

LASS, the numerics

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We summarize our efforts to create a numerical implementation of the Local Analytical Subtraction Scheme (LASS) for obtaining NNLO QCD predictions in electron-positron collisions. We focus on the regularization of double-real radiation contributions, which are typically the most computationally intensive part of NNLO calculations. We describe the structure of the subtraction terms in LASS and our approach to numerically validate them. Using arbitrary precision arithmetic, we demonstrate the proper convergence of individual subtraction terms, overlap removal terms, and spurious singularity cancellations in various singular limits. We show results for the specific case of three-jet production in e^+e^- collisions. Our results validate the correctness of both the LASS formalism and our numerical implementation, setting the stage for full NNLO calculations using this method.

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1. Introduction

We are living in the precision era of LHC physics: precise measurements are common-place today. For example, we refer to the p_\perp distribution for lepton-pairs produced in the Drell-Yan process at low- p_\perp values [1, 2]. High-precision measurements are necessary but not sufficient to drive the physics potential of the LHC to the maximum. In order to harvest the most information equally high-precision predictions are needed from the theory side. Such predictions are needed to successful searches for physics beyond of the standard model as well as provide the key to precision phenomenology.

For high-precision predictions higher-order calculations are needed in both sectors of the standard model: in quantum chromodynamics (QCD) and the electroweak sector. These calculations are cumbersome, technologically challenging and CPU-time hungry. High-accuracy measurements impose yet one more requirement on these calculations, namely not only the formal, but also the numerical accuracy of the computations should be decent enough to meet the expected precision of the predictions.

The QCD corrections are usually sizable at the LHC, which necessitates to go beyond the next-to-leading order (NLO) in the computations. Hence several calculational schemes were constructed for making QCD predictions at the next-to-next-to-leading order (NNLO) accuracy. Although these methods can be used to produce cross section values, the uncertainty of the required numerical integrations is often insufficient for the expected precision. The trivial solution to this problem is the increase of computational time, which however comes with a high price. The CPU time devoted to such calculations has reached millions of hours. As science is a part of society, it cannot be insensitive to problems of society. Energy efficiency is paramount for a sustainable society, which imposes an extra constraint on our calculation schemes. While high theoretical accuracy and numerical precision are expected, those should be reached with as little CPU time as possible. This requirement leads us to investigate any new idea in regards to calculating higher-order corrections promising higher energy efficiency.

One such idea is the Local Analytical Subtraction Scheme (LASS) [3–5]. While the LASS has been developed primarily for electron-positron collisions, the simplicity of its subtraction structure and the resulting integrated terms suggest possible generalization to collisions with initial states involving QCD partons. Despite the theoretical foundation laid out in the original publications, a comprehensive numerical implementation at NNLO remains lacking. Our objective is to address this gap by developing a numerical implementation of the LASS formalism. In the following sections, we outline our efforts to implement and validate the essential ingredients required for the $n + 2$ parton contribution, i.e. the regularized double-real radiation contribution, within the framework of an NNLO QCD calculation.

2. LASS from the practitioner’s point of view

In perturbative QCD the cross section is defined as a series in the strong coupling α_S . The fully differential cross section can formally be written as:

$$d\sigma = d\sigma_{\text{LO}} + d\sigma_{\text{NLO}} + d\sigma_{\text{NNLO}} + \dots, \quad (1)$$

where we labelled the first, non-vanishing contribution as LO (for leading order), and kept the next two orders (NLO and NNLO). The LO contribution is the fully differential Born cross section,

$$d\sigma_{\text{LO}} = d\sigma^{\text{B}}. \quad (2)$$

Beyond LO several different ingredients build up the cross section contribution. These ingredients are categorized according to the number of additional partons present as virtual particles in loops, or real ones in the final state. The correction at NLO is a sum of two terms, the single real and virtual contributions, while at NNLO there are double-real, real-virtual and double-virtual corrections,

$$d\sigma_{\text{NLO}} = d\sigma^{\text{R}} + d\sigma^{\text{V}}, \quad d\sigma_{\text{NNLO}} = d\sigma^{\text{RR}} + d\sigma^{\text{RV}} + d\sigma^{\text{VV}}. \quad (3)$$

In this contribution we are going to focus on the regularized real-emission contributions being the most CPU time consuming elements of N^{th} LO QCD calculations. We shall discuss the case of leptons in the initial state as the LASS is published for electron-positron collisions. We shall use the definitions of Ref. [5], in particular introduce the concept of scaleless energies and angle variables characterizing softness and collinearity of parton emissions in terms of two-particle invariants $s_{ij} = 2p_i \cdot p_j$,

$$e_i = \frac{s_{iq}}{s}, \quad w_{ij} = \frac{s_{ij}}{s_{iq}s_{jq}} = \frac{s_{ij}}{s} \frac{s^2}{s_{iq}s_{jq}}, \quad (4)$$

where q is the total incoming four-momentum, $q^\mu = (\sqrt{s}, 0, 0, 0)$. The same quantities can also be expressed through energies and angles as

$$e_i = \frac{2E_i}{\sqrt{s}}, \quad w_{ij} = \frac{1}{2} (1 - \cos \theta_{ij}). \quad (5)$$

These variables give a convenient way to parametrize the various kinematic singularities at NLO and to define appropriate subtraction terms $\bar{\mathbf{S}}_i R$ in the single-soft ($e_i \rightarrow 0$) and $\bar{\mathbf{C}}_{ij} R$ in the single-collinear ($w_{ij} \rightarrow 0$) limits. In these subtraction terms an operator-like behavior is understood in order to take into account color- (soft) and spin-correlations (collinear) and to replace real-emission dynamics with underlying, non-singular Born multiplied by a factor mimicking the singular behavior of original real radiation squared matrix element (SME) $R = |\mathcal{M}_{n+1}|^2$. The soft and collinear regions are not disjoint. The overlaps should be removed in order to avoid double subtractions. Considering all kinematic singularities the regularized real emission can be cast into a form of

$$R - \sum_i \left[\bar{\mathbf{S}}_i + \sum_{j>i} \bar{\mathbf{C}}_{ij} (1 - \bar{\mathbf{S}}_i - \bar{\mathbf{S}}_j) \right] R, \quad (6)$$

where we used the operator-like actions in the subtraction terms. Because the overlap removal can be achieved by dropping soft terms from two-particle Altarelli-Parisi kernels the regularized real-radiation SME can be greatly simplified and written in the short form $R - K^{(1)}$ where

$$K^{(1)} = \sum_i \bar{\mathbf{S}}_i R + \sum_i \sum_{j>i} \bar{\mathbf{H}} \bar{\mathbf{C}}_{ij} R. \quad (7)$$

The new notation for the modified collinear subtractions stands for the hard-collinear terms with the meaning of removed soft pieces from two-particle Altarelli-Parisi kernels.

At NNLO the set of possible kinematic singularities is enlarged, but it is still possible to characterize and define subtractions using only the scaleless variables previously defined. The single unresolved subtractions are analogous to those in Eq. (6) with R replaced with $RR = |\mathcal{M}_{n+2}|^2$. The double soft ($e_i, e_j \rightarrow 0$) and soft-collinear singularities are regularized as

$$RR - \bar{\mathbf{S}}_{ij}RR, \quad RR - \bar{\mathbf{S}}\bar{\mathbf{C}}_{ijk}RR, \quad (8)$$

while the triple ($w_{ij}, w_{ik}, w_{jk} \rightarrow 0$) and double collinear ($w_{ij}, w_{kl} \rightarrow 0$) singularities are regularized as

$$RR - \bar{\mathbf{C}}_{ijk}RR, \quad RR - \bar{\mathbf{C}}_{ijkl}RR. \quad (9)$$

Overlapping singular regions are also present among subtraction terms for double unresolved limits. To remove these overlaps, in addition to the single hard-collinear limit of Eq. (7) with the replacement $R \rightarrow RR$, we introduce the following combinations, defining the hard-collinear limits:

$$\begin{aligned} \overline{\mathbf{HC}}_{ijk}RR &= \bar{\mathbf{C}}_{ijk} \left(1 - \bar{\mathbf{S}}_{ij} - \bar{\mathbf{S}}_{ik} - \bar{\mathbf{S}}_{jk} \right) RR, \\ \overline{\mathbf{SHC}}_{ijk} \left(1 - \bar{\mathbf{C}}_{ijk} \right) RR &= \bar{\mathbf{S}}\bar{\mathbf{C}}_{ijk} \left(1 - \bar{\mathbf{S}}_{ij} - \bar{\mathbf{S}}_{ik} \right) \left(1 - \bar{\mathbf{C}}_{ijk} \right) RR, \\ \overline{\mathbf{HC}}_{ijkl}RR &= \bar{\mathbf{C}}_{ijkl} \left(1 + \bar{\mathbf{S}}_{ik} + \bar{\mathbf{S}}_{jk} + \bar{\mathbf{S}}_{il} + \bar{\mathbf{S}}_{jl} - \bar{\mathbf{S}}\bar{\mathbf{C}}_{ikl} - \bar{\mathbf{S}}\bar{\mathbf{C}}_{jkl} - \bar{\mathbf{S}}\bar{\mathbf{C}}_{kij} - \bar{\mathbf{S}}\bar{\mathbf{C}}_{lij} \right) RR. \end{aligned} \quad (10)$$

Then the subtraction terms for the regularization of singularities in the double unresolved regions for the double-real radiation SME, $RR - K^{(1)} - K^{(2)}$, can be grouped using the hard-collinear notation as follows:

$$K^{(1)} = \sum_i \bar{\mathbf{S}}_i RR + \sum_i \sum_{j>i} \overline{\mathbf{HC}}_{ij}RR, \quad (11)$$

$$\begin{aligned} K^{(2)} &= \sum_i \sum_{j>i} \bar{\mathbf{S}}_{ij}RR + \sum_i \sum_{j>i} \sum_{k>j} \overline{\mathbf{HC}}_{ijk}RR + \\ &+ \sum_i \sum_{j \neq i} \sum_{\substack{k>j \\ k \neq i}} \overline{\mathbf{SHC}}_{ijk} \left(1 - \bar{\mathbf{C}}_{ijk} \right) RR + \sum_i \sum_{j>i} \sum_{\substack{k>i \\ k \neq j}} \sum_{\substack{l>k \\ l \neq j}} \overline{\mathbf{HC}}_{ijkl}RR. \end{aligned} \quad (12)$$

This construction allows for a *partial* kinematic regularization expressed in the following symbolic form:

$$\left(RR - K^{(1)} \right) \Big|_{\text{singly-unres.}} = \text{finite}, \quad \left(RR - K^{(2)} \right) \Big|_{\text{doubly-unres.}} = \text{finite}. \quad (13)$$

However, as a consequence of $K^{(1)}$ developing spurious singularities in doubly-unresolved and $K^{(2)}$ developing spurious singularities in singly-unresolved regions of phase space,

$$\left(RR - K^{(1)} - K^{(2)} \right) \Big|_{\text{singly- \& doubly-unres.}} \neq \text{finite}. \quad (14)$$

To have full kinematic regularization an additional collection of terms has to be introduced:

$$\left(RR - K^{(1)} - K^{(2)} - K^{(12)} \right) \Big|_{\text{singly- \& doubly-unres.}} = \text{finite}, \quad (15)$$

where the role of the newly introduced contribution, $K^{(12)}$ is two-fold. It regularizes spurious singularities of $K^{(1)}$ in doubly- and those of $K^{(2)}$ in singly-unresolved regions.

In this contribution we focus on the regularization of the double-real radiation. In order not to change the complete NNLO correction, the subtraction terms $K^{(i)}$ ($i = 1, 2$, or 12) will be integrated analytically over the one- or two-particle radiation phase spaces depending on the subtraction terms, and subsequently added to the real-virtual and double virtual contributions with reduced particle multiplicity in final-state. A companion contribution [6] gives details how this is done in the case of the correction with $n + 1$ partons in the final state.

Due to its double role, the definition of the $K^{(12)}$ subtraction is fairly delicate, which is facilitated by the introduction of sectors, where only a subset of the singular regions exists. The sectors are specified by introducing corresponding sector functions

$$\mathcal{Z}_{ijk}, \quad \mathcal{Z}_{ijkl}, \quad (16)$$

whose detailed construction is presented in Sections 3.2 and 3.6 of Ref. [5]. In order not to change the physical cross section, the sector functions must fulfil the important sum rule

$$1 = \sum_{i < j < k} \mathcal{Z}_{ijk} + \sum_{\substack{i, j \\ j > i}} \sum_{\substack{k > i \\ k \neq j}} \sum_{\substack{l > k \\ l \neq j}} \mathcal{Z}_{ijkl}, \quad (17)$$

which offers a way to continuously partition the double-real emission phase space:

$$d\Phi_{n+2} = \sum_{i < j < k} \mathcal{Z}_{ijk} d\Phi_{n+2} + \sum_{\substack{i, j \\ j > i}} \sum_{\substack{k > i \\ k \neq j}} \sum_{\substack{l > k \\ l \neq j}} \mathcal{Z}_{ijkl} d\Phi_{n+2}. \quad (18)$$

The sectors defined by sector function \mathcal{Z}_{ijk} contain singular regions where the subtraction terms

$$\left(\bar{\mathcal{S}}_i, \bar{\mathcal{S}}_j, \bar{\mathcal{S}}_k, \bar{\mathcal{C}}_{ij}, \bar{\mathcal{C}}_{ik}, \bar{\mathcal{C}}_{jk}, \bar{\mathcal{S}}_{ij}, \bar{\mathcal{S}}_{ik}, \bar{\mathcal{S}}_{jk}, \bar{\mathcal{S}}_i \bar{\mathcal{C}}_{jk}, \bar{\mathcal{S}}_j \bar{\mathcal{C}}_{ik}, \bar{\mathcal{S}}_k \bar{\mathcal{C}}_{ij}, \bar{\mathcal{C}}_{ijk} \right) RR \quad (19)$$

becomes singular, while the sectors corresponding to \mathcal{Z}_{ijkl} contain singular regions where

$$\left(\bar{\mathcal{S}}_i, \bar{\mathcal{S}}_j, \bar{\mathcal{S}}_k, \bar{\mathcal{S}}_l, \bar{\mathcal{C}}_{ij}, \bar{\mathcal{C}}_{kl}, \bar{\mathcal{S}}_{ik}, \bar{\mathcal{S}}_{il}, \bar{\mathcal{S}}_{jk}, \bar{\mathcal{S}}_{jl}, \bar{\mathcal{S}}_i \bar{\mathcal{C}}_{kl}, \bar{\mathcal{S}}_j \bar{\mathcal{C}}_{kl}, \bar{\mathcal{S}}_k \bar{\mathcal{C}}_{ij}, \bar{\mathcal{S}}_l \bar{\mathcal{C}}_{ij}, \bar{\mathcal{C}}_{ijkl} \right) RR \quad (20)$$

become singular. The operators are meant to affect not just the dynamics but also these sector functions. The limit(s) corresponding to the subtraction term are strictly taken in the sector function resulting in a factorized form as defined in Eqs. (3.12) to (3.15) of Ref. [3]. There is ample freedom choosing the sector functions permitted as long as the analytic integration of the subtractions is possible.

A subtraction term can contribute in many sectors. The collinear terms necessitate the introduction of reference momenta to define Sudakov parameters. Iterated Catani-Seymour mappings [7] are used to factorize the real-radiation phase space into a phase space with lower multiplicity times a one- or two-body phase space factor. These mappings depend on the sector and the subtraction at hand. To have a working subtraction method the integration over factorized one- or two-body phase spaces should be performed analytically, which is possible if the action of the soft and hard-collinear operators on the sector functions is to collapse them to unity or factorize them with mapped momenta (lower multiplicity) to allow for analytical integrations.

To demonstrate explicitly how these actions take place, consider the definite example of a single-collinear subtraction term $\bar{\mathcal{C}}_{ij}RR$. This subtraction becomes singular in many sectors, namely in

$$\{\mathcal{Z}_{ijk} : k \neq i, j\} \cup \{\mathcal{Z}_{ijkl} : k, l \neq i, j, k < l\}. \quad (21)$$

Thus, the collinear subtractions for the ij splitting pair can be written as:

$$\sum_{k \neq i, j} \left[\bar{\mathcal{C}}_{ij}RR \mathcal{Z}_{ijk} + \sum_{\substack{l > k \\ k, l \neq i, j}} \bar{\mathcal{C}}_{ij}RR \mathcal{Z}_{ijkl} \right] = \sum_{k \neq \bar{j}} \left[\bar{\mathcal{Z}}_{\bar{j}k} \bar{\mathcal{C}}_{ij}RR + \sum_{\substack{l > k \\ k, l \neq \bar{j}}} \bar{\mathcal{Z}}_{kl} \bar{\mathcal{C}}_{ij}RR \right], \quad (22)$$

where the bar over the sector function signifies that it is calculated with mapped momenta and \bar{j} stands for the mother parton (with mapped momentum) for the ij splitting pair. The reference momentum can change from sector to sector, hence the subtraction term cannot be factorized. Still, analytical integration over one-particle radiation phase space is possible with sector functions depending solely on the momenta of the underlying kinematics.

In a similar fashion we can consider the case of a double-soft subtraction term contributing to both three- and four-index sectors

$$\{\mathcal{Z}_{ijk} : k \neq i, j\} \cup \{\mathcal{Z}_{ikjl} : k \neq i, j, l \neq i, j, k\}. \quad (23)$$

Looking at the double-soft operator acting on the sector functions the following limiting behavior is observed (cf. Eqs. (3.4), (C.94) and (C.95) of Ref. [5]):

$$\bar{\mathcal{S}}_{ij} \mathcal{Z}_{ijk} = \frac{\sigma_{ijjk} + \sigma_{ikjk} + (i \leftrightarrow j)}{\sum_{m \neq i} \sum_{n \neq i, j} \sigma_{imjn} + (i \leftrightarrow j)}, \quad \bar{\mathcal{S}}_{ij} \mathcal{Z}_{ikjl} = \frac{\sigma_{ikjl} + \sigma_{jlik}}{\sum_{m \neq i} \sum_{n \neq i, j} \sigma_{imjn} + (i \leftrightarrow j)}. \quad (24)$$

Then the full double-soft subtraction for the ij soft pair can be rewritten as

$$\sum_{k \neq i, j} \left[\bar{\mathcal{S}}_{ij}RR \mathcal{Z}_{ijk} + \sum_{l \neq i, j, k} \bar{\mathcal{S}}_{ij}RR \mathcal{Z}_{ikjl} \right] = \bar{\mathcal{S}}_{ij}RR \left[\bar{\mathcal{S}}_{ij} \mathcal{Z}_{ijk} + \sum_{l \neq i, j, k} \bar{\mathcal{S}}_{ij} \mathcal{Z}_{ikjl} \right] = \bar{\mathcal{S}}_{ij}RR, \quad (25)$$

where factorization was made possible by the subtraction being independent of any reference momentum, hence sector functions. The limit sector function in the square bracket, by construction, also obey a sum rule, thus adding up to one leaving only a sector-independent subtraction term.

3. Numerical checks

As discussed, the construction of the complete LASS scheme requires many steps:

- mappings to underlying kinematics,
- reference momentum choices,
- Sudakov parametrizations,
- spin- and/or color-correlated SME of reduced kinematics,
- overlap removal.

The construction of a complete partonic Monte Carlo program requires careful numerical checks. To this end we selected the specific process of production of three hadronic jets in electron-positron annihilation.¹ Three-jet production at NNLO in QCD has three different classes of subprocesses with the highest multiplicity in the final state:

$$e^+ e^- \rightarrow q\bar{q}r\bar{r}g, \quad e^+ e^- \rightarrow q\bar{q}q\bar{q}g, \quad e^+ e^- \rightarrow q\bar{q}ggg. \quad (26)$$

In order to numerically test the subtraction scheme we created a Fortran 90 program which uses the MPFUN2020 package [8] to evaluate both dynamics and subtractions in arbitrary precision². In order to keep numerical precision consistent we re-implemented the tree-level SMEs describing $e^+ e^- \rightarrow n$ partons ($n \leq 5$) using appropriate data types. We used the same framework to test both the regularized double-real and real-virtual [6] contributions. The latter involves one-loop contributions up to four partons in the final state, so we also re-implemented these using data types and special functions provided by the MPFUN2020 package. The limiting behavior affects Sudakov parameters that appear in the arguments of classical polylogarithms, whose function instances were changed to special variants supporting arbitrary precision.

To perform complete tests of the subtraction terms, all possible singular limits have to be iteratively approached. This is achieved by generating an underlying Born phase space point (also in arbitrary precision) and sequentially increasing the multiplicity of the final state applying the inverted Catani-Seymour mapping.

To characterize the energies (softness) and enclosed angle (collinearity) of the produced pairs a $\lambda \in [0, 1]$ parameter was defined and used. From iteration to iteration appropriately decreasing λ by (half) an order of magnitude³ all possible NNLO limits could be approached.

The code makes it possible to validate subtraction terms at multiple levels:

- Individual subtraction terms compared to the appropriate radiation SME in their restrictive limits.
- Individual overlap terms compared to a single subtraction checking proper overlap removal.
- Individual terms defined to cancel spurious singularities compared to a term from $K^{(1)}$ or $K^{(2)}$ to check cancellation of the spurious singularity.
- Full set of subtractions compared to the radiation SME in all possible physical limits.

In the following we show examples for each of these cases demonstrating the capabilities of our code and of the subtraction scheme. To be definite we selected the most complicated subprocess with three gluons in the final state. To refer to various limits, we use the labelling $e_1^+ e_2^- \rightarrow q_3 \bar{q}_4 g_5 g_6 g_7$.

First we consider the individual subtraction terms for double-soft and triple-collinear emissions that should be compared to the double-real SME in the same limits. In case of correct definition

¹The LASS scheme uses iterated Catani-Seymour mappings with reference (recoil) momentum chosen from the final state, thus it is not suitable for two-jet production in electron-positron annihilation.

²For our tests it was sufficient to use 50 working digits.

³Whenever a soft-collinear limit was probed λ was used to characterize energy of the soft-candidate parton, which was decreased by an order of magnitude, while the square-root of λ was used to characterize collinearity of the collinear pair to maintain proper scaling between these limits.

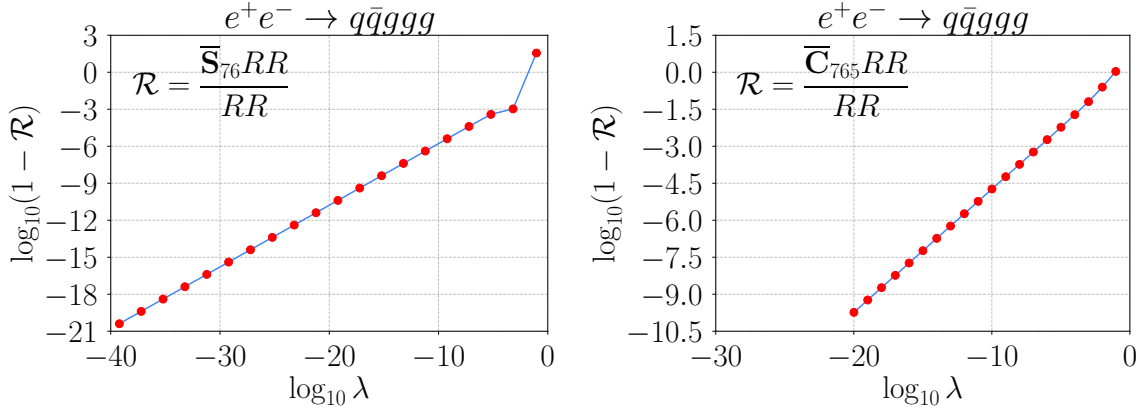


Figure 1: $\bar{S}_{76}RR$ and $\bar{C}_{765}RR$ subtractions compared to SME in their limits, respectively.

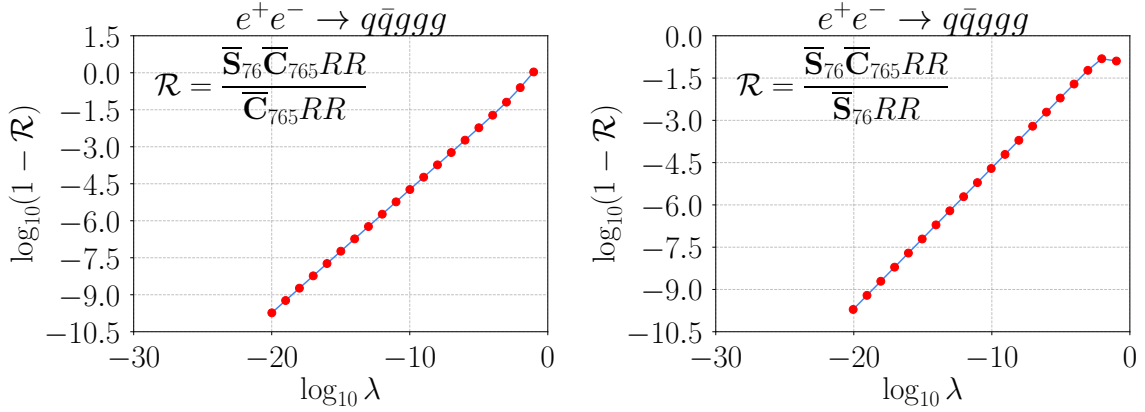


Figure 2: The $\bar{S}_{76}\bar{C}_{765}RR$ compared to the corresponding triple-collinear (left) and double-soft (right) term taking the appropriate limits.

and implementation their ratios should tend to one as we go deeper and deeper into the limit,

$$\frac{\bar{S}_{76}RR}{RR} \xrightarrow{6,7 \rightarrow 0} 1, \quad \frac{\bar{C}_{765}RR}{RR} \xrightarrow{5||6||7} 1. \quad (27)$$

The corresponding convergent set of ratios are depicted on Fig. 1.

There is an overlap between the double-soft and triple-collinear subtractions. If the overlap term is well defined it should cancel with the double-soft(triple-collinear) term in the triple-collinear(double-soft) limits,

$$\frac{\bar{S}_{76}\bar{C}_{765}RR}{\bar{C}_{765}RR} \xrightarrow{6,7 \rightarrow 0} 1, \quad \frac{\bar{S}_{76}\bar{C}_{765}RR}{\bar{S}_{76}RR} \xrightarrow{5||6||7} 1. \quad (28)$$

The corresponding set of points can be seen on Fig. 2.

To illustrate the cancellation of spurious singularities we consider a pair of single- and double-soft subtraction terms. To avoid spurious singularities a strongly-ordered soft term has to be defined

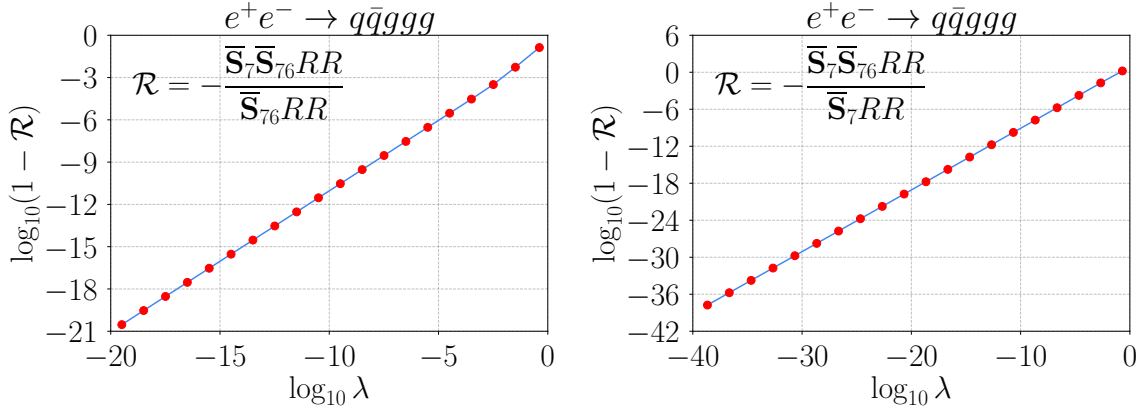


Figure 3: Example convergence for a spurious singularity cancellation involving soft limits.

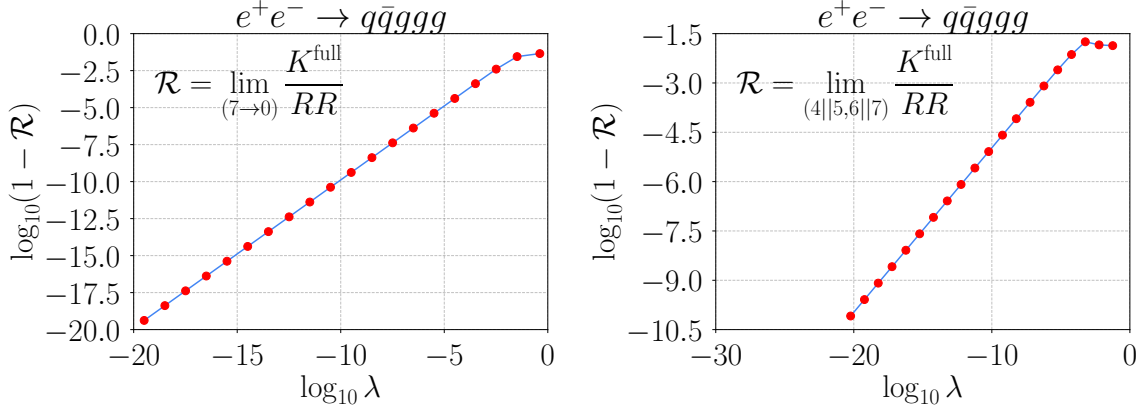


Figure 4: Comparison of the total set of subtractions to the SME in two physical limits: single-soft (left) and double-collinear (right).

with the following conditions:

$$\frac{\bar{S}_7 \bar{S}_{76} RR}{\bar{S}_{76} RR} \xrightarrow{7 \rightarrow 0} -1, \quad \frac{\bar{S}_7 \bar{S}_{76} RR}{\bar{S}_7 RR} \xrightarrow{6,7 \rightarrow 0} -1. \quad (29)$$

The corresponding plots can be seen on Fig. 3. Notice that the ratio approaches minus one in case of spurious singularity cancellations. In our case the corresponding minus signs are defined into the corresponding terms of $K^{(12)}$ which makes it easier in the development phase to group subtraction terms in deep limits to observe cancellations when happening in groups.

Finally, we demonstrate the correctness of the complete set of subtractions by considering a physical limit and comparing to the double-real SME. For illustrative purposes we consider a single-soft and a double-collinear limit:

$$\frac{K^{(1)} + K^{(2)} + K^{(12)}}{RR} = \frac{K^{\text{full}}}{RR} \xrightarrow{7 \rightarrow 0} 1, \quad \frac{K^{(1)} + K^{(2)} + K^{(12)}}{RR} = \frac{K^{\text{full}}}{RR} \xrightarrow{4||5,6||7} 1. \quad (30)$$

The corresponding plots can be seen on Fig. 4.

4. Conclusions

We presented the first numerical implementation of the Local Analytical Subtraction Scheme by illustrating those numerical checks we conducted in order to ensure our numerical implementation is capable of regularizing all those kinematic singularities that can emerge in a computation of a QCD cross section in electron-positron collisions at NNLO accuracy. In our numerical studies we also investigated and proved that subtractions are set up such that they do not only cancel kinematic singularities of the radiation SMEs but also the spurious singularities being inherently present at NNLO. To perform these studies we created an arbitrary-precision numerical framework built around the arbitrary-precision package written in Fortran 90 called MPFUN2020. Our results convinced us that subtractions are coded correctly and our next step towards the computation of a complete QCD prediction at NNLO accuracy for distributions for three-jet production in electron-positron annihilation is to start a numerical integration of these terms with appropriate dynamics in a parton-level Monte-Carlo code.

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