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Hill's stability of the Moon in the spatial elliptic restricted three-body problem

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Abstract
Hill's (G. Hill, 1878) global and nonlinear stability theory has the advantage of being applicable to a great variety of dynamical systems, including those occurring in the solar system. He used his method originally to study the stability of the Moon as influenced by the Earth and the Sun. V. Szebehely (V. Szebehely , 1978) showed that in the model of circular restricted three-body problem the measure of stability for the Earth's Moon is very low. Using the invariant relation of the spatial elliptic restricted three-body problem we show that the measure of stability for the Earth's Moon oscillate above stability critical value.
Keywords: *Hill's stability, Restricted three-body problem*

1 Introduction

Consider a dynamical system with an integral of motion (such as the Jacobi integral in the circular restricted three-body problem) given by

$$v^2 = C - V\left(x, y, z\right),$$

where v is the velocity, V is a generalized potential, and C is the constant of integration. For a given set of initial conditions (x_0, y_0, z_0, v_0) we find

$$C_0 = v_0^2 - V\left(x_0, y_0, z_0\right).$$

The relation

$$v^2 = C_0 - V\left(x, y, z\right)$$

must hold during the motion.

Definition 1 *The equation of* zero velocity surface *(*ZVS*) according to initial conditions* (x_0, y_0, z_0, v_0) *is*

$$C_0 = V\left(x, y, z\right).$$

In general the ZVS separate those points in the space for which

$$C_0 < V(x, y, z)$$

from those for which

$$C_0 \geq V(x, y, z).$$

In the moment $t = 0$ consider a test particle inside a closed ZVS, $C = V(x, y, z)$. If its integration constant C changes slightly, by some outside disturbing effect, then its ZVS will change also. If this new region is still inside a simple closed surface, the stability of the system will not change qualitatively. But if new region does not represent the inside a simple closed surface, then the test particle may depart from the system and its behavior may change suddenly.

Definition 2 *The* C_1 *value of integration constant is a* critical or bifurcation *of the ZVS if at this value the topology of the ZVS changes.*

If the actual value of the constant C is far removed from the bifurcation value C_1, we conclude that the system is more stable than if it is very close since, when $C - C_1 \approx 0$, small perturbations may change the stability characteristics.

Definition 3 *(V. Szebehely , 1978) The difference between the actual value of the integration constant and bifurcation value is a* measure of the stability *of the system:*

$$M_{st} = C - C_1.$$

2 Zero velocity surfaces in the ERTBP

In the elliptic restricted three-body problem (ERTBP) the two massive primaries P_1 and P_2, with masses $m_1$and m_2 revolve on elliptical orbits under their mutual gravitational attraction and the motion of a third, massless body is studied. The orbit of P_2 around P_1, in an inertial system is

$$\|P_1P_2\| = \frac{a\left(1 - e^2\right)}{1 + e\cos f}, \tag{1}$$

where $\|P_1P_2\|$ is the mutual distance, a and e are the semimajor axis and the eccentricity of the elliptical orbit, and f is the true anomaly.

There are several systems of reference that can be used to describe the elliptic restricted three-body problem. In our study a nonuniformly rotating and pulsating coordinate system is used. In this system of reference the origin is in the center of mass of the two massive primaries (Sun and Earth for example), and the $\tilde{\xi}$ axis is directed towards P_2. The $\tilde{\xi}\tilde{\eta}$ coordinate-plane rotates with variable angular velocity, in such a way, that the two massive primaries are always on the $\tilde{\xi}$ axis, and the period of the rotation is 2π. Besides the rotation, the system also pulsates, to keep the primaries in fixed positions $\left(\tilde{\xi}_1 = -\mu, \tilde{\eta}_1 = \tilde{\zeta}_1 = 0, \tilde{\xi}_2 = 1-\mu, \tilde{\eta}_2 = \tilde{\zeta}_2 = 0\right)$. In this system the equations of motion of the third massless particle are:

$$\begin{cases} \tilde{\xi}'' - 2\tilde{\eta}' = \frac{\partial \omega}{\partial \tilde{\xi}}, \\ \tilde{\eta}'' + 2\tilde{\xi}' = \frac{\partial \omega}{\partial \tilde{\eta}}, \\ \tilde{\zeta}'' = \frac{\partial \omega}{\partial \tilde{\zeta}}, \end{cases} \tag{2}$$

where the derivatives are taken with respect to the true anomaly f, and

$$\omega = (1 + e\cos f)^{-1}\,\Omega,$$

with

$$\begin{aligned} \Omega\left(\tilde{\xi}, \tilde{\eta}, \tilde{\zeta}, f\right) &= \frac{1}{2}\left(\tilde{\xi}^2 + \tilde{\eta}^2 - e\tilde{\zeta}^2\cos f\right) + \frac{1-\mu}{\sqrt{\left(\tilde{\xi}+\mu\right)^2 + \tilde{\eta}^2 + \tilde{\zeta}^2}} + \\ &+ \frac{\mu}{\sqrt{\left(\tilde{\xi} - 1 + \mu\right)^2 + \tilde{\eta}^2 + \tilde{\zeta}^2}} + \frac{1}{2}\mu(1-\mu). \end{aligned} \tag{3}$$

Performing the same operations, which in the restricted three-body problem leads to the Jacobi-integral, in the case of the spatial ERTBP we obtain an invariant relation of the form (Z. Mako and F. Szenkovits , 2004)

$$v^2 = 2\omega - e\int_{f_0}^{f} \frac{\tilde{\zeta}^2 \sin h}{1 + e\cos h}dh - 2e\int_{f_0}^{f} \frac{\Omega \sin h}{(1 + e\cos h)^2}dh - C,$$

where v is the velocity of the third massless particle. For a given set of initial conditions $(\tilde{\xi}_0, \tilde{\eta}_0, \tilde{\zeta}_0, v_0, f_0)$ we find

$$C_0 = \frac{2\Omega\left(\tilde{\xi}_0, \tilde{\eta}_0, \tilde{\zeta}_0, f_0\right)}{1 + e\cos f_0} - v_0^2.$$

The zero velocity surfaces in the ERTBP according to initial condition $(\tilde{\xi}_0, \tilde{\eta}_0, \tilde{\zeta}_0, v_0, f_0)$ are

$$\frac{2\Omega}{1+e\cos f} - e\int_{f_0}^{f} \frac{\tilde{\zeta}^2 \sin h}{1+e\cos h} dh - 2e\int_{f_0}^{f} \frac{\Omega \sin h}{(1+e\cos h)^2} dh = C_0. \tag{4}$$

These surfaces delimite the Hill-regions, in which the motion of the third particle is not possible. In three dimension space it means that at every time – or at every value of the true anomaly f – a different set of surfaces of zero velocity are to be constructed. The shape of these ZVSs vary in time. Therefore we might speak about pulsating surfaces of zero velocity.

3 Measure of stability of the Moon in the Sun-Earth-Moon system

In the case of the Sun–Earth system the eccentricity $e = 0.0167$ is small. Due to the variation of f, these regions can pulsate, and near to the critical values they can change they type. The critical points of the pulsating surfaces of zero velocity (4) correspond approximately to the equilibrium solutions of circular restricted three-body problem given by

$$C_i = 2\Omega^\circ(L_i), \qquad i = 1, \ldots, 5, \tag{5}$$

where L_i are the Lagrange-points. For these constants we have

$$3 = C_4 = C_5 \le C_3 \le C_1 \le C_2 \le 4.25$$

in generally, and in the case of the Sun–Earth system the critical value for L_2 between the two primaries is

$$C_2 = 3.000893278,$$

and in L_1, the Lagrange-point outside of the Earth the critical value is

$$C_1 = 3.000889276.$$

For $C > C_2$ the ZVSs delimit three regions where the motion of the small body is possible (Figure 1). Two of these regions are closed around the primaries, the third one is the exterior of the exterior surface. Between these regions the communication is impossible.

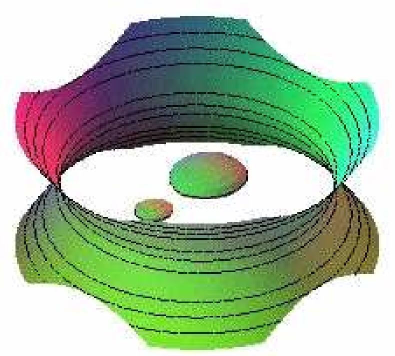

Figure 1: *The ZVSs in ERTBP if $C > C_2$.*

The measure of stability of the Moon in the Sun-Earth-Moon system considered in ERTBP model is

$$\begin{aligned} M_{st}\left(\tilde{\xi}(f), \tilde{\eta}(f), \tilde{\zeta}(f), f\right) &= \frac{2\Omega\left(\tilde{\xi}, \tilde{\eta}, \tilde{\zeta}, f\right)}{1 + e\cos f} - e\int_{f_0}^{f} \frac{\tilde{\zeta}^2 \sin h}{1 + e\cos h} dh \\ &\quad -2e\int_{f_0}^{f} \frac{\Omega\left(\tilde{\xi}, \tilde{\eta}, \tilde{\zeta}, f\right)\sin h}{(1 + e\cos h)^2} dh - C_2, \end{aligned}$$

where $(\tilde{\xi}, \tilde{\eta}, \tilde{\zeta})$ is solution of differential equation (2) at initial conditions of the Moon.

In figure 2 we show the variation of the M_{st} and variation of distance between Earth and Moon, notated by r_2. We observe that the measure of stability for the Moon vary from 0.0004 to 0.00055 and the maximal value is in the pericenter and minimal value is in the apocenter.

4 Conclusion

For the initial conditions of the Moon in the nonuniformly rotating and pulsating coordinate system the Hill region around to Earth is bounded by closed ZVS. The elliptical orbit of the Earth is not change the stability characteristics of the Moon.

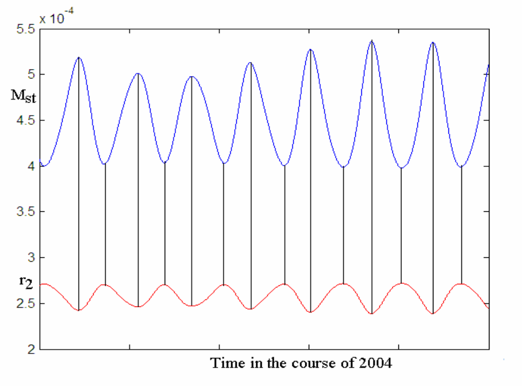

Figure 2: *The variation of measure of stability.*

We have been found that the measure of stability for the Moon vary from 0.0004 to 0.00055. The maximal value is in the pericenter and minimal value is in the apocenter. Comparatively to other satellites around of other planets, the measure of stability for the Moon is very low. For example the measure of stability for Mars's Phobos and Demos are approximately 0.0025 and 0.012.

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