PADEU | PADEU **15**, 191 (2005)
ISBN 963 463 557

Embedding Diagrams for Black Holes

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Abstract
We discuss two types of embedding for some submanifolds of a spacetime containing a black hole into an Euclidean space and a Minkowski spacetime, respectively. We comment on their meaning and usefulness and on the perspectives that the second one opens for further investigations.
Keywords: *black holes; embedding diagrams, geodesics*

1 Introduction

The embedding diagrams for black holes are almost as old as the notion of black hole. Although Oppenheimer and Snyder shown already in the late 30th that if a spherically symmetric star is massive enough, its gravitational collapse cannot be stopped by anything and a singularity of spacetime will be created, the rigorous notion of a black hole was introduced only in the sixtieth by people like Penrose and Hawking who were able to show that such singularities might exist without making assumption of symmetry and showing that all of them have in common the fact that they are surrounded by a special surface, called *event horizon.* The name of *black hole* was coined, as far as we know, by John Wheeler, in the late sixtieth and he was, also, one of the first to consider embedding diagrams, made popular by the classical book Misner et al. (1971). In the classical embedding diagrams, usually, the equatorial plane of the spacetime containing a black hole is embedded into the three dimensional Euclidean space. Quite recently, Donald Marolf considered another kind of diagram, in which another piece of the black hole spacetime is embedded into the three dimensional Minkowski spacetime. In many respects, this embedding is more useful, because it offers more information on the physical peculiarities of the spacetime. The aim of this paper is to review the general notion of embedding and then to shortly discuss the two particular classes we mentioned already.

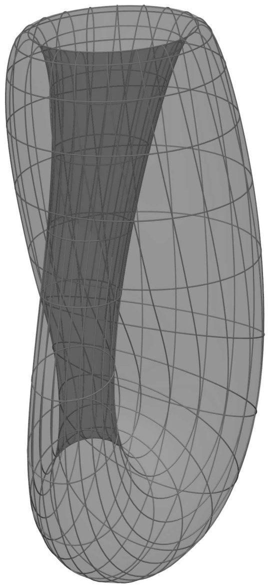
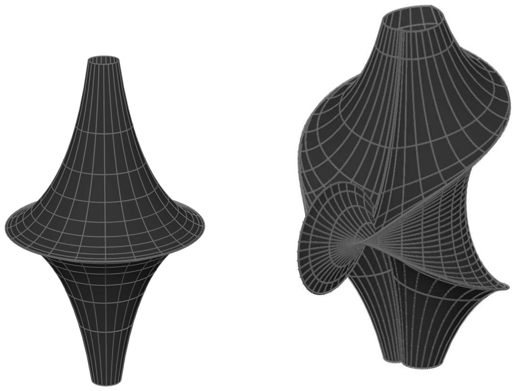


Figure 1: *The Klein's bottle*

2 Embeddings: what and why?

The spacetime containing a black hole, as most of the spacetimes considered in classical general relativity, is four-dimensional. On the other hand, we are living into three dimensional space. Therefore, we don't have an intuitive picture of the entire spacetime. What we can do, nevertheless, is to take lower dimensional pieces of spacetime (in our case they will always be two-dimensional) and represent them as subsets (surfaces in our case) of the three-dimensional Euclidean space we are living in. As a result, we should get subsets of a very particular form. In the case of surfaces, we should have no self-intersections or singular points (for instance corners or edges). A counterexample is the cone, which is not a smooth surface (unless we remove its vertex). In most papers dealing with the problem of embedding diagrams, the existence of embedding is taken as granted. However, this is far from reality. An arbitrary two-dimensional surface cannot be embedded into the Euclidean three-space. For instance, the widely known *bottle of Klein* (see figure 1) cannot be embedded into $\mathbb{R}^3$. As we can see, it has self-intersections. In fact, the Klein's bottle is defined, initially, by factorization, by identifying some subset of $\mathbb{R}^3$ and it can be embedded only in Euclidean spaces of dimension at least four. All we can claim, in the general situation, is that it can



(a) The pseudosphere (b) The surface of Kuen

Figure 2: *Surfaces of constant negative curvature*

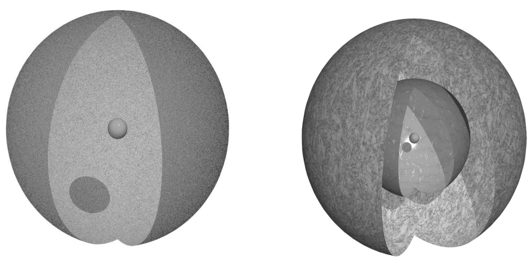
be embedded into $\mathbb{R}^4$. But this is only half of the story. The pieces of spacetime we consider come equipped with a metric, which is induced by the metric of the ambient spacetime. What we would like is to get an embedding that preserves the metrics. In other words, if we put on the embedded surface (provided there is one) the metric induced from the ambient Euclidean or Minkowski space, the embedding should be an isometry. But, as a modern version of a celebrated theorem of John Nash claims, a two-dimensional surface can only be embedded isometrically into $\mathbb{R}^6$ or, if it is endowed with an undefined metric, into the Minkowski spacetime of dimension $6 + 1$. Thus, the existence of embedding into the three-dimensional Euclidean space or into the (2+1)-dimensional Minkowski spacetime seems to be rather the exception than the rule. It should be emphasized, also, that the isometries we are speaking about are, usually, local. Two surfaces can have the same coefficients of the first fundamental form, but their shape can be different. For instance, the surfaces having the same constant curvature are locally isometric, but as the two surfaces from the figure 2 suggest, their shape might be very different.

3 Spherically symmetric black holes

Generally speaking, as we said before, a black hole spacetime is characterized by the existence of a singularity (a point at which at least some of the components of the curvature tensor become infinite), surrounded by an *event horizon*, i.e. a closed surface which has the property that no information (not even light) can escape from its interior towards the infinity. Some black holes (for instance the Reissner-Nordström black hole, or the axially symmetric black hole), may have more than one horizon, although they have, usually, different characteristics (see figure 3). Black holes spacetimes do exist in theory of gravitation different from Einstein's, but we shall confine to this in this paper. As such, the metric of a black hole spacetime should be a solution of *Einstein's field equations*:

$$R_{ij} - \frac{1}{2} g_{ij} R = kT_{ij},$$

where R_{ij} is the Ricci tensor, g_{ij} is the metric, R is the scalar curvature, k is a constant (*Einstein's gravitation constant)* and T_{ij} is the energy-momentum tensor, describing the matter content of the spacetime. We will be interested, in particular, only in



(a) The Schwarzschild BH (b) The Reissner-Nordström BH

Figure 3: *Spherically symmetric Black holes*

two spherically symmetric black hole solution of the Einstein's equations. The first one is the simplest one, the so-called *Schwarzschild solution*, depending on a single parameter, the mass of the body producing the black hole:

$$ds^2 = -\left(1 - \frac{2M}{r}\right) dt^2 + \frac{dr^2}{1 - \frac{2M}{r}} + r^2 d\theta^2 + r^2 \sin^2 \theta d\varphi^2, \tag{1}$$

and the Reissner-Nordström solution, depending on two parameters, the mass and the charge of the body:

$$ds^2 = -\left(1 - \frac{2M}{r} + \frac{Q^2}{r^2}\right) dt^2 + \frac{dr^2}{1 - \frac{2M}{r} + \frac{Q^2}{r^2}} + r^2 d\theta^2 + r^2 \sin^2\theta . d\varphi^2, \tag{2}$$

We have to mention, however, that the equations we just mentioned actually describe the metric in the *exterior* of the black hole or, to be more precise, in the exterior of the event horizon of the black hole. Nevertheless, these metrics have analytical extensions which are valid also on the other side of the horizon. The spacetime obtained through the extension for the case of the Schwarzschild metric is called the *Kruskal spacetime*, after the name of the mathematician who obtained this extension, in the fifties. What it is, usually, embedded, however, is exactly a slice of the exterior part of the black hole, therefore we shall not discuss these extensions.

4 Classical embeddings

Before the Marolf work, what was embedded was a spacelike slice of a black hole spacetime. We shall exemplify on the particular case of the Schwarzschild spacetime. The spacelike part of the spacetime is obtained, in this particular case, by just letting $t = const$. We get, thus, a three-dimensional Riemannian space with the metric given by:

$$ds^2 = \frac{dr^2}{1 - \frac{2M}{r}} + r^2 d\theta^2 + r^2 \sin^2\theta d\varphi^2. \tag{3}$$

This is, however, impossible to visualize, therefore we shall content to embed into the Euclidean space $\mathbb{R}^3$ the "equatorial plane", i.e. we let, also, $\theta = const (= \pi/2)$. We are, left, thus, with a two-dimensional submanifold of the original Schwarzschild spacetime, with the positively-defined metric

$$ds^2 = \frac{dr^2}{1 - \frac{2M}{r}} + r^2 d\varphi^2. \tag{4}$$

To embed this submanifold into the Euclidean space means, in fact, to find a two-dimensional submanifold of $\mathbb{R}^3$ (in other word, a surface in the intuitive space) such that the metric induced on this surface by the metric of the Euclidean space, i.e. its first fundamental form to be exactly the metric (4). As shown in Misner et al. (1971), such a surface can be described, in cylindrical coordinates, by

$$z = \sqrt{8M(r - 2M)} \tag{5}$$

The plot can be seen in the figure 4. The event horizon corresponds to the bottom of the diagram (as it corresponds to $r = 2M$). We will call this kind of embedding, when a spacelike slice of the spacetime is embedded, a *classical embedding diagram.* This

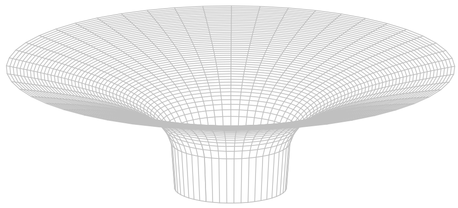
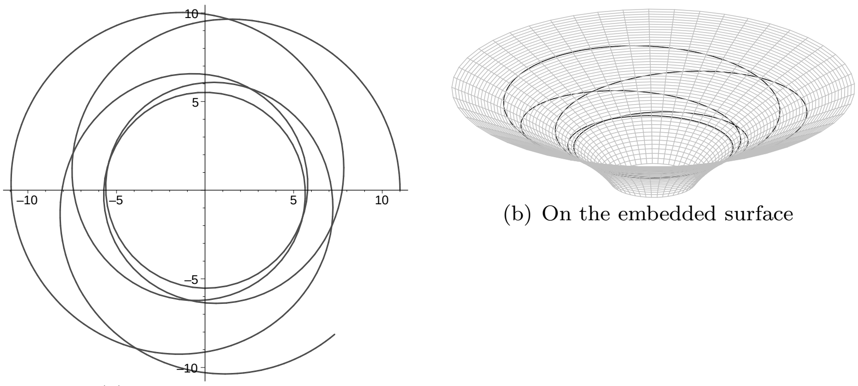


Figure 4: *An embedding diagram for the Schwarzschild black hole for $M = 1$*

kind of diagram is quite useful for the visualization of some phenomena. For instance, in most books of relativity, a picture of an equatorial geodesic in the Schwarzschild spacetime typically looks like that in the figure 5(a) while, instead, it should be



(a) The plane version

(b) On the embedded surface

Figure 5: *A geodesic around a black hole*

viewed as a geodesic on the embedded equatorial plane, i.e. it should look like in the figure 5(b). Thus, the classical embedding diagram does help the intuition and makes things clearer. However, it emphasizes only the curvature of the *space* instead of emphasizing the curvature of the *spacetime*. In other words, generally speaking, what has physical significance is the curvature of the spacetime rather than the curvature of the spacelike section. For instance, the four-dimensional Minkowski spacetime has zero curvature. Nevertheless, we can choose coordinates in such a way that the section $t = const$ are *curved* three-dimensional Riemannian spaces. Still, this has nothing to do with physics, it just reflects a particular choice of coordinates. The four-dimensional spacetime curvature, instead, is either different from zero in any coordinate system, either zero in any coordinate system. It would be useful to have, therefore, also a way of visualizing two-dimensional slices for which the induced metric is non-defined. Clearly, such slices can only be embedded into a Minkowski spacetime. It is exactly what the Marolf's embedding diagrams are doing and we shall dedicate the next section to them.

5 Marolf's embedding diagrams

The new kind of embedding were introduced in Marolf (1999) and discussed in more details in Giblin et al. (2004). As we said previously, the idea is two embed a 2-dimensional submanifold with a Lorentzian metric into the (2+1)-dimensional Minkowski spacetime rather than into the Euclidean space. We mention that, as is the case with the classical embeddings, the construction of a Marolf embedding is quite delicate and it is only possible to be done in special situation, for instance when the spacetime has spherical symmetry and, moreover, it is static. The black hole having this properties are endowed with a metric of the form

$$ds^2 = -\phi dt^2 + \phi^{-1} dr^2 + r^2 d\theta^2 + r^2 \sin^2 \theta d\varphi^2,$$

where ϕ is a function depending only on r, due to the spherical symmetry and to the static character of the spacetime. What is intended is to embed the (t, r)-part of the spacetime, with the metric

$$ds^2 = -\phi dt^2 + \phi^{-1} dr^2.$$

This should be embedded into the (2+1) Minkowski spacetime, with the metric given by

$$ds^2 = -dT^2 + dX^2 + dY^2.$$

In fact, there are some technicalities that we are not going to discuss here (see Giblin et al. (2004)), related to the fact that, usually, we cannot use the same formulae to embed the entire submanifold, therefore we divide the Minkowski spacetime into several regions and then embed different pieces of the submanifold in different region and then we "past" them together to get the overall picture. The trick is, again, to use cylindrical coordinates, but this time they are *hyperbolic* (as the metric of the surface is Lorentzian, rather than Riemannian). In the figure 6 we represent the diagram obtained for the Schwarzschild spacetime (Marolf (1999)). As one can see, it is very different from the classical embedding diagram. In particular, one might have difficulties to locate the event horizon on this diagram which is a smooth surface. It turns out that the horizon correspond to $Y = 0$, while the singularity $r = 0$ is inside the horizon, corresponding to infinite values of T (see Marolf (1999) for the argumentation). The Marolf's diagram is useful because it emphasizes the curvature of spacetime and, also, because on this diagram one can represent the wordlines of particles, instead of geodesic corresponding to constant values of the time coordinate, as is the case for the classical embedding diagrams.

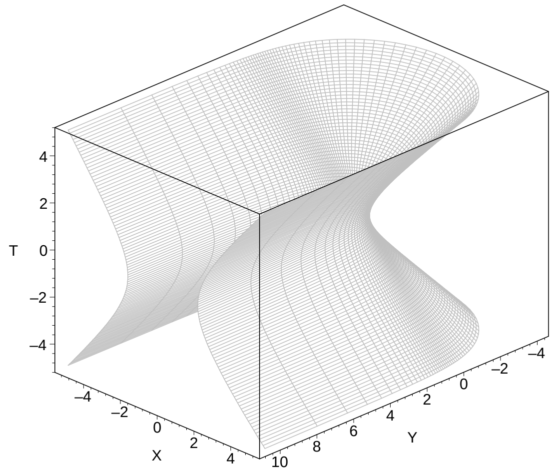


Figure 6: *The Marolf's diagram for the Schwarzschild black hole*

For the Reissner-Nordström black hole (figure 7, Giblin et al. (2004)) the situation is more complicated, because of the presence of two event horizons, corresponding to the two solutions of the equation $\phi(r) = 0$. It turns out that only for the part of the 2-submanifold lying outside the exterior event horizon the embedding is possible and this is the one appearing in the figure.

6 Final notes and perspectives

The embedding diagrams are very useful tools both for teaching general relativity and for a better understanding of different aspects of the geometry and physics of black holes. In particular, the Marolf's diagram should provide a lot of insight. Much remain to be done in this respect. In particular, it would be nice to have, also, such diagrams for relativistic stars and to attempt to study their evolution during the gravitational collapse. A detailed study of the geodesics on these surface is also something that has to be done. Some of these problems will be touched in Blaga (2005).

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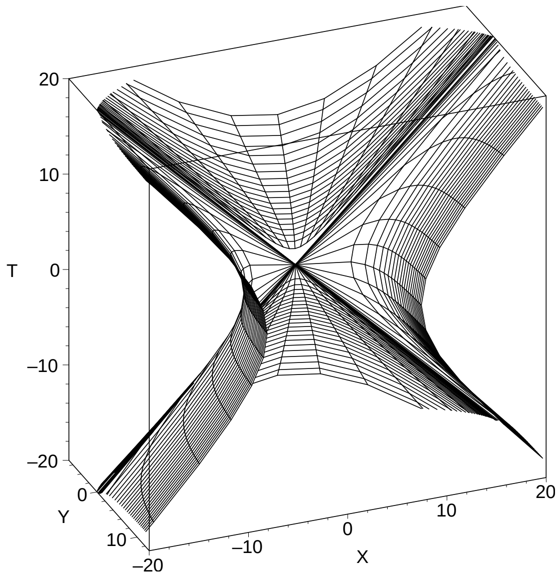

Figure 7: *The Marolf's diagram for the Reissner-Nordström black hole*

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