A NEW DETERMINATION OF THE FUNDAMENTAL FREQUENCIES IN OUR SOLAR SYSTEM

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Abstract In this investigation we integrated the orbits of the planets of our Solar System over 1 billion years (-500 million back and 500 million into the future) based on the Newtonian model of the Solar System including the 8 major planets Mercury to Neptune. For the integration we used the very stable and highly precise Lie-Integration method. The output of the simulation were the osculating orbital elements, stored every 66,6 years. We transformed the data set to Laplace-Lagrange variables and analyzed it using windowed fourier transformation with a windowsize of 10 million years, overlapping with 1 million years. In this paper we present the maximum and minimum values of the orbital elements of the planets and give the time varying fundamental frequencies of all eight planets.

Keywords: Solar System - Fundamental Frequencies - Windowed Fourier Transform

1. Introduction

The numerical simulation of the dynamics of our Solar System on computer systems is a field not older than 50 years. Various people have been working on it: Eckert et al. (1951) integrated the system, using the 5 outer most planets over 3.5×10^2 years. Cohen & Hubbard (1973), Kinoshita & Nakai (1984), Applegate et al. (1986), Sussman & Wisdom (1988), Nobili et al. (1989), Nakai & Kinoshita & (1995) used the same model (5 planets) but varied the stepsize (between 0.5 and 40 days) and increased the integration time of the simulation. Newhall et al. (1983) integrated the whole system of major planets (9) using a very small stepsize (0.25 days), so did Richardson & Walker (1989) (0.5 days), Quinn et al. (1991) (0.75 days), or Sussman and Wisdom (1992) using a stepsize of 7.2 days but integrated the whole system for 10^9 years. Ito et al. (1996) increased the simulation time up to 4.3×10^{10} years but only took the outer four planets into account. Duncan & Lissauer (1998) used Venus to Neptune in their model and integrated the system for 10^9 years.

The main question is still open: How long will our Solar System be stable inspite of its chaotical nature? Are there resonances, which will kick one of our planets from its nowadays known orbit, thus leading to a completely different configuration of our Solar System? Laskar (1990) used a semianalytical solution and showed, that there are no secular variations in the semi major axes. He integrated the Solar System in his paper for 200 Myr years and found secular resonances between the precession periods of Earth and Mars, $2(g_4 - g_3) - (s_4 - s_3)$ and between the main secular frequencies associated with the perihelia and nodes of the planets (Mercury and Jupiter, $(g_1 - g_5) - (s_1 - s_2)$). In this paper we extended the integration time for the full system up to 10^9 years to see the variation of the fundamental frequencies and the possible chaotic nature of our planetary system.

This paper is organized as follows: In the second section we give an overview of the methods used to produce the results outlined in this paper. We introduce the reader into the windowed fourier transform (WFT) - also known as Gabor transform, a special topic from wavelet analysis, and show the mechanism, how we separated the spectral lines in the corresponding power spectrum. In the third section we summarize the evolution of the elements of the planets during 1 billion years of integration time. We present the maximum and minimum values of the eccentricities and semi major axes of the main planets of our Solar System and take a look on the evolution of their characteristic orbital elements in short. The fourth section introduces the frame work of Laplace-Lagrange and defines the fundamental frequencies based on the Laplace-Lagrangian (h, k, p, q) coordinate system. The fifth section reflects the main results of the present work and compares them with those found by Laskar and other results found in literature.

2. Methods

To calculate the motions of the eight major planets we used a standard Newtonian model and integrated the full system of nonlinear equations of motions using the Lie - integration method (Hanslmeier & Dvorak, 1984) in the Cartesian reference frame. Starting from present time we simulated the system 500 million years into the back and 500 million years into the future and collected the positions and velocities referring to the classical orbital elements of all planets every 66,6 years. Thus the time span of 1 billion years resulted in 15 million "observations" of their orbital elements leading to a multivariate time series of 90 million data points, which leads to a set of 720 million real numbers, which is necessary to represent the evolution of our Solar System (eccentricity e_k, semi major axes a_k, inclination i_k, argument of pericenter ω_k, longitude of the ascending node Ω_k and mean anomaly M_k, where $k = 1$ (Mercury), $\ldots, 8$ (Neptune)).

The initial values for the simulation were taken from the JPL (1st August, 1965), the effective computation time just for the integration was about 1 year. To organize and analyze the resulting data set we wrote sophisticated algorithms in *Mathematica* and Fortran. The method used for the frequency analysis of the time series was the approximated windowed fourier transformation (WFT) also known as Gabor Transform and an exponential fitting and optimization algorithm in the power spectrum. For the analysis we split the data set into pieces of equal length – 10 million years per unit, overlapping with 1 million years. Thus we were able to get a time evolution of the frequency space of the system resulting in 1000 data points in time per element, frequency and planet.

To cope with the known problem in Celestial Mechanics, when doing frequency analysis of the orbital elements, namely the mixture between high and low frequencies – resulting from the chaotic structure of the system, we tried various filter methods to smooth the spectrum (Hanning, Hamming and Blackman - Tukey windows) and compared with the respective methods, when smoothing in the time domain, before starting the frequency analysis on the whole data set. In the end we decided to use a linear filter in the time domain, to get rid of high oscillation components. The second problem is, that there are actually no constant frequencies in the orbital elements (because of the nonlinear character of the system, they are time and amplitude dependent). Thus every method based on Fourier analysis will fail, as it was invented for signals of infinite length and a constant frequency domain. This problem can be solved using the WFT: When we split the data set into smaller pieces, we can regard the elements being constant within those lag windows: But using a lag window, which is to small will not cover the frequency range, we are interested in, using a lag window, which is to big, will result in a dispersion of the frequencies in the power spectrum of the signal. So it is a non trivial and difficult task to find the right tuning for the parameters (size of lag windows, overlapping & filtering) to cope with this kind of problems. Other approaches doing frequency analysis in Celestial Mechanics were done by e.g. Laskar (1993) and Chapront (1995).

Our approach used in this paper was to use the WFT on the one hand to cope with the time dependence of the frequencies and to refit the frequency lines in the power spectrum on the other hand using an exponential fitting model. Thus we considered a set of lines around a peak in the power spectrum as belonging to the same line and fitted an exponential curve through it to get a more accurate and not dispersed form of each spectral line. The resulting fitting model was maximized and so we could easily improve the accuracy of the determination of the frequency.

The windowed fourier analysis is the simplest way to extract both the frequency and its respective time evolution of a time series, giving us insights of the evolution of the signal in the time and frequency domain. The background or theory can be found in Wavelet analysis, where the WFT is based on the implementation of Gabor functions. In our approach we approximated the method and used lag windows of equal length of 10 million years (150 000 data points per element and planet) and used a simple but fast FFT procedure to obtain the power spectrum within the window. The overlapping of the windows was 1 million years, thus leading to 1000 frequency spectra for each element and planet. In the next step we used a self written sorting algorithm, which extracted the spectra lines according to their amplitudes and fitted each spectral line within using an exponential model. We calculated the maxima in the models of the first dominant 100 frequencies in each element and planet and searched within for the set of fundamental frequencies according to a reference list given by Bretagnon (1984), Laskar (1992) and Gamsjäger (2002). To check our results, we visualized random samples and overlooked our results to proof the correctness of the automatized identification method of the spectral lines.

3. Evolution of the Orbital Elements

The evolution of the semi major axes of the eight major planets over the integration period is almost constant, which is due to the quasi conservation of energy of every single planet, because of the smallness of the inclinations and eccentricities. This is also a first indicator for the accuracy of the integration method. There are no slopes or gradients in the data set – regarding long time scales. The semi major axes just oscillate around their mean values with small amplitudes. Mercury, the inner most planet moves at 0.39 AU over the whole time span, Neptune – the outer most planet stays at approximately 30 AU. Fig. 1 takes a closer look onto the evolution of the semi major axes of Mercury - the most influenced body in our Solar System 500 million years ago (left graph) vs. 500 million years in the future (right graph). One can see, that the evolution of the semi major axes still lies in the range of present time (see Table 1) but that there are large and chaotic variations, which seem not to follow any periodic behaviour.

In contrast to the nearby constant semi major axis of our planets the eccentricities show large variations with large periods. This effect raises, when going from the outer Solar to the inner Solar System and becomes largest, when arriving at the inner most planet Mercury, where the eccentricity may lie between ~ 0.08 and ~ 0.3. In Fig. 2 one can see the coupling between the eccentric-

Table 1. The maximum and minimum values of the orbital elements of our planets. The semi major axes are given in AU, the eccentricities are numeric, the inclinations are given in degrees.

Planet	a_{max}	a_{min}	e_{max}	e_{min}	i_{max}	i_{min}
Mercury	0.3871	0.3870	0.30120	0.078730	11.40720	0.17599
Venus	0.7233	0.7234	0.07709	0.000020	4.91516	0.00246
Earth	1.00003	0.9998	0.06753	0.000083	4.49496	0.00075
Mars	1.5239	1.5235	0.13110	0.000080	8.60320	0.00291
Jupiter	5.2050	5.2012	0.06188	0.025140	2.06597	0.55867
Saturn	9.5927	9.5128	0.08959	0.007423	2.60186	0.56037
Uranus	19.3351	19.0989	0.07834	0.000095	2.73889	0.42615
Neptune	30.4325	29.9101	0.02316	0.000024	2.38176	0.77977

ities of Earth and Venus due to the 13:8 mean motion resonance and also the coupling between Jupiter and Saturn (due to the 5:2 mean motion resonance) as an example for the outer planetary system: one minima of the first leads to a maxima of the second and vice versa. These resonances stabilize the system over the whole integration time. The mean values and the minima and maxima of the eccentricities can be found in Table 1.

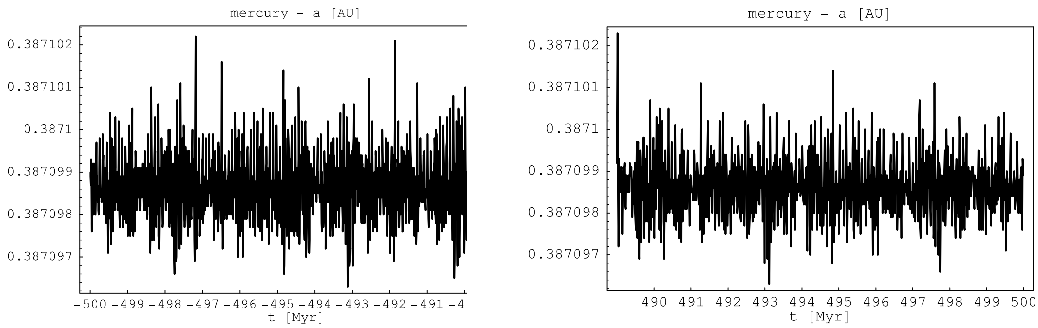

Figure 1. The evolution of the semi major axes of mercury (a_1) 500 million years ago and 500 million years in the future. Although there are chaotic variations around a constant mean value, there is no secular trend, which indicates the stability of the integration method (Lie - integrator).

The inclinations of the orbits of the planets show a similar resonant behaviour like those found in the eccentricities. The influence of the other planets in contrast seems to be more dominant, than e.g. in the eccentricities, the maximum and minimum values of the inclinations of the 8 major planets can also be found in Tab. 1, two representatives of the outer system (Uranus vs. Neptune) are given in Fig. 3 (left graph), another two representatives of the inner system (Mercury vs. Mars) are given in the right graph.

Resonances in our Solar System may stabilize or destabilize the system. Looking to the evolution of the inner and outer planets one can see the coupling of the orbital elements (e and i). Although we are not able to calculate

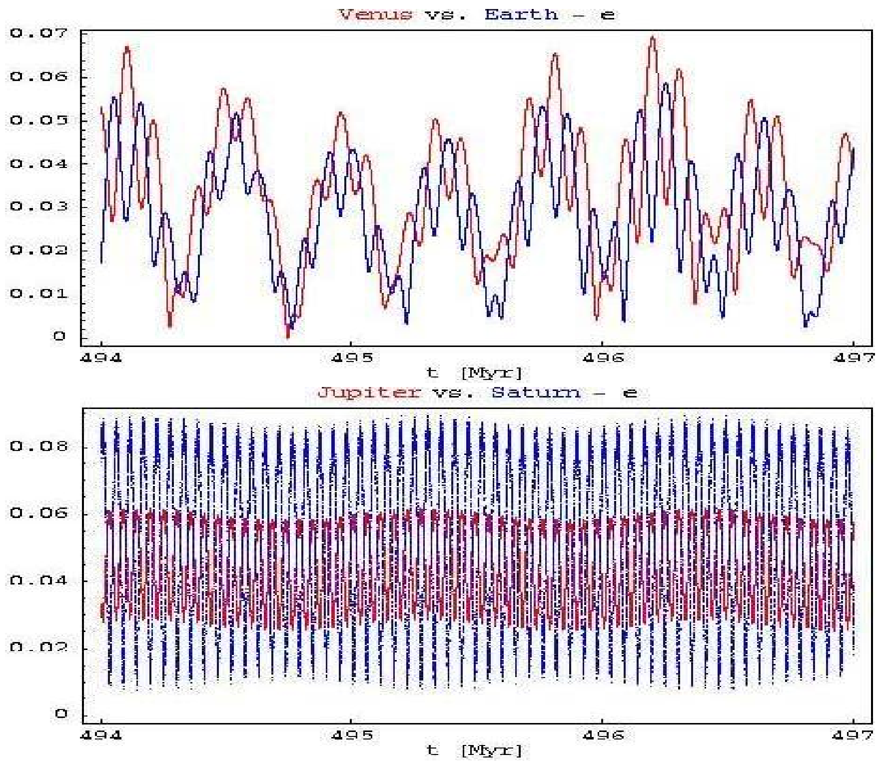

Figure 2. The evolution of the eccentricities of Venus and Earth (upper) and Jupiter and Saturn (lower) over the last 3 million years of the integration time. One can see the coupling between the planets also over the whole time span (upper:the curve with the higher amplitudes belongs to Venus, lower:the curve with the higher amplitudes belongs to Saturn).

the real positions and velocities of all planets for long time scales, it is important to see, that those resonances found last for long time scales. The answer to the question of stability in our Solar System thus needs a better understanding of the resonances in it – stabilizing, as one can see in the coupling effect of the planets or destabilizing, like those found by Laskar (1990).

4. Canonical Elements

The question, if our Solar System is stable or not needs new analytical results and of course a highly accurate and precise numerical investigation of the system. There have been several approaches to derive better and higher order approximations for the analytical part of the solutions. It was first studied by Laplace in the 18th century. He found out, that the semimajor-axes of the planets of our Solar System suffer only from periodic changes up to first order. Poincaré showed, that the formal series of small parameters, like the eccentricities, the inclinations or the masses of the planets are not convergent due to the problem of small divisors.

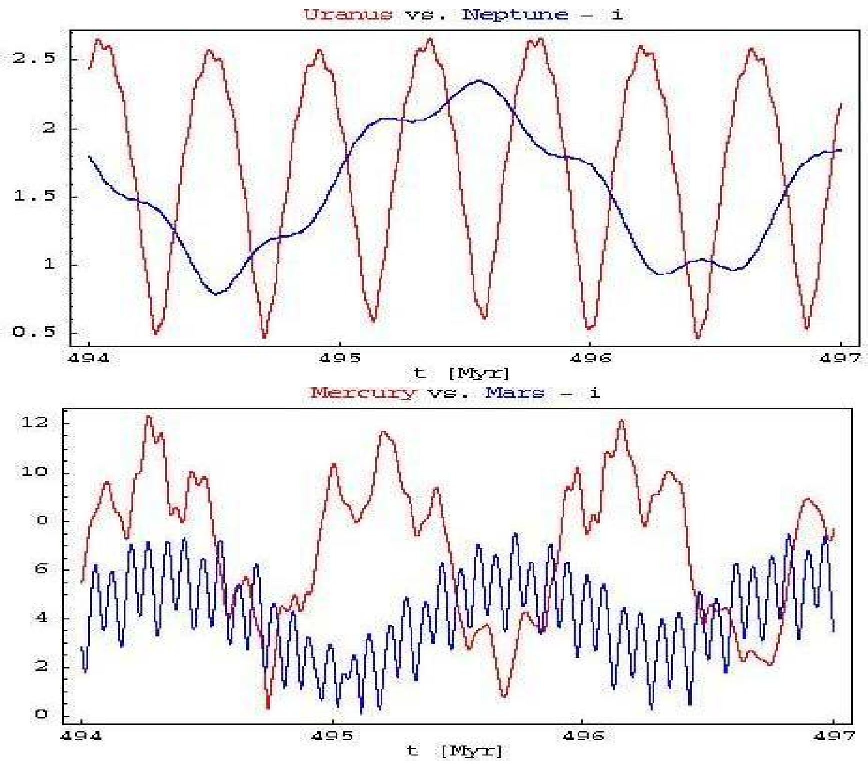

Figure 3. The evolution of the inclinations in time of Uranus and Neptune (upper) and Mercury and Mars (lower) over the last 3 million years of the whole integration time (lower:the curve showing a higher frequency represents Neptune, upper: the curve with the higher amplitudes belongs to Mercury).

Nowadays we are able to find good analytic approximations of the solutions, which allow us to reconstruct the shifting of the proper mode frequencies and the combinations of them, but it is still a problem to give long time predictions of the evolution of our Solar System. Analytical approaches may lead to results, which are good for millions of years and with numerical techniques one may integrate over billion of years, like in this paper. But without the knowing of the structure of the solution, the exact resonance conditions for the inner and outer planets, we will not be able to give a final answer to the question, if our Solar System can be regarded as stable or not. In this chapter we introduce the results of the theory of Laplace-Lagrange. We transform the orbital elements to better ones, canonical and not singular. The benefit is the better treatement when doing frequency analysis in the time depending orbital elements.

Using secular perturbation theory in the N-body system with one heavy mass in the center of gravitiy it is possible to derive the Laplace – Lagrange solution of the system, given in Laplace-Lagrange coordinates (h, k, p, q) defined as:

$$h_j = \sum_{i=1}^{N} e_{j,i} \sin\left(g_i t + \beta_i\right), \quad (1)$$

$$k_j = \sum_{i=1}^{N} e_{j,i} \cos\left(g_i t + \beta_i\right), \quad (2)$$

$$p_j = \sum_{i=1}^{N} I_{j,i} \sin\left(s_i t + \gamma_i\right), \quad (3)$$

$$q_j = \sum_{i=1}^{N} I_{j,i} \cos\left(s_i t + \gamma_i\right), \quad (4)$$

which implies stability for the system for all times assuming small values for the eccentricities $e_{j,i}$ and for the inclinations $I_{j,i}$. The conjugated variables (h_j,k_j) and (p_j, q_j) respectively are the vertical and horizontal components of the eccentricities and the inclinations, so called Lagrange-Laplace coordinates are defined via the relations:

$$h_j = e_j \sin\left(\omega_j + \Omega_j\right),\ k_j = e_j \cos\left(\omega_j + \Omega_j\right), \quad (5)$$

and

$$p_j = \sin\left(I_j/2\right) \sin\Omega_j,\ q_j = \sin\left(I_j\right) \cos\Omega_j. \quad (6)$$

Here ω_j are the arguments of pericenter and Ω_j are the longitudes of the ascending nodes. The quantities g_j and s_j refer to the fundamental frequencies, the quantities β_i and γ_i are the corresponding phases in the solution of the system. The indices (i, j) refer to the bodies in the system (Mercury = 1, . . ., Neptune = 8). The advantage of using this variables is the fact, that they are canonical conjugated to each other and can not become singular. The orbital elements e_j and I_j can be easily derived via the equations:

$$e_j = \sqrt{h_j^2 + k_j^2},\ I_j = 4\sqrt{p_j^2 + q_j^2}. \quad (7)$$

The solution of Laplace-Lagrange given here to introduce the idea of the fundamental frequencies used in the proceding sections, is based on a secular and second order perturbation theory (in e and I) and neglects nonlinear effects, which lead to chaotic phenomena in our Solar System. Looking to equations (1) - (4) one can see that the elements are bounded and somewhat called linearly stable. But this is not true, when going to higher orders of approximations in the analytical formulas.

5. Resulting Fundamental Frequencies

The frequency analysis in the variables (h, k, p, q) show more or less regular periodic behaviour in the evolution of the elements for the outer planets, complex and irregular evolution in the time series of the elements of the inner planets (see Fig. 4 and Fig. 5), overlapping of different frequencies and beats for example in Mars. The parameters h and k are identical but phase-delayed, which is the same for the canonical conjugates p and q. In principle we will find every fundamental frequency of the planets in the frequency spectrum of the other planets, limited due to the fact, that the basic frequencies of the planets of the outer Solar System are more dominant in the spectra of the planets of the inner Solar System, than vice versa and that frequencies, which can be found in (h, k) may be too small to be found in (p, q) and vice versa (note that this effect can not be described by the Lagrange-Laplace solution, given in (1)-(6)). In fact we did the frequency analysis in all four elements using the WFT method. We searched for the fundamental frequencies in the frequency space of all four elements and planets and averaged corresponding ones including their influence according to their amplitudes. To check consistency we compared the results given by the canonical conjugates and found minor neglectible differences between them.

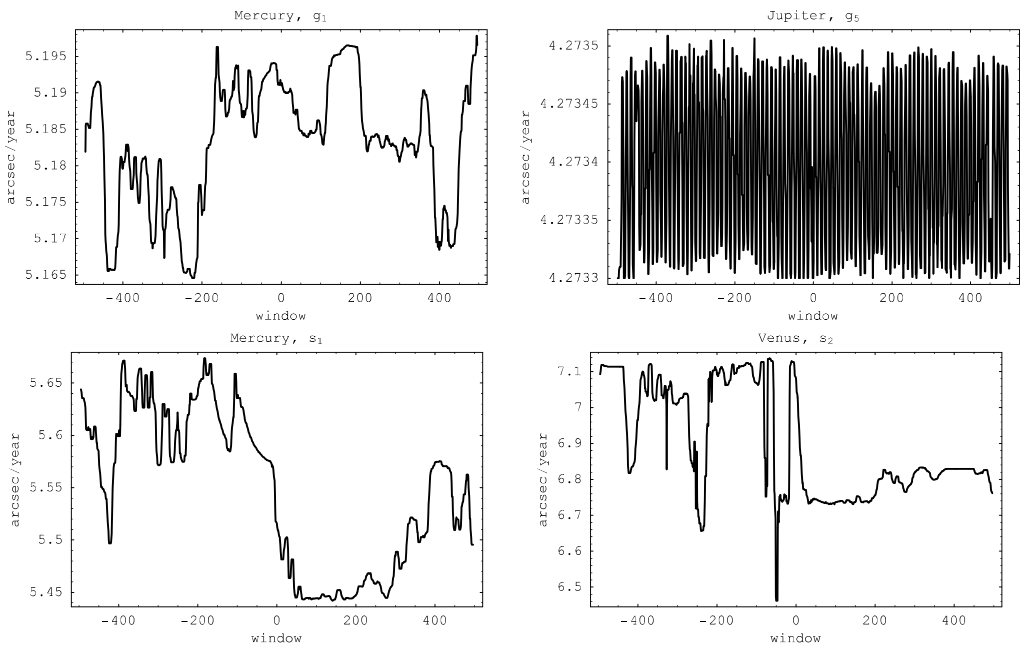

Figure 4. The evolution of the time varying fundamental frequencies g_1 of Mercury (upper left), g_5 of Jupiter (upper right), s_1 of Mercury (lower left) and s_2 of Venus (lower right) over 1 billion years. The samples shown correspond to the critical angle $(g_1 - g_5) - (s_1 - s_2)$.

In Laskar (1990) two angles related to the combinations of the secular frequencies associated with the perihelia and nodes of the planets are responsable for the positive value of the Liapunov exponent in the order of 1/5 million years. Another numerical integration Laskar et al. (1992) confirmed the results found in the previous paper over the time span of 6 million years. The work of Dvorak et. al (2003) has increased the integration time up to 200 million years. Based on an extension of this work we will improve the accuracy of the determination of the critical angles and may find additional ones, when analyzing the fundamental frequency set.

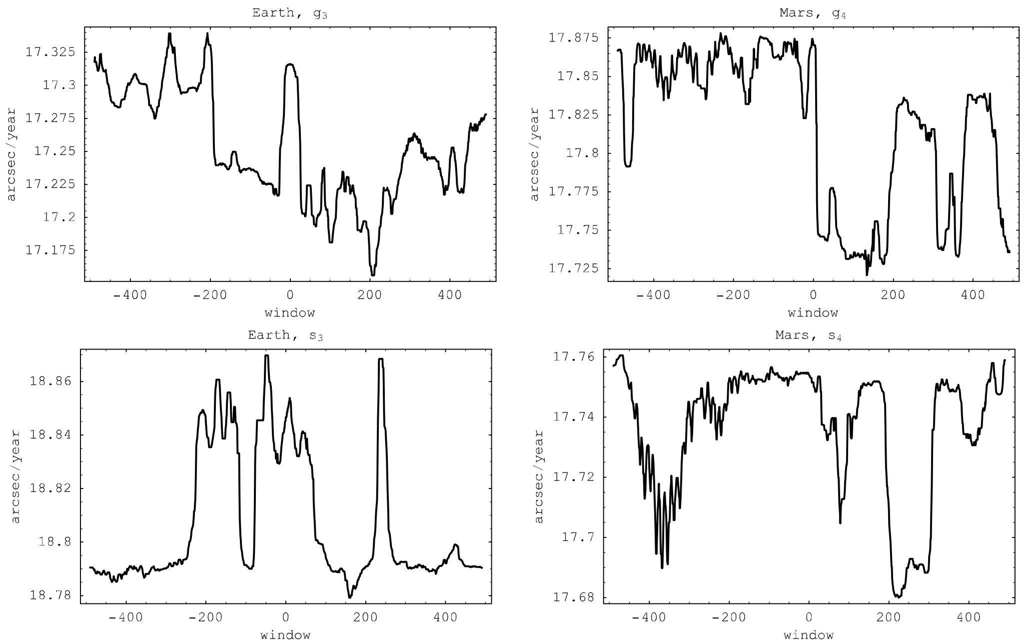

Figure 5. The evolution of the time varying fundamental frequencies g_3 of Earth (upper left), g_4 of Mars (upper right), s_3 of Earth (lower left) and s_4 of Venus (lower right) over 1 billion years in arcseconds per year. The samples shown correspond to the critical angle $2(g_3 - g_4) - (s_3 - s_4)$.

The evolution of the time varying fundamental frequencies g_i of the inner planets over the whole time span can be found in Fig. 6 (upper) and of s_i (lower). The time evolution of the respective frequencies for the outer planets g_i and s_i show no significant variations. The mean values of them over the whole integration time can be found in Tab. 2 (NEW). The standard deviation is small regarding the evolution of the frequencies of the outer planets, it is larger for the inner planetary system. The table compares the results of this work with an analytical work by Lagrange (LAG), a semianalytical approach by Laskar (NGT) and the values found by Gamsjäger (GAMS).

Due to the nonlinear structure of the system, the fundamental frequencies which are constant in the first order approximation of Laplace (see Eq.(1) - (3)) are in reality varying with time. Some of them look like, they are changing randomly (see Fig. 4 and Fig. 5), others look like they follow secular trends or seem to have periodic changes around their mean values. If some of them

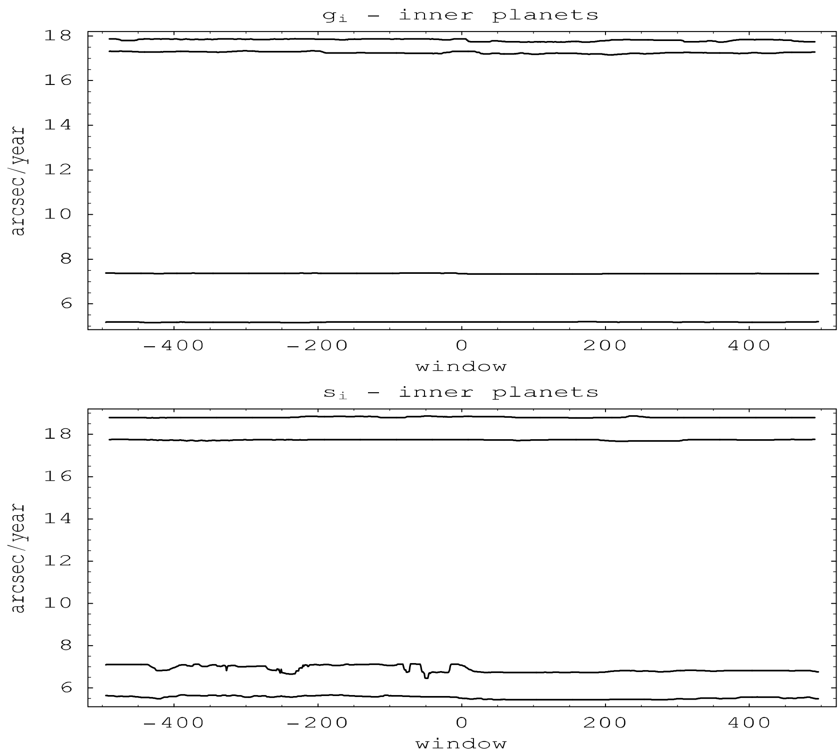

Figure 6. The evolution of the time varying fundamental frequencies g_i (upper panel) and s_i (lower) of the inner planets given in arcseconds per year.

in combination - called critical angles - lead to secular frequencies, their corresponding orbital elements may change from libration to circulation, so that they will cross the separatrix in the phase space - which will directly lead to chaos.

6. Conclusions

Although we were yet not able to confirm the resonant structure of our Solar System, we showed that the system is stable over 1 billion years. There is no planet showing any slightest sign of beeing unstable. The maximum values of the orbital elements also give no evidence, why one of the planets should escape in the next future. There exist a couple of resonances, which stabilize the whole system. The frequency spectrum, particularly the time evolution of the fundamental frequencies of the planets show a very irregular behaviour over the whole time span, if you take a closer look on it. The variances from the mean values are quite big – indicating the chaotical nature of the system. The outer bodies of the system show a more regular behaviour in their time-evolution of the orbital elements and fundamental frequencies (see Fig. 6, lower left and right panels), the inner bodies are highly chaotic but seem to be stabilized by the more massive outer bodies (see Fig. 6, upper left and

Table 2. Fundamental Frequencies of the planets in arcseconds per year. LAG is based on the analytical result of Lagrange (analytical), NGT is the work of Laskar (semianalytical), GAMS presents the results of Gamsjäger (numerical). Our new results are based on windowed frequency analysis with a lag size of 10 million years (1000 lag windows) and average over the elements (h, k, p, q).

Planet	**LAG**	**NGT**	**GAMS**	***NEW***
g_1	5.4615	5.5689	5.2130	5.1832 ± 0.0086
g_2	7.3459	7.4555	7.3343	7.3592 ± 0.0124
g_3	17.3307	17.3769	17.5022	17.2541 ± 0.0419
g_4	18.0042	17.9217	17.8921	17.8176 ± 0.0050
s_1	-5.2007	-5.6043	-5.5010	-5.5467 ± 0.0739
s_2	-6.5701	-7.0530	-6.2230	-6.8978 ± 0.1528
s_3	-18.7455	-18.8499	-18.8574	-18.8069 ± 0.02501
s_4	-17.6358	-17.7614	-17.7167	-17.7363 ± 0.0216
g_5	3.7109	4.2489	4.2567	4.2743 ± 0.00007
g_6	22.2868	27.9606	28.2445	28.2523 ± 0.00006
g_7	2.7014	3.0695	3.0468	3.1075 ± 0.0022
g_8	0.6333	0.6669	0.6727	0.6711 ± 0.00003
s_5	-0.0000	-0.0000	-0.0000	0.0000
s_6	-25.7411	-26.3300	-26.3473	-26.3256 ± 0.00007
s_7	-2.9038	-2.9854	-2.9944	-2.9818 ± 0.00008
s_8	-0.6777	-0.6927	0.7381	-0.6710 ± 0.0001

right panels). The windowed fourier transform is a good tool, when analyzing the time dependent and nonlinear time-evolution of the orbital elements, lag windows of 10 million years overlapping with one million year produced good results. We were not able to confirm the resonances proposed by Laskar (1990) yet, but look forward to find them and maybe additional ones, when using the larger integration time for the simulation of our Solar System.

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