PADEU | PADEU **15**, 73 (2005)
ISBN 963 463 557

Nonlinear waves in solar plasmas

I. Ballai

SPARG, Dept. of Applied Mathematics, University of Sheffield, Hounsfield Road, Hicks Building, Sheffield, S3 7RH, UK

E-mail: `i.ballai@sheffield.ac.uk`

Abstract
Nonlinearity is a direct consequence of large scale dynamics in the solar atmosphere. Here, the nonlinear steepening of waves balanced by dispersion generates solitary waves. Nonlinear waves can also appear in the vicinity of resonances, influencing the efficiency of energy deposition. Here we review recent theoretical breakthroughs that have lead to a greater understanding of many aspects of nonlinear waves arising in homogeneous and inhomogeneous solar plasmas.
Keywords: *Sun: waves, MHD, Sun: wave heating*

1 Introduction

One of the most interesting processes in solar and astrophysical plasmas is the complicated interaction of plasma motions with magnetic fields. These media are highly non-uniform and as a consequence are a natural environment for magnetohydrodynamic (MHD) waves. Waves can transport energy and momentum. When part of their energy or momentum is transferred to the plasma they can heat and accelerate the plasma (e.g. resonant absorption). Waves can carry information about the medium in which they propagate, therefore they can provide a unique tool for plasma diagnostics.

In the present contribution we review two important nonlinear waves arising in inhomogeneous solar plasmas. Firstly, solitary waves arising in structured plasmas (i.e. waveguides) are discussed in different structures and for different dispersions. Secondly, nonlinear waves generated in the vicinity of resonant positions (slow resonance) are revisited and we show how nonlinearity will influence the efficiency of heat deposition.

2 Nonlinear waves in waveguides

One of the basic properties of solar plasma is that is structured, the magnetic field is not distributed smoothly over the surface of the Sun, but it tends to accumulate in entities called *magnetic loops*, the building blocks of the solar corona. These structures can support, e.g. longitudinal wave propagation over long distances. The effect of the structuring is that it introduces dispersion, i.e. a modification in the propagation characteristic of the wave.

Solitons are finite-amplitude waves of permanent shape which owe their existence to the balance between nonlinear wave-steepening and wave dispersion. Nonlinearity appears for waves of finite amplitude and generally is a consequence of large scale dynamics. Dispersion could arise due to two different effects. *Geometrical dispersion* appears for waves propagating in a magnetic guide (flux tube or sheet). This dispersion does not depend on the reaction of the external media and its value is defined by the geometrical scale of the duct (the tube diameter or the thickness of the sheet). Alternatively, waves in open ducts could have dispersion due to the reaction of the external media. It is not always simple to separate these two sources of dispersion in spite of their different behavior. Furthemore, *physical dispersion* appears due to plasma (magnetic) effects (generalized Ohm's law or Finite Larmor Radius (FLR) effects). In general, these two dispersive effects give rise to different dispersive behavior but they have the same result: creation of a new length scale in addition to the natural length scale of the waves, i.e. their wavelength.

Guided waves in solar and space plasmas are investigated in two cases: magnetic slab (Cartesian geometry) and magnetic tube (cylindrical geometry). The dynamics of solitary waves are best described in the so-called *thin flux tube approximation.* For a motion $v(z,t)$ along a tube (slab or cylinder) of cross-sectional area $A(z,t)$, the one-dimensional equations of continuity, longitudinal momentum, isentropic energy and flux conservation are

$$\frac{\partial}{\partial t}(\rho A)+\frac{\partial}{\partial z}\rho v A=0,$$

$$\frac{\partial v}{\partial t}+v\frac{\partial v}{\partial z}=-\frac{1}{\rho}\frac{\partial p}{\partial z},$$

$$\frac{\partial}{\partial t}\left(\frac{p}{\rho^\gamma}\right)+v\frac{\partial}{\partial z}\left(\frac{p}{\rho^\gamma}\right)=0, \quad BA=const,$$

where the quantities $p(z,t)$, $\rho(z,t)$, $B(z,t)$ and $v(z,t)$ are supposed uniform across the tube.

In a magnetic slab of width $2a$ with the magnetic field along the structure, the dispersion relation of slow sausage modes with their wavelength (k^{-1}) much larger than the width of the slab is (Roberts, 1981)

$$\omega/k=c_T-\alpha_1|k|, \quad \alpha_1=\frac{1}{2}\frac{\rho_e}{\rho_e}\left(\frac{c_T}{v_A}\right)^3 a c_T, \tag{1}$$

where c_T is the tube speed (the propagation speed of slow magnetoacoustic waves in an unbounded medium). The $\alpha_1|k|$ term in Eq. (1) arises due to dispersion and in the long wavelength limit is a small quantity. If the amplitude of slow waves becomes large enough, the nonlinear evolution of these waves is described by the Benjamin-Ono (BO) equation written for the z-component of the velocity perturbation (Roberts & Mangeney, 1982; Edwin & Roberts, 1986; Ballai et al., 2002)

$$\frac{\partial v}{\partial t} + c_T \frac{\partial v}{\partial z} + \beta v \frac{\partial v}{\partial z} + \frac{\alpha_1}{\pi} \frac{\partial^2}{\partial z^2} \int \frac{v(z',t)}{z'-z}\, dz' = 0, \tag{2}$$

where β is a coefficient which depends on the characteristic speeds (sound, Alfvén and cusp speeds). The single-soliton solution of this equation is the algebraic soliton,

$$v(z,t) = \frac{A}{1+[(z-st)/L]^2}, \tag{3}$$

where A is the velocity amplitude of the soliton, s and L are the speed and scale of the soliton related by

$$s = c_T + \frac{\beta A}{4}, \quad L = \frac{4\alpha_1}{L\beta}. \tag{4}$$

In a magnetic cylinder with radius R, the dispersion relation of slow surface sausage modes in the long wavelength limit is

$$\omega/k = c_T - \alpha_2 k^2 K_0(|k|\lambda), \tag{5}$$

where $K_0(x)$ is the modified Bessel function of the zeroth-order. The quantities α_2 and λ depend on characteristic speeds and the radius of the tube and λ^2 can be both negative or positive quantity. If these waves steepen into nonlinear waves, their evolution is described by the Leibovich-Roberts (LR) equation (Roberts, 1985),

$$\frac{\partial v}{\partial t} + c_T \frac{\partial v}{\partial z} + \beta v \frac{\partial v}{\partial z} + \frac{\alpha_2}{\pi} \frac{\partial^3}{\partial z^3} \int_{-\infty}^{\infty} \frac{v(z',t)dz'}{[\lambda^2 + (z'-z)^2]^{1/2}} = 0. \tag{6}$$

Although this equation was derived 20 years ago, there is no known analytical solution, however, numerical investigations showed that it has a solitary-like solution (Weisshar, 1989). If the propagation speed of the slow waves inside and outside the tube are approaching each other, the LR equation reduces to a nonlinear wave equation without dispersion which describes shock waves with zero-width. If the internal cusp speed approaches either the external sound or Alfvén speed (supposing a magnetized environment), the LR equation reduces to the Leibovich equation describing nonlinear waves on a cylindrical vortex core. The LR equation is valid provided $\lambda^2 > 0$. If $\lambda^2 < 0$, then slow leaky sausage modes will propagate in the tube draining energy away from the structure. In this case, the LR equation can be modified to describe slow leaky sausage modes as (Ballai & Zhughzda, 2002)

$$\frac{\partial v}{\partial t} + c_T \frac{\partial v}{\partial z} + \beta v \frac{\partial v}{\partial z} + \alpha_1 \frac{\partial^3}{\partial z^3} \left[\int_{-\infty}^{-|\lambda|+z} + \int_{|\lambda|+z}^{\infty} \frac{v(s,t)\, ds}{\sqrt{\lambda^2 + (z-s)^2}} \right] = 0. \tag{7}$$

One of the limitations of these equations is that the solitary wave solution appears only up to some critical amplitude. This amplitude threshold appears because the dispersion relation has a maximum, i.e the maximum value of the dispersion is not enough to smooth out the front of the waves if the amplitude of the waves exceeds a critical threshold value. Thus, the (LR) equation describes the nonlinear behavior of weakly nonlinear slow sausage modes whose phase velocity in the linear limit has an extremum. For solitons with negative dispersion this limitation does not occur, instead they are subject to an aperiodic instability.

There is one aspect which so far has been neglected, and this is related to the dissipative character of the plasma. In fact, the right choice for a dissipative mechanism depends on the location where physical processes are to be studied and also on the physical mechanism itself. For instance, Ohmic dissipation of wave propagation in the solar corona does not result in significant damping (unlike viscosity or thermal conduction) but this dissipative effect must be taken into account when studying effects which require small length scales, e.g. coronal heating.

When dissipation is taken into account, solitary waves will exhibit a slow damping, which means that the energy and momentum of solitary waves are not conserved quantities any longer. The most important dissipative mechanisms are viscosity, thermal and electrical conduction and radiation. If we take into account the first three mechanisms, the solitary wave equations must be supplemented by an extra term proportional to $\partial^2 v/\partial z^2$ which results in an algebraic decay of the soliton. If radiation is considered, the nonlinear equations will have an extra term proportional to v which leads to a slow exponential decay of the solution. Illustrations of when these dissipative terms are added to a nonlinear evolutionary equation are the Leibovich-Roberts-Burgers or the Korteweg-de Vries-Burgers equation.

Dispersion can arise not only due to a geometrical structuring, but also due to the presence of the magnetic field, through, e.g. the Hall term in the generalised Ohm's law. Strictly speaking, Hall MHD is relevant to plasma dynamics occurring on length scales shorter than the ion inertial length, c/ω_i, where c is the speed of light and ω_i is the ion plasma frequency. Inclusion of the Hall term in the magnetohydrodynamic induction equation is known to affect the polarization of waves because it includes the dispersion of Alfvén waves near the ion cyclotron frequency.

The nonlinear wave evolution in the presence of a Hall effect in a viscous plasma has been studied in connection to the acceleration of the solar wind. When the nonlinear steepening of compressional waves is balanced by the broadening of the wavefront caused by the Hall effect, we obtain that the dynamics of solitary waves propagating in a super-radial magnetic field is described by the Korteweg-de Vries-Burgers (KdV-B) equation

$$\frac{\partial v}{\partial t}+c_f\frac{\partial v}{\partial z}+\alpha_1 v\frac{\partial v}{\partial z}-\alpha_2\frac{\partial^3 v}{\partial z^3}-\alpha_3\frac{\partial^2 v}{\partial z^2}=0, \tag{8}$$

where c_f is the phase speed of linear waves and the coefficients α_i depend on characteristic speeds and the angle of propagation with respect to the magnetic field.

Choosing a nearly-parallel propagation, we obtain that solitons arising from the nonlinear steepening of compressional slow waves are able to accelerate the plasma, while solitons which are generated by the nonlinear steepening of fast waves will decelerate the plasma. The speed at which the solar wind is accelerated by means of solitons agrees very well with the observed speeds by UVCS-SOHO at $1.3R_\odot$.

Solitary waves have unique properties which make them special for mathematics and their applications to other fields: *(i) Integrability*: Before the discovery of solitons, mathematicians were under the impression that nonlinear PDEs could not be solved, at least analytically. However, solitons showed us that it is possible to solve PDEs (at least the solitary wave equations) exactly, which gives us a tremendous "window" into what is possible in nonlinearity. *(ii) Nonlinear superposition*: In linear theory, there is a simple way to generate a new solution from known ones, just by multiplying them with a scalar and adding them together. This is known as superposition. Before the discovery of solitons, there was no analogue of this construction for nonlinear equations, but the way that a 2-soliton solution can be viewed as a combination, although not a simple linear combination, of two 1-soliton solution leads us to the recognition that (at least for solitons) there is a nonlinear superposition principle, as well. *(iii) The particle-like behaviour* of solitons leads to a large number of applications. This is true to some extent: there are soliton models for nuclei and the technique known as bosonization allows us to view fermions as being solitons in appropriate situations. Recently, the transport of energy and information along DNA chains was described by the so-called Davydov-solitons. Solitons have also a series of other applications in fields like oceanography, fiber optics, telecommunications and geophysics. Solitary waves carry a large amount of energy, therefore if they are dissipated over short length scales they could provide, e.g. the energy required to heat the coronal plasma (resonant solitary waves).

3 Nonlinear resonant waves

Resonances are ubiquitous every time MHD (magnetohydrodynamic) waves are driven in inhomogeneous plasmas. However in weakly dissipative plasmas (as in the case of solar plasma) driven MHD waves show nearly resonant behaviour, which deviates from the resonant behaviour in ideal plasmas only in thin *dissipative layers* surrounding the ideal resonant positions.

A very important property of these nearly resonant waves is that their damping rate is almost independent of the values of dissipative coefficients. As a result, the damping rate of nearly resonant MHD waves can be many orders of magnitudes larger than the damping rate of MHD waves with the same frequencies in homogeneous plasmas. This property of resonant waves being strongly damped in weakly dissipative plasmas has attracted ample attention from plasma physicists since the transferred energy can be converted into heat (Sakurai et al., 1991; Ruderman et al., 1997a; Ballai et al., 1998a) or it might give valuable information about the density of the plasma and the characteristic scale of inhomogeneity.

Resonant absorption can be considered as an effective process of generating small length scales comparable to the dissipation length scales. The local oscillation modes of an inhomogeneous plasma are represented by continuous spectra for slow MHD and Alfvén waves and a discrete spectrum for fast MHD waves. The resonant absorption occurs when the frequency of a laterally driven oscillation matches the local slow and/or Alfvén wave frequency and a resonant field line is created which transfers energy from the surface disturbance to its environment.

Usually, the importance of the dissipation is characterized by the viscous and magnetic Reynolds numbers (if viscosity and magnetic diffusion are considered as dissipative effects) and we denote by R the total Reynolds number which under solar conditions is a very large number (10^6 in the photosphere and up to 10^{12} in the corona).

Linear theory of resonant absorption has shown that in the vicinity of a resonant position the perturbations have steep gradients and large amplitudes and therefore the linear theory in this region can break down and nonlinear theory has to be considered. Nonlinearity in the dissipative layer was first taken into account in the theory of resonant absorption by Ruderman et al. (1997a) and Ballai et al. (1998a) where they studied the nonlinear evolution of slow resonant MHD waves in the isotropic and anisotropic dissipative layer using a Cartesian geometry. These theories were applied to study the resonant absorption of sound and fast magneto-acoustic waves in solar structures (Ruderman et al., 1997b; Ballai et al., 1998b; Erdélyi & Ballai, 1999). One of their main results was that in contrast to the linear theory, the coefficient of wave energy absorption was dependent on the particular type of dissipation. They have also found that the general tendency of nonlinearity is to decrease the absolute value of the coefficient of wave energy absorption when the wavelength of the incoming wave is much larger than the characteristic scale of the inhomogeneity and nonlinearity is considered weak.

Characteristic quantities used to scale the problem are ϵ (the dimensionless amplitude of perturbations away from the dissipative layer) and the total Reynolds number. One way to determine the importance of nonlinerarity is to calculate the ratio

$$\delta = f\frac{\partial f}{\partial \theta}/\nu\frac{\partial^2 f}{\partial x^2} = \epsilon R^{2/3}, \tag{9}$$

where f is any large variable, i.e. the most singular perturbations (e.g. for slow wave resonance, the most singular are the parallel component of the velocity and magnetic field perturbation). Linear theory works as long as $\delta \ll 1$, i.e. $\epsilon R^{2/3} \ll 1$. For a typical value of $\epsilon \approx 10^{-2}$ to have resonant absorption described by linear theory, we need $R \ll 10^3$ which is in contrast to previously accepted values. Based on these scalings, it is obvious that resonant absorption is a *nonlinear phenomenon.*

In nonlinear theories perturbations cannot be Fourier analysed. However, to be as close as possible to the linear results, we suppose that waves are plane periodic propagating modes with permanent shape, i.e. all perturbations depend only on $\theta = z - Vt$ so they are periodic with respect to θ.

Outside the dissipative layer, the plasma dynamics can be described by the linear ideal MHD system of equations which can be reduced to two coupled first order PDE

$$\frac{\partial u}{\partial x} = \frac{V}{D}\frac{\partial P}{\partial \theta}, \quad \frac{\partial P}{\partial x} = \frac{\rho_0 D_A}{V}\frac{\partial u}{\partial \theta}, \tag{10}$$

where

$$D = \frac{\rho_0 D_A D_C}{V^4 - V^2(v_A^2 + c_S^2) + v_A^2 c_S^2 \cos^2\alpha},$$
$$D_A = V^2 - v_A^2\cos^2\alpha, \quad D_C = (v_A^2 + c_S^2)(V^2 - c_T^2\cos^2\alpha). \tag{11}$$

In the case of cylindrical tube when the equilibrium magnetic field is such that $\mathbf{B}_0 = (0, B_{0\varphi}(r), B_{0z}(r))$ and the wave-vector now has a helical component, therefore the running variable is $\theta = m\varphi + kz - \omega t$. The governing equations outside the dissipative layer are

$$D\frac{\partial ur}{\partial r} = C_1 ur + \omega C_2 r\frac{\partial P}{\partial r},$$
$$\omega r D\frac{\partial^2 P}{\partial r \partial \theta} = c_3 ur - \omega r C_2, \tag{12}$$

where the coefficient functions are given by

$$C_1 = 2\omega^4\frac{B_{0\varphi}}{\nu r} - 2\frac{m f_B B_{0\varphi}}{\nu r^2}D_C, \quad C_2 = \omega^4 - \left(\frac{m^2}{r^2} + k^2\right)D_C,$$

$$C_3 = D\left[\rho_0 D_A\frac{\partial}{\partial \theta^2} + \frac{2B_{0\varphi}}{r}\frac{d}{dr}\left(\frac{B_{0\varphi}}{r}\right)\right] - 4\omega^4\left(\frac{B_{0\varphi}^2}{\nu r}\right)^2 + \frac{4\rho_0 D_C \omega_A^2}{\nu r^2}B_{0\varphi}^2,$$
$$f_B = \frac{m}{r}B_{0\varphi} + kB_{0z},$$

and the coefficients D, D_A and D_C are similar to the equations given by Eq. (11) with V replaced by ω.

In the present study we only focus on the slow resonance given by the condition $V^2 = c_T^2(x)$ or $\omega^2 = \omega_C^2(r)$. The resonant position ($x = x_C$ in Cartesian geometry and $r = r_C$ in cylindrical geometry) is a regular singular point of the system of Eqs. (11)-(12) and as a consequence, the solutions are obtained in form of Fröbenius series. The equilibrium quantities have a slight change across the dissipative layer and they are approximated by the first non-vanishing term in their Taylor expansion. These expansions are valid in a layer wider than the dissipative layer since the characteristic scale of the inhomogeneity is larger than the scale of dissipation.

Inside the dissipative layer, the solutions are obtained in form of asymptotic expansions. In order to connect the solutions in the two regions (inside and outside the dissipative layer) we use the so-called *matched asymptotic expansions* developed by Nayfeh (1981). Both the internal and external solutions have to coincide in the overlap regions.

The dynamics of resonant slow waves in the vicinity of the resonance propagating along the magnetic field is given in cartesian geometry (with isotropic anisotropy) by

$$(x-x_c)\frac{\partial v_\parallel}{\partial\theta}+\Phi_1 v_\parallel\frac{\partial v_\parallel}{\partial\theta}+k\frac{\partial^2 v_\parallel}{\partial x^2}=\Phi_2 P(\theta), \tag{13}$$

and in cylindrical geometry

$$(r-r_c)\frac{\partial v_\parallel}{\partial\theta}+\Psi_1 v_\parallel\frac{\partial v_\parallel}{\partial\theta}+k\frac{\partial^2 v_\parallel}{\partial r^2}=\Psi_2 \mathcal{C}(\theta), \tag{14}$$

where the function $\mathcal{C}(\theta)$ is a sum of the θ-derivative of the total pressure and a function containing the φ-component of the magnetic field.

There are two interesting points to be mentioned. The coefficient Φ_1 and Ψ_1 in Eqs. (13)-(14) are similar to the coefficient of the nonlinear terms found for solitons and it provides a measure of nonlinearity in compressional modes. Secondly, in the coronal case, where all transport coefficients are anisotropic, the nonlinear governing equation is modified in the dissipative term (the third term in the LHS) and instead of having a 2^{nd} order derivative with respect to the transversal coordinate, we have a 2^{nd} order derivative with respect to θ. Eqs. (13) and (14) should be understood in the sense that the nonlinear behaviour of slow waves in the vicinity of the resonance is driven by the variation of the total pressure.

When solving the MHD equations for the entire domain, the resonances are considered as singularities, therefore the evolution of physical quantities in the vicinity of resonances are given as jumps (connection formulae), exactly as the Rankine-Hugeniot relations for shock waves. The jump in a quantity Q across the dissipative layer can be calculated with the aid of

$$[Q]=\lim_{x\to x_C}\left\{Q(x)-Q(-x)\right\}.$$

When connecting the solutions, the jump conditions serve as boundary conditions. In the case of Cartesian geometry, the jumps in the total pressure and the normal component of the velocity are given by

$$[P]=0,\quad [u]=\Omega_1\mathcal{P}\int_{-\infty}^{\infty}\frac{\partial v_\parallel}{\partial\theta}\,dx, \tag{15}$$

and in cylindrical geometry by

$$[P]=\Lambda_1\mathcal{P}\int_{-\infty}^{\infty}v_\parallel\,dr,\quad [u]=\Lambda_2\mathcal{P}\int_{-\infty}^{\infty}\frac{\partial v_\parallel}{\partial\theta}\,dr. \tag{16}$$

Here $\mathcal{P}$ is used for the Cauchy principal part because the integrals are divergent at infinity.

When calculating the efficiency of the resonant absorption (coefficient of wave energy absorption) it is found that the effect of nonlinearity is to decrease the net coefficient of wave absorption. This means that the largest amount of energy stored in nonlinear waves does not go into increasing the absorption rate but into generating a mean flow outside the dissipative layer. This turbulent flow is generated by the absorption of wave momentum in the dissipative layer and its amplitude is determined by the balance of forces created by resonant absorption and shear viscosity. The mean shear flow is a piecewise continuous function of r (e.g. in cylindrical geometry) but its vorticity has a jump given by

$$[v_\varphi = A_1 \int_{-\infty}^{\infty} \left\langle \left(\frac{\partial v_\parallel}{\partial r} \right)^2 \right\rangle dr, \quad [v_z] = A_2 \int_{-\infty}^{\infty} \left\langle \left(\frac{\partial v_\parallel}{\partial r} \right)^2 \right\rangle dr, \tag{17}$$

where the coefficients A_1 and A_2 depend on characteristic speeds, the location, r_C, of the resonance and the dissipative coefficients and $<,>$ is the mean value of a quantity over a period. Estimates of this mean shear flow give us speeds of the order of $0.1 km/s$ in the solar photosphere and a few km/s in the solar corona. Observation of this flow might be a first indirect evidence for resonant absorption in solar plasmas. The properties of generated mean turbulent flow are not fully understood and they are an important topic for further investigations.

The results presented here considered that the equilibrium is static; in reality the plasma is very dynamic, showing motion on all time and space scales. Including an equilibrium steady flow, Ballai & Erdélyi (1998c) obtained the governing equations inside and outside the dissipative layer, as well as the jump conditions across the singularity.

The model described here considered a simplified atmosphere. Possible further investigations could be performed for a more realistic equilibrium (e.g. equilibrium quantities vary not only across the field but also along the field, inclusion of gravity, etc.). The governing equations were obtained in the limit of weak nonlinearity and long wavelength approximation. Recently, Ruderman (2000) considered the analysis of resonantly interacting waves in the limit of strong nonlinearity. He has obtained that the decreasing tendency of the coefficient of wave energy absorption by nonlinearity does not persist in this limit for intermediate values of wave vector. In the long wavelength limit, however, he found that the difference between strong nonlinear and linear limit does not exceed 20%.

Acknowledgement

The author acknowledges the financial support by the British Council, Nuffield Foundation and the great effort of the local organizers to make the meeting a time to remember.

References

Ballai, I., Ruderman, M.S. & Erdélyi, R. 1998, *Phys. Plasmas*, 5, 252.

Ballai, I., Erdélyi, R. & Ruderman, M.S. 1998, *Phys. Plasmas*, 5, 2264.

Ballai, I. & Erdélyi, R. 1998 Sol. Phys., 180, 65

Ballai, I., Erdélyi R., Voitenko, Y. & Goossens, M. 2002, *Phys. Plasmas*, 9, 2593.

Ballai, I. & Zhugzhda, Y.D. 2002, *Phys. Plasmas*, 9, 4280.

Edwin, P & Roberts, B. 1986, *Wave Motion*, 8, 151.

Erdélyi, R. & Ballai, I. 1999, Sol. Phys., 186, 67

Nayfeh, A.H. 1981, *Introduction to Pertubation Techniques*, Wiley-Interscience, New York.

Roberts, B. & Mangeney, A. 1982, *MNRAS*, 198, 7P-11P

Roberts, B. 1981, *Sol. Phys*, 69, 27

Roberts, B. 1985, *Phys. Fluids*, 28, 3280

Ruderman, M.S., Goossens, M. & Hollweg, J.V. 1997a, *Phys. Plasmas*, 4, 75.

Ruderman, M.S., Hollweg, J.V. & Goossens, M. 1997b *Phys. Plasmas*, 4, 92.

Ruderman, M.S. 2000 *J. Plasma Phys.*, 63, 43.

Sakurai, T., Goossens, M. & Hollweg, J.V. 1991a, Sol. Phys., 133, 227.

E. Weisshaar, *Phys. Fluids* **1**, 1406 (1989)