TRIGONOMETRIC CALCULATION OF THE ELEMENTS OF ORBIT OF CELESTIAL BODIES BY TWO TELESCOPES SITUATED IN THE LAGRANGIAN POINTS L_4 AND L_5

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Abstract This paper describes a method to calculate the elements of orbit of a celestial body, detected by two telescopes situated in the Lagrangian points L_4 and L_5 by two satellites. Here the angles between the object and the points L_4 and L_5 are surveyed. Then it is possible to calculate by only these two measurements simultaneous in these points all elements of orbit of the detected object very fast and accurate.

Keywords: NEAs – observation – Lagrange points

1. Introduction

The problem to calculate the elements of the orbit of a detected object was solved in the beginning of the nineteenth century by the methods of J. P. Laplace and C. F. Gauss. We need in order to use these methods at least two observations from *one* point (Earth or satellite). Both methods have the disadventage, that after the first measurement more measurements are necessary in order to approximate the elements of orbit better and better[1]. It is assumed, that one satellite is situated in the Lagrangian Point L_4 and a second one in L_5. The satellites are equipped with telescopes, in order to observe and to measure the angles of objects in the plane of the Ecliptic and also the perpendicular angles to this plane. With these angles it is possible, to calculate the distance of a detected object, to the Earth and the Sun by only one observation. Also it is possible to computer the other orbital elements of such an object by two observations.

2. Calculation of the distance AE by the angles α and β measured in L_4 and L_5

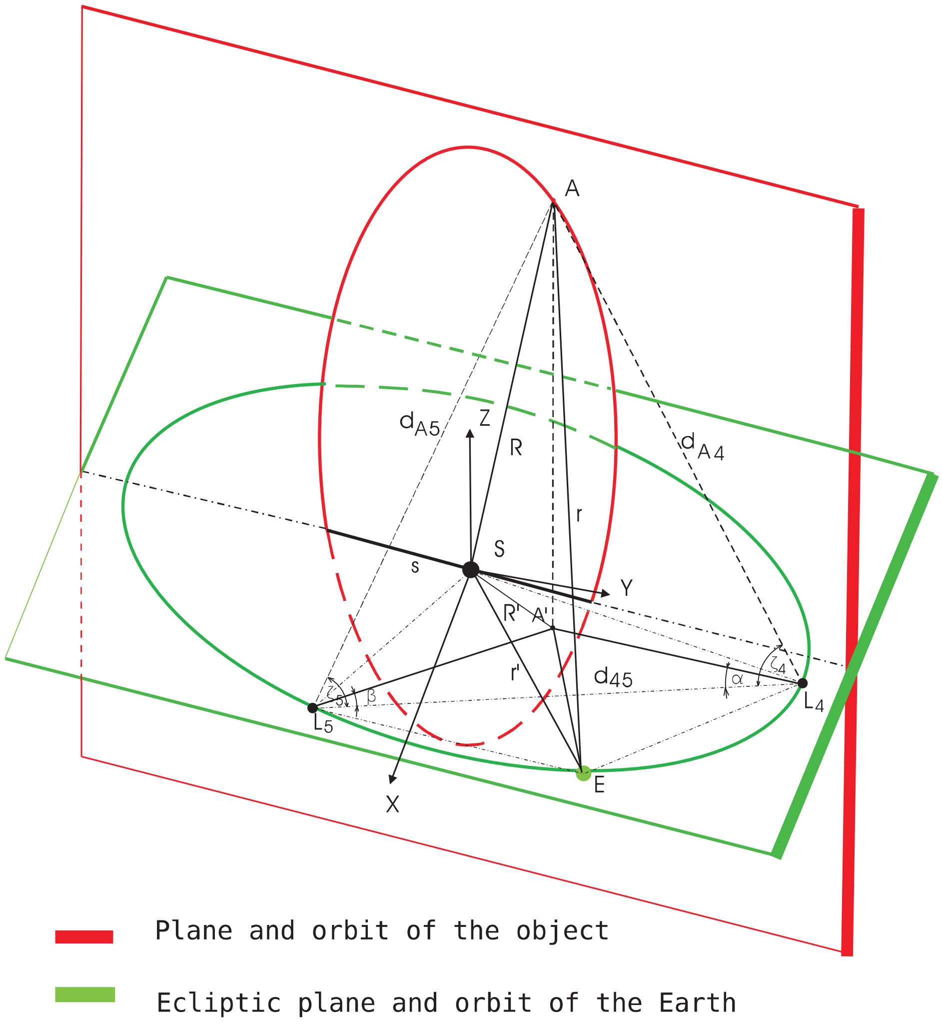

Figure 1. Oblique view

From the points L_4 and L_5 the angles are measured in the plane of the Ecliptic (α in L_4 and β in L_5) and the angles of elevation (ζ_4 at L_4 and ζ_5 at L_5). For the orientation see Fig. 2[2]

2.1 Calculation of the distances d_{A4} and d_{A5} in the triangle $\triangle : AL_4L_5$:

Here the telescope in L_4 (angle α) is orientated with $0°$ in the direction to L_5 and the angle counts clockwise. The telescope in L_5 (angle β) is orientated with $0°$ and counts conterclockwise. Therefore it is possible that α and β can accept values between $0°$ and $360°$ (see Fig. 2).

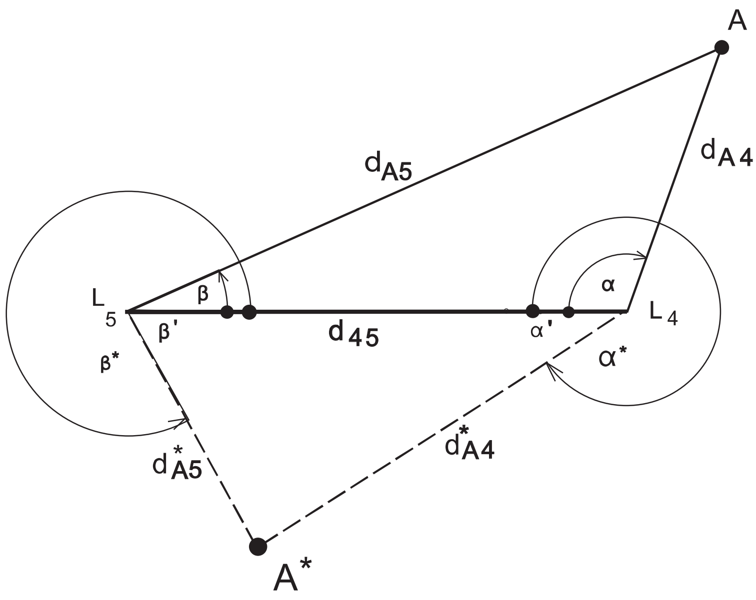

Figure 2. Orientation of the angles

For angles α^* and β^*, between 270° and 360° we calculate with the angles $\alpha' = 360^\circ - \alpha^*$ respectively $\beta' = 360^\circ - \beta^*$.

Known are the distances between the Lagrangian points L_4 and L_5 and also the distances L_4 and L_5 from the Earth ($L_4 - L_5 = d_{45} = \sqrt{3}$ AU). It is assumed that the necessery angles α and β are determined with very high precision simultaneous.

We observe the angles α and β with errors $\Delta\alpha$ and $\Delta\beta$. Also we can observe the angles between the the two Lagrangian points and the Earth. These observations are "surplus measures", because the condition $\alpha + \beta = (\alpha' + 30^\circ) + (\beta' + 30^\circ)$ exists (α' resp. β' are the angles object – Lagrangian point – Earth).

First we have to calculate the distances d_{A4} = distance L_4 – object and d_{A5} = distance L_5 – object. The calculation is done in the plane of the Ecliptic (see Equ. (1) and Equ. (2)).

$$d_{A4} = d_{45} \frac{\sin \beta}{\sin(\alpha + \beta)} \tag{1}$$

and:

$$d_{A5} = d_{45} \frac{\sin \alpha}{\sin(\alpha + \beta)} \tag{2}$$

2.2 Distance AE by mean of the angles α and β .

We calculate now the distance Earth – object r' in the plane of the Ecliptic by the calculated angles η and ζ in the triangle Earth – L_4– object, resp. the trian-

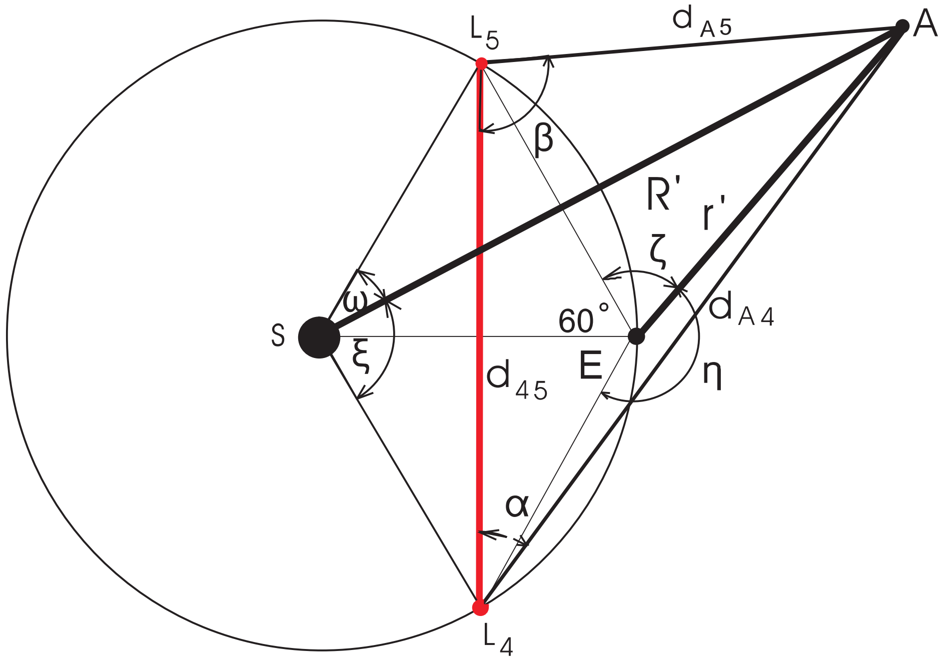

Figure 3. Calculation of the distances d_{A4} and d_{A5}

gle Earth – L_5- object (see Equs. (3) and (4) also see Fig. 3). The real distance Earth – object now we can calculate with the measured angles of elevation ζ_4 in L_4 resp. ζ_5 in L_5.

In the same manner we can calculate the distance R' (Sun – object) by the calculated angles ξ and ω in the Plane of the Ecliptic. (see Equs. (3) and (6) also Fig. 3) Also we can calculate the real distance R by the angles ζ_4 and ζ_5.

It is useful to make these calculations by the distance $L_4 - L_5 = d_{45} = \sqrt{3}$ AU, because this distance is the longest in this configuration and by this kind of calculation we achieve the best values (All values of angles are given in degrees, because the telescopes should show degrees). In order to calculate the angles η and ζ, which are necessary to determine the distance Earth – object resp. the angles ξ and ω for the distance Sun – object , there is a distinction necessary, because the angles α and β can be smaller or bigger than $30°$.

The values of r' in the plane of the ecliptic are:

$$r' = d_{A4} \frac{\sin(\alpha - 30°)}{\sin \eta} \tag{3}$$

and:

$$r' = d_{A5} \frac{\sin(\beta - 30°)}{\sin \zeta} \tag{4}$$

now we can set Equ. (3) = Equ. (4):

$$d_{A4} \frac{\sin(\alpha - 30°)}{\sin \eta} = d_{A5} \frac{\sin(\beta - 30°)}{\sin \zeta} \tag{5}$$

Also we can calculate:

$$R' = d_{A4} \frac{\sin(\alpha + 30^\circ)}{\sin \xi} \tag{6}$$

and

$$R' = d_{A5} \frac{\sin(\beta + 30^\circ)}{\sin \omega} \tag{7}$$

and in similiar manner: Equ. (6) = Equ. (7)

$$d_{A4} \frac{\sin(\alpha + 30^\circ)}{\sin \xi} = d_{A5} \frac{\sin(\beta + 30^\circ)}{\sin \omega} \tag{8}$$

Importing auxiliar values y, h_1,h_2 we get the following equations:

$$y = \frac{\sin \beta}{\sin \alpha} \tag{9}$$

$$h_1 = \frac{\sin(\alpha - 30^\circ)}{\sin(\beta - 30^\circ)} \tag{10}$$

$$h_2 = \frac{\sin(\alpha + 30^\circ)}{\sin(\beta + 30^\circ)} \tag{11}$$

Now we can calculate the following equations for the angles η and ζ by mean of Equ. (3) and Equ. (4)

$$\tan \eta = \frac{\sqrt{3}\, y\, h_1}{y\, h_1 - 2} \tag{12}$$

and

$$\tan \zeta = \frac{\sqrt{3}}{1 - 2\, y\, h_1} \tag{13}$$

and for the angles ξ and ω by mean of Equ. (6) and Equ. (7).

$$\tan \xi = \frac{\sqrt{3}\, y\, h_2}{y\, h_2 - 2} \tag{14}$$

and:

$$\tan \omega = \frac{\sqrt{3}}{1 - 2\, y\, h_2} \tag{15}$$

With these angles from the Equ. (12) or Equ. (13) and Equ. (3) or Equ. (4) now we can calculate r' and R' by the equations Equ. (6) or Equ. (7) by the equations: Equ. (14) or Equ. (15). The possibility to calculate r' and R' by two calculations should be used in every case in order to control the calculations[3].

2.3 Observation of the angles η and ζ from the Earth.

An additional possibility to determine the distance Earth – object directly from the Earth. This are the angles η' and ζ' (see Fig. 3). After reduction of these angles to the center of the Earth we can find the angles η and ζ. From this indirect determination and the values of the angles by measuring from L_4 and L_5now we can calculate an average value and so we have more accurate values.

2.3.1 Another possibility is the calculation with the Cosine theorem.

With the triangles: L_4 – Sun – object, or L_5 – Sun – object it is possible to calculate R' (distance Sun – object, see Fig. 3) Also it is possible to calculate r' (distance Earth - object) with the triangle L_4 – Earth – object, or the triangle L_5 – Earth – object . The distances d_{A4} and d_{A5} are known from the calculations from chapter 1.2. The distance Earth - L_4 and Earth $-L_5$ is known. See Equ. (1) and Equ. (2)

$$r' = \sqrt{d_{A4}^2 + 1 - 2\, d_{A4}\, \cos(\alpha - 30°)} \tag{16}$$

$$R' = \sqrt{d_{A4}^2 + 1 - 2\, d_{A4}\, \cos(\alpha + 30°)} \tag{17}$$

or:

$$r' = \sqrt{d_{A5}^2 + 1 - 2\, d_{A5}\, \cos(\beta - 30°)} \tag{18}$$

$$R' = \sqrt{d_{A5}^2 + 1 - 2\, d_{A5}\, \cos(\beta + 30°)} \tag{19}$$

We do not use this kind of calculation because of the minor precision o the square root.

2.4 Determination of the distance AE = r and R from r' and R' in the plane of the Ecliptic.

Now we can calculate by the measured elevation – angles ζ_4 in L_4 and ζ_5 in L_5 the distances r and R: (See Equ. (25) or Equ. (26) for the distance r and Equ. (28) or Equ. (29) for R)

$$h = d_{A4} \tan \zeta_4 \tag{20}$$

or

$$h = d_{A5} \tan \zeta_5 \tag{21}$$

By the calculation of d_{A4} rsp. d_{A5} from Equ. (1) and Equ. (2) we can find:

$$h = d_{45} \frac{\sin \alpha}{\sin(\alpha + \beta)} \tan \zeta_4 \tag{22}$$

or

$$h = d_{45} \frac{\sin\beta}{\sin(\alpha+\beta)} \tan\zeta_5 \tag{23}$$

and with:

$$r^2 = r'^2 + h^2 \tag{24}$$

$$r = \sqrt{r'^2 + d_{A4}^2 \tan^2 \zeta_4} \tag{25}$$

or

$$r = \sqrt{r'^2 + d_{A5}^2 \tan^2 \zeta_5} \tag{26}$$

and for the distance Sun – object:

$$R^2 = R'^2 + h^2 \tag{27}$$

$$R = \sqrt{R'^2 + d_{A4}^2 \tan^2 \zeta_4} \tag{28}$$

or

$$R = \sqrt{R'^2 + d_{A5}^2 \tan^2 \zeta_5} \tag{29}$$

The values for d_{A4} and d_{A5} ar known from Equ. (1) and Equ. (2).

For small values of the angles ζ_i we cannot find exact values for r and R , but that means only, that the position of the object lies nearly in the plane of the Ecliptic, therefore we can say that the distances at this moment are: $R \sim R'$ and $r \sim r'$. [4]

3. Calculation of the elements of orbit of the object

$$\varphi_A = v_E + (60^\circ - \omega) \tag{30}$$

$$\varphi_E = v_E - (60^\circ - \xi) \tag{31}$$

and

$$\varphi_A = v_E - \frac{1}{2}(\omega - \xi) \tag{32}$$

3.1 Calculation of the rectangular coordinates of the object.

Now it is possible to calculate the rectangular coordinates by the angles ξ and ω and the angle φ_A and (see Fig. 4 and Equ. (32)):

Therefore the rectangular coordinates X_A,Y_A, Z_A are now:

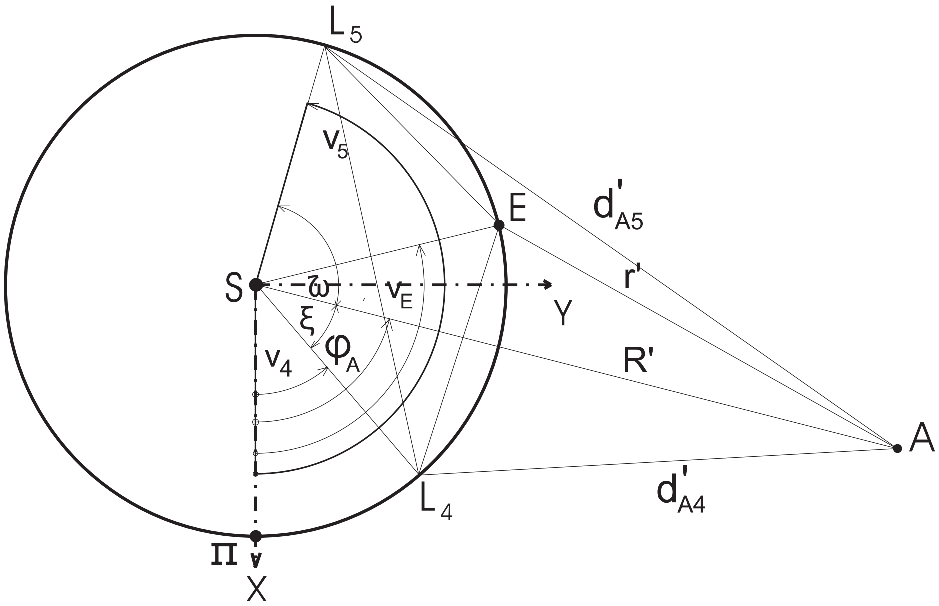

Figure 4. Calculation of the elements of orbit

$$X_A = R' \cos \varphi_A \qquad (33)$$
$$Y_A = R' \sin \varphi_A \qquad (34)$$
$$Z_A = h \qquad (35)$$

The calculation of φ_A can be done by the Mean Anomaly M by Kepler's equation. t_0 is the time of passing the perihel. These values come from the U.S. Naval Observatory, Astronomical Applications Department 2001: Earth's seasons; Equinoxes, Solstices, Perihelon, and Aphelion 1–2-2005 . (e.g for 2005 January 2.at 1h UT).

v_E is the True Anomaly of the earth in the moment of observation.

3.2 Determination of the plane of orbit

From the observation of two or more locations of the asteroid we can calculate by the distances R' and the elevation h above the plane of Ecliptic the equation of the plane of orbit in rectangular and polar coordinates as follows:

Because the Sun as origin of the system lies in the plane of orbit, we have the equation:

$$a\,x + b\,y + z = 0 \qquad (36)$$

With two values of the vector $R\ (X, Y, Z)$ (see Equ. (33) to Equ. (35)) the coefficients of the plane of orbit are:

$$a = \frac{Y_{A1}\,h_2 - Y_{A2}\,h_1}{X_{A1}\,Y_{A2} - X_{A2}\,Y_{A1}} \tag{37}$$

$$b = \frac{X_{A1}\,h_2 - X_{A2}\,h_1}{X_{A1}\,Y_{A2}\,Y_{A1}} \tag{38}$$

To calculate the trace of the plane of orbit with the Plane of the Ecliptic $z = 0$ so the equation of the plane is: $ax + by = 0$, or with the values for a and b:

$$(Y_{A1}\,h_2 - Y_{A2}\,h_1)\,x + (X_{A1}\,h_2 - X_{A2}\,h_1)\,y = 0 \tag{39}$$

After the first observation it is possible to make more measures of the angles. So we do a smoothing of the values with the method of Least Squares. We have $n - 2$ more values then we need. Therefore we can calculate the values of the parameters a and b more and more exactly.

For n measurements the parameters a and b are therefore:

$$a = \frac{\sum_{i=1}^{n} x_i\,y_i \sum_{i=1}^{n} y_i\,z_i - \sum_{i=1}^{n} x_i\,z_i \sum_{i=1}^{n} y_i\,y_i}{D} \tag{40}$$

$$b = \frac{\sum_{i=1}^{n} x_i\,x_i \sum_{i=1}^{n} y_i\,z_i - \sum_{i=1}^{n} x_i\,y_i \sum_{i=1}^{n} x_i\,z_i}{D} \tag{41}$$

$$D = \left(\sum_{i=1}^{n} x_i\,y_i\right)^2 - \sum_{i=1}^{n} x_i\,x_i \sum_{i=1}^{n} y_i\,y_i \tag{42}$$

The gradient of the plane of orbit can calculated from two observations as follows:

$$\vec{N} = \vec{R}_i \times \vec{R}_{i+1} \tag{43}$$

and so we get for the gradient:

$$i = \text{arc cos}\left(\frac{\vec{N}_3}{|\vec{N}|}\right) \tag{44}$$

3.3 Calculation of the other necessary parameters of the orbit

We assume: After the first measurement of the angles further observations follow. By two measures at times t_0 and t_1 with a relatively short difference of time, now it is possible to calculate the velocity of the asteroid as follows:

$$\vec{v}_{12} = \frac{\vec{R}_2 - \vec{R}_1}{t_2 - t_1} \tag{45}$$

and the vector of angular momentum $\vec{C}$:

$$\vec{C} = \vec{R}_{12} \times \vec{v}_{12} \tag{46}$$

Also we calculate the Runge – Lenz Vector $\overrightarrow{P}$:

$$\vec{P} = \vec{v}_{12} \times \vec{C} - k^2 \frac{\vec{r}_{12}}{|\vec{r}_{12}|} \tag{47}$$

The distance of Perihelion ω now is (see Equ. (48))

$$\omega = \text{arc cos} \left(\frac{\vec{P} \cdot \vec{N}}{|\vec{P}||\vec{N}|} \right) \tag{48}$$

for $N \neq 0$. If $N = 0$ ist, the slope of the orbit $i = 0$.

By substituting the Gauss – constant k and the value of the angular momentum $\overrightarrow{C}$ it is possible to calculate by the following equation the parameter p (see Equ. (49)) of the equation of orbit:

$$p = \left(\frac{|C|}{k} \right)^2 \tag{49}$$

From the initial values of $\overrightarrow{R}$ and $\overrightarrow{v_0}$ we can calculate the specific energy E_0 by

$$E_0 = \frac{1}{2} |v_{12}|^2 - \frac{k^2}{|\vec{r}_{12}|} \tag{50}$$

With this value E_0 from Equ. (50) we can calculate the excentricity of the orbit e.

$$e = 1 + 2 \frac{E_0 \, |\vec{C}|^2}{k^4} \tag{51}$$

and the major axis a by Equ. (52)

$$a = -\frac{1}{E} \tag{52}$$

The further values of the orbit: Ω (see Equ. (53)) and ω (see Equ. (54)) we also can calculate by the equations:

$$\Omega = \text{arc cos} \left(\frac{|\vec{N}_3|}{|\vec{N}|} \right) \tag{53}$$

$$\omega = \text{arc cos} \left(\frac{\vec{P} \cdot \vec{N}}{|\vec{P} \, \vec{N}|} \right) \tag{54}$$

and again (see Equ. (55))

$$i = \text{arc}\cos\left(\frac{|\vec{C}_3|}{|\vec{C}|}\right) \tag{55}$$

So we have all elemtes of the orbit of the asteroid by two observations of the angles α and β.

This method was preferred, because the for this method necessary basis is with $\sqrt{3}$ AU the longest basis that could be easy realised. (The ocillation of the Lagrangian Points is very small and could be taken in mind by an error calculation.). Also all observations are free from influences of the earth's atmosphere.

4. Conclusion

By observation of objects by satellites positioned in the Lagrangian points and simultaneous determination of the angles α and β and it is possible to calculate the distances r (Earth – object) and R (Sun – object) only by one calculation. By two simultaneous observations we can calculate all other elements of orbit of the detected methods very exactly.

Notes

1. After a meeting between R. Dvorak and W. Grandl about the possibility to observe objects from the Lagranian points L_4 and L_5 by satellites.
2. Note: This method is not applicable for angles α and $\beta = 90°$, or $270°$. If α and $\beta = 0°$, $\alpha = 180°$, $\beta = 0°$ or $\alpha = 0°$, $\beta = 180°$, or α and $\beta = 180°$, the object is situated on the straight line $L_4 – L_5$ and it is not possible to calculate the position for any angle of elevation ζ_i. (The object lies in a plane, perpendicular to the plane of the Ecliptic, wich includes the straight line $L_4 - L_5$)
3. The possibility to use the triangles Earth – object – L_4 resp. Earth – object – L_5 in order to determine r' should not be used, because the distance Earth – L_4 resp. Earth – L_5 is only 1 AU and therefore smaller then the distance $L_4 - L_5 = \sqrt{3}$ AU.
4. In the case that the position of the object lies in a plane perpendicular to the plane of the Ecliptic through $L_4$and L_5, the angles α and β are 0° or 180° and therefore there exists no possibility to determine the distances r', and R' but we can measure the angles at an other time t_i

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