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The Sun as a laboratory for turbulence theory: the problem of anomalous diffusion

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Abstract
The solar atmosphere offers a unique possibility to study tuubulent motions under conditions presently unattainable in laboratory experiment or even numerical simulations. This short review will focus on one controversial issue in turbulence theory, on which some light can be shed by solar observations: anomalous turbulent diffusion.
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1 Introduction: Random walk and diffusion on the solar surface

Granular and supergranular flows on the solar surface lead to a random motion of tracers, the most important of which are magnetic flux tubes. A simple random walk of stepsize Δx and timestep Δt over a plane is known to lead to an increase of the rms separation r of a tracer from its starting point (or of two tracers from each other) according to the law

$$r^2 = 4Dt \tag{1}$$

where $D = \Delta x^2/2\Delta t$. The time development of a continuous distribution of tracers is then described by a diffusion equation with diffusivity D.

As a first approximation, the advection of tracers by (super)granules may be represented by such a simple random walk/diffusion, identifying Δx with the spatial scale l of the cells and Δt with their lifetime τ.

This is the approach used in Babcock–Leighton-type models of the solar cycle where poloidal fields are brought to the surface in a concentrated form in active regions, and thereafter they are passively transported to the poles by transport processes (diffusion and meridional circulation). The diffusivity in these models is a free parameter: a best fit to the observations yields $D = 600\,\mathrm{km}^2/\mathrm{s}$. Despite the vectorial character of the magnetic field, these 1D models have been remarkably successful in reproducing the observed temporal evolution of the flux distribution. A possible explanation was proposed in their model by Wang et al. (1991): they assume that field lines are vertically oriented throughout much of the convective zone and this essentially reduces the problem to one dimension. Some support for this conjecture has come from the 2D flux transport models of Petrovay & Szakály (1999). Thus, in a first approximation, 1D models may be used for the description of meridional transport, as these fields pervade the convective zone and are continuously reprocessed through it.

The empirically determined value of the diffusivity, $600\,\mathrm{km}^2/\mathrm{s}$, seems to agree with the primitive random walk model if the steps are identified with granular sizes/lifetimes. Supergranulation, however, should lead to a diffusivity that is by an order of magnitude higher than this calibration. The continuous reprocessing of large-scale fields throughout the convective zone offers a plausible explanation for this inconsistency: the empirical value of the diffusivity reflects the turbulent diffusivity in the lower convective zone where the pressure scale height is ∼50 Mm and the turnover time ∼1 month.

An alternative explanation was put forward by Ruzmaikin & Molchanov (1997) who pointed out that, owing to the cellular nature of photospheric flows, identifying cell size l and lifetime τ with random walk steps is an oversimplification. The fact that a tracer cannot leave a cell during the cell's lifetime, even if it was originally placed next to its border, reduces the effective stepsize significantly. The resulting reduction in D is very sensitive to the value of the Strouhal number $\mathrm{St} = \tau v/l$ and is especially strong for $\mathrm{St} \gg 1$. This effect may be sufficient to reduce supergranular diffusivity to the observed value.

2 Anomalous diffusion

The cellular and turbulent nature of the flow implies that a simple random walk cannot account for the motion of magnetic elements. As a consequence, the actual flux redistribution may differ from a simple (Fickian) diffusion process (Isichenko, 1992) and, instead of equation (1) in general one has

$$r = 2Kt^{\zeta}. \tag{2}$$

If $\zeta \neq 1/2$ it is customary to speak of *anomalous* or *non-Fickian diffusion,* $\zeta > 1/2$ corresponding to *superdiffusion,* $\zeta < 1/2$ to *subdiffusion.* As equation (2) means a unique relation between r and t one might formally still write $r = 2K'(r)t^{1/2}$, leading to the concept of a "scale-dependent diffusivity"

$$D(r) = K'^2 = K^{1/\zeta} r^{2-1/\zeta} \tag{3}$$

It is, however, clear that such a concept is in general useless for the description of the evolution of a continuous field where no preferred scale exists. Anomalous diffusion thus cannot be described by a diffusion equation or, indeed, by any partial differential equation.

How can anomalous diffusion come about? One possibility was suggested by Schrijver & Martin (1990). Magnetic flux tubes are located at junctions of a fractal lattice between supergranules, mesogranules and granules. Assuming that limitations exist for the motion of individual flux elements along this lattice, for certain lattice properties subdiffusion may result. They made an attempt to detect subdiffusion by the analysis of observed flux redistribution in the photosphere; however, ζ was not found to differ from 0.5 within the observational uncertanties.

Being a multiscale phenomenon, turbulence can also naturally lead to a "scale-dependent diffusivity". In order to understand the nature of the diffusion process in a turbulent medium let us consider the question how a random continuous velocity field of a given characteristic scale λ (i.e. one level in the turbulent hierarchy) can be best represented by a random walk with steps Δx and Δt. For the best representation one should set $\Delta t = \tau_L$, the Lagrangian correlation time of the flow, as this is just the time after which the advected particle experiences a significant change in its velocity. The distance the particle travels in this time is $\Delta x = v\tau_L = \min(v\tau_E, \lambda)$ where τ_E is the Eulerian correlation time, λ the correlation length, and v the rms velocity. The diffusivity for this random walk will thus depend on the Strouhal number $\mathrm{St} = \tau_E v/\lambda$; assuming a non-cellular flow

$$D = \begin{cases} \tau_E v^2 & \text{if St} < 1 \\ \lambda v & \text{if St} > 1 \end{cases} \tag{4}$$

In a multiscale flow both τ_E and v scale with λ:

$$\tau_E \sim \lambda^z \qquad v^2 \sim \lambda^{\alpha-1} \tag{5}$$

During the random walk, motions on scales exceeding the separation r of two tracers do not contribute to their further separation while all other scales contribute to it. Of these scales, according to equation (3), the smallest one will dominate in the diffusion process for $2 - 1/\zeta < 1$ i.e. $\zeta < 1/2$. In this case, then, the diffusivity will not significantly depend on the separation for all scales above the viscous scale: turbulence can never lead to subdiffusion.

In the case when the relatively largest scale $\lambda \sim r$ dominates, for given values of z and α the anomalous diffusion exponent ζ can be determined by substituting (5) into (4) and equating the resulting scaling exponent of D to $2 - 1/\zeta$, as given by (3). The Strouhal number scales as $\mathrm{St} \sim \lambda^{z+\alpha/2-3/2}$. For high Reynolds numbers, then, the

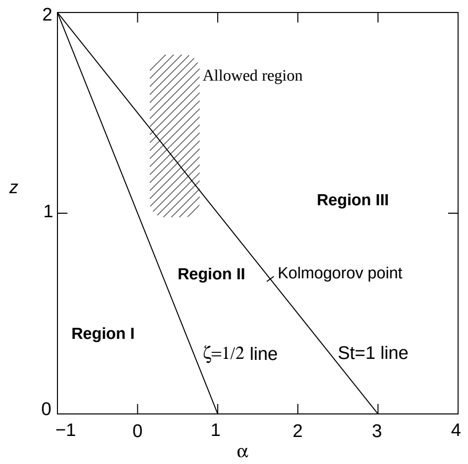

Figure 1: *Regimes of anomalous diffusion on the α–z plane. K is the Kolmogorov point; the shaded area indicates the approximate position of the photospheric flow field in the 1–30 Mm size range. (Non-cellular case)*

sign of (St − 1) at the larger scales depends on the sign of the scaling exponent of St, i.e. the line

$$2z + \alpha - 3 = 0 \tag{6}$$

defines two regimes in the α–z plane (Fig. 1). Above the line, in what is called Region III (Avellaneda & Majda, 1992), one finds $\zeta = 2/(3-\alpha)$ (except in the case of a cellular flow when $\zeta = 1/z$ —cf. the discussion at the end of Sect. 4.1). This Region is clearly superdiffusive for all values of $\alpha > -1$ (or $z < 2$). Below the line, in Region II we have $\zeta = 1/(3-z-\alpha)$, independent of cellularity, as here we have low Strouhal numbers at the large scales. It is then clear that a second dividing line will also exist at

$$z + \alpha = 1 \tag{7}$$

as below that line (Region I) $\zeta < 1/2$ would result, in which case, as we have seen, the smallest scales dominate the diffusion process. Region I is thus characterized by a Fickian diffusion, while Region II is again superdiffusive. Point K in Figure 1 denotes the case of a Kolmogorov spectrum, $\alpha = 5/3$, $z = 2/3$, $\zeta = 3/2$.

3 Observational constraints on anomalous diffusion in the Sun

In order to determine the place of photospheric velocity fields on the α–z diagram, Ruzmaikin et al. (1996) fitted power laws to the spatio-temporal power spectra of photospheric velocity fields with the result $\alpha \sim 1.5$–1.8, $z \sim 0.15$–0.85. This would localize solar turbulence to the neighbourhood of the Kolmogorov point K. However, in Section 2.2.1 above we already stressed the perils of power-law fits to power spectra of solar velocity fields. There is simply no theoretical reason or observational evidence to suggest that these fields should follow a power-law spectrum from supergranular scales down to the resolution limit. Indeed, the well known fact that meso- and supergranular motions have a lower velocity amplitude than granulation, tells us that $\alpha < 1$ in the regime $\lambda > 1\,\mathrm{Mm}$! Using observational estimates for these velocity amplitudes and for correlation times one arrives at much more robust limits that are in plain contradiction to the ones quoted above: $\alpha \sim 0.0$–0.7, $z \sim 0.9$–1.8, leading to $\zeta \sim 0.48$–1.2. These limits in themselves would indicate superdiffusion (shaded area in Fig. 1).

Turbulent erosion models of sunspot decay can also be used to constrain anomalous diffusion in the photosphere (Petrovay, 1998). The size of sunspots spans the granular-supergranular size range that is of interest in this respect, and the fortuitous property of the erosion models that they *do* show a characteristic scale, the radius of the spontaneously formed current sheet, makes it possible to test for ζ by using a scale-dependent diffusion coefficient with the current sheet radius as defining scale. In this way, ζ is found to lie in the range 0.44–0.59, i.e. any deviation from a Fickian diffusion seems to be modest, if present at all. A possible explanation for why the diffusion exponent is lower than suggested by velocity power spectra may be that superdiffusion due to turbulence is offset by subdiffusion effects due to diffusion on a bond lattice.

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