RECURSIVE FORMULAE FOR THE LIE-INTEGRATION OF LINEARIZED EQUATIONS

András Pál
Department of Astronomy
Loránd Eötvös University
Pázmány Péter sétány 1/A
H-1117 Budapest, Hungary
apal@szofi.elte.hu

Abstract Several methods used in celestial mechanics require to solve ordinary differential equations (ODEs) and also derived equations like linearized ones. Lie-integration is known to be one of the fastest ODE-integrators and it is widely applied in long-term investigations. However, an inconvenience of this method is that auxiliary recurrence relations must be deduced which is different for each problem.

We present a lemma which can be used to derive such recurrence relations almost automatically for the linearized equations *if* the relations for the original ODEs are known. This lemma is then applied to the equations of the classical 2-body problem. The knowledge of such relations may imply other chaos detection methods; some concerning (and preliminary) results are also presented.

Keywords: Numerical integration – Lie-integration – Linearized equations

1. Introduction

The integration method based on the Lie-series ([1]) is widely used in celestial mechanics (see [2] and articles refering to it). The basis of this method is to generate the coefficients of the Taylor expansion of the solution by using recurrence relations. Let us write the differential equation to be solved as

$$\dot{x}_i = f_i(\mathbf{x}), \tag{1}$$

where $\mathbf{x}$ is an $\mathbb{R} \to \mathbb{R}^N$ and $\mathbf{f} \equiv (f_1, \ldots, f_N)$ is an $\mathbb{R}^N \to \mathbb{R}^N$ function. Let us also introduce the differential operator

$$D_i := \frac{\partial}{\partial x_i}, \tag{2}$$

and the derivation

$$L_0 := \sum_{i=1}^{n} f_i \frac{\partial}{\partial x_i} \equiv f_i D_i. \tag{3}$$

The latter is known as the Lie-derivation which is also a differential operator: it is linear and Leibnitz's rule stands for it,

$$L_0(ab) = aL_0(b) + bL_0(a). \tag{4}$$

It can easily be proved that the solution of Equ. (1) at a given instance $t + \Delta t$ is formally

$$\mathbf{x}(t + \Delta) = \exp\left(\Delta t \cdot L_0\right) \mathbf{x}(t), \tag{5}$$

where

$$\exp\left(\Delta t \cdot L_0\right) = \sum_{k=0}^{\infty} \frac{\Delta t^k}{k!} L_0^k. \tag{6}$$

Hence, the Lie-integration is the finite approximation (up to the order of M) of the sum in the right-hand side of Equ. (6), namely

$$\mathbf{x}(t + \Delta t) \approx \left(\sum_{k=0}^{N} \frac{\Delta t^k}{k!} L_0^k \right) \mathbf{x}(t) = \sum_{k=0}^{N} \frac{\Delta t^k}{k!} \left(L_0^k \mathbf{x}(t) \right). \tag{7}$$

The proof of Equ. (5) and other related properties of the Lie-derivation can be found in [2].

2. Linearized equations

For numerous chaos detection methods the knowledge of the solution of linearized equations is required. Let us again write the differential equation as

$$\dot{x}_i = f_i(\mathbf{x}). \tag{8}$$

The linearized equations can be written as

$$\dot{\xi}_i = \sum_{m=1}^{n} \xi_m \frac{\partial f_i(\mathbf{x})}{\partial x_m}. \tag{9}$$

Using the above conventions (see Equ. (2)) it can be re-written as:

$$\dot{\xi}_i = \xi_m D_m f_i. \tag{10}$$

Let us introduce the differential operator

$$\partial_i := \frac{\partial}{\partial \xi_i}. \tag{11}$$

Thus the coupled system of equations (both the original and the linearized) is

$$\left.\begin{aligned} \dot{x}_i &= f_i, \\ \dot{\xi}_i &= \xi_m D_m f_i. \end{aligned}\right\} \tag{12}$$

Using the differential operators defined in Equ. (2) and Equ. (11), one can write the Lie operator of Equ. (12) as

$$L = L_0 + L_\ell = f_i D_i + \xi_m D_m f_i \partial_i. \tag{13}$$

Lemma. Using the same notations as above, the Lie-derivatives of ξ_k can be written as

$$L^n \xi_k = \xi_m D_m L^n x_k = \xi_m D_m L_0^n x_k. \tag{14}$$

Proof. Obviously, Equ. (14) is true for $n = 0$:

$$D_m L^0 x_k = D_m x_k = \delta_{mk}, \tag{15}$$

hence

$$\xi_m D_m L^0 x_k = \xi_m \delta_{mk} = \xi_k. \tag{16}$$

Let us suppose that it is true for all $0 \le j \le n$ and calculate the $(n+1)$th Lie derivative of ξ_k:

$$\begin{aligned} L^{n+1}\xi_k &= L\left(\xi_m D_m L^n x_k\right) = \\ &= \left(f_i D_i + \xi_j D_j f_i \partial_i\right)\left(\xi_m D_m L^n x_k\right) = \\ &= f_i D_i \xi_m D_m L^n x_k + \\ &\quad + \xi_j (D_j f_i)[\delta_{im} D_m L^n x_k + \xi_m D_m \partial_i L^n x_k]. \end{aligned} \tag{17}$$

Here the last term ($\xi_m D_m \partial_i L^n x_k$) cancels, because x_k and $L^n x_k$ for all $0 \le n$ do not depend on ξ. So:

$$\begin{aligned} L^{n+1}\xi_k &= f_i D_i \xi_m D_m L^n x_k + \xi_j (D_j f_i) D_i L^n x_k = \\ &= \xi_m f_i D_m D_i L^n x_k + \xi_m (D_m f_i)(D_i L^n x_k) = \\ &= \xi_m \left(f_i D_m + D_m f_i\right)\left(D_i L^n x_k\right) = \\ &= \xi_m D_m (f_i D_i)(L^n x_k) = \\ &= \xi_m D_m L(L^n x_k) = \xi_m D_m L^{n+1} x_k = \xi_m D_m L_0^{n+1} x_k. \end{aligned} \tag{18}$$

Here we have used the Young's theorem:

$$D_m D_i = D_i D_m, \tag{19}$$

and Leibnitz's rule,

$$D_m (f_i D_i) X = D_m f_i (D_i X) = f_i (D_m D_i X) + (D_m f_i)(D_i X), \tag{20}$$

where X can be any function of $\mathbf{x}$, in Equ. (18) $X \equiv L^n x_k$. Thus Equ. (18) is the same relation for $n+1$, as Equ. (14) for n. Continuing the scheme described above, the relation Equ. (14) can be proved for all positive integer values of n.

3. Equations for the two-body problem

The recurrence relations for the Lie-derivatives of the equations of motion of the N-body problem can be found in [2]. Here we present the equations for the two-body problem with almost the same notations. Let us detone the relative coordinates and velocities by $\mathbf{r} = (r_1, r_2, r_3)$ and $\mathbf{w} = (w_1, w_2, w_3)$, respectively and introduce the following variables:

$$\rho := |\mathbf{r}| = \sqrt{r_1^2 + r_2^2 + r_3^2}, \tag{21}$$

$$\phi := \rho^{-3}, \tag{22}$$

$$\Lambda := r_1 w_1 + r_2 w_2 + r_3 w_3 = r_i w_i. \tag{23}$$

The total mass of the system is $M + m$. Using these notations, the equations of motion are

$$\begin{aligned} \dot{r}_i &= w_i, \\ \dot{w}_i &= -G(M+m)\phi r_i. \end{aligned} \tag{24}$$

(Here $G \equiv k^2$, the gravitational constant.) The differential operators D_i and Δ_i are defined as

$$D_i := \frac{\partial}{\partial r_i}, \tag{25}$$

$$\Delta_i := \frac{\partial}{\partial w_i}, \tag{26}$$

and the Lie-operator of Equ. (24) can be written as

$$L_0 = w_i D_i - G(M+m)\phi r_i \Delta_i. \tag{27}$$

It can be proved easily (see [2]) that the recurrence relations are

$$L^{n+1} r_i = L^n w_i, \tag{28}$$

$$L^n \Lambda = \sum_{k=0}^{n} \binom{n}{k} L^k r_i L^{n-k} w_i, \tag{29}$$

$$L^{n+1} w_i = -G(M+m) \sum_{k=0}^{n} \binom{n}{k} L^k \phi L^{n-k} r_i, \tag{30}$$

$$L^{n+1} \phi = \rho^{-2} \sum_{k=0}^{n} F_{nk} L^{n-k} \phi L^k \Lambda, \tag{31}$$

where

$$F_{nk} = (-3)\binom{n}{k} + (-2)\binom{n}{k+1}. \quad (32)$$

Using the lemma Equ. (14) the recurrence relations for the linearized equations of Equ. (24) can be derived *without* knowing these equations explicitly. Let us denote the linearized variables by ξ_i and η_i (respecting to r_i and w_i) and introduce

$$\begin{aligned} \Xi_i &:= (\xi_i \quad \eta_i), && (33) \\ \mathcal{D}_i &:= \begin{pmatrix} D_i \\ \Delta_i \end{pmatrix}, && (34) \\ \Xi\mathcal{D} &= \Xi_i\mathcal{D}_i = \xi_i D_i + \eta_i \Delta_i. && (35) \end{aligned}$$

Since the right-hand sides of the equations Equ. (28) – Equ. (31) are only bilinear in the Lie-derivatives of r_i, Λ, w_i and ϕ, using Leibnitz's rule the recurrence relations for ξ_i, w_i and the auxiliary variables $\Xi\mathcal{D}\phi$ and $\Xi\mathcal{D}\Lambda$ can be calculated automatically:

$$\begin{aligned} L^{n+1}\xi_i &= L^n\eta_i, && (36) \\ \Xi\mathcal{D}L^n\Lambda &= \sum_{k=0}^{n}\binom{n}{k}(L^k\xi_i L^{n-k}w_i + L^k r_i L^{n-k}\eta_i), \\ L^{n+1}\eta_i &= -G(M+m)\sum_{k=0}^{n}\binom{n}{k}\left[(\Xi\mathcal{D}L^k\phi)L^{n-k}r_i + L^k\phi L^{n-k}\xi_i\right], \\ \Xi\mathcal{D}L^{n+1}\phi &= -2\rho^{-2}\xi_i r_i L^{n+1}\phi + \\ &\quad +\rho^{-2}\sum_{k=0}^{n}F_{nk}\left[(\Xi\mathcal{D}L^{n-k}\phi)L^k\Lambda + L^{n-k}\phi(\Xi\mathcal{D}L^k\Lambda)\right]. \end{aligned}$$

For the initialization of the recurrence method, the value of $\Xi\mathcal{D}L^0\phi = \Xi\mathcal{D}\phi$ has to be known:

$$\Xi\mathcal{D}\phi = -3\rho^{-5}\xi_i r_i. \quad (37)$$

References

[1] Gröbner, W., Knapp, H.: 1967, "Contributions to the Method of Lie-Series"

[2] Hansmeier, A., Dvorak, R.: 1984, "Numerical integration with Lie-series" *Astronomy and Astrophysics* **132**, pp. 203-207