Canonical theory of the Kantowski-Sachs cosmological models

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Abstract

We briefly discuss the Hamiltonian formalism of the Kantowski-Sachs space-times with vacuum, anisotropic fluid and two cross-streaming radiation field sources. For these models a cosmological time is introduced. New constraints are found in which the fluid momenta are separated from the rest of the variables. In consequence their Poisson brackets give an Abelian algebra.

Keywords: *Kantowski-Sachs cosmology, canonical gravity*

1 Introduction

The Kantowski-Sachs (hereafter KS) cosmologies have two symmetry properties, the spherical symmetry and the invariance under spatial translations. The vacuum solution for this line element is equivalent to the inner Schwarzschild space-time and exact solutions were also found in presence of some matter fields for homogenous cosmological models. Kantowski and Sachs [Kantowski and Sachs (1966)] provided solutions for dust space-times but later on KS geometries with other matter sources were found, such as scalar fields [Barrow et al. (1997)], prefect fluid [Collins (1977)] and anisotropic fluid [Gergely (1999)] and exotic fluid [Gergely (2002)] models. Here we give a short overview of the Hamiltonian theory of the KS cosmology in the case of vacuum and anisotropic fluid sources. We employ the equivalence of the latter with the model consisting of

two, in- and outgoing radiation streams in a stellar atmosphere. We introduce a cosmological time for the space-time containing colliding radiation streams and introduce new constraints. These results might represent an important step in carrying out a consistent canonical quantization and in building up KS quantum cosmologies.

2   Vacuum Kantowski-Sachs space-times

The line element of KS space-times is given by

$$ds^2 = -d\tau^2 + H\left(\tau\right)dh^2 + R^2\left(\tau\right)d\Omega^2 \ , \tag{1}$$

where τ is the cosmological time, h a radial coordinate and $d\Omega^2$ is the metric on the unit sphere. For vacuum this metric can be written the form

$$ds^2 = -d\eta^2 + b^2\tan^2\eta dh^2 + R^2 d\Omega^2 \ , \tag{2}$$

where we introduced the angle parameter η,

$$\tau = a\left(\eta + \sin\eta\cos\eta\right) + c \qquad a, b, c \in \mathbb{R} \ ,$$

usually employed in homogenous, spherically symmetric cosmologies. With the coordinate transformation $R(\eta) = a\cos^2\eta$ the solution (2) can be cast into the form of the Schwarzschild metric

$$ds^2 = -F(R)dT^2 + F^{-1}(R)dR^2 + R^2 d\Omega^2 \ , \qquad F(R) = (1 - 2M/R) \ , \tag{3}$$

where $R < 2M$ and $T = bh$ are the time- and space-like coordinates, respectively.

The canonical formalism of KS space-times is therefore equivalent to that of the Schwarzschild solution. In the Hamiltonian theory of Schwarzschild black holes we use a foliation consisting of spherical surfaces characterized by a constant time parameter t, which is identified to the Schwarzschild time T [Kuchař (1994)]. The geometry induced on these three-spheres has the form

$$d\sigma^2 = \Lambda^2(r)dr^2 + R^2(r)d\Omega^2 \ ,$$

where the functions Λ and R were chosen as canonical coordinates. Then their conjugated momenta P_Λ and P_R are derived from the action specified for the

Schwarzschild space-time [Kuchař (1994)] written in terms of the canonical variables,

$$P_\Lambda = -N^{-1}R(\dot{R} - R'N^r) \ , \qquad P_R = -N^{-1}[\Lambda(\dot{R} - R'N^r) + R(\dot{\Lambda} - (\Lambda N^r)')] \ .$$

The Legendre transformation of the Lagrangian gives the Hamiltonian

$$\mathcal{H}^G = P_\Lambda \dot{\Lambda} + P_R \dot{R} + NH^G + N^r H_r^G \tag{4}$$

with super-Hamiltonian and supermomentum constraints

$$H^G = R^{-1}P_R P_\Lambda + R^{-2}\Lambda P_\Lambda^2/2 + \Lambda^{-1}RR''$$
$$\qquad - \Lambda^{-2}RR'\Lambda' - \Lambda^{-1}R'^2/2 - \Lambda/2 \ ,$$
$$H_r^G = P_R R' - \Lambda P_\Lambda' \ .$$

The basics of the Hamiltonian formulation do not change if we couple matter fields to gravity. We only have to enlarge the phase space of gravity by including the canonical variables of the matter sources. After decomposing the matter action we can derive the Hamiltonian for the matter fields as well. Then the constraint equations of gravity must be supplemented with those of the matter fields, which gives the full of set constraints on the total system.

3 Kantowski-Sachs cosmologies with anisotropic fluid

Exact solutions for KS space-times with anisotropic fluid sources have also been found in the form

$$ds^2 = -2ae^{L^2}RdL^2 + ae^{L^2}R^{-1}dZ^2 + R^2d\Omega^2 \ , \qquad a = -1 \ , \tag{5}$$

where

$$-R = a(e^{L^2} - 2L\Phi_B) \ , \qquad \Phi_B = B + \int^L e^{x^2}dx \ ,$$

and L and Z are the time and the radial coordinates [Gergely (1999)]. The time dependence of the metric components shows that the KS cosmology with anisotropic fluid is not static. By considering the time evolution of the radial length $R(L)$ and the co-moving energy density of the Universe, we see that the KS Universe has a finite lifetime with an initial and a final singularity.

Anisotropic fluids can be considered as superpositions of two cross-flowing null dust streams [Letelier (1980)]. Thus the action of an anisotropic fluid with the co-moving density ρ, the four velocity U^α of the fluid particles and a vector field X^α describing the direction of the pressure forces is given by

$$S^F = -\frac{1}{2} \int dx^4 \sqrt{-g} \rho (U^\alpha U_\alpha + X^\alpha X_\alpha) \ , \qquad (6)$$

which is both algebraically and dynamically equivalent to the action of the two colliding null dust flows with the four velocities u^α and v^α:

$$S^{2ND} = -\frac{1}{2} \int dx^4 \sqrt{-g} \rho (u^\alpha u_\alpha + v^\alpha v_\alpha) \ . \qquad (7)$$

Here the same energy density is chosen for both the dust components so that the net flow should vanish for the static configuration.

Previously a canonical formalism was developed for two cross-flowing null dust streams coupled to the geometry by [Bičák and Hájíček (2003)]. This however did not solve the problem of the absence of a time standard for the colliding null dusts. However, the anisotropic fluid interpretation of two in- and outgoing null dust streams indicates there may be a possibility to use the same procedure as in the case of the incoherent dust in order to find an internal time for the canonical dynamics of colliding null dust streams.

The action with the constraint equations for the spherically symmetric vacuum solution is to be supplemented with those of the matter fields. If we write of the null vector fields in terms of the coordinates Z and L,

$$u_\alpha = WZ_{,\alpha}/\sqrt{2} + RWL_{,\alpha} \ , \qquad v_\alpha = WZ_{,\alpha}/\sqrt{2} - RWL_{,\alpha}$$

and make the same decomposition for the vector fields U^α and X^α,

$$U_\alpha = WZ_{,\alpha} \ , \qquad X_\alpha = \sqrt{2}RWL_{,\alpha}$$

with $W = (ae^{L^2}/R)^{1/2}$, the matter actions (6) and (7) can be expressed with these coordinates as well. By extremizing these actions with respect to the variables Z, L and the parameter ρ, we obtain equivalent equations of motion for the null dust and fluid models.

We use the coordinates $L(t,r)$ and $Z(t,r)$ as the canonical variables for the matter and derive the canonical momenta conjugated to them form the matter action:

$$P_L = 2a\sqrt{g}R^2 \frac{\rho W^2}{N}(\dot{L} - N^r L') \ , \qquad P_Z = a\sqrt{g}\frac{\rho W^2}{N}(\dot{Z} - N^r Z') \ .$$

As a result of the Legendre transformation of the Lagrangian in the matter actions (6) and (7), we obtain the same Hamiltonian for both types of matter sources

$$\mathcal{H}^M = P_L \dot{L} + P_Z \dot{Z} + N H^M + N^r H^M_Z \ , \tag{8}$$

where the super-Hamiltonian and supermomentum constraints imposed on the matter variables are

$$H^M_\perp = \left[\frac{1}{2\sqrt{g}\rho W^2} \left(P^2_Z + \frac{1}{2aR^2} P^2_L \right) + \sqrt{g} \frac{\rho W^2}{2\Lambda^2} \left((Z')^2 + 2R^2(L')^2 \right) \right] \ ,$$
$$H^M_r = Z' P_Z + L' P_L \ .$$

The Hamiltonian (4) of the vacuum equations, together with Eq. (8) describe the KS space-time with two colliding null dust streams or equivalently, an anisotropic fluid source. The super-Hamiltonian and supermomentum constraints of the total system are given by

$$H_\perp := H^G_\perp + H^M_\perp = 0 \ , \tag{9}$$
$$H_r := H^G_r + H^M_r = 0 \ . \tag{10}$$

These constraints are replaced with an equivalent set by solving the old constraints with respect to P_Z and P_L. Hence the momenta associated with the matter can be separated from the other variables in the constraint equations (9)-(10):

$$H_\uparrow = P_L + h(r; \Lambda, R, L, Z, P_\Lambda, P_R) \ ,$$
$$H_{\uparrow Z} = P_Z + h_Z(r; \Lambda, R, L, Z, P_\Lambda, P_R) \ ,$$

where

$$h = \sqrt{2a} R L'^{-1} \left[\Lambda \sqrt{G} \frac{dZ}{dL} - \sqrt{2} a R^{-1} H^G_r P_L Z' \right] \left[\left(\frac{dZ}{dL} \right)^2 + 2aR^2 H^G_r \right]^{-1}$$

$$h_Z = -\sqrt{2a} R L'^{-1} \left[\Lambda \sqrt{G} - \left(\sqrt{2} a R \right)^{-1} h Z' \right] \ , G^2 = (H^G)^2 - g_{ab} H^G_a H^G_b \ .$$

Since the momenta P_L and P_Z are separated from the rest of the canonical variables, the algebra of the new constraints has strongly vanishing Poisson brackets and the Dirac algebra of the old constraints turns to an Abelian algebra of the new ones [Brown and Kuchař (1995)]. The time variable introduced here can be useful in the description of stellar atmospheres consisting of the in- and outgoing radiation streams. Our result might provide better prospects for the canonical quantization of KS cosmologies with cross-flowing null dust streams.

4   Conclusion

We have studied the Hamiltonian formulation of KS cosmologies. In the case of vacuum, the KS space-time is equivalent to the exterior Schwarzschild solution and we can use the canonical theory developed for Schwarzschild black holes. In static KS space-times with spherical symmetry, filled with an anisotropic fluid, the matter source is equivalent to two cross-streaming radiations. Thus the proper time of the dust particles used in the Hamiltonian treatment of the fluid space-times could also be introduced as a time variable in the canonical formalism of the colliding null dust streams. We have derived a new set of constraints for the fluid or colliding null dust variables as well, in which the canonical momenta of the matter are separated from the rest of the variables. As a result, we have obtained an Abelian constraint algebra. Our treatment could give new possibilities for the discussion of quantum KS cosmologies with anisotropic fluid of colliding null dust streams.

References

Barrow, J.D., Dabrowski, M.P. 1997, Phys. Rev. D, 55, 630

Bičák, J., Kuchař , K.V. 1997, Phys. Rev. D, 56, 4878

Bičák, J., Hájíček, P. 2003, gr-qc/0308013

Brown, J.D.,Kuchař, K.V. 1995, Phys. Rev. D, 51, 5600

Collins, C.B. 1977, J. Math. Phys. 18, 2116

Gergely, L. 1998, Phys. Rev. D, 58, 084030

Gergely, L. 1999, Phys. Rev. D, 59, 104014

Gergely L. 2002, Phys.Rev. D, 65, 127503

Kantowski, R., Sachs,R.K. 1966 J. Math. Phys. 7, 443

Kuchař, K.V. 1994, Phys. Rev. D, 50, 3961

Letelier, P. S. 1980, Phys. Rev. D, 22, 807