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Small radial perturbations of magnetic polytropes

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Abstract
The paper is devoted to a qualitative analysis of the nonlinear, radial oscillations of magnetic polytropes. The magnetic field is assumed to be purely toroidal. The small adiabatic perturbations are investigated using the normal forms method. The nonadiabatic effects, described with the aid of two additional terms, related to the sources of the energy and energy damping, respectively, are analyzed with the aid of the dynamical systems theory. The numerical examples confirm and complete the qualitative investigation.
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1 Introduction

In this paper we study the stability of the polytropic stars in a *weak toroidal magnetic field*. To investigate the radial oscillations of a star we use the radial approximation of the Lorentz force proposed by Monaghan (1968). The study of the nonlinear radial *adiabatic* oscillations of magnetic polytropes is made through the *normal forms method*. The same problem was investigated using the multiple scales method by Das et al (1994). The nonadiabatic effects are described using additional terms connected to the energy production and loss. Their influence on the nonlinear radial pulsation of magnetic polytropes is investigated with the aid of *dynamical systems theory*.

2 Small perturbations of magnetic polytropes

The influence of a weak magnetic field on stellar oscillations can be obtained using a perturbative method, so we consider the Lagrangean perturbation of the hydromagnetic equilibrium equation:

$$\nabla P - \rho\nabla\Phi + \frac{1}{8\pi}\nabla B^2 - \frac{1}{4\pi}\left(\vec{B}\cdot\nabla\right)\vec{B} = 0. \tag{1}$$

where P is the pressure, Φ – the gravitational potential and B – the induction of the magnetic field. After long, but straightforward computations we obtain the equation of small oscillations of a magnetic star:

$$\rho\partial_t^2\xi_i = L_{ij}\xi_j, \tag{2}$$

where the right hand side of the equation is

$$\begin{aligned} L_{ij}\xi_j &= \nabla_i(\Gamma_1 P\nabla_j\xi_j) - (\nabla_j\xi_j)\nabla_i P + (\nabla_i\xi_j)\nabla_j P + \rho\xi_j\nabla_j\nabla_i\Phi + \rho\nabla_i\delta\Phi \\ &- \frac{1}{8\pi}(\nabla_j\xi_j)\nabla_i B^2 - \frac{1}{4\pi}\delta B_k\left(\frac{\partial B_k}{\partial x_i} - \frac{\partial B_i}{\partial x_k}\right) - \frac{1}{4\pi}B_k\left(\frac{\partial\delta B_k}{\partial x_i} - \frac{\partial\delta B_i}{\partial x_k}\right) \\ &- \frac{1}{8\pi}\nabla_i(\xi_k\nabla_k B^2) + \frac{1}{4\pi}\left(\xi_j\nabla_j B_k\nabla_k B_i + \xi_j B_k\nabla_j\nabla_k B_i\right). \end{aligned} \tag{3}$$

To obtain the normal modes equation we consider the peculiar solution of the former equation $\xi(r,t) = \xi(r)e^{i\sigma t}$ (Anand and Kushwaha, 1962).

3 Radial oscillations of a polytropic star in a toroidal magnetic field

The magnetic field destroys the spherical symmetry of the star, but in the first approximation to investigate the problem of stellar oscillation in a weak magnetic field, we can use the radial approximation of the Lorentz force (Monaghan, 1968) which is

$$\int_{-1}^{1}\int_{0}^{2\pi} F(r)d\nu d\varphi = \int_{-1}^{1}\int_{0}^{2\pi}\left(-\frac{1}{8\pi}\nabla B^2 + \frac{(\vec{B}\cdot\nabla)\vec{B}}{4\pi}\right)d\nu d\varphi \tag{4}$$

where $\nu = \cos\theta$, $F(r)$ is the radial approximation of the Lorentz force and B – the induction of the magnetic field.

Assuming that the magnetic field of the star is purely toroidal and that the nonzero component of the induction of the magnetic field is $B_\varphi = \Gamma\rho r\sin\theta$ (Roxburgh, 1966), the equation of radial adiabatic pulsations becomes:

$$\begin{aligned} \rho_0 a\ddot{r}_1 &= -(1+r_1)^2\left[P_0(1+r_1)^{-3\gamma}(1+ar_1'+r_1)^{-\gamma}\right]' - \rho_0 g_0(1+r_1)^{-2} \\ &- \frac{\Gamma^2\rho_0}{6\pi}(1+ar_1'+r_1)^{-1}\left[\rho_0 a^2(1+ar_1'+r_1)^{-1}\right]', \end{aligned} \tag{5}$$

where the dot stands for the time derivative, the prime – for the derivative with respect to the radial variable (here the radius of the unperturbed configuration a), zero index emphasizes that it is considered the value of the function in the unperturbed configuration. The distance to the stars center in the perturbed case, denoted by r, is

$$r = a(1 + r_1) \tag{6}$$

where a is distance to stars center in equilibrium and r_1, the adimensional radius, is the difference between the distances to star's center in unperturbed stellar radius.

Comparing this equation to that derived by Rosseland (1949) for radial adiabatic oscillation of a star without rotation or magnetic field we conclude that the last term from the left hand side appears because of the existence of the magnetic field.

4 Nonlinear radial oscillations of magnetic polytropes

Expanding in Taylor series the right hand side of the equation (5) and keeping the terms up to the first order Roxburgh and Durney (1967) obtained the equation of linear pulsations of magnetic polytropes.

Keeping in the series expansion the terms up to third order in r_1 we obtain the following equation

$$\rho_0 a \ddot{r}_1 = L(r_1) + Q(r_1) + S(r_1), \tag{7}$$

where the functions contain the terms of order one, two, three in r_1. Their expressions are two long to be reproduced here. If we consider that r_1 is

$$r_1 = \xi_1(a) q_1(t), \tag{8}$$

with ξ_1 is the eigenfunction corresponding to the fundamental mode of the equation (5) we obtain

$$\ddot{q}_1 + q_1 = A q_1^2 + B q_1^3, \tag{9}$$

where the variable is proportional to time and the coefficients A and B depend on the polytropic index and on the ratio between the magnetic and gravitational energy. This equation (9) is the equation of the radial nonlinear oscillation of magnetic polytropes obtained by Das et al (1994). They investigated it by using the multiple scales method. We will find its approximate solution using the normal forms method.

5 Adiabatic oscillations of magnetic polytropes through normal forms method

5.1 Short description of the method

We are looking for an approximate solution of the pulsations equation. The small parameter we choose is the amplitude of the initial oscillation, because from observational data it is between 0.05 and 0.15 for classical Cepheids, $0.01-0.08$ for RR Lyrae stars and $0.10-0.30$ for W Vir (Buchler, 1990; Stothers, 1981).

Let $q_1 = \lambda Q_1$, where λ is the initial amplitude and Q_1 the unknown function, then the equation (9) becomes

$$\ddot{Q}_1 + Q_1 = A\lambda Q_1^2 + B\lambda^2 Q_1^3, \tag{10}$$

where $Q_1 = 1.0$ and $\dot{Q}_1 = 0.0$ for the initial moment $t = 0$.

We replace the real unknown function Q_1 with a complex unknown function of complex variable to reduce the order of the differential equation we have to solve. The complex variables $\xi, \bar{\xi}$ are introduced by

$$\xi = \frac{1}{2}(Q_1 - i\dot{Q}_1), \quad \bar{\xi} = \frac{1}{2}(Q_1 + i\dot{Q}_1). \tag{11}$$

In these new variables the equation 10) becomes

$$\dot{\xi} = i\xi - \frac{i\lambda A}{2}(\xi + \bar{\xi})^2 - \frac{i\lambda^2 B}{2}(\xi + \bar{\xi})^3, \tag{12}$$

equation which is written in a simpler form using a power series expansion with respect to a new variable η. Let

$$\xi = \eta + \lambda h_1(\eta, \bar{\eta}) + \lambda^2 h_2(\eta, \bar{\eta}) + \lambda^3 h_3(\eta, \bar{\eta}) + ... \tag{13}$$

and

$$\dot{\eta} = i\eta + \lambda g_1(\eta, \bar{\eta}) + \lambda^2 g_2(\eta, \bar{\eta}) + \lambda^3 g_3(\eta, \bar{\eta}) + ..., \tag{14}$$

where $h_i (i = 1, 2, 3)$ are smooth function in η and $\bar{\eta}$ and $g_i (i = 1, 2, 3)$ contain the resonant terms (*i.e.* $\sim e^{it}$). We mention that after we have determined and replaced the functions which appear in (14), it became the normal form of equation (12).

5.2 Approximate solution of the radial nonlinear pulsations equation

Identifying the coefficients of the terms that contain λ at equal power, we are able to specify the form of the functions g_i and to write down the normal form of equation (12) as

$$\dot{\eta} = i\eta - \lambda^2 i \left(\frac{5A^2}{6} + \frac{3B}{2} \right) \eta^2 \bar{\eta}. \tag{15}$$

with the initial condition $\eta = 0$ for $t = 0$. The solution of this equation is

$$\eta = \frac{1}{2} e^{i\left[t-\lambda^2\left(\frac{5A^2}{24}+\frac{3B}{8}\right)t\right]}, \tag{16}$$

and the oscillations period is $2\pi + O(\lambda^2)$. To simplify the form of the solution we introduce the notation

$$\tau^+ = t - \lambda^2 \left(\frac{5A^2}{24} + \frac{3B}{8}\right) t. \tag{17}$$

The approximate solution of equation (10) is

$$\begin{aligned} Q_1 \quad &= \quad \cos\tau^+ + \lambda\left(-\frac{A}{6}\cos 2\tau^+ + \frac{A}{2}\right) + \frac{\lambda^2}{4}\left(\frac{A^2}{24} - \frac{B}{8}\right)\cos 3\tau^+ \\ &+ \quad \frac{\lambda^3}{8}\left[\left(-\frac{A^3}{540} + \frac{AB}{12}\right)\cos 4\tau^+ + \left(-\frac{7A^3}{216} - \frac{31AB}{12}\right)\cos 2\tau^+ \right. \\ &- \quad \left. \frac{19A^3}{9} - 5AB\right]. \end{aligned} \tag{18}$$

5.3 Concluding remarks

The numerical evaluations of (18) for different values of the polytropic index (n) and ratio between magnetic and gravitational energy (h) revealed us, as expected, that the precision of the computation is highly dependent on the initial amplitude, and less sensitive at the values of n or h.

6 Radial nonlinear nonadiabatic oscillations of magnetic polytropes

The dissipative phenomena are described with the aid of two terms introduced in the equation (10). This idea and the form of these terms belong to Krogdahl (1955), who use them to explain the shape of the light curves observed at Cepheids. The terms added are $\mu\frac{dq_1}{dt}$, with $\mu > 0$, μ being a constant related to the energy sources of the star and $-\frac{\mu}{\lambda}q_1^2\frac{dq_1}{dt}$, where λ is proportional to the energy loss.

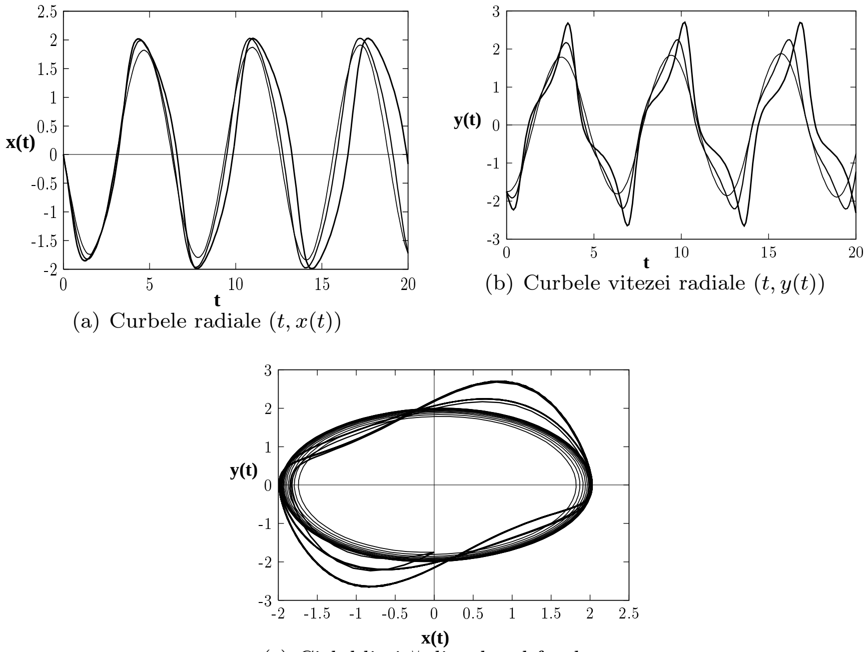
If we substitute $q_1 = \lambda Q_1$, the equation (10), in which we have added the two terms describing the energy production and loss, becomes

$$\ddot{Q_1} + Q_1 = A\lambda {Q_1}^2 + B\lambda^2 {Q_1}^3 + \mu(1 - Q_1^2)\dot{Q}_1, \tag{19}$$

which for $\lambda = 0$ is a van der Pol equation[1]. A qualitative study of the equation (19) was done by Blaga (1998). It reveals that the number of the equilibrium points of

[1]In this case λ is a parameter connected to energy production and dissipation, not to the initial amplitude as in the case of normal form method discussed above.

the equation (19) depends on the polytropic index and the ratio between magnetic and gravitational energy and their nature depends on the energy dissipation through μ. For $\mu > 0$ exists at least one periodic solution. For small positive values of μ the origin is unstable, but there exists a stable limit cycle. From the physical point of view, this means that, no matter which the initial conditions are the solution tends asymptotically to that periodic solution. In the amplitude-frequency relation μ plays no role as long as it is small.



(a) Curbele radiale $(t, x(t))$

(b) Curbele vitezei radiale $(t, y(t))$

(c) Ciclul limită din planul fazelor

Figure 1: *Reprezentarea grafică a soluţiei ecuaţiei pulsaţiilor neliniare neadiabatice pentru* $n = 3$ *şi* $h = 0.004$

We conclude that the presence of the energy sources ($\mu \neq 0$, $\mu > 0$) in a polytropic star does not contradict the existence of periodic orbits for the equation of radial nonlinear pulsations. For $\mu = 0$ (*i.e.* adiabatic pulsations) and $\mu \neq 0$, $\mu > 0$ (*i.e.* nonadiabatic oscillations) the motion in the phase plane looks different. In the first case the solution depends on the initial conditions and in the second case the amplitude of the pulsation is independent of the initial conditions.

We emphasize that for small values of parameters μ şi λ, the magnetic field, through

A and B, the nonadiabatic processes, through the damping coefficient, tend to diminish the frequency of the oscillations. This thing could be observed from the figures 1(a) and 1(b) in which we have represented the radial curves and the radial velocity curves $(t, x(t))$, respectively $(t, y(t))$ for the first order differential system corresponding to the second order differential equation (19). These were obtained solving it numerically for $n = 3$ and $h = 0.004$ (where h is the ratio between the magnetic and gravitational energy) and $\mu \in \{0.05, 0.5, 1.0\}$. In figure 1(c) we have represented the limit cycle. For small values for μ it is symmetric, this quality is lost for bigger values for μ, as could be observed from the radial velocity curves (figure 1(a)).

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