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Generalized eccentric vs. true anomaly parametrizations in the perturbed Keplerian motion

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Abstract
The angular and the radial parts of the dynamics of the perturbed Kepler motion are separable in many important cases. In this paper we study the radial motion and its parametrizations. We develop in detail a generalized eccentric anomaly parametrization and a procedure of computing a generic class of integrals based on the residue theorem. We apply the formalism to determine various contributions to the luminosity of a compact binary.

1 Introduction

The main sources of the gravitational radiation to be detected by Earth-based observatories (LIGO, VIRGO, TAMA and GEO) are compact binaries. The Laser Interferometer Space Antenna (LISA) will also detect gravitational waves from colliding galactic black holes. Compact binaries are composed of black holes and/or neutron stars. In such systems, a highly accurate description would include the spin-spin (SS) interaction [Barker & O'Connell (1979)], the magnetic dipole-magnetic dipole moments (DD) contribution [Ioka & Taniguchi (2000)] and the quadrupole-monopole effect (QM) [Poisson (1998)]. The parametrization of the radial motion for them are presented in [Gergely (2000)], [Vasúth, Keresztes, Mihály, Gergely (2003)], and [Gergely & Keresztes (2003)].

The method for solving a wide class of the perturbed Keplerian radial motions is worked out in [Gergely, Perjés, Vasúth (2000)]. For this purpose two (the true and eccentric anomaly) parametrization were introduced. The method can be further generalized for an even wider class of perturbed Keplerian radial motions (see: [Gergely, Keresztes, Mikóczi (2006)]).

2 The perturbed Kepler motion

The so-called radial equations for SS, DD and QM interactions are given in [Keresztes, Mikóczi , Gergely (2005)]. The radial equation of the generalized perturbed Kepler motion we consider here [Gergely, Keresztes, Mikóczi (2006)] is

$$\dot{r}^2 = \frac{2E}{\mu} + \frac{2Gm}{r} - \frac{L^2}{\mu^2 r^2} + \sum_{i=0}^{p} \frac{\varphi_i(\chi)}{\mu^2 r^i} \tag{1}$$

where $\varphi_i(\chi)$ characterize the perturbative terms and they are periodic functions of the true anomaly:

$$\varphi_i(\chi) = \sum_{j=0}^{\infty} (f_{ij} + g_{ij}\cos\chi)\sin^j\chi \ . \tag{2}$$

The expression $\varphi_i(\chi)$ given above is equivalent with a generic Fourier series [Gergely, Keresztes, Mikóczi (2006)]. The coefficients f_{ij} and g_{ij} can be expressed in terms of the coefficients of the Fourier expansion as well. The last term in Eq. (5) contains the generic perturbing Brumberg force [Brumberg (1991)], the spin-orbit interaction for compact binaries [Rieth and Schafer (1997); Gergely, Perjés, Vasúth (1998)], and the SS, DD and QM contributions. The energy E and angular momentum L refer to the perturbed motion. From the condition $\dot{r}^2 = 0$ we find the turning points $r_{\substack{\max\\\min}}$:

$$r_{\substack{\max\\\min}} = \frac{Gm\mu \pm A_0}{-2E} \pm \frac{1}{2\mu A_0}\sum_{i=0}^{p}\varphi_i^{\pm}(\chi)\left[\frac{\mu(Gm\mu \mp A_0)}{L^2}\right]^{i-2} \ , \tag{3}$$

where $A_0 = \left(G^2m^2\mu^2 + 2EL^2/\mu\right)^{1/2}$ is the magnitude of the Laplace-Runge-Lenz vector belonging to the perturbed motion characterized by E and L. $\varphi_i^- = \varphi_i(0)$, $\varphi_i^+ = \varphi_i(\pi)$ are small coefficients. With these turning points is possible to introduce the generalized true anomaly parametrization of the radial motion,

$r(\chi)$ as

$$\frac{2}{r} = \frac{1+\cos\chi}{r_{\min}} + \frac{1-\cos\chi}{r_{\max}} \ . \tag{4}$$

Integrals of the type

$$\int_0^T \frac{\omega(\chi)}{r^{2+n}} dt \tag{5}$$

frequently occur. Here $\omega(\chi)$ is the same type of periodic function of the true anomaly as $\varphi_i(\chi)$, and T is the radial period of the motion.

For constant values of φ_i and $n \geq 0$ these integrals can be evaluated in terms of a complex true anomaly variable $z = \exp(i\chi)$ by using the residue theorem. The only pole is in the origin ($z = 0$) [Gergely, Perjés, Vasúth (2000)]. For the class $n < 0$ a complex eccentric anomaly parameter can be used and then the poles are in the origin and in

$$w_1 = \left(\frac{Gm\mu^2 - \sqrt{-2\mu EL^2}}{Gm\mu^2 + \sqrt{-2\mu EL^2}}\right)^{1/2} , \tag{6}$$

however the latter occurs only rarely in physical applications (see:[Gergely, Perjés, Vasúth (2000)]).

3 Generalized perturbed Kepler motion with the eccentric anomaly parametrization

We introduce the $r(\xi)$ eccentric anomaly parametrization in the same way as in [Gergely, Perjés, Vasúth (2000)]:

$$2r = (1+\cos\xi)r_{\min} + (1-\cos\xi)r_{\max} \ . \tag{7}$$

We use the relations between the true and eccentric anomaly (4), (7)

$$\cos\chi = \frac{Gm\mu\cos\xi - A_0}{Gm\mu - A_0\cos\xi} , \qquad \sin\chi = \frac{\sqrt{\frac{-2EL^2}{\mu}}\sin\xi}{Gm\mu - A_0\cos\xi} \ . \tag{8}$$

Thus we can express φ_i as the function of ξ:

$$\varphi_i(\xi) = \sum_{j=0}^{\infty}\left(f_{ij} + g_{ij}\frac{Gm\mu\cos\xi - A_0}{Gm\mu - A_0\cos\xi}\right)\left(\frac{-2EL^2\sin\xi}{Gm\mu^2 - A_0\mu\cos\xi}\right)^j \ . \tag{9}$$

Employing the eccentric anomaly parametrization (7), the integrals (5) could be evaluated as

$$\int_0^{2\pi} \frac{\omega(\xi)}{r^{n+1}} \left(\frac{1}{r} \frac{dt}{d\xi} \right) d\xi \ . \tag{10}$$

For $n' \equiv -n-1 \geq 0$ we apply the binomial expansion

$$(2r)^{n'} = \sum_{k=0}^{n'} \binom{k}{n'} r_{\min}^k r_{\max}^{n'-k} (1+\cos\xi)^k (1-\cos\xi)^{n'-k} \ , \tag{11}$$

leading to a polynomial in $\cos\xi$. From the radial equation (5) using the eccentric anomaly parametrization (7) to leading order we obtain:

$$\frac{1}{r}\frac{dt}{d\xi} = \sqrt{\frac{\mu}{-2E}} \left[1 - \frac{E}{2\mu A_0^2 \sin^2\xi} \sum_{i=0}^{p} \left(\Omega_+^i - \Omega_-^i \cos\xi - \frac{2\varphi_i(\xi)}{r^{i-2}} \right) \right] \ , \tag{12}$$

with

$$\Omega_\pm^i = \left(\frac{\mu}{L^2} \right)^{i-2} \left[\varphi_i^+ (Gm\mu - A_0)^{i-2} \pm \varphi_i^- (Gm\mu + A_0)^{i-2} \right] \ . \tag{13}$$

The bracket becomes proportional to $\sin^2\xi$ in (12) if the following two conditions are satisfied:

$$\sum_{i=0}^{p} \frac{\mu^i}{L^{2i}} (Gm\mu \pm A_0)^i (f_{i1} \pm g_{i1}) = 0 \ . \tag{14}$$

With conditions (14) satisfied, using the parametrization (7), the integrand of (10) becomes regular. The conditions are fulfilled in the case of the SS, DD and QM interactions. Introducing the complex variable $w = \exp(i\xi)$, the integral of (10) contains only two poles: the origin and the w_1 (see:[Gergely, Perjés, Vasúth (2000)]). We have proven for the $n < 0$ case:

Theorem: *For all perturbed Kepler motions characterized by the radial equation (5), with periodic perturbing functions $\varphi_i(\xi)$ obeying the conditions (14), and for arbitrary periodic functions $\omega(\xi)$, the integrals (10) are given by the residue in the origin and in the w_1 [Gergely, Perjés, Vasúth (2000)]. on w complex plane.*

In the next section we apply the above method for computing different contributions to the luminosity of compact binaries.

4 Application for some compact binaries

Peters and Mathews [Peters and Mathews (1963)] computed the luminosity of the compact binaries in the Kepler motion (Newtonian case) with orbital parameters:

$$\mathcal{L}_N = \frac{G^4 m^3 \mu^2}{15 c^5 a^5 (1-e^2)^{7/2}} \left(37e^4 + 292e^2 + 96\right) \ , \tag{15}$$

where a the semi-major axis and e the eccentricity. The form of $\varphi_i(\xi)$ can be derived for all of the SS, QM and DD contributions. We can then compute the various contributions to the luminosity ($-\langle dE/dt \rangle$, the time-averaged energy loss over one radial period). These are:

$$\begin{aligned}
\mathcal{L}_{S_1 S_1} &= \frac{G^4 m^2 \mu S_1 S_2}{480 c^7 a^7 (1-e^2)^{10/2}} \Big[c_1 \sin\kappa_1 \sin\kappa_2 \cos 2(\psi_0 - \bar{\psi}) \\
&\quad + c_2 \cos\kappa_1 \cos\kappa_2 + c_3 \cos\gamma \Big] \ , \\
\mathcal{L}_{QM} &= \frac{G^4 m^5 \mu}{60 c^5 a^7 (1-e^2)^{10/2}} \\
&\quad \times \sum_{i=1}^{2} p_i \left[\left(c_4 (3 \sin^2\kappa_i - 2) + c_5 \sin^2\kappa_i \cos\delta_i \right) \right] \ , \\
\mathcal{L}_{DD} &= \frac{G^3 m^2 \mu d_1 d_2}{30 c^5 a^7 (1-e^2)^{10/2}} \left[c_4 \mathcal{A}_0 - c_5 \mathcal{B}_0 \right] \ ,
\end{aligned} \tag{16}$$

where we have denoted by S_i the spin magnitudes, by d_i the magnitudes of the magetic dipoles, by p_i the quadrupole moment scalars and by $\mathcal{A}_0$, $\mathcal{B}_0$, δ_i, γ, ψ_0, $\bar{\psi}$, κ_i auxiliary angular quantities defined in [Gergely (2000)], [Vasúth, Keresztes, Mihály, Gergely (2003)] and [Gergely & Keresztes (2003)]. The constants $c_{1..5}$ are

$$c_i = \sum_{j=0}^{3} C_{ij} e^{2j} \ , \tag{17}$$

with coefficients C_{ij} given in Tab. 1.

5 Summary

We have introduced a generalized eccentric anomaly parametrization for the perturbed Kepler motion. We have proved that even for the generic perturbation

Table 1: *The coefficients in the C_{ij}.*

$_i\backslash^j$	0	1	2	3
1	0	131344	127888	7593
2	-124864	-450656	-215544	-8532
3	42048	154272	75528	3084
4	0	8208	7988	474
5	2600	9376	4479	-177

functions $\varphi_i(\xi)$ there are no new poles as compared to [Gergely, Perjés, Vasúth (2000)]. The method of integration can be widely employed in the case of compact binaries. We have applied the procedure to compute the SS, QM and DD contributions to the luminosity of a compact binary.

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