CHAOS IN SIMPLE DYNAMICAL SYSTEMS – THE SITNIKOV PROBLEM

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Abstract It is well known that in low degrees of freedom dynamical systems chaotic behaviour appears. To examine this phenomenon the Sitnikov problem is a very good example which is a special case of the restricted three-body problem. In this paper we investigate the changing of the phase space structure due to the variation of the initial positions of the primaries in the configuration.

Keywords: dynamical systems – chaos – resonances

1. Introduction

The investigation of the dynamical systems in the past 50 years shows that chaotic behaviour appears not only in difficult, many degrees of freedom systems but in simple configurations as well. Therefore one of the most relevant tasks is to study these simple dynamical systems to understand the chaos, and on the other hand it is a good starting point to investigate the more difficult problems.

Out of the simplest and most interesting system in celestial mechanics is the Sitnikov problem. Essentially, it is a special case of the restricted three-body problem. Namely there are two equal masses m_1 and m_2 revolving in Keplerian orbits around each other, a the third massles body m_3 moves on an axis perpendicular to the plane of the primaries through its barycenter.

Mac Millan (1913) [6] showed that in the circular problem, when the primaries revolve on circular orbit, the problem is integrable and the solution is expressed by elliptic integrals. The motion of the masless body is more various when we allow the two primaries to move in eccentric orbits. In this case quasi-periodic and chaotic orbits appear beside the periodic ones. The solution of the problem was first given by Sitnikov in 1960 [9], after that many authors examined the existence of periodic orbits in this configuration. The first mapping model was derived by Liu and Sun (1990) [5], who showed that for small eccentricities the phase space becomes very complex. Perdios and Markellos (1998) [8] studied the stability and the bifurcations in the Sitnikov problem. Dvorak (1993) [1] investigated numerically the problem by using Poincaré's surfaces of section. Martínez Alfaro an Chilart (1993) found that for certain eccentricities the fixed points $z = \dot{z} = 0$ in the center becomes unstable. Kallrath et al. (1997) [4] explored the phase space in detail laying emphasis on resonances. For small eccentricities Hagel (1992) [3] and Faruque (2003) [2] applied perturbation methods and gave an analitical approximation to the problem.

In this study we investigate the phase space structure for different initial conditions, eminently for different initial positions of the primaries. For the visualization of the results we use Poincaré's surfaces of section.

2. Equation of motion

As mentioned above, we investigate the motion of a massles body which moves along a line perpendiclar to the plane of the primaries through their baricenter (see Fig. 1). By introducing suitable units we can write the equation of motion. We choose the total mass of the primaries as mass unit, the rotating period equal to 2π, the semi-major axis of the orbit of the primaries as distance unit (m_1 and m_2), so the Gaussian constant becomes 1. Then the equation of motion of the massles body is

$$\ddot{z} = -\frac{1}{r^3} z, \tag{1}$$

where

$$r = \sqrt{R^2 + z^2}, \qquad R = 1 - e \cos E. \tag{2}$$

R is the distance between the primaries , z is the distance of the massles body from the plane of primaries, e is the eccentricity, and E is the eccentric anomaly, which depends on the time according to Kepler's equation:

$$t - \tau = E - e \sin E. \tag{3}$$

The $\tau = 0$ phase constant corresponds to the pericenter passage at $t = 0$. Since the problem is only one degree of freedom, we can introduce the true anomaly v as for independent variable instead of the time. (See [4].)

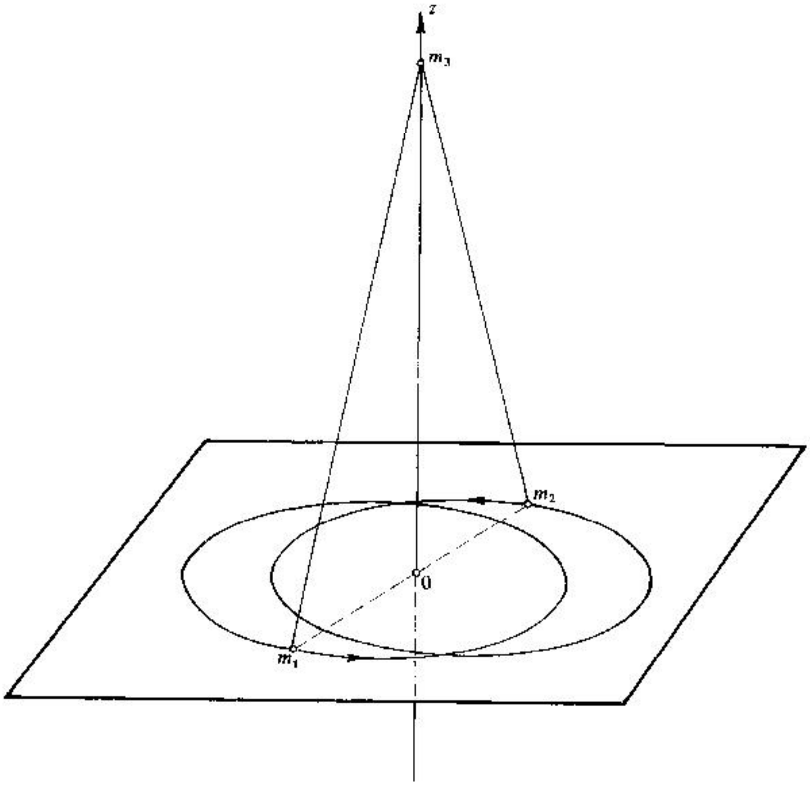

Figure 1. The Sitnikov problem.

3. Structure of the phase space

We studied the Sitnikov problem for different initial conditions. On the phase portraits we plotted many trajectories corresponding to different initial conditions. The set of initial conditions was $z_0 = 0.15 - 1.8$ with $\Delta z = 0.05$, and the initial velocities were $\dot{z} = 0$ in all cases. We choosed the integration time to be 10000 periods of the primaries.

The circular case is equivalent to the two center problem, which was solved already by Euler in 1764. In this case, when the third mass has bounded motion, the solutions are periodic or quasi-periodic depending on the initial conditions. The trajectories corresponding to these latter give close curves on a convenient surface of section in the phase space. Such curves are shown in Fig. 2.

In the eccentric case we have more various phenomena in the phase space. It is well known that increasing the parameter e the structure of the $z - \dot{z}$ space is also changing (Kallrath et al., 1997) [4]. For initial conditions close to the plane of the primaries the solutions are quasi-periodic motions by invariant cueves on the surface of section (see Fig. 3). However, small islands appear

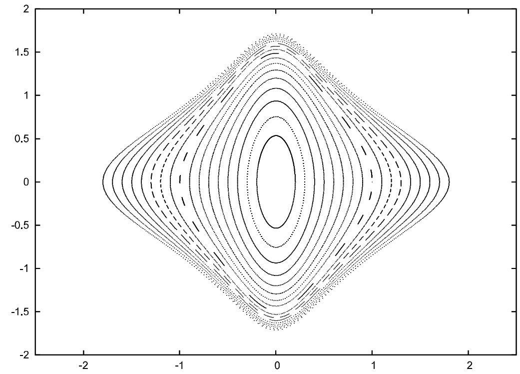

Figure 2. Invariant curves in the circular case of the Sitnikov problem. There are 17 initial conditions, $z_0 = 0.2 - -1.8,\ \Delta z = 0.1$.

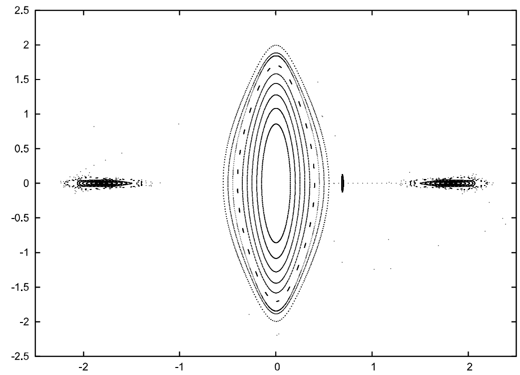

Figure 3. $e = 0.4$ and $v_0 = 0°$ (pericenter passage). The islands outside the invariant curves correspond to the 1:1 and 2:1 mean motion resonances. Between the islands there are escaping trajectories.

for particular initial distances outsid these invariant curves. These formations correspond to reasonances with the primaries. The massles body escapes from the system in the region between the islands (see Fig. 3).

In this paper we investigated the changing of the phase portraits when the primaries are not in the pericenter at $t = 0$. We calculated the motions for four initial positions of the primaries $v_0 = 45°$, $90°$, $135°$ and $180°$. Fig. 4 and Fig. 6 show the results.

The eccentricity of the binary was 0.4 (see Fig. 4). It can be seen that the 2:1 mean motion resonance (the two small islands in the Fig. 3) remains in all cases except $v_0 = 180°$. For example in the case $v_0 = 45°$ these islands dissolve to three smaller ones (see Fig. 4, top left panel). In addition, chaotic motion appears close to the separatrices. In Fig. 4 the bottom right panel shows a quite distinct picture. Except for initial conditions close to the baricenter, there are chaotic motions.

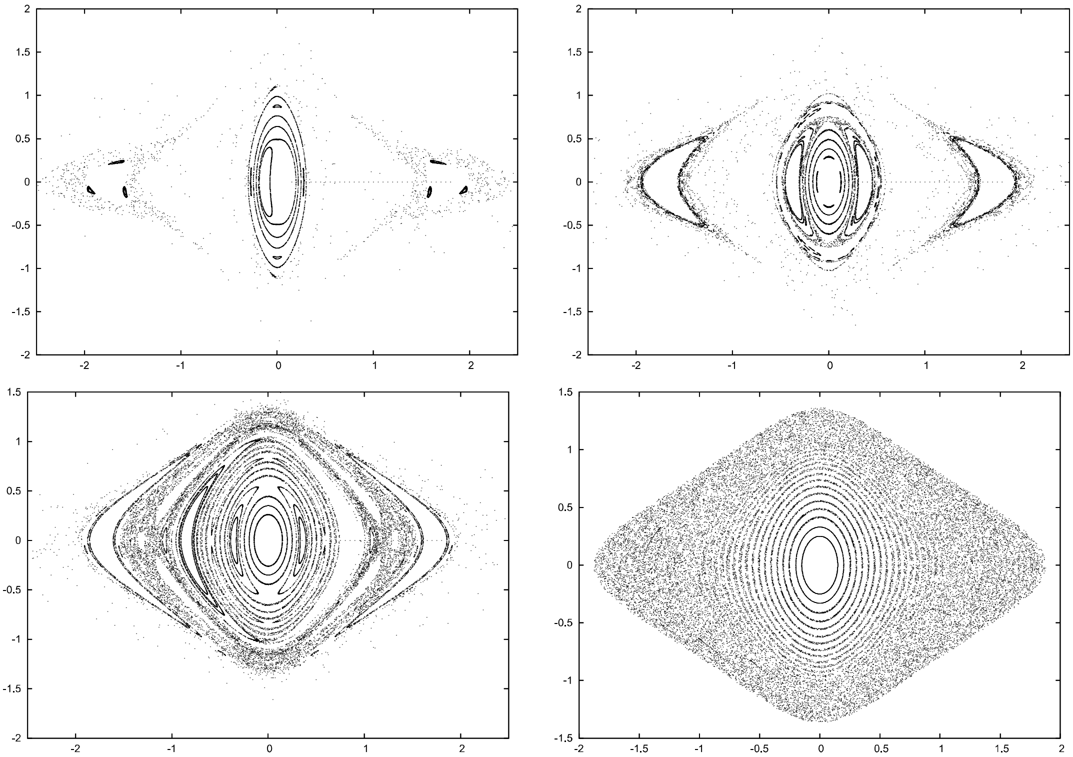

Figure 4. $e = 0.4$. Top left panel: $v_0 = 45^\circ$, the islands split up at the edge of the figure and the trajectories appear close to the separatrix. Top right panel: $v_0 = 90^\circ$, here it can be seen that a separatrix appears also between the invariant curves, and the 2:1 resonance is more unstable. Bottom left panel: $v_0 = 135^\circ$, the area of the chaotic region increases between the invariant curves and the 2:1 resonance. Bottom right panel: $v_0 = 180^\circ$, the total phase space is chaotic except some initial conditions close to the primaries' plane.

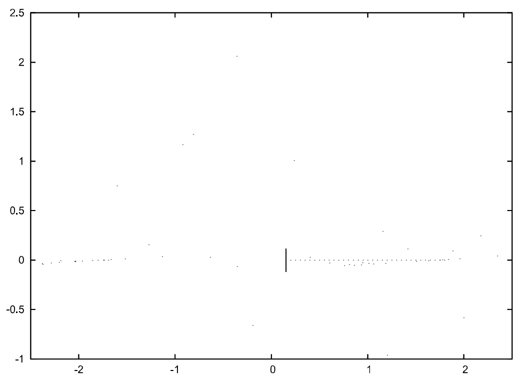

Figure 5. $e = 0.8$ and $v_0 = 0^\circ$. It is an interesting phase space portrait of the problem. Close to this value of the eccentricity (0.8) the fixed point in the middle becomes unstable. This was studied by Martínez Alfaro and Chiralt in 1993 [7]. In our case there is only one invariant curve for the initial distance $z_0 = 0.15$. In [7] the center is unstable for $e = 0.8558625$.

Fig. 5 shows the case where the eccentricity of the primaries was 0.8 and $v_0 = 0^\circ$. This is an interesting phase space portrait, because the parameter e

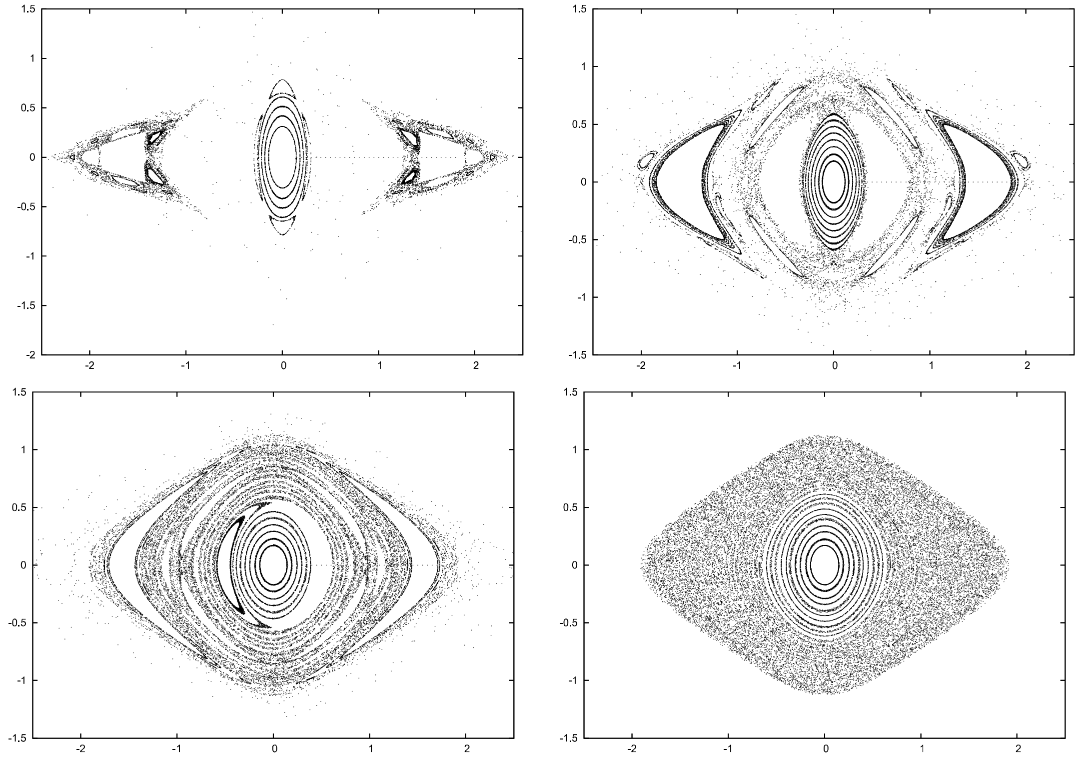

Figure 6. The eccentricity of the primaries is 0.8. Top left panel: $v_0 = 45°$, the structure of the phase space is very similar to that of Fig. 4. However, in this panel the resonances are more complex. Top right panel: $v_0 = 90°$, the chaotic domain grows between the islands. Bottom left panel: $v_0 = 135°$, there are many islands in the chaotic sea. Bottom right panel: $v_0 = 180°$, the phase space is mostly chaotic.

is very close to the value where the stable fixed point becomes unstable in the center of the $z - \dot{z}$ plane [7]. We can see only one island which corresponds to $z_0 = 0.15$ initial distance from the primaries orbital plane.

The four panels in Fig. 6 where $e = 0.8$ are similar to those of Fig. 4. There are resonances and for this larger the eccentricity there are stronger separatrices. The phase space is also very chaotic when the primaries are in the apocenter at $t = 0$.

4. Concluding remarks

We investigated the phase space of the Sitnikov problem for different initial conditions. Four initial positions of the primaries were studied beside the pericenter passage. There are closed curves on the surfaces of section corresponding to quasi-periodic orbits and small islands which means resonances. These small islands break up with varying the initial true anomaly, or higher order resonances appear. It is important to note that increasing the initial value of v more and more chaotic motion occur close to the separatrices. Finally, if the initial true anomaly is $180°$ then almost the whole phase space is chaotic.

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