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MHD Waveguide Modes in Structured Magnetic Flux Tubes

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Abstract
The propagation of MHD waves in a structured magnetic flux tube embedded within a straight vertical magnetic environment are studied analytically. The motivation behind this study comes from the observations of damped loop oscillations showing that only part of the loop is homogeneous. The magnetic tube considered contains two characteristically distinct parts, namely an internal straight core, and a surrounding uniformly twisted magnetic annulus envelope. The general dispersion relation is derived. Modes of oscillations are examined from this general dispersion relation that is suitable for obtaining information on not just oscillations but some instability properties of this complex tube structure. Both short and long wavelength approximations are considered analytically for the symmetrical mode and with small twist.

1 Introduction

The recent launch of sophisticated satellites such as SOHO and TRACE, that have imaging telescopes on-board, have lead to very detailed data and a surge in the development of the theory of oscillations of MHD waveguides. Numerous oscillatory periods within the Sun's surface have been observed and, in an attempt to account for these periods, numerous theoretical models developed (see reviews e.g. Aschwanden (2003); Roberts (2004); Roberts & Nakariakov (2003)). We consider one specific area of the many theoretical models: the effects of magnetic structuring. The role of magnetic structuring on wave existence and propagation has been studied in some detail (see e.g. De Pontieu et al. (2004); Erdélyi et al. (2004), James et al. (2003), Mendoza-Briceno et al. (2004), Ruderman & Erdélyi (2003), Taroyan et al. (2004)). Here I discuss previous literature on this specific aspect of coronal seismology. Gravity effects are ignored, the emphasis being on the role of the magnetic structuring and, from the basic MHD equations, dispersion relations are found and specific analysis carried out.

2 Waves in a Strongly Inhomogeneous Medium

When the wavelength is greater than or approximately equal to the length-scale the inhomogeneous nature of the medium determines the behavior of any disturbances. Within the Sun and the solar atmosphere the principal causes of inhomogeneity are gravity and the magnetic field. Gravity creates a vertical stratification in plasma pressure and the magnetic field can cause the plasma pressure to increase in a direction normal to the field. These stratifications introduce significant effects, namely amplification (of the wave amplitude (increase or decrease) as it propagates), evanescence (regions in which waves, otherwise oscillating spatially, may decay exponentially) and surface modes (a discontinuity in the medium (magnetic, pressure,...) may give rise to decaying surface waves and will modify the 'body' waves).

2.1 A Magnetic Interface

Ignoring effects due to gravity, we are now interested in the effect of magnetic structuring. Assume that in the basic state the plasma is permeated by a magnetic field $B_0(x)\hat{\mathbf{z}}$, working in Cartesian coordinates x, y, z. Then the pressure and density are structured by the x-dependence of the magnetic field and the basic states are found to be

$$p_0 = p_0(x), \quad \rho_0 = \rho_0(x), \quad \frac{d}{dx}\left(p_0 + \frac{B_0^2}{2\mu}\right) = 0, \tag{1}$$

for the pressure, density and total (gas + magnetic) pressure. Linearised perturbations from this state are taken and the continuity, momentum, induction and energy equations, after Fourier analysis, give a single ODE for $\hat{v}_x(x)$. Consider the magnetic interface:

$$B_0(x) = \begin{cases} B_e, & x > 0, \\ B_0, & x < 0, \end{cases} \tag{2}$$

with B_0 and B_e both constants. Considering pressure continuity at $x = 0$ leads to

$$\frac{d^2\hat{v_x}}{dx^2} - m_0^2\hat{v_x}(x) = 0 \quad \text{for} \quad x < 0, \tag{3}$$

and

$$\frac{d^2\hat{v_x}}{dx^2} - m_e^2\hat{v_x}(x) = 0 \quad \text{for} \quad x > 0. \tag{4}$$

where

$$m_0^2(x) = \frac{(k^2 c_0^2(x) - \omega^2)(k^2 v_A^2(x) - \omega^2)}{(c_0^2(x) + v_A^2(x))(k^2 c_T^2(x) - \omega^2)}, \quad c_T^2(x) = \frac{c_0^2(x) v_A^2(x)}{c_0^2(x) + v_A^2(x)}, \tag{5}$$

and m_e^2 is defined in a similar way to m_0^2 except that the Alfvén and sound speeds appropriate to $x > 0$ are taken.
It is the presence of the discontinuity in $B_0(x)$ that is responsible for the existence of surface waves which may arise if m_0^2 and m_e^2 are both real and positive. Solving (3) and (4) for $\hat{v}_x(x)$ gives

$$\hat{v}_x(x) = \begin{cases} \alpha_e \, e^{-m_e x}, & x > 0, \\ \alpha_0 \, e^{m_0 x}, & x < 0. \end{cases} \tag{6}$$

In writing this solution we are excluding laterally propagating waves, so only the surface modes arise. Then, using the continuity of both $\hat{v_x}(x)$ and of the total pressure perturbation $\hat{p_T}(x)$ across the interface $x = 0$ the general dispersion relation for the magnetic interface is

$$\rho_0 (k^2 v_A^2 - \omega^2) m_e + \rho_e (k^2 v_{Ae}^2 - \omega^2) m_0 = 0, \tag{7}$$

valid for m_0 and m_e both positive and for $l = 0$. From this one is able to expand, simplify and conclude that the existence of the magnetic interface supports the propagation of surface waves (see e.g. Roberts (1981a)).

2.2 Waves in a Magnetic Slab

First consider a magnetic slab (Edwin & Roberts (1982)) with zero field surrounding it so that

$$B_0(x) = \begin{cases} B_0, & |x| < x_0, \\ 0, & |x| > x_0, \end{cases} \tag{8}$$

with pressure p_0 and density ρ_0 inside the slab, p_e and ρ_e outside. The two regions are related by

$$p_e = p_0 + \frac{B_0^2}{2\mu}, \quad \rho_e = \left(\frac{c_0^2 + \frac{1}{2}\gamma v_A^2}{c_e^2}) \right) \rho_0 \tag{9}$$

where c_o and c_e are the sound speeds inside and outside the slab and v_A is the Alfvén speed in the slab. Again, attention is confined to two-dimensional disturbances so velocity perturbation component v_y and wavenumber l are supposed zero. Again, the boundary conditions, $\hat{v}_x(x)$ and $\hat{p}_T(x)$ continuous across the boundary $x = \pm x_0$ are used and the general dispersion relation recovered as:

$$(k^2 v_A^2 - \omega^2) m_e = \left(\frac{\rho_e}{\rho_0} \right) \omega^2 m_0 \begin{pmatrix} tanh \\ coth \end{pmatrix} m_0 x_0, \tag{10}$$

valid for $\omega^2 < k^2 c_e^2$. Further analysis discovers the existence of slow magnetoacoustic waves, both as a body wave and as a surface wave and fast magnetoacoustic waves only if the slab is cooler than the surrounding plasma also as body and surface waves. When the slab is considered thin in comparison to the wavelength (long wavelength approximation that is of interest for photospheric and coronal conditions) the kink mode vibrates as a single thin string and sausage mode vibrates as both surface and body waves.
Expanding the above case to one of a slab embedded in a magnetic environment certain differences were discovered. The dispersion relation for this case is found to be:

$$\rho_e(k^2 v_{Ae}^2 - \omega^2) m_0 \begin{pmatrix} tanh \\ coth \end{pmatrix} m_0 x_0 + \rho_0 (k^2 v_A^2 - \omega^2) m_e = 0, \tag{11}$$

for $m_e > 0$. First, considering incompressible motions, m_0 and m_e both tend to k and thus the modes are Alfvén surface waves. The sausage and kink modes both exist but their general behaviour regarding phase speeds exchanges for v_A greater or less than v_{Ae}. For compressible motions, the transcendental nature of (11) makes the analysis more difficult but when taking the slender slab (long wavelength) approximation the situation become slightly easier. It is supposed that $m_0 x_0 \to 0$ as $k x_0 \to 0$ so that $\tanh(m_0 x_0) \simeq m_0 x_0$ for $k x_0 \ll 1$. Equation (11) then reduces to

$$\rho_0(k^2 v_A^2 - \omega^2) m_e + \rho_e (k^2 v_{Ae}^2 - \omega^2) m_0^2 x_0 = 0, \tag{12}$$

which, for the tanh function (corresponding to the sausage mode) gives solutions indicating that the sausage modes are only weakly affected by the external field. For the kink mode (coth) the dispersion relation becomes

$$\rho_e(k^2 v_{Ae}^2 - \omega^2) + \rho_0 (k^2 v_A^2 - \omega^2) m_e x_0 = 0 \tag{13}$$

which, when solved, shows the external field to dictate the behaviour for a sufficiently thin slab. Only one mode is shown to exist.

For a finite slab, applicable to coronal structures which have been observed as about one tenth wide as they are long, some other simplifying assumptions are made. In the limit of large $k x_0$ both kink and sausage modes result in

$$m_0 \rho_e (k^2 v_{Ae}^2 - \omega^2) + m_e \rho_0 (k^2 v_A^2 - \omega^2) = 0 \tag{14}$$

for the dispersion relation which can be reduced to a cubic and solved. Two cases are considered, firstly a small non-zero external field then a large external field with small plasma beta. Sausage and kink modes are both found to exist and behaviour for various situations are discussed.

2.3 Wave Propagation in a Magnetic Cylinder

After observations in $H\alpha$ and soft and hard X-rays, it became quite clear that cylinders would be a better approximation to the tubular structures seen in the corona and photosphere (Edwin & Roberts (1983)).

With the equilibrium configuration, in cylindrical coordinates $(r,\ \theta,\ z)$, taken as

$$\rho,\ p,\ \mathbf{B}_0 = \begin{cases} \rho_0,\ p_0,\ (0,0,B_0), & r < a \\ \rho_e,\ p_e,\ (0,0,B_e), & r > a \end{cases} \tag{15}$$

using Fourier analysis and continuity across $r = a$ and needing bounded solutions at $r = 0$ the dispersion relation was found. For surface waves ($m_0^2 > 0$)

$$\rho_0(k^2 v_A^2 - \omega^2) m_e \frac{K_n^{'}(m_e a)}{K_n(m_e a)} = \rho_e(k^2 v_{Ae}^2 - \omega^2) m_0 \frac{I_n^{'}(m_0 a)}{I_n(m_0 a)}, \tag{16}$$

and for body waves ($m_0 = -n_0 < 0$)

$$\rho_0(k^2 v_A^2 - \omega^2) m_e \frac{K_n^{'}(m_e a)}{K_n(m_e a)} = \rho_e(k^2 v_{Ae}^2 - \omega^2) n_0 \frac{J_n^{'}(n_0 a)}{J_n(n_0 a)}, \tag{17}$$

where K_n, I_n, J_n, are Bessel functions and the dash denotes the derivative with respect to the argument. Three cases of study were chosen for the complex array of modes given by equations (16) and (17).
Incompressible modes. In the incompressible limit $(c_0^2 \to \infty, c_e^2 \to \infty), m_0$ and m_e become $|k|$. The kink and sausage modes are then given explicitly and it is noted that the phase speed for the kink mode is not monotonic as a function of k but has a maximum/minimum and the sausage mode is monotonically increasing/decreasing. This max/min feature of the kink mode is absent in the slab case so can be deduced to be a reflection of the geometry of the magnetic field.
Photospheric tubes. The dispersion relations were solved for photospheric values, paying particular attention to the slender tube case and not including stratification. The kink mode with phase-speed close to c_k is considered and it is noticed that the corresponding equation is of the general form previously discussed. This suggests this mode may propagate nonlinearly as a solitary wave. By similar argument it is concluded that two kink (surface) modes exist, a slow and a fast mode, the slow mode having phase-speed close to c_T.
Coronal loops. The coronal conditions, i.e. $v_{Ae},\ v_A > c_e,\ c_0$, imply that there are no longer surface waves present but two classes of body waves can occur - fast, which is of particular interest, and slow body waves, which i shall not mention in detail. Fast kink modes, when $v_{Ae} > v_A$, are sustained in dense loops with periods on an Alfvénic timescale and it is found that these body waves have a low wavenumber cut off implying that only wavelengths shorter than the diameter of a loop can propagate

freely. The sausage mode, however, has a much shorter period, approximately one tenth of that of the kink mode. Sausage and kink fast modes exist only in high density loops. However, the slow modes appear in both high and low density cylinders.

2.4 Wave Propagation in Twisted Magnetic Cylinder

Granular shear motions in the photosphere can introduce a twist to the flux tube and prominences often appear to have twisted field lines so the study of this modification to the straight flux tube is of some importance (Bennett et al. (1999)). Twisted tubes have been studied before but only in terms of stability. Bennett et al. (1999) investigates the details of different modes in a uniformly twisted flux tube embedded in a straight magnetic field given by:

$$\mathbf{B} = \begin{cases} (0, Ar, B_0), & r < a, \\ (0\,,\,0\,,\,B_e), & r > a. \end{cases} \tag{18}$$

The plasma is taken as incompressible, with the field and plasma pressure being structured in the radial direction. Again, using continuity of total pressure p_T and perturbation velocity v_r across $r = a$ and searching for a bounded solution at $r = 0$ and as $r \to \infty$ leads to the recovery of dispersion relation

$$\frac{(\omega^2 - \omega_{A0}^2)\dfrac{xI_m^{'}(x)}{I_m(x)} - 2m\omega_{A0}\dfrac{A}{\sqrt{\mu\rho_0}}}{(\omega^2 - \omega_{A0}^2)^2 - 4\omega_{A0}^2\dfrac{A^2}{\mu\rho_0}} = \frac{\dfrac{x_1K_m^{'}(x_1)}{K_m(x_1)}}{\dfrac{\rho_e}{\rho_0}(\omega^2 - \omega_{Ae}^2) + \dfrac{A^2}{\mu\rho_0}\dfrac{x_1K_m^{'}(x_1)}{K_m(x_1)}}. \tag{19}$$

In this equation the dash, as before, denotes derivative with respect to the argument of the bessel function, ω_{A0} and ω_{Ae} are the internal and external alfvén speeds, $x = m_0a$, $x_1 = |k_z|a$ and

$$m_0^2 = k_z^2\left(1 - \frac{4A^2\omega_{A0}^2}{\mu\rho_0(\omega^2 - \omega_{A0}^2)^2}\right). \tag{20}$$

This is the dispersion relation for waves in an incompressible magnetic tube with uniform twist embedded in a straight magnetic environment. For an incompressible tube with no twist there are no body waves but when twist is introduced body waves appear. It was found that as twist is increased the body modes cover a wider range of the phase speeds and become more distinct. As the value of k_za increases the distribution of the body waves decreases showing that long wavelength modes display body mode features. A dual nature is discovered where body waves exist for long wavelengths but surface wave characteristics are displayed for shorter wavelengths. When $m = 1$, the kink modes, it is noted that the phase speed of the body modes tends to infinity as $k_za \to 0$, so waves with larger wavelengths propagate with larger phase speeds. Bennett et al. (1999) explored further the cases of large and small k_za. These approximations, when plotted and compared to the full dispersion relation solved numerically, provide a useful check and cast additional light on the behaviour of the various modes.

2.5 Wave Propagation in Twisted Magnetic Annulus

The interpretation of damping of loop oscillations by resonant absorption given by eg. Robbrecht et al. (2001); Ruderman & Roberts (2002); Goossens et al. (2002) indicates that the coronal flux tubes are homogeneous for only a percentage of their radius (∼5−45%) being inhomogeneous, in the first approximation, both in the centre and in the surrounding plasma. To model this we consider, not only the structuring of a filament surrounded by a coaxial shell as already considered by, among others, Mikhalyaev & Solov'ev (2004), but also added twist that makes the annulus inhomogeneous. We consider a uniformly twisted magnetic annulus embedded in a vertical straight magnetic field.

$$\mathbf{B} = \begin{cases} \mathbf{B}_i = (0\,,\,0\,,\,B_i), & r < a, \\ \mathbf{B}_a = (0, Br, B_0), & a < r < R, \\ \mathbf{B}_e = (0\,,\,0\,,\,B_e), & r > R. \end{cases} \tag{21}$$

For equilibrium state there is no background flow and gravity effects are neglected. Careful analysis is used to find the dispersion relation in terms of the Bessel functions $I_m(z), K_m(z)$ and their derivatives:

$$\frac{\Xi_{aK} - \Xi_i + \Xi_{aK}\Xi_i\frac{A_0^2}{\mu}}{\Xi_{aI} - \Xi_i + \Xi_{aI}\Xi_i\frac{A_0^2}{\mu}}\,\frac{K_m(m_o a)}{I_m(m_o a)} = \frac{K_m(m_o R)}{I_m(m_o R)}\,\frac{\Xi_{RK} - \Xi_e + \Xi_{RK}\Xi_e\frac{A_0^2}{\mu}}{\Xi_{RI} - \Xi_e + \Xi_{RI}\Xi_e\frac{A_0^2}{\mu}}. \tag{22}$$

where

$$\Xi_i = \frac{|k|a\, I_m'(|k|a)}{\rho_i(\omega^2-\omega_{Ai}^2)I_m(|k|a)}, \qquad \Xi_e = \frac{|k|R\, K_m'(|k|R)}{\rho_e(\omega^2-\omega_{Ae}^2)K_m(|k|R)}$$

and

$$\Xi_{\alpha X} = \frac{(\omega^2-\omega_{A0}^2)\dfrac{m_0\alpha X_m'(m_0\alpha)}{X_m(m_0\alpha)} - \dfrac{2mB\omega_{A0}}{\sqrt{\mu\rho_0}}}{\rho_0(\omega^2-\omega_{A0}^2)^2 - \dfrac{4B^2\omega_{A0}^2}{\mu}} \qquad \text{for} \qquad \begin{array}{l} \alpha = a,\ R, \\ X = \text{Bessel fncs}\ I,\ K. \end{array}$$

From this equation, setting first $a = 0$ then setting twist to zero and $a = 0$ one can corroborate this result by recovering the general dispersion relations from Bennett et al. (1999) and from Edwin & Roberts (1983).
Setting $m = 0$, for the sausage modes, let us consider both the long/short wavelength approximations for slender annulus ($|k_z|(R-a) \ll 1$).

Long wavelength. Let us denote $m_0 a = x, \;\; m_0 R = y$. Letting $ka < kR \ll 1$, the dispersion relation reduces to:

$$\frac{K_0(x)}{I_0(x)} \frac{1 + \dfrac{c_{a\theta}}{2ic_{Az}} \dfrac{K_0^{'}(x)}{K_0(x)}}{1 + \dfrac{c_{a\theta}}{2ic_{Az}} \dfrac{I_1(x)}{I_0(x)}} = \frac{1 + \dfrac{c_{R\theta}}{2ic_{Az}} \dfrac{K_0^{'}(y)}{K_0(y)}}{1 + \dfrac{c_{R\theta}}{2ic_{Az}} \dfrac{I_1(y)}{I_0(y)}} \frac{K_0(y)}{I_0(y)}. \tag{23}$$

If twist is small it follows from equation(23), after expansion of the Bessel functions and noting that $x, \; y \;\ll 1$:

$$1 + \frac{1}{2ic_{Az}} \left(\frac{c_{a\theta}}{xK_0(x)} - \frac{c_{R\theta}}{yK_0(y)} \right) = \frac{K_0(x)}{K_0(y)}, \tag{24}$$

and, since $y = \frac{R}{a} x$ and $c_{R\theta} = \frac{R}{a} c_{a\theta}$ we obtain

$$xK_0(x) = -\frac{c_{a\theta}}{2ic_{Az}}, \tag{25}$$

a transcendental equation determining x. We are now left to determine c_{ph} from

$$c_{ph}^2 = c_{Az}^2 \left(1 \pm \frac{2c_{a\theta}/c_{Az}}{\sqrt{x^2 - k^2a^2}} \right).$$

Short wavelength ($1 \ll ka < kR$). We introduce the following notation.

$$c_0 = \rho_0(c_{ph}^2 - c_{Az}^2), \quad c_i = \rho_i(c_{ph}^2 - v_{Ai}^2), \quad c_e = \rho_e(cph^2 - v_{Ae}^2),$$
$$c_a = \rho_0 c_\theta^2, \quad c_R = \rho_0 c_{R\theta}^2 \quad \text{and} \quad \varphi_0 = \tfrac{K_0(ka)}{I_0(ka)} \tfrac{I_0(kR)}{K_0(kR)}$$

After some algebra we find that:

$$\varphi_0 = \frac{c_e - c_0}{c_e + c_0} \frac{c_i - c_0}{c_i + c_0} \left(1 - \frac{1}{x} \frac{c_i^2 - c_i c_0 - 2c_0 c_a}{c_i^2 - c_0^2} \; + \; \frac{1}{y} \frac{c_e^2 - c_e c_0 - 2c_0 c_R}{c_e^2 - c_0^2} \right). \tag{26}$$

This is of too high order in c_{ph} to be solvable. However, it is possible to further simplify for the slender annulus limit. So, for $k(R-a) \ll 1$, we find

$$\varphi_0 \approx 1 \; + \; 2(R-a)m_0, \tag{27}$$

and using the fact that $x, y \gg 1$ (short wavelength) and $k(R-a) \ll 1$ (slender annulus) equation (26) can be reduced to give

$$(c_e + c_0)(c_i + c_0) = (c_e - c_0)(c_i - c_0) \tag{28}$$

which yields the solutions

$$c_{ph}^2 = c_{Az}^2, \qquad c_{ph}^2 = \frac{\rho_i v_{Ai}^2 + \rho_e v_{Ae}^2}{\rho_i + \rho_e} = c_k^2, \tag{29}$$

a similar result to that found by Bennett et al. (1999).

It remains, in the near future, to study, for these dispersion relations, c_{ph} as a function of wavenumber $k_z a$ and to investigate the existence and behaviour of the various modes. Further study will hopefully lead to examining the case of a twisted annulus without the discontinuity in magnetic field at the inner annulus boundary in the hope to extend understanding of heating processes within the solar atmosphere.

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