\lim_{n\to\infty} \frac{(r^n z)^\alpha}{\ell(r^n)} \overline{F}(r^n z) = p_0(z), \quad z \in C_{p_0}, \tag{1}$$

where the limit p_0 is not identically 0. Then the appearing function p_0 is necessarily log-periodic, i.e. $p_0(rx) = p_0(x)$, and since F is monotone, $p_0(x)x^{-\alpha}$ is nonincreasing. Then $\overline{F}$ is called *regularly log-periodic*. A stronger assumption is

$$\overline{F}(x) = \ell(x) x^{-\alpha} p_0(x), \qquad \text{with } p_0 \in \mathcal{P}_r,$$

which follows from (1) if p_0 is continuous.

Let $U(x) = \sum_{n=0}^\infty F^{*n}(x)$ be the corresponding renewal function, where $*n$ stands for the usual convolution power. If (1) holds then a slight generalization of [15, Theorem 2] (with the identical proof) shows that

$$\lim_{n\to\infty} \frac{U(r^n z)\ell(r^n)}{(r^n z)^\alpha} = p_1(z),$$

where p_1 can be determined explicitly, see [15, Theorem 2].

For finer results we first need the asymptotic behavior of the characteristic function of X. In what follows, oscillatory integrals appear naturally. The notation $\int_0^{\infty-}$ means that the integral is understood as improper Riemann integral, and not as Lebesgue integral on $[0,\infty)$.

Assume that

$$\begin{aligned} \overline{F}(x) &= \frac{\ell(x)}{x^\alpha} h(x), \\ F(-x) &= \frac{\ell(x)}{x^\alpha} k(x),\ x > 0, \end{aligned} \tag{2}$$

where $\alpha \in (0,2)$, the function ℓ is a slowly varying, and h and k are either identically 0, or positive bounded functions with strictly positive infimum, and at least one of them is not identically zero. Let

$$\varphi(t) = \mathbb{E}e^{\mathrm{i}tX} = \int_{\mathbb{R}} e^{\mathrm{i}tx} \mathrm{d}F(x).$$

We write $\Re$ for the real part and $\Im$ for the imaginary part.

Theorem 1. *Assume that* (2) *holds. If* $\alpha \in (0,1)$ *then*

$$\limsup_{t\to 0} \frac{|1-\varphi(t)|}{|t|^\alpha \ell(1/|t|)} < \infty.$$

Furthermore, if $h(x)x^{-\alpha}$ *and* $k(x)x^{-\alpha}$ *in* (2) *are ultimately nonincreasing then as* $t \to 0$

$$1-\varphi(t) \sim -\mathrm{i}\mathrm{sgn}(t)\, |t|^\alpha \ell(1/|t|) p_2(t),$$

where

$$p_2(t) = \int_0^{\infty-} y^{-\alpha} \left[h(y/|t|) e^{\mathrm{i}y\mathrm{sgn}(t)} - k(y/|t|) e^{-\mathrm{i}y\mathrm{sgn}(t)} \right] \mathrm{d}y.$$

If $\alpha \in (1,2)$ *then as* $t \to 0$

$$1 + \mathrm{i}t\mathbb{E}X - \varphi(t) \sim -\mathrm{i}\mathrm{sgn}(t)\, |t|^\alpha \ell(1/|t|) p_2(t),$$

where

$$p_2(t) = \int_0^{\infty} y^{-\alpha} \left[h(y/|t|)(e^{\mathrm{i}y\mathrm{sgn}(t)} - 1) - k(y/|t|)(e^{-\mathrm{i}y\mathrm{sgn}(t)} - 1) \right] \mathrm{d}y.$$

If $\alpha = 1$

$$\limsup_{t\to 0} \frac{\Re(1-\varphi(t))}{|t|\ell(1/|t|)} < \infty \quad \text{and} \quad \limsup_{t\to 0} \frac{|\Im\varphi(t)|}{|t| L(1/|t|)} < \infty,$$

where

$$L(x) = \int_1^x \left[\overline{F}(u) + F(-u) \right] \mathrm{d}u$$

is a slowly varying function such that $L(x)/\ell(x) \to \infty$ *as* $x \to \infty$*. In the one-sided case, i.e. if* $k \equiv 0$ *then*

$$|\Im\varphi(t)| \sim |t| L(1/|t|),$$

also holds. Furthermore if $h(x)/x$ *and* $k(x)/x$ *are ultimately nonincreasing then*

$$\Re(1-\varphi(t)) \sim |t|\ell(1/|t|) \int_0^{\infty-} \frac{\sin y}{y} \left(h(y/|t|) + k(y/|t|) \right) \mathrm{d}y.$$

Finally, for any $\alpha \in (0,2)$

$$\liminf_{t\to 0} \frac{\Re(1-\varphi(t))}{|t|^\alpha \ell(1/|t|)} > 0.$$

Remark 1. For $\alpha \in (0,1)$ some monotonicity conditions are needed for the finiteness of the improper integral in p_2. Indeed, it is easy to construct examples such that $\int_0^{\infty-} \ell(x) x^{-\alpha} \cos x \, \mathrm{d}x$ does not exist and $\lim_{x\to\infty} \ell(x) = 1$. On the other hand, for $\alpha > 1$ the function p_2 is defined as a Lebesgue integral.

We note that the $\alpha = 1$ case is more complicated, as usual. The main difficulty is that the order of the real and imaginary parts are different and in general, the imaginary part is larger. However, for symmetric distributions the imaginary part disappears. For a treatment of $\alpha = 1$ in the regular variation case we refer to [1]. See also Lemma 2 by Erickson [9], or Pitman [18]. For the corresponding result in the regularly varying case see Theorem 2.6.5 in Ibragimov and Linnik [11], for results on more general integral transform see also Theorem 4.1.5 in Bingham et al. [5].

Let X be a random variable with distribution function F. Assume that

$$\begin{gathered}\overline{F}(x) = \ell(x) x^{-\alpha} p_R(x), \ F(-x) = \widetilde{\ell}(x) x^{-\alpha} p_L(x), \ \ell(x) \sim \widetilde{\ell}(x), \\ \ell, \widetilde{\ell} \text{ slowly varying}, \ \alpha \in (0,2), \ p_R, p_L \in \mathcal{P}_r \cup \{0\}, \ p_L + p_R \neq 0.\end{gathered} \tag{3}$$

Notice that, due to the logarithmic periodicity of p_R and p_L the functions $p_R(x)x^{-\alpha}$ and $p_L(x)x^{-\alpha}$ are both nonincreasing. Therefore the following is an immediate consequence of Theorem 1.

Corollary 1. *Assume that (3) holds, and if $\mathbb{E}|X| < \infty$ then $\mathbb{E}X = 0$. Then, for $\alpha \neq 1$, as $t \to 0$*

$$1 - \varphi(t) \sim -\mathrm{i}\mathrm{sgn}(t)\, |t|^\alpha \ell(1/|t|) p_2(t),$$

where

$$p_2(t) = \begin{cases} \int_0^{\infty -} y^{-\alpha} \left[p_R(\frac{y}{|t|}) e^{\mathrm{i}y\mathrm{sgn}(t)} - p_L(\frac{y}{|t|}) e^{-\mathrm{i}y\mathrm{sgn}(t)} \right] \mathrm{d}y, & \alpha < 1, \\ \int_0^{\infty} y^{-\alpha} \left[p_R(\frac{y}{|t|}) (e^{\mathrm{i}y\mathrm{sgn}(t)} - 1) - p_L(\frac{y}{|t|}) (e^{-\mathrm{i}y\mathrm{sgn}(t)} - 1) \right] \mathrm{d}y, & \alpha > 1. \end{cases}$$

While for $\alpha = 1$

$$\Re(1 - \varphi(t)) \sim |t| \ell(1/|t|) \int_0^\infty \frac{\sin y}{y} (p_R(y/|t|) + p_L(y/|t|))\, \mathrm{d}y.$$

3 Semistable laws

Semistable laws are limits of centered and normed sums of iid random variables along subsequences k_n for which

$$k_n < k_{n+1} \text{ for } n \geq 1 \text{ and } \lim_{n \to \infty} \frac{k_{n+1}}{k_n} = c > 1 \tag{4}$$

hold. Since $c = 1$ corresponds to the stable case ([16, Theorem 2]), we assume that $c > 1$. In what follows we let c be as defined in (4).

The characteristic function of a non-Gaussian semistable random variable V has the form

$$\psi(t) = \mathbb{E} e^{\mathrm{i}tV} = \exp\left\{ \mathrm{i}ta + \int_{-\infty}^{\infty} (e^{\mathrm{i}tx} - 1 - \mathrm{i}tx \mathbb{I}\{|x| \leq 1\})\, \Lambda(\mathrm{d}x) \right\}, \tag{5}$$

with $\mathbb{I}\{\cdot\}$ standing for the indicator function, where $a \in \mathbb{R}$, and for the Lévy measure Λ, we have $\Lambda((x,\infty)) = M_R(x) x^{-\alpha}$, $\Lambda((-\infty,-x)) = M_L(x) x^{-\alpha}$, where $M_R, M_L \in \mathcal{P}_{c^{1/\alpha}} \cup \{0\}$, such that not both of them are 0. We further assume that V is nonstable, that is either M_R or M_L is not constant.

In the following $X, X_1, X_2, \ldots$ are iid random variables with distribution function $F(x) = \mathbb{P}(X \leq x)$. Let $S_n = X_1 + \ldots + X_n$ denote the partial sum. We fix a semistable random variable $V = V(R, M)$ with distribution function G and characteristic function ψ in (5). The random variable X belongs to the domain of geometric partial attraction of the semistable law G if there is a subsequence k_n for which (4) holds, and a norming and a centering sequence A_n, C_n, such that

$$\frac{\sum_{i=1}^{k_n} X_i - C_{k_n}}{A_{k_n}} \to^d V, \tag{6}$$

where $\to^d$ means convergence in distribution. By [16, Theorem 3], without loss of generality we may assume that

$$A_n = n^{1/\alpha} \ell_1(n), \quad C_n = n \int_{1/n}^{1-1/n} Q(s) \, \mathrm{d}s, \tag{7}$$

with some slowly varying function ℓ_1, where $Q(s) = \inf\{x : F(x) \geq s\}$, $s \in (0,1)$ is the quantile function of F.

In order to characterize the domain of geometric partial attraction we need some further definitions. As $k_{n+1}/k_n \to c > 1$, for any x large enough there is a unique k_n such that $A_{k_n} \leq x < A_{k_{n+1}}$. Define

$$\delta(x) = \frac{x}{A_{k_n}}.$$

Note that the definition of δ does depend on the norming sequence. Finally, let

$$x^{-\alpha} \ell(x) = \sup\{t : t^{-1/\alpha} \ell_1(1/t) > x\}.$$

Then $A_x = A(x) = x^{1/\alpha} \ell_1(x)$ and $B(y) = y^\alpha / \ell(y)$ are asymptotic inverses of each other, i.e.

$$A(B(x)) \sim B(A(x)) \sim x \quad \text{as } x \to \infty, \tag{8}$$

and $x^{1/\alpha} \ell_1(x) \sim \inf\{y : x^{-1} \geq y^{-\alpha} \ell(y)\}$. Thus ℓ and ℓ_1 asymptotically determines each other. For properties of asymptotic inverse of regularly varying functions we refer to [5, Section 1.7].

By Corollary 3 in [16] (6) holds on the subsequence k_n with norming sequence A_{k_n} if and only if

$$\begin{aligned} \overline{F}(x) &= \frac{\ell(x)}{x^\alpha} [M_R(\delta(x)) + h_R(x)], \\ F(-x) &= \frac{\ell(x)}{x^\alpha} [M_L(\delta(x)) + h_L(x)], \end{aligned} \tag{9}$$

where h_R, h_L are right-continuous functions such that $\lim_{n\to\infty} h_{R/L}(A_{k_n} x) = 0$, whenever x is a continuity point of $M_{R/L}$. Moreover, if $M_{R/L}$ is continuous, then $\lim_{x\to\infty} h_{R/L}(x) = 0$.

Clearly, (9) implies (2). Thus if F belongs to the domain of geometric partial attraction of a semistable law, then Theorem 1 applies.

Conditions (3) and (9) are similar, but the δ function in (9) complicates the asymptotics. In the special case $\ell_1 \equiv 1$ and $k_n = \lfloor c^n \rfloor$, the function $\delta(x)$ can be replaced by x in (9). Then (3) with $\ell \sim 1$ is equivalent to (9) with $h_{R/L}(x) \to 0$ as $x \to \infty$. In general, (3) is a stronger condition.

Lemma 1. *Assume (3). Then there exists a subsequence (k_n) satisfying (4) with $c = r^\alpha$ such that (9) holds with $M_R = p_R$ and $M_L = p_L$.*

Proof. Recall the definition of A and B. Define $k_n = B(c^{n/\alpha})$. For notational ease we suppress the integer part. Since B is regularly varying with index α, condition (4) holds. By (8) we have $A_{k_n} \sim c^{n/\alpha}$. Writing

$$\overline{F}(x) = \frac{\ell(x)}{x^\alpha} \left[p_R(\delta(x)) + (p_R(x) - p_R(\delta(x))) \right],$$

we only have to show that $\lim_{n\to\infty} h_R(A_{k_n} x) = 0$ holds whenever x is a continuity point of p_R, for $h_R(x) = p_R(x) - p_R(\delta(x))$. For simplicity fix $x \in (1, c^{1/\alpha})$ to be a continuity point of p_R. Then $A_{k_n} \le A_{k_n} x < A_{k_n+1}$ for large n, thus $\delta(A_{k_n} x) = A_{k_n} x / A_{k_n} = x$. On the other hand, by the logarithmic periodicity of p_R

$$p_R(A_{k_n} x) = p_R(c^{-n/\alpha} A_{k_n} x) \to p_R(x),$$

which implies that $h_R(A_{k_n} x) \to 0$. Clearly, the same argument works for $F(-x)$. $\square$

It is easy to give examples that show that the converse is not true. Choose $\alpha = 1$, $c = 2$, $\ell(x) = \ell_1(x) = \log_2 x$, $k_n = 2^n$, $p_R = 2^{\{\log_2 x\}}$, $p_L \equiv 0$, where $\log_2$ stands for the base-2 logarithm, and $\{\cdot\}$ is the fractional part. Define for $x > 3$

$$\overline{F}(x) = 2^{-\lfloor \log_2 x - \log_2 \log_2 x \rfloor} = \frac{\log_2 x}{x} 2^{\{\log_2 x - \log_2 \log_2 x\}}.$$

Some lengthy but straightforward calculation shows that (9) holds, but (3) does not.

For $x > 0$ (large) we define the position parameter as

$$\gamma_x = \gamma(x) = \frac{x}{k_n}, \quad \text{where } k_{n-1} < x \le k_n. \tag{10}$$

We say u_n *circularly converges to* $u \in (c^{-1}, 1]$, $u_n \stackrel{cir}{\to} u$, if $u \in (c^{-1}, 1)$ and $u_n \to u$ in the usual sense, or $u = 1$ and (u_n) has limit points c^{-1}, or 1, or both. From Theorem 1 [8] we see that (6) holds along a subsequence $(n_r)_{r=1}^\infty$ (instead of k_n) if and only if $\gamma_{n_r} \stackrel{cir}{\to} \lambda \in (c^{-1}, 1]$ as $r \to \infty$. In this case, by [8, Theorem 1] (or directly from the relation $-R_\lambda(x) = \lim_{r\to\infty} n_r \overline{F}(A_{n_r} x)$) the Lévy measure of the limit

$$\begin{aligned} \Lambda_\lambda((x,\infty)) &= x^{-\alpha} M_R(\lambda^{1/\alpha} x) \\ \Lambda_\lambda((-\infty,-x)) &= x^{-\alpha} M_L(\lambda^{1/\alpha} x), \quad x > 0. \end{aligned}$$

For any $\lambda > 0$ let V_λ be a semistable random variable with characteristic and distribution function

$$\begin{aligned} \psi_\lambda(t) &= \mathbb{E} e^{\mathrm{i} t V_\lambda} = \exp\left\{ \mathrm{i} t a_\lambda + \int_{-\infty}^{\infty} \left(e^{\mathrm{i} t x} - 1 - \mathrm{i} t x \mathbb{I}\{|x| \le 1\} \right) \Lambda_\lambda(\mathrm{d}x) \right\} \\ G_\lambda(x) &= \mathbb{P}(V_\lambda \le x), \end{aligned} \tag{11}$$

where $a_\lambda \in \mathbb{R}$, for its precise form see [8, Theorem 1]. Thus, whenever $\gamma_{n_r} \stackrel{cir}{\to} \lambda$,

$$\frac{\sum_{i=1}^{n_r} X_i - C_{n_r}}{A_{n_r}} \to^d V_\lambda \quad \text{as } r \to \infty.$$

To ease notation we define Λ_λ, G_λ for any $\lambda > 0$, but note that $\Lambda_{c\lambda} \equiv \Lambda_\lambda$, $G_{c\lambda} \equiv G_\lambda$, so these functions, distributions are different for $\lambda \in (c^{-1}, 1]$.

Let $X, X_1, X_2, \ldots$ be iid random variables with distribution function F such that (9) holds. Csörgő and Megyesi [8, Theorem 2] showed the following merging result:

$$\lim_{n\to\infty} \sup_{x\in\mathbb{R}} \left| \mathbb{P}\left(\frac{S_n - C_n}{A_n} \leq x \right) - G_{\gamma_n}(x) \right| = 0. \tag{12}$$

The main theorem in [7] implies that G_λ is C^∞, in particular its density function g_λ exists.

4 Local limit theorems for semistable laws

We prove local limit theorems for the distributions in the domain of geometric partial attraction of semistable laws. As usual we have to distinguish between lattice and nonlattice distributions. We first consider the lattice case.

A random variable, or its distribution is called lattice, if it is concentrated on the set $\{a + h\mathbb{Z}\}$ for some $a \in \mathbb{R}$ and $h > 0$. If $a = 0$ the distribution is called arithmetic, or centered lattice. The largest possible h is the span of the lattice distribution. We assume that $a = 0$ and $h = 1$, i.e. the distribution is integer valued with span 1. We prove the analogue of Gnedenko's Local Limit Theorem ([5, Theorem 8.4.1], [11, Theorem 4.2.1]). The statement can be readily extended to the general lattice case.

Theorem 2. *Let $X, X_1, \ldots$ be integer valued iid random variables with span 1, such that (9) holds. Then*

$$\lim_{n\to\infty} \sup_{k} |A_n \mathbb{P}(S_n = k) - g_{\gamma_n}((k - C_n)/A_n)| = 0.$$

The Fourier analytic proof relies on the inversion formula

$$\mathbb{P}(S_n = k) = \frac{1}{2\pi} \int_{-\pi}^{\pi} e^{-\mathrm{i}tk} \varphi(t)^n \, \mathrm{d}t, \tag{13}$$

and on the merging result (12).

In the nonlattice case we extend Stone's local limit theorem [20], see also [5, Theorem 8.4.2].

Theorem 3. *Let $X, X_1, \ldots$ be iid nonlattice random variables such that (9) holds. Then for any $h > 0$*

$$\lim_{n\to\infty} \sup_{x} \left| \frac{A_n}{2h} \mathbb{P}(S_n \in (x - h, x + h]) - g_{\gamma_n}((x - C_n)/A_n) \right| = 0.$$

The difficulty in the nonlattice setup is the lack of a simple inversion formula as (13). Instead, in the usual Fourier inversion formula one has to take limits. The standard trick to overcome this is to add a small continuous random variable with compactly supported characteristic function. Fix $T > 0$ and let Y be a random variable with density and characteristic function

$$j(x) = \frac{1 - \cos(Tx)}{\pi T x^2}, \quad \eta(t) = \begin{cases} 1 - \frac{|t|}{T}, & \text{for } t \in [-T, T], \\ 0, & \text{otherwise.} \end{cases} \tag{14}$$

Then the inversion formula gives

$$\mathbb{P}(S_n + Y \in (x-h, x+h]) = \frac{h}{\pi} \int_{-T}^{T} \frac{\sin th}{th} e^{-\mathrm{i}tx} \varphi^n(t) \left(1 - \frac{|t|}{T}\right) \mathrm{d}t. \tag{15}$$

Having this formula the proof goes as in the lattice case, only at the end we have to get rid of the small perturbation.

5 Strong renewal theorem in the semistable setting

In what follows, we consider only nonnegative random variables with infinite mean in the domain of geometric partial attraction of a semistable law. In particular, $\alpha \in (0,1]$. For $\alpha \in (0,1)$ there is no need for centering, i.e. in (7) we choose $C_n \equiv 0$.

Using the local limit theorems, we obtain the analogue of [10, Theorem 1.1] in the semistable setting, that is assuming (9). Unlike in [10], we cannot use the precise asymptotic of $(1-\varphi(t))^{-1}$. Instead, we heavily exploit the LLT, namely Theorems 2 and 3 together with the asymptotic of $(1-\varphi(t))^{-1}$ obtained in Theorem 1.

We start with the arithmetic case, and assume that X is integer valued with span 1. With the same notation as in [10] introduce the renewal sequence

$$u_n = \sum_{k=0}^{\infty} \mathbb{P}(S_k = n) = \frac{1}{\pi} \Re \int_0^{\pi} (1-\varphi(t))^{-1} e^{-\mathrm{i}nt} \, \mathrm{d}t, \tag{16}$$

where we used the inversion (13).

Theorem 4. *Assume that X is a nonnegative integer valued random variable with span 1 and (9) holds with $\alpha \in (1/2, 1)$. Set $B(x) = x^{\alpha} \ell(x)^{-1}$. Then*

$$\lim_{n\to\infty} \left| n^{1-\alpha} \ell(n) u_n - \alpha \int_0^{\infty} g_{\gamma(B(n)x^{-\alpha})}(x) \, x^{-\alpha} \, \mathrm{d}x \right| = 0.$$

The estimate of the main term above holds in the whole range $\alpha \in (0,1)$, and it is treated separately in the following statement. It is the analogue of Lemma 2.2.1 in [10].

Lemma 2. *Assume that X is a nonnegative integer valued random variable with span 1 and (9) holds with $\alpha \in (0,1)$. For any $L > 1$*

$$\limsup_{n\to\infty} \left| n^{1-\alpha} \ell(n) \sum_{k=B(n/L^2)}^{B(nL)} \mathbb{P}(S_k = n) - \alpha \int_{L^{-1}}^{L^2} g_{\gamma(B(n)y^{-\alpha})}(y) y^{-\alpha} \, \mathrm{d}y \right| \le L^{-1}.$$

Recall that the renewal function is denoted by $U(y) := \sum_n F^{n*}(y)$. The next result gives the SRT in the semistable nonarithmetic case.

Theorem 5. *Assume that X is a nonnegative nonarithmetic random variable and (9) holds with $\alpha \in (1/2, 1)$. Set $B(x) = x^{\alpha} \ell(x)^{-1}$. Then for any $h > 0$,*

$$\lim_{y\to\infty} \left| \frac{y^{1-\alpha} \ell(y)}{2h} \left(U(y+h) - U(y-h) \right) - \alpha \int_0^{\infty} g_{\gamma(B(y)x^{-\alpha})}(x) \, x^{-\alpha} \, \mathrm{d}x \right| = 0.$$

In the nonarithmetic lattice case without loss of generality we assume that X has span 1, and that $X \in a + \mathbb{N}$, where $a \in (0,1)$ is irrational. The proof of Theorem 5 in this case is essentially the same as the proof of Theorem 4, except for the treatment of the leading term. To make this precise we introduce the following notation.

Let $\widetilde{X} = X - a$ denote the centered version of X, and $\widetilde{S}_k = S_k - ka$. Fix $0 < h < 1/2$. Define

$$I_{k,y} = \begin{cases} 1, & \text{if } (y - ka - h, y - ka + h] \text{ contains an integer}, \\ 0, & \text{otherwise}, \end{cases}$$

and let $\langle y - ka \rangle$ denote the unique integer in the interval $(y - ka - h, y - ka + h]$ if $I_{k,y} = 1$, and 0 otherwise. Then

$$\mathbb{P}(S_k \in (y-h, y+h]) = \mathbb{P}(\widetilde{S}_k \in (y - ka - h, y - ka + h]) = I_{k,y} \mathbb{P}(\widetilde{S}_k = \langle y - ka \rangle)$$

and

$$U(y+h) - U(y-h) = \sum_{k=0}^{\infty} \mathbb{P}(S_k \in (y-h, y+h]) = \sum_{k=0}^{\infty} I_{k,y} \mathbb{P}(\widetilde{S}_k = \langle y - ka \rangle). \tag{17}$$

Lemma 3. *Assume that X is a nonnegative, nonarithmetic and lattice with span 1. Suppose that (9) holds with $\alpha \in (0,1)$. Then for any $h \in (0, 1/2)$*

$$\limsup_{y \to \infty} \left| y^{1-\alpha} \ell(y) \sum_{k=B(y/L^2)}^{B(yL)} I_{k,y} \mathbb{P}(\widetilde{S}_k = \langle y - ka \rangle) - 2h\alpha \int_{L^{-1}}^{L^2} g_{\gamma(B(y)x^{-\alpha})}(x) x^{-\alpha} \, \mathrm{d}x \right| \le L^{-1}.$$

In the proof of the nonlattice case of Theorem 5 we first apply the ideas of the arithmetic case to the smoothed version as in (15), then 'unsmooth' the limit.

The case $\alpha = 1$ is different, already in the regularly varying framework. However, the difference is more apparent in the semistable setup, since the usual limit result holds. We assume that $\mathbb{E}X = \infty$, because if it was finite, the classical renewal theorem would work. Our results are special cases of Lemma 67 in Uchiyama [21] and Corollary 2 in [22]. For completeness, we state the results.

Now, instead of (16) we use the inversion formula

$$u_n = \frac{2}{\pi} \int_0^{\pi} W(t) \cos nt \, \mathrm{d}t, \tag{18}$$

see Lemma 3.1.1 in [10] or (2.5) in [9], where

$$W(t) = \Re \frac{1}{1 - \varphi(t)} = \frac{\Re(1 - \varphi(t))}{|1 - \varphi(t)|^2}.$$

The expectation 'almost exists' in the sense that the truncated first moment

$$L(x) = \int_1^x \overline{F}(u) \mathrm{d}u$$

is slowly varying. The key ingredient is Lemma 1 in [22], the slow variation of the integral of W. The regularly varying version is Lemma 3 in [9].

Lemma 4 (Lemma 1 in [22]). *Assume that X is a nonnegative random variable such that (9) holds with $\alpha = 1$, and $\mathbb{E}X = \infty$. Then as $x \to \infty$*

$$\int_0^{1/x} W(t)\,\mathrm{d}t \sim L(x)^{-1}\frac{\pi}{2}.$$

The arithmetic version of the next result is a special case of Lemma 67 in [21], and the nonarithmetic version is a special case of Corollary 2 in [22]. The proof is based on Lemma 4 and on the argument in [9].

Theorem 6. *Assume that X is a nonnegative random variable such that (9) holds with $\alpha = 1$, and $\mathbb{E}X = \infty$. If X is integer valued with span 1 then*

$$\lim_{n\to\infty} L(n)u_n = 1,$$

while if X is nonarithmetic then for any $h > 0$

$$\lim_{y\to\infty} L(y)(U(y+h) - U(y-h)) = 2h.$$

6 Renewal function in the semistable setting

In this section we determine the asymptotic of $U(y)$, as $y \to \infty$ for any $\alpha \in (0,1)$. This time we will not exploit the LLT, but simply the merging result (28) in terms of the characteristic function. In short, the basic observation is that the semistable limit theorem, equivalently the merging result (28), is the only thing one needs to obtain the asymptotic of $U(y)$ for both arithmetic and nonarithmetic semistable distributions. This type of argument is not needed (although it makes sense) to obtain the asymptotic of $U(y)$ in the regularly varying (stable) setting where Karamata's Tauberian theorem gives immediate results.

Recall that G_{γ_k} is the semistable distribution defined in (11). We note that

$$\int_0^\infty G_{\gamma(B(y)x^{-\alpha})}(x)x^{-\alpha-1}\,\mathrm{d}x < \infty.$$

At ∞ this is clear, while at 0 this follows from the fact that $G_\gamma(x)$ is exponentially small around 0, see Theorem 1 by Bingham [4] (or Lemma 2 in [15]).

Theorem 7. *Assume that X is a nonnegative random variable and (9) holds with $\alpha \in (0,1)$. Set $B(x) = x^\alpha \ell(x)^{-1}$. Then*

$$\lim_{y\to\infty}\left| y^{-\alpha}\ell(y)U(y) - \alpha\int_0^\infty G_{\gamma(B(y)x^{-\alpha})}(x)x^{-\alpha-1}\,\mathrm{d}x\right| = 0.$$

As a consequence of Theorem 6 we obtain for $\alpha = 1$ the following.

Corollary 2. *Assume that X is a nonnegative random variable such that (9) holds with $\alpha = 1$ and $\mathbb{E}X = \infty$. Then, as $y \to \infty$*

$$U(y) \sim \frac{y}{L(y)}.$$

It is natural to expect that under some additional assumption the SRT in Theorems 4 and 5 remains true for $\alpha \in (0,1/2]$. The problem to find the necessary and sufficient conditions for the SRT in the regularly varying setup was open for more than 50 years, and was solved recently by Caravenna and Doney [6]. In the regularly varying setup, already in the first papers [10, 9] it was pointed out that for $\alpha \in (0,1/2]$ the results hold with lim inf instead of lim, moreover the exceptional set is negligible in the sense that has density 0.

We do not know what happens for $\alpha \in (0,1/2]$. We only point out the essential difficulty to obtain further asymptotics. By Lemma 2 for any $\alpha \in (0,1)$

$$\liminf_{n\to\infty} \left[n^{1-\alpha}\ell(n)\, u_n - \alpha \int_0^\infty g_{\gamma(B(n)y^{-\alpha})}(y) y^{-\alpha}\,\mathrm{d}y \right] \geq 0. \tag{19}$$

In the regularly varying case (19), together with Theorem 7, is enough to conclude that for $\alpha \in (0,1/2]$ the liminf in (19) is 0, moreover the limit exists and equals 0 except in a set of density 0; see [10, Theorem 1.1], [9, Theorem 2], or [5, Theorem 8.6.6]. If G is any distribution function of a nonnegative random variable with density g, then simply

$$\alpha \int_0^\infty G(x) x^{-\alpha-1}\,\mathrm{d}x = \int_0^\infty g(x) x^{-\alpha}\,\mathrm{d}x.$$

In our case the distribution function itself depends on x, thus the argument above does not work.

7 Proofs

7.1 Proof of Theorem 1

Case 1: $\alpha \in (0,1)$. Integration by parts shows

$$\begin{aligned} 1 - \varphi(t) &= \int_{[0,\infty)} (e^{\mathrm{i}tx} - 1)\,\mathrm{d}\overline{F}(x) + \int_{(0,\infty)} (e^{-\mathrm{i}tx} - 1)\,\mathrm{d}F(-x) \\ &= -\mathrm{i}t \left(\int_0^{\infty-} \overline{F}(x) e^{\mathrm{i}tx}\,\mathrm{d}x - \int_0^{\infty-} F(-x) e^{-\mathrm{i}tx}\,\mathrm{d}x \right) \\ &= -\mathrm{i}\mathrm{sgn}(t)|t|^\alpha \int_0^{\infty-} \ell(\frac{y}{|t|}) y^{-\alpha} \left(h(\frac{y}{|t|}) e^{\mathrm{i}\mathrm{sgn}(t) y} - k(\frac{y}{|t|}) e^{-\mathrm{i}\mathrm{sgn}(t) y} \right) \mathrm{d}y. \end{aligned} \tag{20}$$

To ease notation we write $x = |t|^{-1}$. We consider the first term in the integral above, and assume $t > 0$. For any $0 < a < b < \infty$ by the uniform convergence theorem for slowly varying functions as $x \to \infty$

$$\frac{1}{\ell(x)} \int_a^b h(yx)\ell(yx) y^{-\alpha} e^{\mathrm{i}y} \mathrm{d}y - \int_a^b h(yx) y^{-\alpha} e^{\mathrm{i}y} \mathrm{d}y \to 0.$$

Next we show that the contribution of the integral on $(0,a)$, and on (b,∞) is negligible. Indeed, by Karamata's theorem

$$\begin{aligned} \left| \int_0^a h(yx)\ell(yx) y^{-\alpha} e^{\mathrm{i}y} \mathrm{d}y \right| &\leq C\, x^{\alpha-1} \int_0^{ax} \ell(u) u^{-\alpha}\,\mathrm{d}u \\ &\sim C\, a^{1-\alpha} \ell(x) \quad \text{as } x \to \infty. \end{aligned} \tag{21}$$

In the following $C > 0$ is always a finite positive constant, which may be different from line to line, and its actual value is not important for us. On (b, ∞) we consider only the real part. Since the function $\overline{F}(x) = \ell(x)h(x)x^{-\alpha}$ is nonincreasing, by the second mean value theorem for definite integrals we obtain

$$\begin{aligned}\left| \int_b^{\infty-} h(yx)\ell(yx)y^{-\alpha} \cos y \,\mathrm{d}y \right| &\le h(bx)\ell(bx)b^{-\alpha} \sup_{z>b} \left| \int_b^z \cos y \,\mathrm{d}y \right| \\ &\le C\, \ell(x) b^{-\alpha}.\end{aligned} \tag{22}$$

Clearly, the inequalities (21) and (22) hold true for the second term in (20), therefore

$$\left| \frac{1-\varphi(t)}{|t|^\alpha \ell(1/|t|)} \right| \le C \left(a^{1-\alpha} + b^{-\alpha} + \int_a^b y^{-\alpha} \,\mathrm{d}y \right),$$

showing the first part of the theorem.

For the more precise asymptotic first note that with the extra monotonicity condition the function p_2 is well-defined. This follows from the Leibniz criterion for the finiteness of an alternating series, recalling the fact that $h(y)y^{-\alpha}$ and $k(y)y^{-\alpha}$ are ultimately nonincreasing. Moreover, the inequalities (21) and (22) hold true with $\ell(x) \equiv 1$. Therefore

$$\begin{aligned}&\left| \frac{1}{\ell(x)} \int_0^{\infty-} h(xy)\ell(xy)y^{-\alpha} e^{\mathrm{i}y} \,\mathrm{d}y - \int_0^{\infty-} h(xy)y^{-\alpha}e^{\mathrm{i}y} \,\mathrm{d}y \right| \\ &\le C(a^{1-\alpha} + b^{-\alpha}) + \left| \frac{1}{\ell(x)} \int_a^b h(yx)\ell(yx)y^{-\alpha}e^{\mathrm{i}y} \,\mathrm{d}y - \int_a^b h(yx)y^{-\alpha}e^{\mathrm{i}y} \,\mathrm{d}y \right|,\end{aligned}$$

and the statement follows by letting $x = 1/|t| \to \infty$, then $a \to 0$ and $b \to \infty$.

Case 2: $\alpha \in (1, 2)$. In this case $\mathbb{E}X$ exists, and by subtracting, and using that $\mathbb{E}e^{\mathrm{i}tX} = 1 + \mathrm{i}t\mathbb{E}X + o(t)$ as $t \downarrow 0$, we may and do assume that $\mathbb{E}X = 0$. Similarly as in (20)

$$\begin{aligned}1 - \varphi(t) &= \int_{\mathbb{R}} \left(1 - e^{\mathrm{i}tx} + \mathrm{i}tx \right) \mathrm{d}F(x) \\ &= -\mathrm{i}\mathrm{sgn}(t)|t|^\alpha \int_0^\infty \frac{\ell(y/|t|)}{y^\alpha} \left((e^{\mathrm{i}\mathrm{sgn}(t)y} - 1)h(y/t) - (e^{-\mathrm{i}\mathrm{sgn}(t)y} - 1)k(y/t) \right) \mathrm{d}y.\end{aligned}$$

As above, for any $0 < a < b < \infty$ as $x = |t|^{-1} \to \infty$

$$\frac{1}{\ell(x)} \int_a^b h(yx)\ell(yx)y^{-\alpha}(e^{\mathrm{i}y} - 1) \,\mathrm{d}y - \int_a^b h(yx)y^{-\alpha}(e^{\mathrm{i}y} - 1) \,\mathrm{d}y \to 0.$$

Next we show that the contribution of the integral on $(0, a)$ and on (b, ∞) is negligible. For y small enough $e^{\mathrm{i}y} - 1 \sim \mathrm{i}y$, thus by Karamata's theorem

$$\begin{aligned}\left| \int_0^a h(yx)\ell(yx)y^{-\alpha}(e^{\mathrm{i}y} - 1) \,\mathrm{d}y \right| &\le C \int_0^a \ell(yx)y^{1-\alpha} \,\mathrm{d}y \\ &\sim C\, a^{2-\alpha}\ell(x) \quad \text{as } x \to \infty.\end{aligned} \tag{23}$$

Similarly, on (b,∞) we have

$$\left| \int_b^\infty h(yx)\ell(yx)y^{-\alpha}(e^{\mathrm{i}y}-1)\,\mathrm{d}y \right| \le C\ell(x)b^{1-\alpha}. \tag{24}$$

Since the inequalities (23) and (24) hold with $\ell(x)\equiv 1$, therefore

$$\begin{aligned}
&\left| \frac{1}{\ell(x)} \int_0^\infty y^{-\alpha}h(xy)\ell(xy)(e^{\mathrm{i}y}-1)\,\mathrm{d}y - \int_0^\infty y^{-\alpha}h(xy)(e^{\mathrm{i}y}-1)\,\mathrm{d}y \right| \\
&\le C\left(a^{2-\alpha}+b^{1-\alpha}\right) + \left| \frac{1}{\ell(x)} \int_a^b h(yx)\ell(yx)y^{-\alpha}e^{\mathrm{i}y}\,\mathrm{d}y - \int_a^b h(yx)y^{-\alpha}e^{\mathrm{i}y}\,\mathrm{d}y \right|,
\end{aligned}$$

and statement follows by letting $x=1/|t|\to\infty$, then $a\to 0$ and $b\to\infty$.

Case 3: $\alpha=1$. In this case the calculations are more troublesome. Using that

$$\int_{(-1,1]} x\,\mathrm{d}F(x) = \int_0^1 [\overline{F}(x)-F(-x)]\,\mathrm{d}x - \overline{F}(1)+F(-1)$$

and that $e^{\mathrm{i}tx}-1-\mathrm{i}tx=O(t^2)$ for $x\in[-1,1]$, straightforward calculation shows

$$\begin{aligned}
1-\varphi(t) &= \int_{\mathbb{R}} (1-e^{\mathrm{i}tx})\,\mathrm{d}F(x) \\
&= -\mathrm{i}t \int_1^{\infty-} \left(\overline{F}(x)e^{\mathrm{i}tx} - F(-x)e^{-\mathrm{i}tx}\right)\mathrm{d}x - \mathrm{i}t\int_0^1 [\overline{F}(x)-F(-x)]\mathrm{d}x + O(t^2) \\
&= -\mathrm{i}\,\mathrm{sgn}(t)|t| \int_{|t|}^{\infty-} \frac{\ell(y/|t|)}{y}\left[h(y/|t|)e^{\mathrm{i}\,\mathrm{sgn}(t)y} - k(y/|t|)e^{-\mathrm{i}\,\mathrm{sgn}(t)y}\right]\mathrm{d}y \\
&\quad - \mathrm{i}t\int_0^1 [\overline{F}(x)-F(-x)]\,\mathrm{d}x + O(t^2).
\end{aligned} \tag{25}$$

In this case the order of the real and imaginary parts are different. As $\sin y\sim y$ at 0, using the arguments in (21) and (22) we have

$$\left| \frac{1}{\ell(1/|t|)} \int_{|t|}^{\infty-} \frac{\sin y}{y}\ell(y/|t|)h(y/|t|)\,\mathrm{d}y - \int_0^b \frac{\sin y}{y}h(y/|t|)\,\mathrm{d}y \right| \le Cb^{-1},$$

for t small enough, for some $C>0$. Moreover, if $h(y)y^{-1}$ is ultimately monotone this can be strengthened to

$$\left| \frac{1}{\ell(1/|t|)} \int_{|t|}^{\infty-} \frac{\sin y}{y}\ell(y/|t|)h(y/|t|)\,\mathrm{d}y - \int_0^{\infty-} \frac{\sin y}{y}h(y/|t|)\,\mathrm{d}y \right| \to 0$$

as $t\to 0$. Thus the statement for the real part follows.

For the imaginary part in (25) we obtain as in (22)

$$\left| \int_1^{\infty-} \frac{\cos y}{y}\ell(y/|t|)h(y/|t|)\,\mathrm{d}y \right| \le C\ell(1/|t|),$$

while

$$\int_{|t|}^{1} \frac{\cos y}{y} \ell(y/|t|) h(y/|t|)\, \mathrm{d}y \sim \int_{1}^{1/|t|} \frac{\ell(y)h(y)}{y}\, \mathrm{d}y =: L_h(1/|t|).$$

If h is nonzero then $L_h(x)/\ell(x) \to \infty$ as $x \to \infty$. To see this write

$$\liminf_{x\to\infty} \frac{L_h(x)}{\ell(x)} \geq \liminf_{x\to\infty} \int_{\varepsilon x}^{x} \frac{\ell(u)}{\ell(x)} \frac{h(u)}{u} \mathrm{d}u \geq \inf h \, \log \varepsilon^{-1},$$

as $\varepsilon \downarrow 0$ the claim follows. Moreover, L_h is slowly varying. Indeed, for $\lambda > 1$ fixed

$$\begin{aligned} L_h(\lambda x) - L_h(x) &= \int_x^{\lambda x} \frac{\ell(u)h(u)}{u} \mathrm{d}u \\ &\sim \ell(x) \int_x^{\lambda x} \frac{h(u)}{u} \mathrm{d}u \leq \ell(x) \log \lambda \, \sup h. \end{aligned}$$

Since $\ell(x)/L_h(x) \to 0$, we have $L_h(\lambda x)/L_h(x) \to 1$, that is, L_h is slowly varying. The same argument shows that L is slowly varying too. (We note that in (2.6.34) in [11] it is wrongly stated that $L_h(x) \sim \ell(x) \log x$.) The bound for the imaginary part follows from the inequality $L_h(x) \leq CL(x)$. Finally, if $k \equiv 0$ then $L_h \equiv L$.

Strict positivity of the real part. The following argument works for any $\alpha \in (0, 2)$. Let $a_0 > 0$ be a small number, chosen later. Using that $\sin y > 2y/\pi$ for $y \in (0, \pi/2)$ we have

$$\begin{aligned} \Re(1 - \varphi(t)) &= \int_0^\infty (1 - \cos tx) \mathrm{d}F(x) \\ &= \int_0^\infty 2 \sin^2 \frac{tx}{2} \, \mathrm{d}F(x) \\ &\geq 2 \int_{a_0/t}^{\pi/t} \left(\frac{tx}{\pi} \right)^2 \mathrm{d}F(x) \\ &\geq \frac{2}{\pi^2} a_0^2 \left[\overline{F}(a_0/t) - \overline{F}(\pi/t) \right] \\ &\geq t^\alpha \ell(1/t) \frac{2a_0^2}{\pi^2} \left[\frac{h(a_0/t)}{a_0^\alpha} \frac{\ell(a_0/t)}{\ell(1/t)} - \frac{h(\pi/t)}{\pi^\alpha} \frac{\ell(\pi/t)}{\ell(1/t)} \right]. \end{aligned}$$

Since ℓ is slowly varying $\ell(\lambda/t)/\ell(1/t) \to 1$ for any λ, therefore the expression in the bracket is strictly positive for $a_0 > 0$ small enough.

7.2 Local limit theorems

Before the proof of the LLTs we collect some important facts on the characteristic function φ, which we use later.

Lemma 5. *Let X be an integer valued random variable with span 1 such that (2) holds. Let $\varphi(t) = \mathbb{E}e^{\mathrm{i}tX}$ denote its characteristic function. Then there exist positive numbers ν_1, ν_2, ν_3 such that* *(i) if $\alpha \in (0,2)$ then $|\varphi(t)| \le e^{-\nu_1 |t|^\alpha \ell(1/|t|)}$, for $t \in [-\pi, \pi]$.*

(ii) if $\alpha \in (0,1)$ then $|(1-\varphi(t))^{-1}| \le \nu_2 |t|^{-\alpha} \ell(1/t)^{-1}$, for $t \in [-\pi, \pi]$;

(iii) if $\alpha \in (0,1)$ then $|\varphi(t+h) - \varphi(t)| \le \nu_3 |h|^\alpha \ell(1/|h|)$, for $t \in \mathbb{R}$, $h \in [-1,1]$, and if $\alpha = 1$ then $|\varphi(t+h) - \varphi(t)| \le \nu_3 |h| L(1/|h|)$.

In the nonlattice case (i)–(iii) remain valid and (i)–(ii) can be extended to any compact interval.

Proof Using that $\varphi(t) = e^{\Re \log \varphi(t)}$, and $\log \varphi(t) \sim \varphi(t) - 1$ around zero, the first three statements follows from Theorem 1 for $|t|$ small. Possibly changing the constant, we can extend the inequality to the desired interval.

The fourth inequality follows from (2) together with a classical argument; see, for instance, [10, Proof of Lemma 3.3.2] or Lemma 5 in [9]. □

Proof of Theorem 2 Using the inversion formula (13) we have

$$\mathbb{P}(S_n = k) = \frac{1}{2\pi A_n} \int_{-A_n\pi}^{A_n\pi} e^{-\mathrm{i}tk/A_n} \varphi(t/A_n)^n \,\mathrm{d}t.$$

By the density inversion theorem the limiting density can be written as

$$g_\lambda(x) = \frac{1}{2\pi} \int_{-\infty}^{\infty} e^{-\mathrm{i}tx} \psi_\lambda(t) \,\mathrm{d}t. \tag{26}$$

Thus

$$2\pi \left| A_n \mathbb{P}(S_n = k) - g_{\gamma_n}((k - C_n)/A_n) \right| \le I_1 + I_2 + I_3 + I_4,$$

where

$$\begin{aligned}
I_1 &= \int_{-K}^{K} \left| e^{-\mathrm{i}tC_n/A_n} \varphi(t/A_n)^n - \psi_{\gamma_n}(t) \right| \mathrm{d}t \\
I_2 &= \int_{K \le |t| \le \varepsilon A_n} |\varphi(t/A_n)|^n \,\mathrm{d}t \\
I_3 &= \int_{\varepsilon A_n \le |t| \le \pi A_n} |\varphi(t/A_n)|^n \,\mathrm{d}t \\
I_4 &= \int_{|t| > K} |\psi_{\gamma_n}(t)| \,\mathrm{d}t,
\end{aligned} \tag{27}$$

where $K > 0$ is a large constant.

By Theorem 3.1 in [14] the merging relation (12) holds if and only if for any $t \in \mathbb{R}$ as $n \to \infty$

$$\mathbb{E} e^{\mathrm{i}t(S_n - C_n)/A_n} - \mathbb{E} e^{\mathrm{i}tV_{\gamma_n}} = e^{-\mathrm{i}tC_n/A_n} \varphi(t/A_n)^n - \psi_{\gamma_n}(t) \to 0. \tag{28}$$

Moreover, since both $((S_n - C_n)/A_n)_n$ and $(V_{\gamma_n})_n$ are tight, the convergence in (28) is uniform on any finite interval $[-K, K]$. Therefore $I_1 \to 0$ as $n \to \infty$ for any $K > 0$.

To estimate I_2 we use Lemma 5 (i) together with the Potter bounds. Using the inverse relation (8) we have

$$\begin{aligned} n(t/A_n)^{\alpha}\ell(A_n/t) &= nt^{\alpha}\frac{\ell(A_n/t)}{\ell(A_n)}\frac{\ell(A_n)}{A_n^{\alpha}} \\ &\sim t^{\alpha}\frac{\ell(A_n/t)}{\ell(A_n)} \geq 2^{-1}t^{\alpha'}, \end{aligned}$$

for any $\alpha' \in (0,\alpha)$, where the last inequality follows from the Potter bounds. Therefore, for $\varepsilon > 0$ small enough

$$I_2 \leq \int_K^{\infty} e^{-2^{-1}\nu_1 t^{\alpha'}} \mathrm{d}t,$$

which goes to 0 as $K \to \infty$.

Since X is lattice with span 1

$$|\varphi(t)| \leq a < 1 \quad \text{for some } \ a \in (0,1) \ \text{ for } \ |t| \in [\varepsilon, \pi]. \tag{29}$$

Therefore $I_3 \leq 2\pi A_n a^n$, while $\psi_{\gamma_n}(t)$ is uniformly integrable by (7) in [7], implying that $\lim_{K\to\infty} I_4 = 0$. □

Proof of Theorem 3 We only sketch the proof, because the arguments needed to extend Stone's original proof to the semistable case are essentially contained in the proof of Theorem 2.

Changing variables and using (15) and (26), the difference

$$2\pi \left| \frac{A_n}{2h}\mathbb{P}(S_n + Y \in (x-h, x+h]) - g_{\gamma_n}((x-C_n)/A_n) \right|$$

can be bounded exactly as in (27), with TA_n instead of πA_n in I_3. Now, I_1, I_2, and I_4 can be treated the same way as in the lattice case, while for I_3 we use that by the nonlattice condition $\sup_{|t|\in[\varepsilon,T]} |\varphi(t)| < 1$ for any $\varepsilon > 0$ and $T > 0$. Thus as $n \to \infty$

$$\sup_{x\in\mathbb{R}} 2\pi \left| \frac{A_n}{2h}\mathbb{P}(S_n + Y \in (x-h, x+h]) - g_{\gamma_n}((x-C_n)/A_n) \right| \to 0. \tag{30}$$

Using that Y concentrates at 0 as $T \to \infty$, one can get rid of the Y above as in [20]. For completeness and later use, we include the argument here. Let $h > 0$ be fixed, and let $\delta > 0$. Putting $h^+ = (1+\delta)h$ we have by the independence of Y and S_n,

$$\mathbb{P}(S_n \in (x-h, x+h]) \leq \frac{1}{\mathbb{P}(|Y| \leq \delta h)}\mathbb{P}(S_n + Y \in (x-h^+, x+h^+]). \tag{31}$$

Thus

$$\begin{aligned} &\frac{A_n}{2h}\mathbb{P}(S_n \in (x-h, x+h]) - g_{\gamma_n}((x-C_n)/A_n) \\ &\leq \left(\frac{A_n}{2h^+}\mathbb{P}(S_n + Y \in (x-h^+, x+h^+]) - g_{\gamma_n}((x-C_n)/A_n) \right) \\ &\quad + \frac{A_n}{2h^+}\mathbb{P}(S_n + Y \in (x-h^+, x+h^+]) \left[\frac{h^+}{h\mathbb{P}(|Y| \leq \delta h)} - 1 \right]. \end{aligned}$$

By (30) the first summand tends to 0 as $n \to \infty$ for any δ and T. Using (30) again, and that $\sup_{\lambda>0, x\in\mathbb{R}} g_\lambda(x) < \infty$,

$$\sup_{x\in\mathbb{R}} \frac{A_n}{2h^+} \mathbb{P}(S_n + Y \in (x - h^+, x + h^+]) < \infty.$$

Therefore, choosing first $\delta > 0$ small then T large we obtain

$$\limsup_{n\to\infty} \sup_{x\in\mathbb{R}} \frac{A_n}{2h} \mathbb{P}(S_n \in (x - h, x + h]) - g_{\gamma_n}((x - C_n)/A_n) \le 0. \tag{32}$$

For the lower bound, putting $h^- = (1 - \delta)h$, using also (32)

$$\begin{aligned}
&\mathbb{P}(S_n + Y \in (x - h^-, x + h^-]) = \int_{\mathbb{R}} \mathbb{P}(S_n + u \in (x - h^-, x + h^-]) j(u) \mathrm{d}u \\
&\le \mathbb{P}(S_n \in (x - h, x + h]) \mathbb{P}(|Y| \le \delta h) + 2 \sup_{\lambda>0, x\in\mathbb{R}} g_\lambda(x) \frac{2h}{A_n} \mathbb{P}(|Y| > \delta h).
\end{aligned}$$

Therefore, with $C = 4 \sup_{\lambda>0, x\in\mathbb{R}} g_\lambda(x)$

$$\mathbb{P}(S_n \in (x - h, x + h]) \ge \frac{\mathbb{P}(S_n + Y \in (x - h^-, x + h^-])}{\mathbb{P}(|Y| \le \delta h)} - Ch \frac{\mathbb{P}(|Y| > \delta h)}{A_n \mathbb{P}(|Y| \le \delta h)}.$$

Thus

$$\begin{aligned}
&\frac{A_n}{2h} \mathbb{P}(S_n \in (x - h, x + h]) - g_{\gamma_n}((x - C_n)/A_n) \\
&\ge \frac{A_n}{2h^-} \mathbb{P}(S_n + Y \in (x - h^-, x + h^-]) - g_{\gamma_n}((x - C_n)/A_n) \\
&\quad + \frac{A_n}{2h^-} \mathbb{P}(S_n + Y \in (x - h^-, x + h^-]) \left(\frac{h^-}{h \mathbb{P}(|Y| \le \delta h)} - 1 \right) - C \frac{\mathbb{P}(|Y| > \delta h)}{\mathbb{P}(|Y| \le \delta h)}.
\end{aligned}$$

Choosing again first $\delta > 0$ small and then $T > 0$ large we obtain

$$\liminf_{n\to\infty} \inf_{x\in\mathbb{R}} \frac{A_n}{2h} \mathbb{P}(S_n \in (x - h, x + h]) - g_{\gamma_n}((x - C_n)/A_n) \ge 0,$$

completing the proof.

For later use, we note that the argument implies that for any $\varepsilon > 0$ there exists $T > 0$ such that for n large enough

$$\sup_{x\in\mathbb{R}} A_n |\mathbb{P}(S_n + Y \in (x - h, x + h]) - \mathbb{P}(S_n \in (x - h, x + h])| \le \varepsilon. \tag{33}$$

$\square$

7.3 Strong renewal theorems

We need a continuity property of the densities $g_\lambda(x)$, in λ. Recall the definition of the constant $c > 0$ in (4). In the following result the interval $[c^{-2}, c]$ could be replaced by any compact interval of $(0, \infty)$. For our purpose anything larger than $(c^{-1}, 1]$ would suffice.

Lemma 6. *There exists $\nu_4 > 0$ such that for any $\lambda_1, \lambda_2 \in [c^{-2}, c]$*

$$\sup_{x \in \mathbb{R}} |g_{\lambda_1}(x) - g_{\lambda_2}(x)| \leq \nu_4 \, |\lambda_1 - \lambda_2|.$$

Moreover,

$$\sup_{\lambda \in (c^{-1}, 1]} \sup_{x \in \mathbb{R}} \frac{\partial}{\partial x} g_\lambda(x) < \infty. \tag{34}$$

Proof Introduce the notation $\psi_\lambda(t) = \mathbb{E} e^{\mathrm{i}tV_\lambda} = e^{y_\lambda(t)}$. By formula (2.6) in [12]

$$y_\lambda(t) = \lambda y_1(t/\lambda^{1/\alpha}) - \mathrm{i}tc_\lambda, \tag{35}$$

with

$$c_\lambda = \lambda^{(\alpha-1)/\alpha} \int_1^{1/\lambda} [\psi_2(s) - \psi_1(s)] \, \mathrm{d}s,$$

where $\psi_1(s) = \inf\{-x : M_L(x)x^{-\alpha} > s\}$, $\psi_2(s) = \inf\{-x : M_R(x)x^{-\alpha} > s\}$. For any $\lambda > 0$ the function $e^{\lambda y_1(t)}$, $t \in \mathbb{R}$, is a characteristic function. Let $G(x; \lambda)$ denote its distribution function, i.e. $e^{\lambda y_1(t)} = \int_{\mathbb{R}} e^{\mathrm{i}tx} G(\mathrm{d}x; \lambda)$. Csörgő [7] proved that these functions are infinitely many times differentiable with respect to both variables. Let $g(x; \lambda)$ be the density of $G(x; \lambda)$.

Using the density inversion formula and (35) we obtain

$$\begin{aligned} g_\lambda(x) &= \frac{1}{2\pi} \int_{-\infty}^{\infty} e^{-\mathrm{i}tx} e^{y_\lambda(t)} \mathrm{d}t \\ &= \lambda^{1/\alpha} \frac{1}{2\pi} \int_{-\infty}^{\infty} e^{-\mathrm{i}s\lambda^{1/\alpha}(x + c_\lambda)} e^{\lambda y_1(s)} \mathrm{d}s \\ &= \lambda^{1/\alpha} g\left(\lambda^{1/\alpha}(x + c_\lambda); \lambda\right). \end{aligned} \tag{36}$$

By Lemmas 1 and 2 in [7] for each j, k

$$\sup_{\lambda \in [c^{-2}, c]} \sup_{x \in \mathbb{R}} \left| \frac{\partial^{j+k}}{\partial x^j \partial \lambda^k} G(x; \lambda) \right| < \infty, \tag{37}$$

which implies that for some constant $C > 0$, for any $\lambda_1, \lambda_2 \in [c^{-2}, c]$

$$|g(x; \lambda_1) - g(x; \lambda_2)| \leq C|\lambda_1 - \lambda_2|.$$

Using (36)

$$\begin{aligned} g_{\lambda_1}(x) - g_{\lambda_2}(x) &= \lambda_1^{1/\alpha} \left[g\big(\lambda_1^{1/\alpha}(x + c_{\lambda_1}), \lambda_1\big) - g\big(\lambda_1^{1/\alpha}(x + c_{\lambda_1}), \lambda_2\big) \right] \\ &+ \lambda_1^{1/\alpha} \left[g\big(\lambda_1^{1/\alpha}(x + c_{\lambda_1}), \lambda_2\big) - g\big(\lambda_2^{1/\alpha}(x + c_{\lambda_2}), \lambda_2\big) \right] \\ &+ \big(\lambda_1^{1/\alpha} - \lambda_2^{1/\alpha}\big) g\big(\lambda_2^{1/\alpha}(x + c_{\lambda_2}); \lambda_2\big). \end{aligned}$$

Using (37) with $j = k = 1$, $j = 2$, $k = 0$, and $j = 1$, $k = 0$ respectively, and for the second term using also that c_λ is Lipschitz in $\lambda \in [c^{-2}, c]$, we obtain

$$|g_{\lambda_1}(x) - g_{\lambda_2}(x)| \leq C|\lambda_1 - \lambda_2|,$$

as claimed. The uniform boundedness of the derivatives in (34) follows simply from (36) and (37). □

Proof of Theorem 4 We estimate u_n via (16). This is possible due to Theorem 1, which ensures that $\Re \int_0^\pi (1 - \varphi(t))^{-1} \mathrm{d}t$ is well defined. Let $L > 1$ be a large fixed number. To ease notation, we suppress the $\lfloor \cdot \rfloor$ notation. Write

$$\begin{aligned}
\pi u_n &= \Re \int_0^\pi (1 - \varphi(t))^{-1} e^{-\mathrm{i}nt} \,\mathrm{d}t \\
&= \left(\sum_{k < B(n/L^2)} + \sum_{k = B(n/L^2)}^{B(nL)} + \sum_{k > B(nL)} \right) \Re \int_0^\pi \varphi(t)^k e^{-\mathrm{i}nt} \,\mathrm{d}t \\
&=: I_1 + I_2 + I_3.
\end{aligned}$$

First, by Lemma 2,

$$\limsup_{n \to \infty} \left| n^{1-\alpha} \ell(n) I_2 - \pi \alpha \int_{L^{-1}}^{L^2} g_{\gamma(B(n)x^{-\alpha})}(x) x^{-\alpha} \,\mathrm{d}x \right| \leq \frac{\pi}{L}. \tag{38}$$

Next we handle I_3. By Theorem 2 for k large enough

$$\sup_n \mathbb{P}(S_k = n) \leq C\, A_k^{-1},$$

with $C = 1 + \sup_{\gamma, x} g_\gamma(x)$. Therefore, using Karamata's theorem, the inverse relation (8) and Potter's bounds we obtain for any $\varepsilon > 0$

$$\begin{aligned}
I_3 &\leq \pi \sum_{k \geq B(nL)} C\, A_k^{-1} \\
&\sim C\pi \frac{\alpha}{1 - \alpha} B(nL)^{1 - 1/\alpha} \ell_1(B(nL))^{-1} \\
&\leq C n^{\alpha - 1} \ell(n)^{-1} L^{\alpha + \varepsilon - 1}
\end{aligned} \tag{39}$$

Note that the estimate works for $\alpha \in (0, 1)$, the assumption $\alpha > 1/2$ is not needed at this point.

It remains to estimate I_1. We have

$$\begin{aligned}
|I_1| &\leq \left| \sum_{k < B(n/L^2)} \int_0^{L/n} \varphi(t)^k e^{-\mathrm{i}nt} \,\mathrm{d}t \right| + \left| \sum_{k < B(n/L^2)} \int_{L/n}^\pi \varphi(t)^k e^{-\mathrm{i}nt} \,\mathrm{d}t \right| \\
&=: |I_1^1| + |I_1^2| =: |I_1^1| + \left| \sum_{k < B(n/L^2)} I_1^{2,k} \right|.
\end{aligned}$$

Clearly, $|I_1^1| \le B(n/L^2)\cdot L/n$ and using Potter's bounds, for any $\alpha' < \alpha$ for n large enough

$$|I_1^1| \le 2n^{\alpha-1}\ell(n)^{-1}L^{-(2\alpha'-1)}. \tag{40}$$

Next, similarly to [10, Section 3.5], note that

$$\begin{aligned} I_1^{2,k} &= \frac{1}{2}\Big(\int_{\pi-\pi/n}^{\pi} + \int_{L/n}^{(L+\pi)/n}\Big)\varphi(t)^k\, e^{-\mathrm{i}nt}\,\mathrm{d}t \\ &\quad + \frac{1}{2}\int_{(L+\pi)/n}^{\pi}\Big(\varphi(t)^k - \varphi(t-\pi/n)^k\Big)\, e^{-\mathrm{i}nt}\,\mathrm{d}t \\ &=: J_1^k + J_2^k. \end{aligned} \tag{41}$$

Since, $|J_1^k| \le \pi/n$, for any $\alpha' < \alpha$ for large n,

$$\left|\sum_{k<B(n/L^2)} J_1^k\right| \le B(n/L^2)\frac{\pi}{n} \le 2n^{\alpha-1}\ell(n)^{-1}L^{-2\alpha'}. \tag{42}$$

Using Lemma 5 (iii)

$$\begin{aligned} \Big|\varphi(t)^k - \varphi(t-\pi/n)^k\Big| &\le |\varphi(t)-\varphi(t-\pi/n)|\sum_{j=0}^{k-1}|\varphi(t)^j|\,|\varphi(t-\pi/n)^{k-j-1}| \\ &\le 2\nu_3\pi^\alpha\, n^{-\alpha}\ell(n)k\,(|\varphi(t-\pi/n)^{k-1}| + |\varphi(t)^{k-1}|). \end{aligned}$$

Thus,

$$\left|\sum_{k<B(n/L^2)} J_2^k\right| \le Cn^{-\alpha}\ell(n)\sum_{k=0}^{B(n/L^2)} k\int_{L/n}^{\pi}|\varphi(t)|^k\,\mathrm{d}t. \tag{43}$$

Recall that $\lim_{k\to\infty}\frac{k\ell(A_k)}{(A_k)^\alpha} = 1$. Using Lemma 5 (i), change of variables $y \to tA_k$, and Potter's bound we obtain

$$\begin{aligned} \int_{L/n}^{\pi}|\varphi(t)|^k\,\mathrm{d}t &\le \int_{L/n}^{\pi} e^{-\nu_1 k t^\alpha \ell(1/t)}\,\mathrm{d}t \\ &\le \frac{1}{A_k}\int_{LA_k/n}^{\pi A_k} e^{-\nu_1 y^\alpha\, k\, A_k^{-\alpha}\ell(A_k/y)}\,\mathrm{d}y \\ &\le \frac{1}{A_k}\int_{LA_k/n}^{\pi A_k} e^{-\frac{\nu_1}{2}y^\alpha \ell(A_k/y)\ell(A_k)^{-1}}\,\mathrm{d}y \\ &\le \frac{1}{A_k}\int_0^\infty e^{-C(y^{\alpha-\delta}+y^{\alpha+\delta})}\,\mathrm{d}y \le \frac{C}{A_k}, \end{aligned} \tag{44}$$

for any $\delta > 0$ and some $C > 0$. Recall (8). Substituting the bound (44) into (43), using Karamata's theorem and that $\alpha > 1/2$, we have

$$\begin{aligned} \left|\sum_{k=0}^{B(n/L^2)} J_2^k\right| &\le Cn^{-\alpha}\ell(n)\sum_{k=0}^{B(n/L^2)}\frac{k}{A_k} \le Cn^{-\alpha}\ell(n)\frac{B(n/L^2)^{2-\frac{1}{\alpha}}}{\ell_1(B(n/L^2))} \\ &\le C\frac{n^{\alpha-1}}{\ell(n)}L^{2-4\alpha'}, \end{aligned} \tag{45}$$

with $\alpha' \in (1/2, \alpha)$.

It is worth to note that this is the only part in the proof where we use that $\alpha > 1/2$. Seemingly, in (40) we also use this fact, but in that argument we can enlarge the power of L in $B(n/L^2)$ to work for smaller α.

Putting (42) and (45) together, recalling that $\alpha' < \alpha \in (1/2, 1)$

$$|I_1^2| = \left| \sum_{k<B(n/L^2)} I_1^{2,k} \right| \leq C n^{\alpha-1} \ell(n)^{-1} L^{-2\alpha'},$$

which combined with (40) implies that for any $\alpha' < \alpha$

$$|I_1| \leq C n^{\alpha-1} \ell(n)^{-1} L^{1-2\alpha'}. \tag{46}$$

To finish the proof we have to show that

$$\int_0^\infty \sup_{\lambda \in (c^{-1},1]} g_\lambda(y) y^{-\alpha} \,\mathrm{d}y < \infty. \tag{47}$$

This follows from Theorem 1 by Bingham [4] (see also Lemma 2 in [15]). By (47) we have

$$\lim_{L\to\infty} \left(\int_0^{L^{-1}} + \int_{L^2}^\infty \right) g_{\gamma(B(n)x^{-\alpha})}(x) x^{-\alpha} \,\mathrm{d}x = 0.$$

Letting $L \to \infty$ we see that the latter limit together with (38), (39), and (46) imply the statement. □

Proof of Lemma 2 With the same notation as in Theorem 2, we write

$$\begin{aligned}
\frac{1}{\pi} \sum_{k=B(n/L^2)}^{B(nL)} \Re \int_0^\pi \varphi(t)^k \, e^{-\mathrm{i}nt} \,\mathrm{d}t &= \sum_{k=B(n/L^2)}^{B(nL)} \mathbb{P}(S_k = n) \\
&= \sum_{k=B(n/L^2)}^{B(nL)} \frac{g_{\gamma_k}(n/A_k)}{A_k} + \sum_{k=B(n/L^2)}^{B(nL)} \frac{1}{A_k} [A_k \mathbb{P}(S_k = n) - g_{\gamma_k}(n/A_k)].
\end{aligned}$$

By Theorem 2, recalling that $C_n \equiv 0$ in our case, for any $\varepsilon > 0$, for n large enough, for all $k \geq B(n/L^2)$ we have

$$|A_k \mathbb{P}(S_k = n) - g_{\gamma_k}(n/A_k)| < \varepsilon.$$

Hence, using (7), Karamata's theorem and Potter's bound, for any $\alpha' < \alpha$, similarly as in (39)

$$\begin{aligned}
\sum_{k=B(n/L^2)}^{B(nL)} \frac{1}{A_k} |A_k \mathbb{P}(S_k = n) - g_{\gamma_k}(n/A_k)| &\leq \sum_{k=B(n/L^2)}^{\infty} \frac{\varepsilon}{A_k} \\
&\leq \frac{2\alpha\varepsilon}{1-\alpha} n^{\alpha-1} \ell(n)^{-1} \, L^{2-2\alpha'},
\end{aligned} \tag{48}$$

where in the last inequality we also used the inverse relation $A(B(n)) \sim n$ in (8). For n large enough and L fixed, we can take ε so small that

$$\sum_{k=B(n/L^2)}^{B(nL)} \frac{1}{A_k} |A_k \mathbb{P}(S_k = n) - g_{\gamma_k}(n/A_k)| \le 2^{-1}\, n^{\alpha-1} \ell(n)^{-1} L^{-1}. \tag{49}$$

Next, we write $\sum_{k=B(n/L^2)}^{B(nL)} A_k^{-1} g_{\gamma_k}(n/A_k)$ as a Riemann sum proceeding as in [10, Lemma 2.2.1] (see also [5, Proof of Th. 8.6.6]). More precisely, set $x_k = k\frac{\ell(n)}{n^\alpha}$. By definition, A_k is the asymptotic inverse of $n \to \frac{n^\alpha}{\ell(n)} = \frac{k}{x_k}$. Thus

$$L^{-2\alpha-\delta} \le B(n/L^2)\frac{\ell(n)}{n^\alpha} \le x_k \le B(nL)\frac{\ell(n)}{n^\alpha} \le L^{\alpha+\delta} \tag{50}$$

with $\delta > 0$ arbitrarily small. Using the uniform convergence theorem and the inverse relation $B(A_n) \sim A(B(n)) \sim n$ (as in [5, Proof of Th. 8.6.6]), we have $x_k^{-1/\alpha} \sim \frac{n}{A_k}$ as $k, n \to \infty$, uniformly in the relevant range of k, n. By (50) this is equivalent to

$$\lim_{n\to\infty} \sup_{B(n/L^2)\le k\le B(nL)} \left| x_k^{-1/\alpha} - \frac{n}{A_k} \right| = 0. \tag{51}$$

Since $x_{k+1} - x_k = \frac{\ell(n)}{n^\alpha}$ and $k = B(n)x_k$

$$\begin{aligned}\sum_{k=B(n/L^2)}^{B(nL)} \frac{g_{\gamma_k}(n/A_k)}{A_k} &= \frac{n^\alpha}{n\ell(n)} \sum_{k=B(n/L^2)}^{B(nL)} \frac{n}{A_k} g_{\gamma_k}(n/A_k)\frac{\ell(n)}{n^\alpha} \\ &\sim \frac{n^{\alpha-1}}{\ell(n)} \sum_{L^{-2\alpha}<x_k<L^\alpha} x_k^{-1/\alpha} g_{\gamma(x_k B(n))}(x_k^{-1/\alpha})\,(x_{k+1} - x_k)\,,\end{aligned}$$

where in the last line we have used that by (51) and by (34) we have as $n \to \infty$

$$\sup_{B(n/L^2)\le k\le B(nL)} |g_{\gamma(x_k B(n))}(n/A_k)) - g_{\gamma(x_k B(n))}(x_k^{-1/\alpha})| \to 0.$$

To finish the proof it is enough to show that

$$f_n(x) := x^{-1/\alpha} g_{\gamma(xB(n))}(x^{-1/\alpha}) \tag{52}$$

is uniformly Lipschitz on $[L^{-2\alpha}, L^\alpha]$. Indeed, for uniformly Lipschitz f_n the convergence of the Riemann sums follows, i.e.

$$\begin{aligned}&\sum_{k=B(n/L^2)}^{B(nL)} x_k^{-\frac{1}{\alpha}} g_{\gamma(x_k B(n))}(x_k^{-\frac{1}{\alpha}})\,(x_{k+1} - x_k) \\ &\sim \int_{L^{-2\alpha}}^{L^\alpha} x^{-\frac{1}{\alpha}} g_{\gamma(B(n)x)}\big(x^{-\frac{1}{\alpha}}\big)\mathrm{d}x \\ &= \alpha \int_{L^{-1}}^{L^2} g_{\gamma(B(n)y^{-\alpha})}(y) y^{-\alpha}\mathrm{d}y.\end{aligned}$$

This together with (49) implies the statement.

Therefore, it only remains to show that the sequence (f_n) in (52) is uniformly Lipschitz on any compact subset of $(0,\infty)$. Recall (10) and for $x>0$ large set $b(x)$ to be the unique index for which $k_{b(x)-1} < x \le k_{b(x)}$. Then $\gamma_x = x/k_{b(x)}$. For some large M fix the interval $I=[c^{-M},c^M]$, and let $h>0$ be small enough such that $1+hc^M \le \sqrt{c}$. Then $B(n)(x+h) = B(n)x(1+h/x) \le B(n)x\sqrt{c}$, which implies that $b(B(n)(x+h))$ is either $b(B(n)x)$, or $b(B(n)x)+1$. Both cases can be handled similarly, we consider only the former. Then

$$\gamma(B(n)(x+h)) = \frac{B(n)(x+h)}{k_{b(B(n)x)}} = \gamma(B(n)x) + h\frac{B(n)}{k_{b(B(n)x)}}.$$

The factor of h is $O(1)$ since

$$\frac{B(n)}{k_{b(B(n)x)}} = x^{-1}\frac{B(n)x}{k_{b(B(n)x)}},$$

where $x\in I$ and the second factor is less than, or equal to 1. Thus by Lemma 6 the result follows. □

Before proceeding to the proof of Theorem 5, we prove Lemma 3.

Proof of Lemma 3 Recall that $h\in(0,1/2)$ is fixed. Proceeding as in the proof of Lemma 2, the conclusion follows once we show that as $y\to\infty$,

$$y^{1-\alpha}\ell(y)\sum_{k=B(y/L^2)}^{B(yL)} (I_{k,y}-2h)\frac{g_{\gamma_k}(y/A_k)}{A_k} \to 0. \tag{53}$$

Let R_a denote the irrational rotation with $-a$, i.e.

$$R_a : \mathbb{R}/\mathbb{Z} \to [0,1),\ y\mapsto y-a \mod 1.$$

Note that

$$I_{k,y} = 1_{[0,h)\cup(1-h,1)} \circ R_a^k(y).$$

Let $\varepsilon>0$ be arbitrary. Because of the unique ergodicity property of R_a (see, for instance, [17, Section 5]), there exists $N=N_\varepsilon$ such that for any $n\ge N$

$$\sup_{m,y}\left|\frac{\sum_{k=m+1}^{m+n} I_{k,y}}{n} - 2h\right| = \sup_{m,y}\left|\frac{1}{n}|\{1\le j\le n : R_a^{j+m}(y)\in[0,h]\cup(1-h,1)\}| - 2h\right| \le \varepsilon. \tag{54}$$

Divide the interval $[B(y/L^2),B(yL)]$ into blocks $[k_j,k_{j+1})$ of length N. Let

$$n_y = \left\lfloor \frac{\lfloor B(yL)\rfloor - \lceil B(y/L^2)\rceil}{N}\right\rfloor$$

and define

$$k_j = \lceil B(y/L^2)\rceil + jN, \quad j=0,1,2,\dots,n_y-1, \quad k_{n_y} = \lfloor B(yL)\rfloor + 1. \tag{55}$$

Then each block $[k_j, k_{j+1})$ has length N, except the last one, which might be longer, but at most of size $2N$.

By Lemma 6,

$$\lim_{y\to\infty} \sup_{B(y/L^2)\le k\le B(yL)} \left| g_{\gamma_{k+1}}(y/A_{k+1}) - g_{\gamma_k}(y/A_k) \right| = 0. \tag{56}$$

Thus, for arbitrarily small ε_0 there exists y sufficiently large, such that for any $j = 0, 1, \ldots,$ $n_y - 1$

$$|g_{\gamma_k}(y/A_k)) - g_{\gamma_{k_j}}(y/A_{k_j})| \le \varepsilon_0 \quad \text{ for every } k \in \{k_j, \ldots, k_{j+1}\}.$$

Next, using properties of slowly varying function, we have that for arbitrarily small ε_1, there exists y large enough, such that for any $j = 0, 1, \ldots, n_y - 1$

$$\frac{1}{A_{k_j}} - \frac{1}{A_k} \le \left(\varepsilon_1 + \frac{N}{\alpha k_j} \right) \frac{1}{A_{k_j}} \text{ for every } k \in \{k_j, \ldots, k_{j+1}\}.$$

As N is fixed and $y \to \infty$, for any $\varepsilon_2 > 0$ there exists y large enough such that $N/k_j \le \varepsilon_2$. Therefore, with $\varepsilon_3 = \varepsilon_0 + \varepsilon_1 + \varepsilon_2$, for every $k \in \{k_j, \ldots, k_{j+1} - 1\}$,

$$\left| \frac{g_{\gamma_k}(y/A_k)}{A_k} - \frac{g_{\gamma_{k_j}}(y/A_{k_j})}{A_{k_j}} \right| \le \frac{\varepsilon_3}{A_{k_j}}. \tag{57}$$

Now,

$$\begin{aligned}
&\sum_{k=B(y/L^2)}^{B(yL)} (I_{k,y} - 2h) \frac{g_{\gamma_k}(y/A_k)}{A_k} \\
&= \sum_{j=0}^{n_y-1} \sum_{k=k_j}^{k_{j+1}-1} (I_{k,y} - 2h) \frac{g_{\gamma_{k_j}}(y/A_{k_j})}{A_{k_j}} + \sum_{j=0}^{n_y-1} \sum_{k=k_j}^{k_{j+1}-1} 2h \left(\frac{g_{\gamma_{k_j}}(y/A_{k_j})}{A_{k_j}} - \frac{g_{\gamma_k}(y/A_k)}{A_k} \right) \\
&=: S_1 + S_2.
\end{aligned}$$

Using (54), (57), and that $A_{k_j} \sim A_k$ uniformly in $k \in \{k_j, \ldots, k_{j+1}\}$, we have

$$\begin{aligned}
|S_1| &\le \varepsilon \sum_{j=0}^{n_y-1} (k_{j+1} - k_j) \frac{g_{\gamma_{k_j}}(y/A_{k_j})}{A_{k_j}} \\
&\le \varepsilon \sum_{j=0}^{n_y-1} \sum_{k=k_j}^{k_{j+1}-1} \frac{g_{\gamma_k}(y/A_k) + \varepsilon_3}{A_k}.
\end{aligned} \tag{58}$$

While for S_2 by (57) we obtain

$$|S_2| \le \sum_{k=B(y/L^2)}^{B(yL)} 2h \frac{\varepsilon_3}{A_k}. \tag{59}$$

Since ε and ε_3 are as small as we want, (58) and (59) imply

$$\lim_{y\to\infty} \frac{\sum_{k=B(y/L^2)}^{B(yL)} (I_{k,y} - 2h) A_k^{-1} g_{\gamma_k}(y/A_k)}{\sum_{k=B(y/L^2)}^{B(yL)} A_k^{-1} g_{\gamma_k}(y/A_k)} = 0,$$

thus (53) follows. □

The proof below goes by and large as the proof of Theorem 4. In the lattice case we combine Theorem 4 with Lemma 3. In the nonlattice case, we use Theorem 3 and the inversion formula (15) used in the proof of Theorem 3, along with the approximation equations (31) and (33). At some extent, our strategy resembles the one in [9] (suitable for the usual stable/regular variation setting), but we do not use it a such.

Proof of Theorem 5

Nonarithmetic, lattice case. We continue from (17) and split the sum into I_1, I_2, I_3 exactly as in the proof of Theorem 4. The terms I_1 and I_3 are negligible, which follows exactly as in the proof of Theorem 4. The asymptotic of the term I_2, which gives the exact term, follows from Lemma 3.

Nonlattice case. We start from

$$\begin{aligned} U(y+h) - U(y-h) &= \sum_{k=0}^{\infty} \mathbb{P}(S_k \in (y-h, y+h]) \\ &= \left(\sum_{k<B(y/L^2)} + \sum_{k=B(y/L^2)}^{B(yL)} + \sum_{k>B(yL)} \right) \mathbb{P}(S_k \in (y-h, y+h]) \\ &=: E_1 + E_2 + E_3. \end{aligned}$$

For E_2 and E_3, using (33), (48) (choosing ε small enough) and (15),

$$\begin{aligned} E_2 + E_3 &= \left(\sum_{k=B(y/L^2)}^{B(yL)} + \sum_{k>B(yL)} \right) \mathbb{P}(S_k + Y \in (y-h, y+h]) + O\left(\frac{y^{\alpha-1}}{\ell(y) L} \right) \\ &= \left(\sum_{k=B(y/L^2)}^{B(yL)} + \sum_{k>B(yL)} \right) \frac{h}{\pi} \int_{-T}^{T} \frac{\sin th}{th} e^{-\mathrm{i}ty} \varphi(t)^k \left(1 - \frac{|t|}{T}\right) \mathrm{d}t + O\left(\frac{y^{\alpha-1}}{\ell(y) L} \right) \\ &=: I_2 + I_3 + O\left(L^{-1} y^{\alpha-1} \ell(y)^{-1} \right). \end{aligned}$$

The terms I_2 and I_3 can be treated as their analogues in the proof of Theorem 4 just writing x instead of n and T instead of π. We skip the details, and continue with E_1.

Using (31) with $h^+ = (1+\delta)h$, for $\delta > 0$ and also (15),

$$\begin{aligned} E_1 &\le \frac{1}{\mathbb{P}(|Y| \le \delta h)} \sum_{k<B(y/L^2)} \mathbb{P}(S_k + Y \in (y - h^+, y + h^+]) \\ &= \frac{h^+}{\mathbb{P}(|Y| \le \delta h)\pi} \sum_{k<B(y/L^2)} \int_{-T}^{T} \frac{\sin th^+}{th^+} e^{-\mathrm{i}ty} \varphi(t)^k \left(1 - \frac{|t|}{T}\right) \mathrm{d}t \\ &=: \frac{h^+}{\mathbb{P}(|Y| \le \delta h)\pi} I_1. \end{aligned}$$

To ease notation put

$$\beta(t) = \frac{\sin(th^+)}{th^+}\left(1 - \frac{|t|}{T}\right).$$

Then β is uniformly Lipschitz on $[-T, T]$, thus there is a constant C for which

$$|\beta(t) - \beta(t+s)| \le C\,s \quad \text{for any } t, t+s \in [-T, T]. \tag{60}$$

Splitting I_1 further as in the arithmetic case, let

$$\begin{aligned} I_1 &= \sum_{k<B(y/L^2)} \int_{-T}^{T} \beta(t)\varphi(t)^k e^{-\mathrm{i}ty}\,\mathrm{d}t \\ &= \sum_{k<B(y/L^2)} \left(\int_{|t|\le L/y} + \int_{|t|\in(L/y,T)} \right) \beta(t)\varphi(t)^k e^{-\mathrm{i}ty}\,\mathrm{d}t =: I_1^1 + I_1^2. \end{aligned}$$

As in (40) we obtain that for any $\alpha' < \alpha$ for x large enough

$$|I_1^1| \le 2y^{\alpha-1}\ell(y)^{-1}L^{-(2\alpha'-1)}. \tag{61}$$

To estimate I_1^2, as in the arithmetic case (see also the proof of (5.11) in [9]) write

$$\begin{aligned} \int_{L/y}^{T} \beta(t)\varphi(t)^k e^{-\mathrm{i}ty}\,\mathrm{d}t &= \frac{1}{2}\left(\int_{T-\pi/y}^{T} + \int_{L/y}^{(L+\pi)/y} \right) \beta(t)\varphi(t)^k e^{-\mathrm{i}ty}\,\mathrm{d}t \\ &+ \frac{1}{2}\int_{L/y}^{T-\pi/y} e^{-\mathrm{i}ty}\left[\beta(t)\varphi(t)^k - \beta(t+\pi/y)\varphi(t+\pi/y)^k\right]\mathrm{d}t. \end{aligned}$$

Using (60) and Lemma 5 (iii), as in the arithmetic case we obtain that for any $\alpha' < \alpha$ for x large enough

$$|I_1^2| \le C y^{\alpha-1}\ell(y)^{-1}L^{-2\alpha'}.$$

Combining with (61) we have

$$\lim_{L\to\infty}\limsup_{y\to\infty} |I_1| \frac{y^{1-\alpha}}{\ell(y)} = 0,$$

proving the statement. □

7.4 Renewal function asymptotics

Proof of Theorem 7 We first assume that X is integer valued with span 1. Let $L > 1$ be a fixed large number. Using (16)

$$\begin{aligned} \mathbb{P}(S_k \le n) &= \sum_{\ell=0}^{n} \mathbb{P}(S_k = \ell) \\ &= \frac{1}{2\pi}\int_{-\pi}^{\pi} \sum_{\ell=0}^{n} e^{-\mathrm{i}\ell t}\varphi(t)^k\,\mathrm{d}t \\ &= \frac{1}{2\pi}\int_{-\pi}^{\pi} \frac{1-e^{-\mathrm{i}(n+1)t}}{1-e^{-\mathrm{i}t}}\varphi(t)^k\,\mathrm{d}t, \end{aligned}$$

thus

$$U(n)=\sum_{k=0}^{\infty}\mathbb{P}(S_k\le n)=\frac{1}{2\pi}\int_{-\pi}^{\pi}\frac{1-e^{-\mathrm{i}(n+1)t}}{1-e^{-\mathrm{i}t}}\frac{1}{1-\varphi(t)}\mathrm{d}t.$$

First we show that the main contribution in $U(n)$ comes from the integral on $[(nL)^{-1},L/n]$. Indeed, for $|t|\ge L/n$, using Lemma 5 (ii)

$$\begin{aligned}\left|\int_{\frac{L}{n}\le|t|\le\pi}\frac{1-e^{-\mathrm{i}(n+1)t}}{1-e^{-\mathrm{i}t}}\frac{1}{1-\varphi(t)}\mathrm{d}t\right| &\le C\int_{L/n}^{\pi}\frac{1}{t}t^{-\alpha}\ell(1/t)^{-1}\mathrm{d}t\\ &\le C\frac{n^{\alpha}}{\ell(n)}L^{-\alpha},\end{aligned}\tag{62}$$

while for $|t|\le 1/(nL)$

$$\begin{aligned}\left|\int_{|t|\le(nL)^{-1}}\frac{1-e^{-\mathrm{i}(n+1)t}}{1-e^{-\mathrm{i}t}}\frac{1}{1-\varphi(t)}\mathrm{d}t\right| &\le C\int_{0}^{(nL)^{-1}}nt^{-\alpha}\ell(1/t)^{-1}\mathrm{d}t\\ &\le C\frac{n^{\alpha}}{\ell(n)}L^{\alpha-1}.\end{aligned}\tag{63}$$

Therefore we need to consider the integral on $[(nL)^{-1},L/n]$. Write

$$\begin{aligned}&\int_{\frac{1}{Ln}\le|t|\le\frac{L}{n}}\frac{1-e^{-\mathrm{i}(n+1)t}}{1-e^{-\mathrm{i}t}}\left[\sum_{k=0}^{B(n/\sqrt{L})}+\sum_{k=B(n/\sqrt{L})}^{B(nL^2)}+\sum_{k>B(nL^2)}\right]\varphi(t)^k\,\mathrm{d}t\\ &=: I_1+I_2+I_3.\end{aligned}$$

The arguments below are somewhat similar to the ones in the proof of Theorem 4, but simplified. For the first term for n large enough

$$|I_1|\le C\int_{1/(Ln)}^{L/n}\frac{1}{t}B(n/\sqrt{L})\mathrm{d}t\le CB(n/\sqrt{L})\log L\le C\frac{n^{\alpha}}{\ell(n)}L^{-\alpha/2}\log L,$$

while for the third using Lemma 5 (i) and (ii) and the uniform convergence theorem for slowly varying functions we obtain for n large enough

$$\begin{aligned}|I_3| &\le C\int_{\frac{1}{Ln}}^{\frac{L}{n}}\frac{1}{t}e^{-\nu_1 t^{\alpha}\ell(1/t)B(nL^2)}t^{-\alpha}\ell(1/t)^{-1}\mathrm{d}t\\ &\le C\frac{1}{\ell(n)}\int_{\frac{1}{Ln}}^{\frac{L}{n}}t^{-\alpha-1}e^{-\frac{\nu_1}{2}(nL^2t)^{\alpha}}\mathrm{d}t\\ &\le C\frac{1}{\ell(n)}\int_{\frac{1}{Ln}}^{1}t^{-\alpha-1}\,\mathrm{d}t\,e^{-\frac{\nu_1}{2}L^{\alpha}}\\ &\le C\frac{n^{\alpha}}{\ell(n)}L^{\alpha}e^{-\frac{\nu_1}{2}L^{\alpha}}.\end{aligned}$$

It remains to estimate I_2. For $B(n/\sqrt{L}) \leq k \leq B(nL^2)$ uniformly in k as $n \to \infty$ we have

$$\int_{\frac{1}{Ln} \leq |t| \leq \frac{L}{n}} \frac{1 - e^{-\mathrm{i}(n+1)t}}{1 - e^{-\mathrm{i}t}} \varphi(t)^k \, \mathrm{d}t \sim \int_{\frac{1}{Ln} \leq |t| \leq \frac{L}{n}} \frac{1 - e^{-\mathrm{i}(n+1)t}}{\mathrm{i}t} \varphi(t)^k \, \mathrm{d}t =: I_2^k.$$

Changing variables and using the usual inversion formula for characteristic functions

$$\begin{aligned}
I_2^k &= \int_{\frac{A_k}{Ln} \leq |u| \leq \frac{LA_k}{n}} \frac{1 - e^{-\mathrm{i}\frac{n+1}{A_k}u}}{\mathrm{i}u} \varphi(u/A_k)^k \, \mathrm{d}u \\
&= \int_{-\infty}^{\infty} \frac{1 - e^{-\mathrm{i}\frac{n+1}{A_k}u}}{\mathrm{i}u} \psi_{\gamma_k}(u) \, \mathrm{d}u \\
&\quad - \left(\int_{|u| \leq \frac{A_k}{Ln}} + \int_{|u| \geq \frac{LA_k}{n}} \right) \frac{1 - e^{-\mathrm{i}\frac{n+1}{A_k}u}}{\mathrm{i}u} \psi_{\gamma_k}(u) \mathrm{d}u \\
&\quad + \int_{\frac{A_k}{Ln} \leq |u| \leq \frac{LA_k}{n}} \frac{1 - e^{-\mathrm{i}\frac{n+1}{A_k}u}}{\mathrm{i}u} \Big(\varphi(u/A_k)^k - \psi_{\gamma_k}(u) \Big) \, \mathrm{d}u \\
&= G_{\gamma_k} \left(\frac{n+1}{A_k} \right) - J_1^k - J_2^k + J_3^k.
\end{aligned}$$

Since A_k/n ranges from $L^{-1/2}$ to L^2, it can be shown as in (50) that for any fixed L the interval $[A_k/(Ln), LA_k/n]$ for $B(n/\sqrt{L}) \leq k \leq B(nL^2)$ is bounded away both from 0 and from ∞ uniformly in k. The merging relation implies that (28) holds, therefore

$$\lim_{n \to \infty} \sup_{B(n/\sqrt{L}) \leq k \leq B(nL^2)} J_3^k = 0. \tag{64}$$

Since G_γ has a density g_γ, the characteristic function ψ_γ is integrable, and as $n \to \infty$

$$\frac{LA(B(n/\sqrt{L}))}{n} \sim \sqrt{L},$$

which tends to ∞ as $L \to \infty$, we have that

$$\lim_{L \to \infty} \limsup_{n \to \infty} \sup_{B(n/\sqrt{L}) \leq k \leq B(nL^2)} J_2^k = 0. \tag{65}$$

Finally, for J_1^k note that for L large

$$\left| 1 - e^{-\mathrm{i}\frac{n+1}{A_k}u} \right| \leq 2 \frac{n+1}{A_k} |u|$$

whenever $|u| \leq A_k/(Ln)$. Thus

$$|J_1^k| \leq 2 \frac{n+1}{A_k} \frac{A_k}{Ln} \leq \frac{3}{L}. \tag{66}$$

Putting together (64), (65), and (66), we obtain that for any $\varepsilon > 0$ we can choose L large enough such that for n large enough

$$\sup_{B(n/\sqrt{L}) \leq k \leq B(nL^2)} \left| I_2^k - G_{\gamma_k}\left(\frac{n+1}{A_k}\right) \right| \leq \varepsilon. \tag{67}$$

Finally, as in the proof of Lemma 2 we obtain that

$$\begin{aligned} \sum_{k=B(n/\sqrt{L})}^{B(nL^2)} G_{\gamma_k}\left(\frac{n+1}{A_k}\right) &\sim \frac{n^\alpha}{\ell(n)} \int_{L^{-\alpha/2}}^{L^{2\alpha}} G_{\gamma(B(n)x)}(x^{-1/\alpha})\,\mathrm{d}x \\ &= \frac{n^\alpha}{\ell(n)} \alpha \int_{L^{-2}}^{\sqrt{L}} G_{\gamma(B(n)u^{-\alpha})}(u) u^{-\alpha-1}\,\mathrm{d}u. \end{aligned}$$

This completes the proof in the arithmetic case.

The nonarithmetic case is similar. The only difference in this case is the expression of the inversion formula. As in (15) (with Y defined in (14)),

$$\mathbb{P}(S_k + Y \leq y) = \frac{1}{2\pi} \int_{-T}^{T} \frac{1 - e^{-\mathrm{i}yt}}{\mathrm{i}t} \varphi(t)^k (1 - |t|/T)\,\mathrm{d}t$$

which gives

$$\sum_{k=0}^{\infty} \mathbb{P}(S_k + Y \leq y) = \frac{1}{2\pi} \int_{-T}^{T} \frac{1 - e^{-\mathrm{i}yt}}{\mathrm{i}t} \frac{1}{1 - \varphi(t)} (1 - |t|/T)\,\mathrm{d}t.$$

Proceeding as in the argument above in the integer valued case with y instead of n, T instead of π and $\mathrm{i}t$ instead of $1 - e^{-\mathrm{i}t}$, we obtain the analogues of (62), (63), and (67). Putting these together,

$$\lim_{y\to\infty} \left| y^{-\alpha}\ell(y) \sum_{k=0}^{\infty} \mathbb{P}(S_k + Y \leq y) - \alpha \int_0^\infty G_{\gamma(B(y)x^{-\alpha})}(x) x^{-\alpha-1}\,\mathrm{d}x \right| = 0.$$

To complete, we need to get rid of Y in the above equation. This can be done using (33). $\square$

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