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Weak gravitational lensing in brane-worlds

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Abstract
We derive the deflection angle of light rays caused by a brane black hole with mass m and tidal charge q in the weak lensing approach, up to the second order in perturbation theory. We point out when the newly derived second order contributions become important.
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1 Introduction

The possibility of allowing gravitation to exist in a more than four-dimensional non-compact space-time [Randall and Sundrum (1999)], while keeping the other interactions locked in four space-time dimensions, has raised interesting new perspectives in the solvability of the hierarchy problem and in cosmological evolution. This hypothesis has led to alternative explanations for both dark energy (see for example [Deffayet (2001))] and dark matter ([Mak and Harko (2004)], [Pal at al. (2005)] and [Pal (2005)]). The simplest so-called brane-world model is five-dimensional. Gravitational dynamics on the four-dimensional brane is governed by a modified Einstein equation, derived in full generality in [Gergely (2003)].

Gravitational lensing is one of the means by which the existence of brane-worlds can in principle be tested. A recent review in the topic can be found in

[Majumdar and Mukherjee (2005)]. In the context of brane-worlds, both weak [Kar and Sinha (2003)], [Majumdar and Mukherjee (2004)] and strong [Whisker (2005)] gravitational lensing were discussed.

Black holes on the brane are described by the *tidal* charged black holes, derived in [Dadhich et al. (2000)]:

$$ds^2 = -f(r)\,dt^2 + f^{-1}(r)\,dr^2 + r^2\left(d\theta^2 + \sin^2\theta d\varphi^2\right) . \tag{1}$$

The metric function f is given as

$$f(r) = 1 - \frac{2m}{r} + \frac{q}{r^2} . \tag{2}$$

These black holes are characterized by two parameters: their mass m and their tidal charge q. The latter arises from the bulk Weyl curvature (more exactly, from its "electric" part as compared to the brane normal).

Formally the metric (1) is the Reissner-Nordström solution of a spherically symmetric Einstein-Maxwell system in general relativity. There, however the place of the tidal charge q is taken by the square of the *electric* charge Q. Thus $q = Q^2$ is always positive, when the metric (1) describes the spherically symmetric exterior of an electrically charged object in general relativity. By contrast, in brane-world theories the metric (1) allows for any q.

The case $q > 0$ is in full analogy with the general relativistic Reissner-Nordström solution. For $q < m^2$ it describes tidal charged black holes with two horizons at $r_h = m \pm \sqrt{(m^2 - q)}$, both below the Schwarzschild radius. For $q = m^2$ the two horizons coincide at $r_h = m$ (this is the analogue of the extremal Reissner-Nordström black hole). In these cases it is evident that the gravitational deflection of light and gravitational lensing is decreased by q. Finally there is a new possibility forbidden in general relativity due to physical considerations on the smallness of the electric charge. This is $q > m^2$ for which the metric (1) describes a naked singularity. Such a situation can arise whenever the mass m of the brane object is of small enough, compared to the effect of the bulk black hole generating Weyl curvature, and as such, tidal charge. Due to its nature, the tidal charge q should be a more or less global property of the brane, which can contain many black holes of mass $m \geq \sqrt{q}$ and several naked singularities with mass $m < \sqrt{q}$.

For any $q < 0$ there is only one horizon, at $r_h = m + \sqrt{(m^2 + |q|)}$. For these black holes, gravity is increased on the brane by the presence of the tidal charge [Dadhich et al. (2000)]. Light deflection and gravitational lensing are stronger than for the Schwarzschild solution.

The metric (1) also describes compact stellar objects. In this case one does not have to worry about the existence or location of horizons, as they would lie inside the star, where an interior solution should replace the metric (1). The generic feature that a positive (negative) tidal charge is weakening (strengthening) gravitation on the brane, is kept.

In this paper we derive the deflection angle of light rays caused by brane black holes with tidal charge (1). Generalizing previous approaches [Kar and Sinha (2003)], [Majumdar and Mukherjee (2004)], we carry on this computation up to the second order in the weak lensing parameters. As the metric (1) is static, we consider only the second order gravielectric contributions, but no gravimagnetic contributions, which are of the same order and would appear due to the movement of the brane black holes. Gravimagnetic effects in the general relativistic approach were considered in [Schäfer and Bartelmanm (2005)].

2 Light propagation

Light follows null geodesics of the metric (1). Its equations of motion can be derived either from the geodesic equations, or from the Lagrangian given by $2\mathcal{L} = \left(ds^2/d\lambda^2\right)$ [Straumann (2004)] (λ being a parameter of the null geodesic curve). Due to spherical and reflectional symmetry across the equatorial plane, $\theta = \pi/2$ can be chosen. Thus

$$0 = 2\mathcal{L} = -f(r)\,\dot{t}^2 + f^{-1}(r)\,\dot{r}^2 + r^2\dot{\varphi}^2\,. \tag{3}$$

(A dot represents derivative with respect to λ.) The cyclic variables t and φ lead to the constants of motion E and L

$$E = f\dot{t}\,, \qquad L = r^2\dot{\varphi}\,. \tag{4}$$

By inserting these into Eq. (3), passing to the new radial variable $u = 1/r$ and introducing φ as a dependent variable, we obtain

$$\left(u'\right)^2 = \frac{E^2}{L^2} - u^2 f\,(u)\,, \tag{5}$$

where a prime refers to differentiation with respect to φ.

Unless $u' = 0$ (representing a circular photon orbit), differentiation of Eq. (5) gives

$$u'' = -uf - \frac{u^2}{2}\frac{df}{du}\,, \tag{6}$$

For $f = 1$, when there is no gravitation at all (the metric (1) becomes flat), the above equation simplifies to $u'' + u = 0$, which is solved for $u = u_0 = b^{-1}\cos\varphi$. The impact parameter b represents the closest approach of the star on the straight line orbit obtained by disregarding the gravitational impact of the star (this is the viewpoint an asymptotic observer will take, as the metric (1) is asymptotically flat). The polar angle φ is measured from the line pointing from the centre of the star towards the point of closest approach. With $u' = 0$ at the point of closest approach, given in the asymptotic limit by $u = b^{-1}$, Eq. (5) with $m = 0 = q$ gives $b = L/E$.

3 Perturbative solution

Eq. (6), written in detail, gives

$$u'' + u = 3mu^2 - 2qu^3 \ . \tag{7}$$

For studying weak lensing, we look for a perturbative solution in series of the small parameters

$$\varepsilon = mb^{-1} \qquad \text{and} \qquad \eta = qb^{-2} \tag{8}$$

in the form

$$u = b^{-1}\cos\varphi + \varepsilon u_1 + \eta v_1 + \varepsilon^2 u_2 + \eta^2 v_2 + \varepsilon\eta w_2 + \mathcal{O}\left(\varepsilon^3, \eta^3, \varepsilon\eta^2, \varepsilon^2\eta\right) \ . \tag{9}$$

The index on the unknown functions u_1, u_2, v_1, v_2 and w_2 counts the perturbative order in which they appear. By inserting Eq. (9) into the weak lensing equation (7) we obtain the relevant differential equations for the unknown functions. Up to the second order in both small parameters these are:

$$\varepsilon \ : \quad u_1'' + u_1 = 3b^{-1}\cos^2\varphi \,, \tag{10}$$

$$\eta \ : \quad v_1'' + v_1 = -2b^{-1}\cos^3\varphi \,, \tag{11}$$

$$\varepsilon^2 \ : \quad u_2'' + u_2 = 3u_1\left[u_1\left(m - 2qb^{-1}\cos\varphi\right) + 2\cos\varphi\right] \ , \tag{12}$$

$$\eta^2 \ : \quad v_2'' + v_2 = 3v_1\left[v_1\left(m - 2qb^{-1}\cos\varphi\right) - 2\cos^2\varphi\right] \ , \tag{13}$$

$$\varepsilon\eta \ : \quad w_2'' + w_2 = 6\left[u_1 v_1\left(m - 2qb^{-1}\cos\varphi\right) + v_1\cos\varphi - u_1\cos^2\varphi\right] . \tag{14}$$

The first order equations are solved for

$$u_1 = \frac{b^{-1}}{2}\left(3 - \cos 2\varphi\right) \ , \tag{15}$$

$$v_1 = -\frac{b^{-1}}{16}\left(9\cos\varphi - \cos 3\varphi + 12\varphi\sin\varphi\right) \ . \tag{16}$$

Thus, both mu_1 and mv_1 are of order ε, while both $qb^{-1}u_1$ and $qb^{-1}v_1$ are of order η. In consequence, all these terms drop out from Eqs. (12)-(14), which are then solved for

$$u_2 = \frac{3b^{-1}}{16}\left(10\cos\varphi + \cos 3\varphi + 20\varphi\sin\varphi\right) , \tag{17}$$

$$v_2 = \frac{b^{-1}}{256}\big(192\cos\varphi - 48\cos 3\varphi + \cos 5\varphi + 384\varphi\sin\varphi - 36\varphi\sin 3\varphi - 72\varphi^2\cos\varphi\big) , \tag{18}$$

$$w_2 = \frac{b^{-1}}{16}\left(-87 + 40\cos 2\varphi - \cos 4\varphi + 12\varphi\sin 2\varphi\right) . \tag{19}$$

With this, we have found the generic solution of Eq. (7), up to the second order in both small parameters.

Far away from the lensing object $u = 0$ and $\varphi = \pi/2 + \delta\varphi/2$, where $\delta\varphi$ represents the angle with which the light ray is bent by the object with mass m and tidal charge q. In our second-order approach it has the form:

$$\delta\varphi = \varepsilon\alpha_1 + \eta\beta_1 + \varepsilon^2\alpha_2 + \eta^2\beta_2 + \varepsilon\eta\gamma_2 + \mathcal{O}\left(\varepsilon^3, \eta^3, \varepsilon\eta^2, \varepsilon^2\eta\right) . \tag{20}$$

A power series expansion of the solution (9) then gives the coefficients of the above expansion, and the deflection angle becomes:

$$\delta\varphi = 4\varepsilon - \frac{3\pi}{4}\eta + \frac{15\pi}{4}\varepsilon^2 + \frac{105\pi}{64}\eta^2 - 16\varepsilon\eta . \tag{21}$$

The first three terms of this expansion were already given in [Briët and Hobill (2005)] for the Reissner-Nordström black hole. There, however the argument that η is of ϵ^2 order was advanced. In brane-worlds there is no a priori reason for considering only small values of the tidal charge, thus we have computed the deflection angle (21) containing all possible contributions up to second order in both parameters.

The deflection angle however is given in terms of the Minkowskian impact parameter b. It would be useful to write this in term of the distance of minimal approach r_{min} as well. The minimal approach is found by inserting the values $u = 1/r_{\mathrm{min}}$ and $\varphi = 0$ in Eq. (9):

$$r_{\mathrm{min}} = b\left(1 - \varepsilon + \frac{1}{2}\eta - \frac{17}{16}\varepsilon^2 - \frac{81}{256}\eta^2 + 2\varepsilon\eta\right) . \tag{22}$$

Inverting this formula gives to second order accuracy (the small parameters being now $m/r_{\rm min}$ and $q/r_{\rm min}^2$):

$$\frac{1}{b} = \frac{1}{r_{\rm min}} \left(1 - \frac{m}{r_{\rm min}} + \frac{q}{2r_{\rm min}^2} - \frac{m^2}{16r_{\rm min}^2} + \frac{47q^2}{256r_{\rm min}^4} + \frac{mq}{2r_{\rm min}^3} \right) . \tag{23}$$

As the deflection angle consists only of first and second order contributions, the above formula is needed only to first order for expressing $\delta\varphi$ in terms of the minimal approach:

$$\delta\varphi = \frac{4m}{r_{\rm min}} - \frac{3\pi q}{4r_{\rm min}^2} + \frac{(15\pi - 16)\, m^2}{4r_{\rm min}^2} + \frac{57\pi q^2}{64r_{\rm min}^4} + \frac{(3\pi - 28)\, mq}{2r_{\rm min}^3} . \tag{24}$$

The first three terms again agree with the ones given in [Briët and Hobill (2005)], for $q = Q^2$.

4 Concluding remarks

In this paper we have computed the deflection angle caused by a tidal charged brane black hole / naked singularity / star, up to second order in the two small parameters, related to the mass and tidal charge of the lensing object.

As already remarked in [Sereno (2003)], the electric charge of the Reissner-Nordström black hole decreases the deflection angle, as compared to the Schwarzschild case. The same is true for a positive tidal charge. In brane-worlds, however there is no upper limit for q as compared to m. Thus for small mass brane black holes / naked singularities / stars the condition $16mr_{\rm min} = 3\pi q$ could be obeyed. In this case the first order contributions to the deflection angle cancel and the three second order terms of $\delta\varphi$ give the leading effect to the weak lensing.

Furthermore, $16mr_{\rm min} < 3\pi q$ could be obeyed, leading to a *negative* deflection angle, at least to first order. That would mean that rather than magnifying distant light sources, such a lensing object will demagnify them.

By contrast, a negative tidal charge can considerably increase the lensing effect. Therefore a negative tidal charge could be responsible at least for part of the lensing effects attributed at present to dark matter.

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References

Briët, J., Hobill, D. 2005 Gravitational lensing by charged black holes, University of Calgary preprint

Dadhich, N., Maartens, R., Papadopoulos, P., Rezania, V. 2000 Phys. Lett. B, 487, 1

Deffayet, C. 2001 Phys. Lett. B, 502, 199

Gergely, L. Á. 2003 Phys. Rev. D, 68, 124011

Kar, S. Sinha, M. 2003 Gen. Rel. Grav. 35, 10

Majumdar, A. S., Mukherjee, N. astro-ph/0403405

Majumdar, A. S., Mukherjee, N. 2005 Int. J. Mod. Phys D, 14, 1095 [astro-ph/0503473]

Mak, M. K., Harko, T. 2004 Phys. Rev. D, 70, 024010

Pal, S., Bharadwaj, S., Kar, S. 2005 Phys. Lett. B, 609, 194

Pal, S. 2005 astro-ph/0512494

Randall, L., Sundrum, R. 1999 Phys. Rev. Lett.83, 4690

Schäfer, B. M., Bartelmann, M. 2005 astro-ph/0502208

Sereno, M. 2003 Phys. Rev. D, 67, 064007

Straumann, N. 2004 General Relativity (Springer), ch. 3

Whisker, R. 2005 Phys. Rev. D, 71, 064004