Model order reduction method for sampled-data Lur'e-type systems: a Laplace and z-domain approach

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Summary. This work presents a model order reduction technique in case when a discrete-time control law is applied to a Lur'e-type system, i.e., to a linear dynamical system with a nonlinear feedback component. It is shown that the Laplace and z-transform of the piecewise-continuous response functions can be determined analytically, which allows model order reduction to be performed on the linear component of the closed control loop. Simulations of a balancing problem verify that the proposed method can accurately model the sampling-related complex vibration phenomena and also the nonlinear characteristics of the system.

Introduction

The analysis and design of digitally controlled systems can often be a challenging task due to the nonlinear properties of the plant and the piecewise-smooth dynamics of the controller, as the control input is only updated at the sampling instants. Sampling and using zero-order hold (ZOH) introduces time delay in the feedback loop, causing infinite-dimensional dynamical behavior with multi-frequency vibrations, even in 1 DoF systems [1]. These phenomena can only be observed in continuous time; thus, using traditional discrete-time approaches leads to inaccurate models [2]. This work aims to provide a reduced-order continuous-time model for the whole closed control loop, which can exhibit the dominant behavior of the original sampled-data system with the nonlinear plant.
Consider an inverted pendulum with stationary pivot point (see Figure 1(a)) described by the following equation of motion

$$\ddot{\varphi}(t) - \tfrac{3g}{2\ell}\sin(\varphi(t)) = \tfrac{3}{m\ell^2}M(t), \tag{1}$$

where φ is the angular position, m and ℓ denotes the mass and length of the pendulum, respectively, g is the gravitational acceleration, and $M(t)$ is the excitation torque, which is determined by a discrete-time state-feedback controller as follows

$$M(t) = K_r r[k] - k_1\varphi(k\tau) - k_2\dot{\varphi}(k\tau) \quad \text{with} \quad k\tau \le t < (k+1)\tau \quad \text{and} \quad k = 0, 1, \dots, \tag{2}$$

where $r[k]$ is the discrete-time reference and K_r, k_1, k_2 are the control parameters, and τ is the sampling time.
According to Figure 1(b), the pendulum can be considered as a Lur'e-type system [3, 4], i.e., a linear state-space model with

$$\mathbf{x}(t) \equiv \begin{bmatrix}\varphi(t)\\ \dot{\varphi}(t)\end{bmatrix}, \quad y(t) \equiv \varphi(t), \quad \mathbf{A} = \begin{bmatrix}0 & 1\\ \frac{3g}{2\ell} & 0\end{bmatrix}, \quad \mathbf{B} = \begin{bmatrix}0\\ \frac{3}{m\ell^2}\end{bmatrix}, \quad \mathbf{C} = \begin{bmatrix}1 & 0\end{bmatrix}, \quad \text{and} \quad \mathbf{K} = \begin{bmatrix}k_1 & k_2\end{bmatrix}, \tag{3}$$

and a nonlinear feedback component represented by $\psi(\varphi) = \frac{mg\ell}{2}(\sin(\varphi) - \varphi)$.

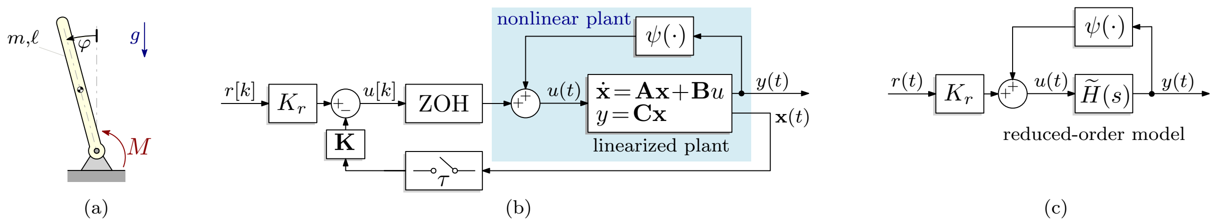

Figure 1: (a) Schematic model of the inverted pendulum. (b) Block diagram of the Lur'e-type plant controled by a discrete-time state-feedback control law. (c) Block diagram of the reduced-order model with a continuous-time transfer function $\widetilde{H}(s)$.

Reduced-order modeling

The aim of the model-order reduction is to find a reasonably low-order continuous-time system represented by the transfer function $\widetilde{H}(s)$ in Figure 1(c) to approximate the overall characteristics of the original system. The main idea of the proposed method is that the step response of the linearized system without the feedback component $\psi(\cdot)$ can be represented analytically using LTI (linear time invariant) systems theory. Let $U^*(z) = \mathcal{Z}\{u[k]\}(z)$ denote the z-transform of the discrete-time control input $u[k]$, which propagates through the ZOH and the plant. The continuous-time output signal can be represented in the Laplace-domain with $z = \mathrm{e}^{s\tau}$ as

$$\mathcal{L}\{y(t)\}(s) = U^*(\mathrm{e}^{s\tau})H_{\mathrm{ZOH}}(s)H_{\mathrm{p}}(s) \tag{4}$$

where $H_{\mathrm{ZOH}}(s)$ and $H_{\mathrm{p}}(s)$ are the transfer functions of the ZOH and the plant, respectively, and $\mathcal{L}$ denotes the Laplace transform [2].

The continuous-time transfer function $H(s)$ of the closed control loop can be obtained using step-invariant transform to satisfy the following equation

$$\mathcal{L}\{y(t)\}(s) = \tfrac{1}{s}H(s). \tag{5}$$

Combining Equations (4) and (5) leads to a transfer function $H(s)$ with an infinite number of poles as the $z = \mathrm{e}^{s\tau}$ substitution is periodic along the imaginary axis. A reduced-order model represented by the transfer function $\widetilde{H}(s)$ is provided with the following conditions according to Reference [2]:

1. $\widetilde{H}(s)$ is a rational function of s,
2. the poles of $\widetilde{H}(s)$ match the poles of $H(s)$,
3. the residues of $\widetilde{H}(s)$ at its poles match the residues of $H(s)$,
4. the static gain of $\widetilde{H}(s)$ matches $H(s)$.

The first condition ensures that the system can be represented by a constant-coefficient linear ODE, the second and third conditions ensure matching dynamic properties and the fourth condition can be simply expressed as $\widetilde{H}(0) = H(0)$.

Results

A numerical example is presented with parameters $m = 1\,\mathrm{kg}$, $\ell = 0.2\,\mathrm{m}$, $g \approx 10\,\frac{\mathrm{m}}{\mathrm{s}^2}$, $\tau = 0.05\,\mathrm{s}$, $K_r = 18.49\,\frac{\mathrm{Nm}}{\mathrm{rad}}$, $k_1 = 19.49\,\frac{\mathrm{Nm}}{\mathrm{rad}}$ and $k_2 = 0.5290\,\frac{\mathrm{Nms}}{\mathrm{rad}}$. A 4th-order approximation can be obtained using the proposed method as

$$\widetilde{H}(s) = 0.0007893 + \frac{381.153 - 0.385066s}{s^2 + 4.21819s + 3203.47} + \frac{0.346652s - 314.01}{s^2 + 4.21819s + 4779.79}. \tag{6}$$

Figure 2(a) compares $H(s)$ with the the reduced-order transfer function $\widetilde{H}(s)$ in the Laplace domain. It is clearly visible that the sampling-related infinite-dimensional dynamics lead to repeated poles along the frequency axis, while the reduced-order model only approximates the dominant poles. Finally, the model is completed with the nonlinear feedback component $\psi(\cdot)$, resulting in a reduced-order Lur'e-type model, whose step response is shown in Figure 2(b).

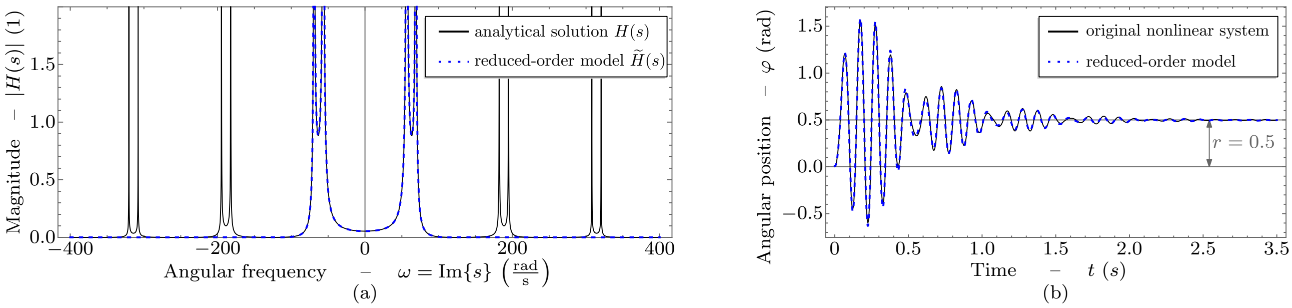

Figure 2: (a) Transfer functions of the exact and the reduced-order model of the closed control loop evaluated at $\mathrm{Re}\{s\} = -2.109$. (b) Step responses of the original system and the reduced-order model with Lur'e-type nonlinearity.

Conclusions

This work presented a model order reduction method for Lur'e-type nonlinear systems controlled by discrete-time control law. It is shown that sampling results in infinite-dimensional dynamics, which can be approximated by a reduced-order continuous-time model in the Laplace domain. Numerical simulations of a balancing problem illustrate the proposed method, and show that the reduced-order model accurately matches the original system both in time and Laplace domain.

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