PADEU PADEU **17**, 235 (2006)
ISBN 963 463 557, ISSN 0238-2423

Gravitational Waveforms from Compact Binary Systems

János Majár

KFKI Research Institute for Particle and Nuclear Physics, Budapest 114, P.O.Box 49, H-1525 Hungary

E-mail: majar@rmki.kfki.hu

Abstract
Here we present the basics of the method for determining the polarization states h_+ and $h_\times$ of the detectable gravitational waves emitted by a compact spinning binary system. The waveform and the dynamics of the binary are described with the use of the post-Newtonian (PN) approximation up to 1.5 PN relative order, related to the leading order newtonian expressions. Beyond point mass effects we investigate the influence of the rotation of the bodies on the waveform to linear order, in the case of eccentric orbits.
Keywords: *gravitational waves, PN, compact binary*

1 Introduction

Compact stars forming binary systems are promising sources of gravitational radiation which detection is expected by the gravitational wave observatories, i.e. LIGO, LISA, VIRGO, TAMA and GEO600. For the extraction of the true signal from the noisy output of the detectors accurate knowledge of the gravitational waveforms emitted by the binary is required. Thus, the construction of ready to use templates for gravitational waves is an important challenge in the investigation of detectable wave signals.

Many works have determined the form of the detectable gravitational wave signals emitted by compact binaries formally in terms of the dynamical properties of motion (Kidder, 1995; Will and Wiseman, 1996; Apostolatos et al., 1994; Blanchet, 2001), but there exist fewer results in the literature where the detectable waveform is computed explicitly in terms of time or a useful parameter.

Here we present the method to describe the time evolution of the detectable gravitational waveform of a binary system. To fully integrate the problem one can use the so-called generalized true anomaly parametrization of the orbit (for further details see (Gergely et al., 2000)). Using this parametrization of the radial motion one can express the contributions to the gravitational wave polarizations h_+ and $h_\times$ up to 1.5 PN order in the case of eccentric orbits.

Sec.II is an introduction to the world of detectable gravitational waves. We introduce the basic quantities and the formalism used by the theory of detection of these gravitational waves. The description of the motion is shown in Sec.III, where we present the method to determine the evolution of the elements of the motion. In Sec.IV we show how to evaluate the polarizaton states of the detectable gravitational waves. In the last section we collect the steps needed to determine explicitly the dynamics of the waves. We use the $c = G = 1$ convention.

2 Gravitational waves in linearized gravity

The true signal of a laser-interferometric gravitational wave detector can be expressed by the linear combination of two polarization states h_+ and $h_\times$:

$$h(t) = F_+(\alpha, \beta, \xi)h_+(t) + F_\times(\alpha, \beta, \xi)h_\times(t) \ , \tag{1}$$

where F_+ and $F_\times$ are the so-called beam-pattern functions depending on the direction of the source (angles α, β) and the polarization angle (ξ).

The polarization states h_+ and $h_\times$ can be projected from the transverse-traceless tensor h^{ij}_{TT} describing the perturbations of the metric using an appropriate gauge. To be able to describe this projection we introduce the orthonormal triad ($\mathbf{N}$, $\mathbf{p}$, $\mathbf{q}$). Vector $\mathbf{N}$ is the direction of the line of sight, and we choose $\mathbf{p}$ to lie in the direction of the node line (the intersection of the orbital plane of the source and the so-called plane of the sky, the plane perpendicular to $\mathbf{N}$), and $\mathbf{q} = \mathbf{N} \times \mathbf{p}$. This way

$$h_+ = \frac{1}{2}(p_i p_j - q_i q_j)h^{ij}_{TT} \ , \qquad h_\times = \frac{1}{2}(p_i q_j - q_i p_j)h^{ij}_{TT} \ . \tag{2}$$

In this work we present the basics of the method of this projection and the determination of h_+ and $h_\times$ in terms of time or an appropriate parameter.

3 Description of the motion

The description of the projection method leads to the clearest expressions in the comoving system fixed to the orbital plane and the separation vector $\mathbf{r}$. The z axis of this coordinate system is fixed to the direction of the Newtonian angular momentum $\mathbf{L}_N := \mu\mathbf{r}\times\mathbf{v}$ ($\mathbf{v}$ denotes the relative velocity vector) which is perpendicular to the orbital plane, and the x axis to the separation vector.

To be able to determine the dynamics of the elements of motion needed we introduce an invariant coordinate system which do not move. Since the total angular momentum $\mathbf{J}$ is constant up to 2 PN order we fix the z axis of this system to it. We choose the x and y axes of the invariant system in a way that the form of the vector $\mathbf{N}$ is

$$\mathbf{N} = \begin{pmatrix} \sin\gamma \\ 0 \\ \cos\gamma \end{pmatrix} \tag{3}$$

in this system, where γ is the constant angle between $\mathbf{J}$ and $\mathbf{N}$.

The transition between the comoving and invariant systems is described with so-called Euler-angles. With these angles the separation vector in the invariant system has the form

$$\mathbf{r} = r\begin{pmatrix} \cos\Phi\cos\Psi - \cos\iota\sin\Phi\sin\Psi \\ \sin\Phi\cos\Psi + \cos\iota\cos\Phi\sin\Psi \\ \sin\iota\sin\Psi \end{pmatrix}\ , \tag{4}$$

where r is the length of the separation vector, ι is the angle between $\mathbf{L_N}$ and $\mathbf{J}$ and Φ shows the direction of the intersection of the orbital and invariant planes (determined by $\mathbf{L_N}$ and $\mathbf{J}$). This way Φ represents the precession of $\mathbf{L_N}$ over $\mathbf{J}$. Ψ describes the evolution of the separation vector in the orbital plane.

In this case every vector $\mathbf{u}$ which is given in the invariant system, in the comoving system will become

$$\mathbf{u}' = R_z(\Phi)R_x(\iota)R_z(\Psi)\mathbf{u}\ , \tag{5}$$

where the matrix $R_{x_i}(\alpha)$ denotes the rotation about the x_i axis with angle α. To be able to determine the dynamics of the system we have to evaluate the equations for the length of the separation vector and the Euler-angles.

The basic equation of the description of the motion is the radial equation of the motion evaluated from the Lagrangian formalism

$$\dot{r}^2 = \frac{2E}{\mu} + \frac{2m}{r} - \frac{L^2}{\mu^2 r^2} + \frac{2E\mathbf{L}\sigma}{m\mu^2 r^2} - \frac{2(2\mathbf{LS} + \mathbf{L}\sigma)}{\mu r^3} \tag{6}$$

where E and L are constants of the motion, μ is the reduced and m is the total mass of the system, and $\mathbf{S} := \mathbf{S}_1 + \mathbf{S}_2$ and $\sigma := m_2/m_1\mathbf{S}_1 + m_1/m_2\mathbf{S}_2$.

To describe the time evolution of the elements of the motion up to 1.5 PN order one has to integrate the spin precession equations linearly in spin

$$\dot{\mathbf{S}}_1 = (4 + 3\frac{m_2}{m_1})\frac{G}{2c^2r^3}\mathbf{J} \times \mathbf{S}_1 \tag{7}$$

$$\dot{\mathbf{S}}_2 = (4 + 3\frac{m_1}{m_2})\frac{G}{2c^2r^3}\mathbf{J} \times \mathbf{S}_2 \ . \tag{8}$$

From the Lagrangian formalism of the motion one can determine the form of the total angular momentum

$$\mathbf{J} = \mathbf{L}_N + \mathbf{L}_{SO} + \mathbf{S}_1 + \mathbf{S}_2 \tag{9}$$

where the explicit form of the spin-orbit angular momentum can be found in (Gergely et al., 1998).

Using the form of the relative velocity vector in the comoving system and the components of the total angular momentum in the invariant system (since in the invariant system its first and second components vanish and the third one is constant) one can determine all the equations needed to evaluate the time evolution of every elements of the motion which is needed to determine the time dependence of the polarization states.

We choose a parametrization of the orbit $\mathbf{r} = \mathbf{r}(\chi)$, which gives a generalization of the Keplerian true anomaly parametrization, see (Gergely et al., 2000). With the use of this parametrization we can integrate r, solve the spin precession equations, evaluate the length and the components of the relative velocity and the total angular momentum vectors too. After all we can determine the parameter dependence of the angular variables.

4 Determining waveform polarization states

To be able to evaluate the projection of the polarization states we need to determine the components of the $\mathbf{N}$, $\mathbf{p}$, $\mathbf{q}$ triad in terms of the elements of the motion.

At first we take a look at vector $\mathbf{N}$. Its form in the invariant system is given before, see Eq.(3), and in the comoving one it changes to

$$\mathbf{N} = \begin{pmatrix} \cos\Psi\cos\Phi\sin\gamma - \sin\Psi\cos\iota_N\sin\Phi\sin\gamma + \sin\Psi\sin\iota_N\cos\gamma \\ -\sin\Psi\cos\Phi\sin\gamma - \cos\Psi\cos\iota_N\sin\Phi\sin\gamma + \cos\Psi\sin\iota_N\cos\gamma \\ \sin\iota_N\sin\Phi\sin\gamma + \cos\iota_N\cos\gamma \end{pmatrix} , \tag{10}$$

where we used the inverse of the transformation law Eq.(5).

Since there are three conditions for the vector $\mathbf{p}$, it can be determined easily in the comoving system. It is a unit vector which is perpendicular to $\mathbf{N}$ and $\mathbf{L}_N$. After using all the conditions, the form of $\mathbf{p}$ becomes

$$\mathbf{p} = \frac{1}{N}\begin{pmatrix} \sin\Psi\cos\Phi\sin\gamma + \cos\Psi\cos\iota_N\sin\Phi\sin\gamma - \cos\Psi\sin\iota_N\cos\gamma \\ \cos\Psi\cos\Phi\sin\gamma - \sin\Psi\cos\iota_N\sin\Phi\sin\gamma + \sin\Psi\sin\iota_N\cos\gamma \\ 0 \end{pmatrix}, \quad (11)$$

where

$$N = \sqrt{N_x^2 + N_y^2} = \sqrt{1 - N_z^2} = \sqrt{1 - (\sin\iota_N\sin\Phi\sin\gamma + \cos\iota_N\cos\gamma)^2}\ , \quad (12)$$

and we can calculate the components of vector $\mathbf{q}$ with

$$\mathbf{q} = \mathbf{N} \times \mathbf{p}\ . \quad (13)$$

In the post-Newtonian approximation the transverse-traceless tensor can be decomposed into terms corresponding to different PN orders and effects, see Eqs.(3.21) in (Kidder, 1995). Using this result and Eqs.(2) we can evaluate the contributions to the polarization states. Our notation is similar to the one given in (Kidder, 1995):

$$h_{\substack{+\\ \times}} = \frac{2\mu}{D}\left[h_{\substack{+\\ \times}}{}^{N} + h_{\substack{+\\ \times}}{}^{0.5} + h_{\substack{+\\ \times}}{}^{1} + h_{\substack{+\\ \times}}{}^{1SO} + h_{\substack{+\\ \times}}{}^{1.5} + h_{\substack{+\\ \times}}{}^{1.5SO}\right]\ , \quad (14)$$

where D is the distance between the detector and the source. h^N terms denote the quadrupole (or Newtonian) expressions, $h^{0.5}$, h^1 and $h^{1.5}$ are corrections corresponding to higher PN orders, h^{1SO} and $h^{1.5SO}$ are the spin-orbit terms. These contributions can be derived with the use of the formal expressions given in (Kidder, 1995) and (Will and Wiseman, 1996).

Since the expressions of the terms according to different orders and effects are rather long we give only two examples of them, namely the Newtonian and the lowest order spin-orbit contributions in the case of the "plus" polarization state:

$$h_+^N = \left(\dot{r}^2 - \frac{M}{r}\right)(p_x^2 - q_x^2) + 2v_\perp\dot{r}(p_xp_y - q_xq_y) + v_\perp^2(p_y^2 - q_y^2)\ ,$$

$$h_+^{1SO} = \frac{m_2 + m_1}{r^2 m_2}\left[(\mathbf{q}\mathbf{S}_1)p_x + (\mathbf{p}\mathbf{S}_1)q_x\right] - \frac{m_2 + m_1}{r^2 m_1}\left[(\mathbf{q}\mathbf{S}_2)p_x + (\mathbf{p}\mathbf{S}_2)q_x\right]\ . \quad (15)$$

5 Conclusions and remarks

After all the method is the following. After determining and solving the equations of motion the results can be inserted into the components of the **N**, **p**, **q** triad. With the evaluation of the components of the relative velocity and spin vectors we have all the quantities which are to be substituted into the expressions of h_+ and $h_\times$. This way (neglecting higher-order spin corrections) we get the evolution of the polarization states of the detectable gravitational waveform.

To be able to find the effects of the rotation of the bodies and the eccentricity of the orbit one may use this method in the spinless and circular orbit cases too. The circular orbit case has the advantage that it can be integrated explicitly in time, however the meaning of the circular orbit is highly nontrivial in a perturbative sense.

In the future the knowledge of the form of the detectable gravitational waves emitted by a compact binary can be a starting point of a measurement method for determining the main features of such compact binaries with gravitational wave spectroscopy. Besides astronomy and radioastronomy the detection of these gravitational waves may become an important tool in the exploration of our universe.

Acknowledgement

This work was supported by OTKA no. TS044665 and F049429 grants.

References

L.E.Kidder, Phys. Rev. D**52**, 821 (1995).

C.M.Will and A.G.Wiseman, Phys. Rev. D**54**, 4813 (1996).

T.A.Apostolatos, C.Cutler, G.J.Sussman, K.S.Thorne, Phys. Rev. D**49**, 6274 (1994).

L.Gergely, Z.Perjés, M.Vasúth, Phys. Rev. D**58**, 124001 (1998).

L.Gergely, Z.Perjés, M.Vasúth, ApJS 126, **79** (2000).

L.Blanchet, Proc. of the Como School on Gravitational Waves in Astrophysics (2001).