Moments of the stationary distribution of subcritical multitype Galton–Watson processes with immigration

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Abstract

A necessary and sufficient condition for the existence of moments of the stationary distribution of a subcritical multitype GWI process was obtained by Szűcs [15]. In this short note we give a simple proof of this result.

Keywords: branching process with immigration, stationary distribution, moments
2000 MSC: 60J80, 60G10

1. Introduction

Let $(\mathbf{X}_n)_{n \geq 0} = (X_{n,1}, \ldots, X_{n,d})_{n \geq 0}$ be a d-type Galton–Watson process with immigration (GWI process), defined as

$$\begin{aligned} \mathbf{X}_n &= \sum_{j=1}^{d} \sum_{i=1}^{X_{n-1,j}} \mathbf{A}_{n,i;j} + \mathbf{B}_n, \quad n = 1, \ldots, \\ \mathbf{X}_0 &= \mathbf{0} = (0, \ldots, 0), \end{aligned} \tag{1}$$

where $\mathbf{A}_{n,i;j}, \mathbf{B}_n$, $n \geq 1$, $i \geq 1$, $j \in \{1, \ldots, d\}$, are independent d-dimensional random vectors with integer coordinates such that $\{\mathbf{A}_{n,i;j} : n \geq 1, i \geq 1\}$ is an identically distributed sequence of random vectors for each $j \in \{1, \ldots, d\}$, and $\mathbf{B}_n$, $n \geq 1$, are identically distributed. Here $X_{n,j}$ is the number of j-type individuals in generation n, $\mathbf{A}_{n,i;j}$ is the number of offsprings produced by the ith individual of type j in generation $n-1$, and $\mathbf{B}_n$ is the number of immigrants in generation n. To exclude trivialities we always assume that $\mathbb{P}(\mathbf{B}_1 = \mathbf{0}) < 1$. In the present paper we are only interested in properties of the stationary distribution, therefore the initial distribution of the process is irrelevant. For simplicity, we choose $\mathbf{X}_0 = \mathbf{0}$. In what follows, vectors, both deterministic and random, are denoted by boldface letters, and are meant as d-dimensional row vectors.

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The simplest branching process, the Galton–Watson process was introduced by Sir Francis Galton in 1873 to model the evolution of British family names. Since then, the theory of branching processes has evolved to describe more complex systems. Nowadays, branching processes play an important role in models of genetics, molecular biology, physics and computer science. As a main reference on branching processes and its applications we refer to the classical books by Athreya and Ney [1], by Mode [10] and by Haccou et al. [6].

Multitype Galton–Watson processes with immigration were introduced and studied by Quine [12]. Recently, Cerf and Dalmau [5] obtained probabilistic representations of the Perron–Frobenius eigenvector of the mean matrix of a multitype Galton–Watson process. Multitype GWI processes in random environment were investigated by Roitershtein [13], Roitershtein and Zhong [14], Wang [16], to mention just a few.

Quine [12] gave a necessary and sufficient condition for the existence of a stationary distribution in the subcritical case assuming that the mean matrix M is irreducible and aperiodic (also called positively regular process). Mode [10, Theorem 7.1] also provided a sufficient condition for the existence of a stationary distribution. Under the same assumption on M the complete answer for the existence of a limiting stationary distribution was obtained by Kaplan [9], considering not only subcritical but critical processes as well. Without any structural assumption on the mean matrix, in the subcritical case Szűcs [15] proved necessary and sufficient conditions.

However, interestingly enough, properties of the stationary distribution, in particular, existence of its moments, are much less investigated. Even in the single type case, we are only aware of the recent results by Buraczewski and Dyszewski [4] on Galton–Watson processes in random environment without immigration, from which the existence of certain moments can be deduced, see Lemma 3.1 in [4], and by Basrak and Kevei [3, Lemma 1] on GWI processes in random environment. In the multitype setup, an explicit formula for the variance was obtained already by Quine [12] and for the third moment very recently by Barczy et al. [2, Lemma 1]. Using Foster–Lyapunov technique Szűcs [15] proved necessary and sufficient condition for the existence of general, i.e. not necessarily integer moments of the stationary distribution. Moreover, in [15] exponential ergodicity was also proved.

The aim of the present note is to obtain a simple short proof of the necessary and sufficient condition for the existence of the moments of any order $\alpha > 0$ of the stationary distribution, which was proved in [15, Theorem 4]. Instead of the Foster–Lyapunov technique we use the infinite sum representation of the stationary distribution.

While the powerful Foster–Lyapunov technique is intrinsically related to Markov processes, our probabilistic proof only relies on the infinite sum representation of the stationary distribution. Therefore, our method might be applicable in a non-Markovian setting, for example in the theory of age-dependent (or Bellman–Harris) branching processes, see Chapter 4 in [1]. Pakes and Kaplan [11] obtained a necessary and sufficient condition for the existence of a nondegenerate limiting distribution with a compound renewal immigration component. The generating function of the limiting distribution is given as a unique solution of a renewal-type integral equation. In a more general setting, the limiting distribution of random processes with immigration was investigated recently by Iksanov et al. [8]. In the latter paper the limiting process, called *stationary random process with immigration*, is given as an infinite sum of iid stochastic processes translated by an independent random walk. To the best of our knowledge, moment properties of the limiting distribution were not investigated in detail. However, these problems are beyond the scope of the present paper.

2. Results

We assume that the offspring means are finite, and let M denote the offspring mean matrix

$$M = \begin{pmatrix} \mathbb{E}\mathbf{A}_{1,1;1} \\ \vdots \\ \mathbb{E}\mathbf{A}_{1,1;d} \end{pmatrix} = \begin{pmatrix} m_{1,1} & \dots & m_{1,d} \\ \vdots & \ddots & \vdots \\ m_{d,1} & \dots & m_{d,d} \end{pmatrix}, \tag{2}$$

where $m_{i,j}$ is the mean offsprings of type j produced by an individual of type i. Let ρ denote the spectral radius of M, and assume that the process is subcritical, i.e. $\rho < 1$. Without immigration a subcritical process dies out almost surely exponentially fast, therefore immigration is necessary to obtain nontrivial stationary distribution.

Let $\|\mathbf{x}\| = \sum_{j=1}^{d} |x_j|$ denote the ℓ^1 norm of a vector in $\mathbb{R}^d$ and also the generated matrix norm $\|A\| = \sup_{\|\mathbf{x}\|=1} \|\mathbf{x}A\|$. Then $\|A\|$ is the maximum absolute row sum, see e.g. Examples 5.6.4 and 5.6.5 in Horn and Johnson [7] (note that we multiply from the left). To ease notation we introduce the random operators θ_n as

$$\theta_n \circ \mathbf{k} = \sum_{j=1}^{d} \sum_{i=1}^{k_j} \mathbf{A}_{n,i;j}, \quad \mathbf{k} = (k_1, \dots, k_d), \tag{3}$$

where the empty sum is 0, i.e. $\theta_n \circ \mathbf{0} = 0$. We slightly abuse the notation writing $\theta_n \circ (\mathbf{k}_1 + \mathbf{k}_2) = \theta_n \circ \mathbf{k}_1 + \theta_n \circ \mathbf{k}_2$, where on the right-hand side the two summands are independent. Further, write $\Pi_n = \theta_1 \circ \dots \circ \theta_n$, for $n \geq 1$, and $\Pi_0 = \mathrm{Id}$. With this notation (1) can be written as $\mathbf{X}_n = \theta_n \circ \mathbf{X}_{n-1} + \mathbf{B}_n$. Iteration gives that the stationary distribution can be represented in distribution as

$$\mathbf{Y} = (Y_1, \dots, Y_d) = \mathbf{B}_1 + \theta_1 \circ \mathbf{B}_2 + \theta_1 \circ \theta_2 \circ \mathbf{B}_3 + \dots = \sum_{i=0}^{\infty} \Pi_i \circ \mathbf{B}_{i+1}, \tag{4}$$

provided that the infinite sum converges almost surely. Quine [12] showed that the latter holds if and only if the immigration has finite logarithmic moment, i.e. $\mathbb{E} \log(\|\mathbf{B}_1\| + 1) < \infty$. In our main result we assume that $\mathbb{E}\|\mathbf{B}_1\|^\alpha < \infty$ for some $\alpha > 0$, which implies $\mathbb{E} \log(\|\mathbf{B}_1\| + 1) < \infty$, thus the stationary distribution indeed exists. Formula (4) corresponds to formula (16) in [12] in terms of generating functions.

Note that there might be types which does not appear in the stationary distribution. Indeed, if a type-1 particle never immigrates and it cannot be a descendant of an immigrant then it does not appear in the stationary distribution, and its offspring distribution does not matter. More precisely, introduce the set of essential types $\mathcal{E} \subset \{1, \dots, d\}$ as

$$\mathcal{E} = \{i : \mathbb{P}(Y_i = 0) < 1\}.$$

It is easy to see that type i is essential if and only if the ith coordinate of the vector $\mathbb{E}(\min\{\mathbf{B}_1, \mathbf{1}\})\,(M^0 + \ldots + M^{d-1})$ is strictly positive, where $\mathbf{1} = (1, \dots, 1)$ and the minimum is taken coordinatewise. (This is necessary, as the expectation of the immigrants is not assumed to be finite.)

Now we can state our main result.

Theorem. *Let* $(\mathbf{X}_n)_{n\geq 0}$ *be a d-type, subcritical Galton–Watson process with immigration and let* $\mathbf{Y}$ *be defined in* (4)*. For any* $\alpha > 0$ *the following are equivalent:*

(i) $\mathbb{E}\|\mathbf{A}_{1,1;i}\|^{\max\{\alpha,1\}} < \infty$ *for all* $i \in \mathcal{E}$*, and* $\mathbb{E}\|\mathbf{B}_1\|^\alpha < \infty$*;*

(ii) $\mathbb{E}\|\mathbf{Y}\|^\alpha < \infty$*.*

In particular, (ii) implies that each component of $\mathbf{Y}$ has finite moment of order α.

Note that by the subcriticality assumption $\mathbb{E}\|\mathbf{A}_{1,1;i}\| < \infty$ for any $i \in \{1, \dots, d\}$. Thus, even for $\alpha < 1$ we assume the existence of the offspring mean. However, the immigration distribution might have infinite mean.

We emphasize that we do not need any structural assumption (positive regularity, irreducibility) on the mean matrix. We use directly the infinite sum representation in (4).

3. Proof

We can simply discard the non-essential types, therefore without loss of generality we assume that $\mathcal{E} = \{1, \dots, d\}$.

Implication (ii) $\Rightarrow$ (i) follows easily from representation (4). Indeed, $\mathbf{Y} \geq \mathbf{B}_1$ (meant coordinatewise), thus

$$\mathbb{E}\|\mathbf{B}_1\|^\alpha \leq \mathbb{E}\|\mathbf{Y}\|^\alpha < \infty.$$

Since $1 \in \mathcal{E}$, a type-1 particle appears as an immigrant or as a descendant of an immigrant, meaning that for some $i \in \{1, \dots, d\}$ the event

$$A = \{\theta_2 \circ \ldots \theta_i \circ \mathbf{B}_{i+1} \geq (1, 0, \dots, 0)\}$$

has positive probability and it is independent of the vector $\mathbf{A}_{1,1;1}$. Therefore $\mathbf{Y} \geq \mathbb{I}\{A\}\mathbf{A}_{1,1;1}$, thus

$$\mathbb{E}\|\mathbf{Y}\|^\alpha \geq \mathbb{E}\left(\mathbb{I}\{A\}\|\mathbf{A}_{1,1;1}\|^\alpha\right) = \mathbb{P}(A)\,\mathbb{E}\|\mathbf{A}_{1,1;1}\|^\alpha,$$

implying that $\mathbb{E}\|\mathbf{A}_{1,1;1}\|^\alpha < \infty$. Clearly, the proof works for any type i.

In what follows we prove the implication (i) $\Rightarrow$ (ii).

3.1. The case $\alpha \geq 1$

Let $\alpha \geq 1$ be fixed. First we prove the theorem under the additional assumption that

$$\|M\| < 1, \tag{5}$$

that is the row sums are less than 1. Let $\mu_j = \sum_{\ell=1}^d m_{j,\ell}$, $j \in \{1, \ldots, d\}$.

We prove that

$$M_\alpha(k) = \mathbb{E}\|\Pi_k \circ \mathbf{B}_{k+1}\|^\alpha \tag{6}$$

tends to 0 exponentially fast.

Recall (3). To ease notation we suppress the lower index. We have

$$\|\theta \circ \mathbf{k}\| = \sum_{j=1}^d \sum_{i=1}^{k_j} \|\mathbf{A}_{i;j}\| = \sum_{j=1}^d \sum_{i=1}^{k_j} \sum_{\ell=1}^d A_{i;j}^{(\ell)} =: \sum_{j=1}^d S_{k_j;j},$$

where $\|\mathbf{A}_{i;j}\| = \sum_{\ell=1}^d A_{i;j}^{(\ell)}$ is the number of all the offsprings of the ith individual of type j, with $A_{i;j}^{(\ell)}$ being the number of ℓ-type offsprings, and $S_{k;j} = \sum_{i=1}^k \|\mathbf{A}_{i;j}\|$ is the sum of k iid *scalar* random variables, with mean $\mathbb{E}\|\mathbf{A}_{1;j}\| = \sum_{\ell=1}^d m_{j,\ell} = \mu_j < 1$.

For any $\overline{\mu} \in (\max_{1\leq j \leq d} \mu_j^\alpha, 1)$ there exists k_0' such that

$$\mathbb{E}\left(\frac{S_{k;j}}{k}\right)^\alpha < \overline{\mu} \quad \text{for all } k \geq k_0',\ j \in \{1, \ldots, d\}. \tag{7}$$

Indeed, by the strong law of large numbers

$$\lim_{n\to\infty} \frac{S_{n;j}}{n} = \mu_j \quad \text{a.s.}$$

Therefore, the moment convergence theorem implies

$$\lim_{n\to\infty} \mathbb{E}\left(\frac{S_{n;j}}{n}\right)^\alpha = \mu_j^\alpha,$$

provided we show that the sequence $((S_{n;j}/n)^\alpha)_n$ is uniformly integrable. By the convexity of x^α, for any $y > 0$

$$\begin{aligned}\mathbb{E}\left[\mathbb{I}\left\{\left(\frac{S_{n;j}}{n}\right)^\alpha > y\right\}\left(\frac{S_{n;j}}{n}\right)^\alpha\right] &\leq \mathbb{E}\left[\mathbb{I}\left\{S_{n;j} > ny^{1/\alpha}\right\}\frac{\sum_{i=1}^n \|\mathbf{A}_{i;j}\|^\alpha}{n}\right] \\ &= \mathbb{E}\left[\mathbb{I}\left\{S_{n;j} > ny^{1/\alpha}\right\}\|\mathbf{A}_{1;j}\|^\alpha\right],\end{aligned}$$

where $\mathbb{I}\{\cdot\}$ stands for the indicator function. Markov's inequality gives

$$\mathbb{P}(S_{n;j} > ny^{1/\alpha}) \leq \frac{\mathbb{E}S_{n;j}}{ny^{1/\alpha}} = \mu_j \, y^{-1/\alpha},$$

and since $\mathbb{E}\|\mathbf{A}_{1;j}\|^\alpha < \infty$ the uniform integrability, and thus (7) follows.

Put

$$\max_{1\leq j\leq d} \max_{k<k_0'} \mathbb{E}\left(\frac{S_{k;j}}{k}\right)^\alpha = c_0. \tag{8}$$

Since the function x^α is convex, we have

$$\begin{aligned}\left(\frac{\sum_{j=1}^d S_{k_j;j}}{k_1+\ldots+k_d}\right)^\alpha &= \left(\sum_{j=1}^d \frac{k_j}{k_1+\ldots+k_d}\frac{S_{k_j;j}}{k_j}\right)^\alpha \\ &\leq \sum_{j=1}^d \frac{k_j}{k_1+\ldots+k_d}\left(\frac{S_{k_j;j}}{k_j}\right)^\alpha.\end{aligned}$$

Combined with (7) and (8) this implies for $\mathbf{k} \neq \mathbf{0}$

$$\begin{aligned}\mathbb{E}\|\theta\circ\mathbf{k}\|^\alpha &= \|\mathbf{k}\|^\alpha \mathbb{E}\left(\frac{\|\theta\circ\mathbf{k}\|}{\|\mathbf{k}\|}\right)^\alpha \\ &\leq \|\mathbf{k}\|^\alpha \sum_{j=1}^d \frac{k_j}{\|\mathbf{k}\|}\mathbb{E}\left(\frac{S_{k_j;j}}{k_j}\right)^\alpha \\ &\leq \|\mathbf{k}\|^\alpha \sum_{j=1}^d \frac{k_j}{\|\mathbf{k}\|}\left[\mathbb{I}\{k_j \geq k_0'\}\overline{\mu} + \mathbb{I}\{k_j < k_0'\}c_0\right] \\ &\leq \|\mathbf{k}\|^\alpha\left(\overline{\mu} + \frac{k_0' c_0 d}{\|\mathbf{k}\|}\right).\end{aligned}$$

Choosing $k_0 > 2k_0' c_0 d/(1-\overline{\mu})$ we obtain that

$$\mathbb{E}\|\theta\circ\mathbf{k}\|^\alpha \leq \mu\|\mathbf{k}\|^\alpha \quad \text{whenever } \|\mathbf{k}\| \geq k_0, \tag{9}$$

with $\mu = (1+\overline{\mu})/2 < 1$.

Put

$$c_1 = \max_{\|\mathbf{k}\|<k_0} \mathbb{E}\|\theta\circ\mathbf{k}\|^\alpha$$

and let $\mathbf{Z}$ be a random vector with nonnegative integer components, independent of the $\mathbf{A}$'s. Then, by (9)

$$\begin{aligned}\mathbb{E}\|\theta\circ\mathbf{Z}\|^\alpha &= \sum_{\mathbf{k}} \mathbb{P}(\mathbf{Z}=\mathbf{k})\mathbb{E}\|\theta\circ\mathbf{k}\|^\alpha \\ &= \sum_{\mathbf{k}:\|\mathbf{k}\|\geq k_0} \mathbb{P}(\mathbf{Z}=\mathbf{k})\mathbb{E}\|\theta\circ\mathbf{k}\|^\alpha + \sum_{\mathbf{k}:0<\|\mathbf{k}\|<k_0} \mathbb{P}(\mathbf{Z}=\mathbf{k})\mathbb{E}\|\theta\circ\mathbf{k}\|^\alpha \\ &\leq \sum_{\mathbf{k}:\|\mathbf{k}\|\geq k_0} \mathbb{P}(\mathbf{Z}=\mathbf{k})\mu\|\mathbf{k}\|^\alpha + \sum_{\mathbf{k}:0<\|\mathbf{k}\|<k_0} \mathbb{P}(\mathbf{Z}=\mathbf{k})c_1 \\ &\leq \mu\mathbb{E}\|\mathbf{Z}\|^\alpha + c_1\,\mathbb{E}\|\mathbf{Z}\|.\end{aligned}$$

Turning back to $M_\alpha(k)$ in (6), we obtain the recursion

$$M_\alpha(k) \le \mu M_\alpha(k-1) + c_1 M_1(k-1), \tag{10}$$

with $\mu < 1$. Note that $M_1(k)$ is the expectation of the total number of individuals in generation k in a multitype Galton–Watson process without immigration, starting with $\mathbf{B}_{k+1}$ at generation 0. Therefore $M_1(k) = \|\mathbb{E}\mathbf{B}_1 M^k\| \le \|\mathbb{E}\mathbf{B}_1\| \, \|M\|^k$, which goes to 0 exponentially fast. Thus, recursion (10) implies that

$$\begin{aligned} M_\alpha(k) &\le \mu^k \mathbb{E}\|\mathbf{B}_1\|^\alpha + c_1 \left(M_1(k-1) + \mu M_1(k-2) + \ldots + \mu^{k-1} M_1(0) \right) \\ &\le \mu^k \mathbb{E}\|\mathbf{B}_1\|^\alpha + c_1 \|\mathbb{E}\mathbf{B}_1\| \left(\|M\|^{k-1} + \mu \|M\|^{k-2} + \ldots + \mu^{k-1} \right) \\ &\le \mu^k \mathbb{E}\|\mathbf{B}_1\|^\alpha + c_1 \|\mathbb{E}\mathbf{B}_1\| k (\max\{\|M\|, \mu\})^{k-1}. \end{aligned}$$

Therefore, with $\nu = (1 + \max\{\|M\|, \mu\})/2$ and

$$C = \mathbb{E}\|\mathbf{B}_1\|^\alpha + \frac{\mathbb{E}\|\mathbf{B}_1\| c_1}{\max\{\|M\|, \mu\}} \sup_{k \ge 0} k \left(\frac{\max\{\|M\|, \mu\}}{\nu} \right)^k,$$

we obtain

$$M_\alpha(k) \le C \nu^k \quad \text{for all } k \ge 0. \tag{11}$$

The statement now follows from Minkowski's inequality, as

$$\begin{aligned} (\mathbb{E}\|\mathbf{Y}\|^\alpha)^{1/\alpha} &= \left(\mathbb{E} \Big\| \sum_{k=0}^\infty \Pi_k \circ B_{k+1} \Big\|^\alpha \right)^{1/\alpha} \\ &\le \sum_{k=0}^\infty M_\alpha(k)^{1/\alpha} < \infty. \end{aligned}$$

The additional assumption in (5) can be omitted easily. By Gelfand's formula for the spectral radius we have

$$\lim_{k \to \infty} \|M^k\|^{1/k} = \rho,$$

which is strictly less than 1, by subcriticality. Thus, there exists r such that $\|M^r\| < 1$. The matrix M^r is the mean matrix of the offspring distribution corresponding to Π_r, i.e. when we sample the process only in every rth generation. Therefore, the previous argument gives that $M_\alpha(rk+i)$ in (6) is exponentially small for each $i \in \{0, 1, \ldots, r-1\}$. Clearly, then $M_\alpha(k)$ is also exponentially small, that is (11) holds with some $C > 0$ and $\nu < 1$. Then result follows from Minkowski's inequality as above.

3.2. The case $\alpha < 1$

This case is in fact simpler, but needs to be treated differently.

First, assume again that (5) holds. We use the same notations as above. Now x^α is concave, thus by Jensen's inequality for any $\mathbf{k}$

$$\begin{aligned}\mathbb{E}\|\theta \circ \mathbf{k}\|^\alpha &= \mathbb{E}\Big(\sum_{j=1}^d S_{k_j;j}\Big)^\alpha \leq \Big(\mathbb{E}\sum_{j=1}^d S_{k_j;j}\Big)^\alpha \\ &= \Big(\sum_{j=1}^d k_j \mu_j\Big)^\alpha \leq \mu \|\mathbf{k}\|^\alpha,\end{aligned}$$

with $\mu = \max_{1\leq j\leq d} \mu_j^\alpha$, implying

$$\begin{aligned}\mathbb{E}\|\theta \circ \mathbf{Z}\|^\alpha &= \sum_{\mathbf{k}} \mathbb{P}(\mathbf{Z} = \mathbf{k})\mathbb{E}\|\theta \circ \mathbf{k}\|^\alpha \\ &\leq \sum_{\mathbf{k}} \mathbb{P}(\mathbf{Z} = \mathbf{k})\mu\|\mathbf{k}\|^\alpha \leq \mu\mathbb{E}\|\mathbf{Z}\|^\alpha.\end{aligned}$$

Therefore, $M_\alpha(k)$ in (6) is exponentially small. By subadditivity we have

$$\mathbb{E}\|\mathbf{Y}\|^\alpha = \mathbb{E}\Big\|\sum_{k=0}^\infty \Pi_k \circ B_{k+1}\Big\|^\alpha \leq \sum_{k=0}^\infty M_\alpha(k) < \infty,$$

as claimed.

Condition (5) can be omitted the same way as before.

Acknowledgement. We thank Mátyás Barczy and the two anonymous referees for their constructive remarks on the paper. PK is supported by the János Bolyai Research Scholarship of the Hungarian Academy of Sciences, by the NKFIH grant FK124141, and by the EU-funded Hungarian grant EFOP-3.6.1-16-2016-00008. PW is supported by the Ministry of Innovation and Technology grant UNKP-19-2-SZTE-139, by the EU-funded Hungarian grant EFOP-3.6.2-16-2017-00015, and by the Ministry of Human Capacities, Hungary grant 20391-3/2018/FEKUSTRAT.

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