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Wave Propagation In Stratified One Dimensional Waveguides

J. Hargreaves

Solar Physics and upper-Atmosphere Research Group, Department of Applied Mathematics, University of Sheffield, S3 7RH, Sheffield, U.K.

E-mail: `j.hargreaves@shef.ac.uk`

Abstract
We investigate the effect of stratification by gravity on the propagation of linear magnetohydrodynamic (MHD) waves along a magnetic flux tube. For a quiescent environment linear wave propagation is governed by a Klein-Gordon equation. We consider the presence of a background flow in the tube and show the response to various atmospheric footpoint drivers. We solve for three drivers corresponding to a monochromatic source, a delta-function pulse and a sinusoidal pulse. This work is motivated by the vast amount of recent observational evidence supporting the existence of waves and flows in solar MHD waveguides.

1 Introduction

The effect of stratification on the propagation of MHD waves is important, especially in the lower atmosphere. These effects, of course, have been considered previously (e.g Rae and Roberts, 1982). In this paper we consider take a look at the effect of flows upon wave propagation in such atmospheres.

Recent observational evidence gives strong support to the long discussed possibility of flows in the solar atmosphgere. Waves will be influenced and possibly even generated by their presence. A detailed derivation of steady flow effects on MHD waveguides can be found e.g. in Terra-Homem, Erdélyi & Ballai (2003). The identification of a fast wind originating from the polar coronal holes has been established for many years (e.g., Watanabe, 1975; Gloeckner and Geiss, 1998). Observations from SOHO and TRACE have indicated the presence of steady flows in the south polar coronal hole and the equatorial quiet Sun-region (Buchlin and Hassler, 2000). Bi-flows, associated with tens of thousands of small scale explosions, have been found in the chromosphere (e.g., Innes et al., 1997; Perez et al. 1999; Sarro et al., 1999; Teriaca et al. 1999; Roussev et al. 2001a,b,c, etc.). Background flows have been noted in arched isolated magnetic flux tubes, steady flows have been observed in slender magnetic flux tubes and in return flows from spicules. Thus there is a clear need to study the effect of such flows on the wave characteristics in the solar atmosphere, as flows may influence not just the dynamic properties of wave propagation, but also wave dissipation (Erdélyi 1996; Erdélyi & Goossens 1996; etc.).

2 Wave Propagation in an Elastic Tube

The Sun's photospheric region exhibits enormous complexity, being both highly dynamic and highly structured, with magnetic structuring over many scales. Further, stratification due to gravity is important and must be taken into account. The use of slender flux tube theory (wavelengths much greater than tube width) allows an analytical description (see Roberts and Webb, 1978). Consider a general slender elastic tube in a stratified atmosphere (e.g. Rae and Roberts 1982; Erdélyi & Fedun 2004). The atmosphere is stratified such that

$$p_0' = -g\rho_0(z), \tag{1}$$

where the prime ($'$) shows a derivative with respect to the z direction. The governing equations of the system are given by the linearised equations for continuity, momentum and energy

$$\frac{\partial}{\partial t}\left(\rho_1 A_0 + \rho_0 A_1\right) + \left(\rho_0 A_0 u_1\right)' = 0, \tag{2}$$

$$\rho_0 \frac{\partial u_1}{\partial t} = -p' - \rho g, \tag{3}$$

$$\frac{\partial p}{\partial t} + p_0' u_1 = c_0^2 \left(\frac{\partial \rho}{\partial t} + \rho_0' u_1\right), \tag{4}$$

where perturbed quantities are functions of z and t and equilibrium quantities are functions of z only. It is assumed that vertical motions are dominate and thus are considering sausage modes only.

With the introduction of $c(z)$ (Lighthill, 1978)

$$\frac{1}{c^2} = \frac{1}{c_0^2} + \frac{\rho_0}{A_0}\left(\frac{\partial A_1}{\partial p_1}\right)_{p_1=0}, \tag{5}$$

then $Q(t,z)$ (a scaled velocity) can be seen to satisfy

$$\frac{\partial^2 Q}{\partial t^2} - c^2(z)\frac{\partial^2 Q}{\partial z^2} + \Omega_0^2(z)Q = 0, \tag{6}$$

where

$$\begin{aligned}\Omega_0^2 \quad &= \quad N_0^2 + c^2\left[\frac{1}{2}\left(\frac{\rho_0'}{\rho_0} + \frac{A_0'}{A_0} + \frac{c_0^{2\prime}}{c_0^2}\right)' + \frac{1}{4}\left(\frac{\rho_0'}{\rho_0} + \frac{A_0'}{A_0} + \frac{c_0^{2\prime}}{c_0^2}\right)^2 + \right.\\ &\left. + \left(\frac{g}{c_0^2} - \frac{A_0'}{A_0}\right)' + \left(\frac{g}{c_0^2} - \frac{A_0'}{A_0}\right)\left(\frac{\rho_0'}{\rho_0} + \frac{c_0^{2\prime}}{c_0^2} + \frac{g}{c_0^2}\right)\right],\end{aligned} \tag{7}$$

and N_0^2, given by

$$N_0^2 = -g\left(\frac{\rho_0'}{\rho_0} + \frac{g}{c_0^2}\right), \tag{8}$$

is the square of the Brunt-Väisälä frequency. Notice that (6) is a Klein-Gordon type differential equation, with all its attendant features (see Rae and Roberts 1982, Roberts 2004).

2.1 Addition of a Background Flow

We extend the earlier work of Rae and Roberts by the addition of a background flow inside the tube. A full treatment, with the flow, U_0, varying with height, is as yet unavailable since the problem becomes rapidly intractable. Here we consider the simpler, but physically less realistic, case of steady flow inside the tube, such that $U_0'(z) = 0$. The addition of the flow modifies the governing equations and following a similar method, $Q(t,z)$ can be shown to satisfy

$$\frac{D^2 Q}{Dt^2} - U_0\left(\frac{A_0'}{A_0} + \frac{c_0^{2\prime}}{c_0^2}\right)\frac{DQ}{Dt} - c^2(z)\frac{\partial^2 Q}{\partial z^2} + \Omega_0^2(z)Q + S = 0, \tag{9}$$

where

$$\begin{aligned}\Omega_0^2 \quad &= \quad N_0^2 + (c^2 - U_0^2)\left[\frac{1}{2}\left(\frac{\rho_0'}{\rho_0} + \frac{A_0'}{A_0} + \frac{c_0^{2\prime}}{c_0^2}\right)' \frac{1}{4}\left(\frac{\rho_0'}{\rho_0} + \frac{A_0'}{A_0} + \frac{c_0^{2\prime}}{c_0^2}\right)^2 + \right.\\ &\left. + \left(\frac{g}{c_0^2} - \frac{A_0'}{A_0}\right)' + \left(\frac{g}{c_0^2} - \frac{A_0'}{A_0}\right)\left(\frac{\rho_0'}{\rho_0} + \frac{c_0^{2\prime}}{c_0^2} + \frac{g}{c_0^2}\right)\right],\end{aligned} \tag{10}$$

and

$$S = \frac{-1}{\rho_0 R^{\frac{1}{2}}}\left[\left(\frac{c^2 U_0}{A_0}(\rho_0' A_1 + A_0'\rho_1)\right)' + \frac{gU_0 c^2}{c_0^2} A_0(\rho_0' A_1 + A_0'\rho_1)\right], \tag{11}$$

and where

$$\frac{D}{Dt} = \frac{\partial}{\partial t} + U_0 \frac{\partial}{\partial z}. \tag{12}$$

Equation (9) describes the wave motion in a general elastic tube with the presence of a steady background flow. It should be noted that Eq. (9) is not of the Klein-Gordon type and also note the term S, given by (11), is a function of the perturbations ρ_1 and A_1. We are seeking a result that is a function of the scaled velocity Q only and thus to proceed we take S as being negligible.

2.1.1 Basic Formation of Solution

In order to demonstrate some of the features of (9) we consider further simplifications to a straight ($A_0' = 0$) and rigid tube ($c = c_0$). Further, if the gas inside the tube is considered isothermal (c_0 is a constant) then (9) reduces to that describing isothermal acoustic-gravity waves with a background flow

$$\frac{D^2 Q}{Dt^2} - c_0^2(z)\frac{\partial^2 Q}{\partial z^2} + \Omega_a^2(z) Q = 0, \tag{13}$$

where

$$\Omega_a^2 = \frac{c_0^2 - U_0^2}{4\Lambda_0^2}, \tag{14}$$

where Ω_a^2 is a modified acoustic cut-off frequency, and Λ_0 is the isothermal scale height, given by $\Lambda_0 = p_0/\rho_0 g$. If we consider Fourier forms of solution, $e^{i(\omega t - kt)}$, we obtain the following dispersion relation

$$\omega_D^2 = k^2 c_0^2 + \Omega_a^2, \tag{15}$$

where $\omega_D = \omega + U_0 k$ is the *Doppler-shifted* frequency.

To solve (13) analytically we consider Laplace transformations in time. Since we consider an isothermal medium, c_0 and Ω_a are constant. A wave source is introduced into the atmosphere by applying a driver, $A(t)$, at $z = 0$. Thus the atmosphere is bounded and it is assumed that the atmosphere is initially at rest with no wave motions or derivatives of wave motions present. Further it is assumed there are no wave motions at $z = \infty$. The problem is a homogeneous Klein-Gordon type equation with inhomogeneous boundary conditions. We can thus obtain the general solution, given by

$$Q(t,z) = A_0((t-(t_c-t_u))H(t-(t_c-t_u)) + \int_0^t A_0(t+t_u-\tau)\cdot H(t+t_u-\tau)\cdot P(\tau,z)d\tau, \quad (16)$$

where

$$P(\tau,z) = -\frac{z}{2\Lambda_0}\frac{J_1\left(a\sqrt{\tau^2-t_c^2}\right)}{\sqrt{\tau^2-t_c^2}} \times H(\tau-t_c), \quad (17)$$

and

$$a = \sqrt{\frac{\Omega_a^2}{c_0^2}(c_0^2-U_0^2)} = \frac{c_0^2-U_0^2}{2\Lambda_0 c_0}, \qquad t_c = \frac{c_0 z}{c_0^2-U_0^2}, \qquad t_u = \frac{U_0 z}{c_0^2-U_0^2}, \quad (18)$$

where $\Omega_a^2 = (c_0^2-U_0^2)/4\Lambda_0^2$, a can thus be thought of as an acoustic cut-off frequency analogous to $\omega_a(=c_0/2\Lambda_0)$, but having been modified by the presence of the flow, from this point on referred to as the *flow acoustic cut-off*. Note the fact the as $U_0 \to 0$, $\Omega_a \to \omega_a, a \to \omega_a$, $t_c \to z/c_0$ and $t_u \to 0$, thus we return exactly to the Sutmann et al case. We consider only the case for $U_0 > 0$ in this analysis.

We now solve the general solution for variuos prescribed drivers, namely a monochromatic driver, an impulsive driver and finally a sinusoidal driver, each applied at $z = 0$.

3 Monochromatic Source

As per Sutmann et al. (1998) we consider first the boundary conditions to be satisfied by a source of monochromatic acoustic waves, generated continuously with the frequency ω and amplitude Q_0. Thus we have

$$A_0(t) = Q_0 e^{-i\omega t}. \quad (19)$$

Substituting this boundary condition into (16) we obtain

$$Q(t,z) = Q_0\left[e^{-i\omega(t-(t_c-t_u))} \times H(t-(t_c-t_u)) + \int_0^t e^{-i\omega(t+t_u-\tau)}P(\tau,z)d\tau\right]. \quad (20)$$

Notice the removal of the Heaviside function from (16), this is due to the fact that in the interval $(0,t)$, $H(t+t_u-\tau) = 1$ since we consider t_u to be positive. Let the integral in this expression be given by $I = I_1 - I_2$ where

$$I_1 = -e^{-i\omega(t+t_u)}\frac{z}{2\Lambda_0}\int_0^\infty \frac{J_1\left(a\sqrt{\tau^2-t_c^2}\right)}{\sqrt{\tau^2-t_c^2}}H(\tau-t_c)e^{i\omega\tau}d\tau, \quad (21)$$

and I_2 being the same, save for the limits being now (t, ∞). The integral I_1 may be evaluated directly, however the integral I_2 cannot be evaluated analytically unless we apply the condition $\tau >> t_c$ and thus $t >> t_c$. Thus we take

$$I_2 \approx -e^{-i\omega(t+t_u)} \frac{z}{2\Lambda_0} \int_t^\infty \frac{J_1(a\tau)}{\tau} e^{i\omega\tau} d\tau. \tag{22}$$

We have applied the condition that τ is large and so $J_1(a\tau)$ can be expanded asymptotically, thus

$$J_1(a\tau) \approx \sqrt{\frac{2}{\pi a\tau}} \cos\left(a\tau - \frac{3\pi}{4}\right), \tag{23}$$

and by applying the exponential form of the cosine function we obtain

$$I_2 = -e^{-i\omega(t+t_u)} \frac{z}{4\Lambda_0} \sqrt{\frac{2}{\pi a}} \left[\int_t^\infty \frac{e^{i(a\tau-3\pi/4)} e^{i\omega t}}{\tau^{3/4}} d\tau + \int_t^\infty \frac{e^{-i(a\tau-3\pi/4)} e^{i\omega t}}{\tau^{3/4}} d\tau\right]. \tag{24}$$

Which may be evaluated (see Sutmann et al. (1998) Appendix B for a detailed discussion) to give

$$I_2 = \frac{z}{2\Lambda_0} \sqrt{\frac{2}{\pi a}} \frac{1}{t^{3/2}} \frac{1}{a^2 - \omega^2} e^{-i\omega t_u} \left[a \sin\left(at - \frac{3\pi}{4}\right) + i\omega \cos\left(at - \frac{3\pi}{4}\right)\right]. \tag{25}$$

Substituting $I_{1,2}$ into (20), we obtain

$$\begin{aligned} Q(t,z) &= Q_0 e^{-i\omega t_u} \Bigg[e^{-i(\omega t + t_c\sqrt{\omega^2 - a^2})} - \\ &- \frac{z}{2\Lambda_0} \sqrt{\frac{2}{\pi a}} \frac{1}{t^{3/2}} \frac{1}{a^2 - \omega^2} \left(a \sin\theta + i\omega \cos\theta\right) \Bigg], \end{aligned} \tag{26}$$

where $\theta = (at - 3\pi/4)$, being an expression fully describing the atmosphere responding to a source of monochromatic acoustic waves, valid for $w \neq a$. Following the discussion in Sutmann et al. (1998) we see the velocity $Q(t, z)$ is again comprised of two ocscillations superimposed upon each other. Firstly, the *forced atmospheric osciallations* at the driving frequency ω, representing well-known prpoagating acoustic waves for $\omega > a$ and evanescent waves for $\omega < a$. Secondly, we have the *free atmospheric oscillations* at the frequency a, given by the second term on the RHS, decaying with $t^{-3/2}$ and increasing linearly with z.

4 Delta Function Pulse

Here we consider the source to be a single pulse of δ- function form. Thus the boundary condition is

$$A_0(t) = Q_0'\delta(t). \tag{27}$$

The δ-function has a non dimensional argument, hence Q_0' has different dimensions to Q_0. Thus we introduce a non-dimensional time, t', by $t = Rt'$, where $R = 2\pi/a$. Letting $Q_0 = Q_0'/R$ and by assuming $t' >> t_c/R > t_u/R$, the fist term on the RHS of (16) disappears and we obtain

$$Q(t', z) = \int_0^{t'R} Q_0 R\delta(t' - \tau')P(\tau' R, z)d\tau', \tag{28}$$

thus by employing a fundamental property of the δ-function and rewriting for $Q(t, z)$ we obtain

$$\begin{aligned} Q(t, z) &= Q_0 R P(t, z), \\ &= Q_0 R \frac{z}{2\Lambda_0} \frac{J_1\left(a\sqrt{t^2 - t_c^2}\right)}{\sqrt{t^2 - t_c^2}}, \end{aligned} \tag{29}$$

and since $t' >> t_c/R$ the Bessel function reduces to $J_1(at)$ which can be evaluated asymptotically as before, giving

$$Q(t, z) = -Q_0 R \frac{z}{2\Lambda_0} \sqrt{\frac{2}{\pi a}} \frac{1}{t^{3/2}} \cos\theta, \tag{30}$$

where $\theta = (at - 3\pi/4)$, being very similar to the no-flow case in from apart from the change in frequencies.

5 Sinusoidal Pulse

The final case we consider is a sinusoidal pulse generated at the lower boundary and lasting for one wave period P, where $P = 2\pi/\omega$. Thus the boundary condition is given by

$$A_0(t) = Q_0 \left[H(t) - H(t - P)\right] e^{-i\omega t}. \tag{31}$$

Considering the first part of (16) we see that since we are mainly concerned with $t >> t_c$ and our analysis is for $c_0 > U_0$ we can see that the arguments of both Heaviside functions are positive and so they cancel each other out. Thus we have

$$Q(t,z) = Q_0 \int_0^t P(\tau,z)H(t+t_u-\tau)e^{-i\omega(t+t_u-\tau)}d\tau - \\ -Q_0 \int_0^t P(\tau,z)H(t+t_u-P-\tau)e^{-i\omega(t+t_u-\tau)}d\tau. \quad (32)$$

In the interval $(0,t)$ the first Heaviside function is unity since we assume t_u positive, however two cases can be considered for the second, depending on the relative values of P and t_u. For the case of $P < t_u$ then in the interval $(0,t)$ the Heaviside function is unity and so the two terms cancel resulting in $Q(t,z) = 0$, implying there is no atmospheric response to the sinusoidal pulse. For $P > t_u$ then we see that the limits of integration may be changed to be between $(0, t-b)$, where $b = P - t_u$. Thus we can write (32) as

$$Q(t,z) = Q_0 \int_0^t P(\tau,z)e^{-i\omega(t+t_u-\tau)}d\tau - Q_0 \int_0^{t-b} P(\tau,z)e^{-i\omega(t+t_u-\tau)}d\tau, \quad (33)$$

and thus combining the two integrals together by changing the limits and rewriting it in the form two integrals, I_1 and I_2 so $Q(t,z) = I_1 - I_2$, giving

$$Q(t,z) = Q_0 \int_{t-b}^{\infty} P(\tau,z)e^{-i\omega(t+t_u-\tau)}d\tau - Q_0 \int_t^{\infty} P(\tau,z)e^{-i\omega(t+t_u-\tau)}d\tau, \quad (34)$$

and solving asymptotically and then by parts and taking the first term only, as in Section 3, giving

$$I_1 = \frac{z}{2\Lambda_0}\sqrt{\frac{2}{\pi a}}\frac{1}{(t-b)^{3/2}}\frac{1}{a^2-\omega^2}e^{-i\omega t_u} \times \\ \times \left[a \sin\left(a(t-b) - \frac{3\pi}{4}\right) + i\omega \cos\left(a(t-b) - \frac{3\pi}{4}\right)\right], \quad (35)$$

and

$$I_2 = \frac{z}{2\Lambda_0}\sqrt{\frac{2}{\pi a}}\frac{1}{t^{3/2}}\frac{1}{a^2-\omega^2}e^{-i\omega t_u}\left[a \sin\left(at - \frac{3\pi}{4}\right) + i\omega \cos\left(at - \frac{3\pi}{4}\right)\right]. \quad (36)$$

Since we are dealing with large times we can expand the $1/(t-b)^{3/2}$ in equation (35) binomially and taking the first term only the combined result is

$$Q(t,z) = \frac{z}{2\Lambda_0}\sqrt{\frac{2}{\pi a}}\frac{1}{t^{3/2}}\frac{1}{a^2-\omega^2}\left\{a\left[\sin\left(a(t-b)-\frac{3\pi}{4}\right)-e^{-iwt_u}\sin\theta\right]+\right.$$
$$\left.+i\omega\left[\cos\left(a(t-b)-\frac{3\pi}{4}\right)-e^{-iwt_u}\cos\theta\right]\right\}, \quad (37)$$

where $\theta = (at - 3\pi/4)$.

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