# MULTIPLICITY AND LATTICE COHOMOLOGY OF PLANE CURVE SINGULARITIES

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The lattice cohomology and the graded root of an isolated curve singularity were recently introduced in [3]. The lattice cohomology is a categorification of the $\delta$–invariant. The hope is that for plane curve singularities it encodes subtle information about the analytic structure and concrete analytic invariants.

The present paper is a positive result in this direction: we prove that the multiplicity of an irreducible plane curve singularity can be recovered from its lattice cohomology (or, from its graded root). In fact, we give four distinct proofs of this statement, each of them emphasizing a rather different aspect of the theory of plane curve germs.

With these proofs we also create new bridges between the abstract analytic type and the embedded topological type of the germ. In particular, we provide a new characterization of the Apéry set of the semigroup of the germ in terms of embedded data.

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## 1. **INTRODUCTION**

### 1.1.

Probably the oldest and most classical part of algebraic geometry is the theory of plane curves incorporating the theory of their singularities as well. The theory of complex plane curve germs $(C, 0) \subset (\mathbb{C}^2, 0)$ amalgamates the local analytic theory with low dimensional topology via their local embedded topological types. According to this symbiosis, their invariants also reflect this unity between analytic and topological data. Several classical invariants are read from the embedded topological type (i.e. from the corresponding algebraic link $C \cap S^3_\epsilon \subset S^3_\epsilon$, $0 < \epsilon \ll 1$), while several others from the structure of the

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local ring. Though the theory grows continuously and intensively, still there are many-many old and new open problems and conjectures regarding e.g. the classification problems, deformations, adjacencies. The newer theory of general curve singularities (with arbitrarily large embedded dimension) is even more exciting with even more unanswered questions.

Therefore, any new invariant of such germs creates a new possibility to attack these open problems, or to illuminate the theory deeper.

Such a new invariant is the *lattice cohomology* associated with any isolated (complex) curve singularity [3]. This is a multigraded $\mathbb{Z}$–module $\mathbb{H}^* = \oplus_{q\geq 0}\mathbb{H}^q$ such that each $\mathbb{H}^q$ is a $2\mathbb{Z}$–graded $\mathbb{Z}[U]$–module. Formally it has many similarities with the monopole cohomology, or with the Heegaard Floer homology of compact 3–manifolds. (For the $HF$–theory see the long series of articles of Ozsváth and Szabó, e.g. [20]. Some more connections will be mentioned below). It is one of the members of a series of 'lattice cohomologies' associated with singularities, like the lattice cohomology associated with the topological type of a normal surface singularity [16], or its analytic counterpart [1], which has an extension even for higher dimensional germs [2].

The lattice cohomology associated with a curve singularity is defined via its local algebra (see section 4 here). In the case of an irreducible curve singularity $(C, 0)$ it can be derived from the numerical semigroup of values $\mathcal{S}_C \subset \mathbb{Z}_{\geq 0}$ as well (see sections 2 and 3 below).

In the case of plane curve singularities, the semigroup $\mathcal{S}_C$ is a complete embedded topological invariant, hence $\mathbb{H}^*$ is in fact an output of the embedded topological type of $C \cap S^3_\epsilon \subset S^3_\epsilon$.

In the irreducible case it turns out that $\mathbb{H}^{>0} = 0$, hence all the information is coded in the $\mathbb{Z}[U]$–module $\mathbb{H}^0$. This module $\mathbb{H}^0$ has an enhanced version as well, the *graded root* $\mathfrak{R}$ of $(C, 0)$ (which, in principle, is a stronger invariant than $\mathbb{H}^0$), see section 2. In the case of plane curves, $\mathbb{H}^0$ has a deep connection with the Heegaard Floer Link homology associated with the link $C \cap S^3_\epsilon \subset S^3_\epsilon$, see [19].

### 1.2.

It is natural to test in the case of irreducible plane curve singularities whether one can recover certain analytic invariants from $\mathbb{H}^0$ (or, from $\mathfrak{R}$). In this direction, the first analytic invariant which should be tested is the *multiplicity* of the germ (the smallest degree of a monomial in its expansion). It is the starting point and the subject of several results regarding the analytic classifications of germs and the topological characterizations of certain analytic invariants (e.g. Artin's topological characterization of the multiplicity of rational surface singularities). Or, it appears in famous conjectures like Zariski's conjecture which predicts that the multiplicity of an isolated hypersurface singularity can be recovered from the embedded topological type. (This is positively answered for plane curve singularities.)

Our main result is the following:

THEOREM 1.1 (NON-TECHNICAL VERSION). *The multiplicity of an irreducible plane curve singularity* $(C,0) \subset (\mathbb{C}^2,0)$ *can be recovered from the graded* $\mathbb{Z}[U]$*–module structure of the lattice cohomology* $\mathbb{H}^0$ *(or, from the graded root* $\mathfrak{R}$*) associated with* $(C,0)$*.*

Here some comments are in order. In order to understand better the connection between $\mathbb{H}^0$ (or $\mathfrak{R}$) and the multiplicity $\mathbf{m}$, it is worth describing both of them in terms of the semigroup of values $\mathcal{S}_C \subset \mathbb{Z}_{\geq 0}$. The multiplicity is the smallest element of $\mathcal{S}_C \setminus \{0\}$, and, as we already mentioned, both $\mathbb{H}^0$ and $\mathfrak{R}$ are outputs of $\mathcal{S}_C$. In particular, the multiplicity of an irreducible plane curve singularity can be recovered from its embedded topological type, and also from its semigroup, and here we ask if it can be recovered from $\mathbb{H}^0$.

Here it is important to emphasize that any numerical semigroup (monoid) $\mathcal{S} \subset \mathbb{Z}_{\geq 0}$ can be realized as the semigroup of values of a certain (non unique) curve singularity, e.g. of the curve with local algebra $\mathbb{C}[\mathcal{S}]$, see e.g. [24]. Hence, in fact, our question regarding the multiplicity can even be generalized into the following: can the smallest non-zero element of a semigroup $\mathcal{S} \subset \mathbb{Z}_{\geq 0}$ be recovered from the lattice cohomology associated with $\mathcal{S}$ (defined combinatorially as in subsection 5.2(d))? In the case of plane curve singularities, the semigroup has an additional (Gorenstein) symmetry property, so we might restrict this question to symmetric semigroups as well. [For any semigroup $\mathcal{S} \subset \mathbb{Z}_{\geq 0}$ with $\mathbb{Z}_{\geq 0} \setminus \mathcal{S}$ finite there exists a smallest element $\mathbf{c}$ of $\mathcal{S}$, called the conductor, such that $\mathbf{c} + \mathbb{Z}_{\geq 0} \subset \mathcal{S}$. The semigroup is (Gorenstein) symmetric if $s \in \mathcal{S} \Leftrightarrow \mathbf{c} - 1 - s \notin \mathcal{S}$.]

The point is that the answer to both of these general questions is negative. That is, there are pairs of irreducible curve singularities (not necessarily plane curve germs) with the same graded root and lattice cohomology, but with different semigroups and different multiplicities, see Example 1.2 below. (In particular, in the above theorem, the fact that we consider only plane curve singularities is not a weakness of the statement, but a necessary assumption.)

*Example* 1.2. Let $a \gg 0$ be a big positive integer and consider the following semigroups (below '$\bullet$' represent the elements of $\mathcal{S}$, while '$\circ$' denote the gaps $\overline{\mathcal{S}} := \mathbb{Z}_{\geq 0} \setminus \mathcal{S}$). In both cases $\mathbf{c} = 2a + 10$. The multiplicity in the first case is $\mathbf{m} = a$ and in the second case it is $\mathbf{m} = a + 1$.

| $\ell$ | 0 | 1 | ... | $a-1$ | $a$ | $a+1$ | $a+2$ | $a+3$ | $a+4$ | $a+5$ | $a+6$ | $a+7$ | $a+8$ | $a+9$ | $a+10$ | ... | $2a+8$ | $2a+9$ | $2a+10$ |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| $\mathcal{S}_1$ | $\bullet$ | $\circ$ | ... | $\circ$ | $\bullet$ | $\circ$ | $\circ$ | $\bullet$ | $\bullet$ | $\circ$ | $\circ$ | $\bullet$ | $\bullet$ | $\circ$ | $\bullet$ | ... | $\bullet$ | $\circ$ | $\bullet$ |
| $\mathfrak{h}_1$ | 0 | 1 | ... | 1 | 1 | 2 | 2 | 2 | 3 | 4 | 4 | 4 | 5 | 6 | 6 | ... | $a+4$ | $a+5$ | $a+5$ |
| $w_{0,1}$ | 0 | 1 | ... | $3-a$ | $2-a$ | $3-a$ | $2-a$ | $1-a$ | $2-a$ | $3-a$ | $2-a$ | $1-a$ | $2-a$ | $3-a$ | $2-a$ | ... | 0 | 1 | 0 |

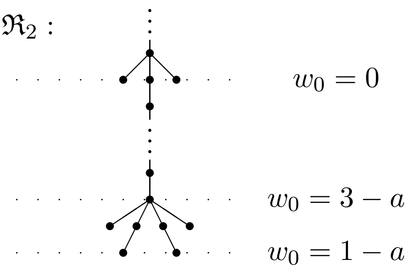
and

| $\ell$ | 0 | 1 | ... | $a-1$ | $a$ | $a+1$ | $a+2$ | $a+3$ | $a+4$ | $a+5$ | $a+6$ | $a+7$ | $a+8$ | $a+9$ | $a+10$ | ... | $2a+8$ | $2a+9$ | $2a+10$ |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| $\mathcal{S}_2$ | $\bullet$ | $\circ$ | ... | $\circ$ | $\circ$ | $\bullet$ | $\bullet$ | $\circ$ | $\bullet$ | $\circ$ | $\bullet$ | $\circ$ | $\circ$ | $\bullet$ | $\bullet$ | ... | $\bullet$ | $\circ$ | $\bullet$ |
| $\mathfrak{h}_2$ | 0 | 1 | ... | 1 | 1 | 1 | 2 | 3 | 3 | 4 | 4 | 5 | 5 | 5 | 6 | ... | $a+4$ | $a+5$ | $a+5$ |
| $w_{0,2}$ | 0 | 1 | ... | $3-a$ | $2-a$ | $1-a$ | $2-a$ | $3-a$ | $2-a$ | $3-a$ | $2-a$ | $3-a$ | $2-a$ | $1-a$ | $2-a$ | ... | 0 | 1 | 0 |

In the tables above we inserted the Hilbert functions $\mathfrak{h}$ and weight functions $w_0$ as well. $\mathfrak{h}(\ell)$ is determined from the numerical semigroup $\mathcal{S}$ as follows: $\mathfrak{h}(\ell) = \#\{s \in \mathcal{S} \,:\, s < \ell\}$. The mapping $\ell \mapsto w_0(\ell) := 2\mathfrak{h}(\ell) - \ell$ is the weight function used in the definition of $\mathbb{H}^*$ and $\mathfrak{R}$, see section 3.

One can check that both semigroups determine the same graded root:

$\mathfrak{R}_1 \cong \mathfrak{R}_2$ :




(For details regarding graded roots, and the steps how one recovers $\mathbb{H}^0$ from them, see section 3.)

Let $(C, 0) \subset (\mathbb{C}^2, 0)$ be an irreducible plane curve singularity with multiplicity $\mathbf{m}$ and semigroup $\mathcal{S}_C$.

Technically speaking, the multiplicity of $(C, 0)$ will be recovered from the 'local minimum points' of the weight function $w_0$, or from the degrees of the local minimum points of the graded root. In fact, we have the following statement.

PROPOSITION 1.3. *In the lattice cohomology of the irreducible plane curve $(C, 0)$ the local minimum points of the weight function $w_0$ correspond to the local minimum points of the associated graded root and their weights can be recovered from the $\mathbb{Z}[U]$–module structure of $\mathbb{H}^0$. (Cf. Propositions 2.8 and 2.12 and Remark 3.1.)*

We need to distinguish the two cases when $(C, 0)$ is smooth or not.

By [3, Example 4.6.1] $(C, 0)$ is smooth if and only if $\mathfrak{R}$ has exactly one local minimum point (by the notation of Example 2.5 below, if $\mathfrak{R} = \mathfrak{R}_0$), or equivalently, if the reduced lattice cohomology $\mathbb{H}^0_{red}$ is zero. In this case $\mathbf{m}=1$. In all other cases $\mathfrak{R}$ has at least two local minimum points, $\mathbb{H}^0_{red}\neq 0$ and $\mathbf{m}\geq 2$. In particular, smoothness can be clearly identified from both $\mathfrak{R}$ and $\mathbb{H}^0$.

In the sequel we always assume that $(C,0)$ is not smooth, hence $\mathbf{m}\geq 2$. Moreover, $\mathcal{S}_C\neq \mathbb{Z}_{\geq 0}$, hence its conductor $\mathbf{c}\geq 2$.

In this non-smooth case, there are two 'standard' local minimum points, corresponding to the points $s=0$ and $s=\mathbf{c}$, with values $w_0(0)=w_0(\mathbf{c})=0$. (We might call them 'trivial' ones.) All the other local minimum points are located strictly between 0 and $\mathbf{c}$. By the Gorenstein symmetry one has $\mathbf{c}=2\delta$, where $\delta=\#\{\mathbb{Z}_{\geq 0}\setminus \mathcal{S}_C\}$ is the delta invariant of $(C,0)$. The weight function and hence the local minimum points are symmetric with respect to the involution $s\leftrightarrow \mathbf{c}-s$. Moreover, it turns out that the values $w_0(s)$ of the local minima $s$ are always non-positive (see subsection 5.2)

The multiplicity $\mathbf{m}=\min\{\mathcal{S}_C\setminus\{0\}\}$ is also a local minimum point.

The characterization of the multiplicity $\mathbf{m}$ is the following.

THEOREM 1.4. *Assume that $(C,0)$ is a non-smooth irreducible plane curve germ. Let $s\notin\{0,\mathbf{c}\}$ be a local minimum of the weight function $w_0$. Then*

$$w_0(s)\leq 2-\mathbf{m},$$

*and equality holds (at least) for the local minimum point $s=\mathbf{m}$.*

As we already mentioned, the multiplicity $\mathbf{m}=\min\{\mathcal{S}_C\setminus\{0\}\}$ is a local minimum point, in fact, it is the first non-trivial one with respect to the ordering of the integers. The difficulty is that the local minimum points of the graded root (i.e. their 'end-leaves') are not marked by the corresponding semigroup elements which generate them, so from the leaves (local minima) of the graded root one cannot read off the order how they appear in the semigroup. This is exactly the core of the problem: we have to show the following property (P): the degree of the 'highest-degree' non-trivial leaf of the root is the weight of the first non-trivial local minimum point of $\mathcal{S}_C$.

The fact that this property (P) does not hold in general (for non-plane germs) can already be seen in Example 1.2, in the case of $\mathcal{S}_2$. (However, for $\mathcal{S}_1$ it holds.)

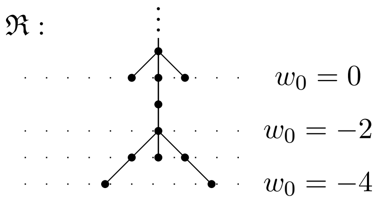
In the next example, we provide a graded root, which can be realized by a symmetric semigroup, but for any symmetric semigroup realization the above property (P) fails.

*Example* 1.5. The above statement is not true for an arbitrary symmetric semigroup (hence, for an arbitrary curve singularity). For example, consider the symmetric semigroup $\mathcal{S} = \mathbb{Z}_{\geq 0}\langle 6, 7, 9\rangle$ with multiplicity $\mathbf{m} = 6$ and conductor $\mathbf{c} = 18$. The weights are given in the diagram below.

| $\ell$ | 0 | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 | 10 | 11 | 12 | 13 | 14 | 15 | 16 | 17 | 18 |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| $\mathcal{S}_C$ | $\bullet$ | $\circ$ | $\circ$ | $\circ$ | $\circ$ | $\circ$ | $\bullet$ | $\bullet$ | $\circ$ | $\bullet$ | $\circ$ | $\circ$ | $\bullet$ | $\bullet$ | $\bullet$ | $\bullet$ | $\bullet$ | $\circ$ | $\bullet$ |
| $\mathfrak{h}$ | 0 | 1 | 1 | 1 | 1 | 1 | 1 | 2 | 3 | 3 | 4 | 4 | 4 | 5 | 6 | 7 | 8 | 9 | 9 |
| $w_0$ | 0 | 1 | 0 | $-1$ | $-2$ | $-3$ | $-4$ | $-3$ | $-2$ | $-3$ | $-2$ | $-3$ | $-4$ | $-3$ | $-2$ | $-1$ | 0 | 1 | 0 |

From the diagram we read that the local minimum points are $\{0, 6, 9, 12, 18\}$. The highest non-trivial local minimum value is $-3$, it is realized for $s = 9$, while $w_0(\mathbf{m}) = w_0(6) = 2 - \mathbf{m} = -4$.

The graded root is the following:




One can see also on $\mathfrak{R}$ that there is the non-trivial local minimum with degree $-3 \not\leq 2 - 6$.

At the start of our investigation we thought that the proof of the Main Theorem 1.4 will be 'easy', we 'just' have to show that the local minimum $w_0$–values are non-increasing for local minimum points $0 \leq s \leq \delta$ and (symmetrically) non-decreasing for local minimum points $\delta \leq s \leq 2\delta = \mathbf{c}$. Well, it turned out that this is not the case (hence we faced and had to investigate a more complex structure). Already Example 1.5 and Example 1.2 (case $\mathcal{S}_2$) provide examples of (non-necessarily plane) curve singularities when this monotoneity of the local minimum values fails.

But, it fails even for plane curve singularities, even for irreducible germs with one Puiseux pair.

*Example* 1.6. The weights of the local minimum points of $w_0$ are not necessarily monotone non-increasing between 0 and $\delta$ even for irreducible plane curve singularities. For example if we take the singularity $(\{x^{11} - y^{14} = 0\}, 0)$, then we get the following weight function:

| $\ell$ | ... | 54 | 55 | 56 | 57 | 58 | 59 | ... |
|---|---|---|---|---|---|---|---|---|
| $\mathcal{S}_C$ | ... | $\circ$ | $\bullet$ | $\bullet$ | $\circ$ | $\bullet$ | $\circ$ | ... |
| $w_0$ | ... | $-26$ | $-27$ | $-26$ | $-25$ | $-26$ | $-25$ | ... |

Here 55 and 58 are both local minima with $w(55) = w(58) - 1$ but $58 < \delta = 65$.

In fact, one can construct for any $n \in \mathbb{N}$ an irreducible plane curve singularity for which, in the sequence of the weights of the local minimum points between 0 and $\delta$, there is a monotone increasing sequence of length at least $n$.

## 1.3.

In the body of the paper we provide four different proofs for the Main Theorem. They reflect three different aspects of the theory of irreducible plane curve singularities. For details regarding plane curve singularities see [5, 23, 24, 25].

The first two proofs are based on the combinatorics of the semigroup $\mathcal{S}_C$ (symmetry and some kind of 'distribution' property of the first generators).

The third proof uses the Apéry set of $\mathcal{S}_C$ associated with its element $\mathbf{m}$ [4]. Here we compare invariants associated with $(C, 0)$ and the germ $(C', 0)$ obtained as the strict transform after one blow–up. Their Apéry sets are related by a classical theorem of Apéry, but here we have to investigate further deeper properties.

In fact the first three proofs are rather elementary. Most of the preparatory parts are used only in the fourth proof.

In the fourth proof we investigate the embedded topological type: we use the minimal embedded resolution graph $\Gamma$ associated with $(C, 0) \subset (\mathbb{C}^2, 0)$. We rewrite the weight function in terms of a Riemann–Roch expression of certain universal cycles of the resolution, and we use again the comparison between $(C, 0)$ and $(C', 0)$.

## 1.4.

A considerable material, motivated by the fourth proof, but which might have an independent interest, connects the embedded topology with the abstract analytic setup. Namely, we establish a series of new statements connecting the combinatorics of the Lipman cone $\mathcal{S}_\Gamma$ associated with the minimal embedded resolution graph $\Gamma$ with the combinatorics of the semigroup $\mathcal{S}_C$ associated with the normalization of $(C, 0)$. The connection is realized via a sequence of universal cycles $\{x(\ell)\}_\ell$ (which was introduced earlier in the context of almost rational normal surface singularities in [14]). In fact, in order to establish this bridge, we needed to create and develop a little theory, which will be useful in further studies and generalizations.

The last section is also part of this discussion, it proves a new characterization of the Apéry set of the abstract semigroup $\mathcal{S}_C$ in terms of the embedded resolution graph $\Gamma$ and the universal cycles $\{x(\ell)\}_\ell$. This characterization reveals certain similarity between the third and fourth proofs.

1.5.

We proved our theorem for *irreducible* plane curve singularities. However, we believe that the statement extends for non-irreducible germs as well.

CONJECTURE 1.7. *The statement of Theorem 1.1 is true for any plane curve singularity.*

Since in this general case the structure of the semigroup of values is much more complicated, one definitely needs some new techniques and ideas to establish the proof in the extended case.

1.6.

The structure of the paper is the following. In section 2 we recall the definition of the lattice cohomology and the graded root and some needed properties. Additionally, we also prove some equivalences connecting their local minimum points and values. Section 3 discusses the lattice cohomology of curve singularities. The case of plane curve singularities is treated in section 4. Here we review and prove several statements regarding the embedded topology, its lattice, the structure of the Lipman cone, universal cycles, generalized Laufer algorithms. We also re-express the weight function (originally defined from the normalization of $(C, 0)$ via the Hilbert function, or from $\mathcal{S}_C$) in terms of embedded topological data. In section 5 we discuss the Apéry set of a semigroup (associated with one of its elements), with a special emphasis on plane germs. Sections 6,7,8 contain the four proofs. Section 9 contains the new characterization of the Apéry set of $\mathcal{S}_C$ in terms of $\Gamma$.

## 2. BASIC PROPERTIES OF LATTICE COHOMOLOGY AND THE GRADED ROOT

### 2.1. Lattice cohomology associated with a weight function [14, 16]

2.1.1. **Lattice with partial ordering.** We consider a free $\mathbb{Z}$–module, with a fixed basis $\{E_i\}_{i\in\mathcal{I}}$, denoted by $\mathbb{Z}^r$ (hence $r = \#\{\mathcal{I}\}$). There is a natural partial ordering $\leq$ of $\mathbb{Z}^r$ (and of $\mathbb{R}^r$) defined coordinatewise induced by the fixed basis: for $\ell_1, \ell_2 \in \mathbb{Z}^r$ with $\ell_j = \sum_i \ell_{ji} E_i$ ($j = \{1, 2\}$) one says that $\ell_1 \leq \ell_2$ if $\ell_{1i} \leq \ell_{2i}$ for all $i$.

The lattice cohomology construction associates a graded $\mathbb{Z}[U]$–module with the pair $(\mathbb{Z}^r, \{E_i\}_i)$ and a set of compatible weight functions, see [14, 15, 16, 18]. In fact, the grading is $2\mathbb{Z}$–valued; we prefer this convention in order to keep the compatibility with the Heegaard Floer (Link) theory, see [19]. (The definition of a set of compatible weight functions will be given below in 2.1.)

2.1.2. $\mathbb{Z}[U]$**–modules.** Consider the graded $\mathbb{Z}[U]$–module $\mathbb{Z}[U, U^{-1}]$, and (following [20]) denote by $\mathcal{T}_0^+$ its quotient by the submodule $U \cdot \mathbb{Z}[U]$. This has a grading in such a way that $\deg(U^{-d}) = 2d$ ($d \geq 0$). Similarly, for any $n \geq 1$, the quotient of $U^{-(n-1)} \cdot \mathbb{Z}[U]$ by $U \cdot \mathbb{Z}[U]$ (with the same grading) defines the graded module $\mathcal{T}_0(n)$. Hence, $\mathcal{T}_0(n)$, as a $\mathbb{Z}$–module, is freely generated by $1, U^{-1}, \ldots, U^{-(n-1)}$, and has finite $\mathbb{Z}$–rank $n$.

More generally, for any graded $\mathbb{Z}[U]$–module $P$ with $d$–homogeneous elements $P_d$, and for any $m \in \mathbb{Z}$, we denote by $P[m]$ the same module graded in such a way that $P[m]_{d+m} = P_d$. Then define the modules $\mathcal{T}_m^+ := \mathcal{T}_0^+[m]$ and $\mathcal{T}_m(n) := \mathcal{T}_0(n)[m]$. Hence, for $m \in \mathbb{Z}$, $\mathcal{T}_{2m}^+ = \mathbb{Z}\langle U^{-m}, U^{-m-1}, \ldots \rangle$ as a $\mathbb{Z}$–module.

2.1.3. **The cubical decomposition.** $\mathbb{Z}^r \otimes \mathbb{R} = \mathbb{R}^r$ has a natural cellular decomposition into cubes. The set of zero–dimensional cubes consists of the lattice points $\mathbb{Z}^r$. Any $\ell \in \mathbb{Z}^r$ and any subset $I \subset \mathcal{I}$ of cardinality $q$ define a $q$–dimensional cube $(\ell, I)$, having vertices $\{\ell + \sum_{i \in I'} E_i\}_{I'}$, where $I'$ runs over all subsets of $I$. The set of $q$–dimensional cubes is denoted by $\mathcal{Q}_q$ ($0 \leq q \leq r$). (We regard these cubes as *closed* cubes, which contain their boundaries as well.)

Next we consider a set of compatible *weight functions* $w = \{w_q\}_q$.

*Definition* 2.1. A set of functions $w_q : \mathcal{Q}_q \to \mathbb{Z}$ ($0 \leq q \leq r$) is called a *set of compatible weight functions* if the following hold:

(a) for any integer $k \in \mathbb{Z}$, the set $w_0^{-1}(\,(-\infty, k]\,)$ is finite;

(b) for any $\square_q \in \mathcal{Q}_q$ and for any of its faces $\square_{q-1} \in \mathcal{Q}_{q-1}$ one has $w_q(\square_q) \geq w_{q-1}(\square_{q-1})$.

In the present paper we will only give $w_0 : \mathcal{Q}_0 = \mathbb{Z}^r \to \mathbb{Z}$ and define the others as follows:

$$\text{(1)} \quad \text{for any } q\text{–cube } \square_q \in \mathcal{Q}_q : \; w_q(\square_q) = \max\{w_0(\ell) \,:\, \ell \text{ is a vertex of } \square_q\}.$$

When the dimension is clear from the context we will omit the index $q$ from our notation.

2.1.4. **The $S_n$ spaces and the lattice cohomology.** For each $n \in \mathbb{Z}$ we define $S_n = S_n(w) \subset \mathbb{R}^r$ as the union of all the cubes $\square$ (of any dimension) with $w(\square) \le n$. Clearly, $S_n = \emptyset$, whenever $n < m_w := \min\{w_0(\ell) : \ell \in \mathbb{Z}^r\}$. For any $q \ge 0$, set

$$\mathbb{H}^q(\mathbb{R}^r, w) := \bigoplus_{n \ge m_w} H^q(S_n, \mathbb{Z}) \quad \text{and} \quad \mathbb{H}^q_{red}(\mathbb{R}^r, w) := \bigoplus_{n \ge m_w} \widetilde{H}^q(S_n, \mathbb{Z}).$$

Then $\mathbb{H}^q$ is $\mathbb{Z}$ (in fact, $2\mathbb{Z}$)–graded, the $2n$–homogeneous elements $\mathbb{H}^q_{2n}$ consist of $H^q(S_n, \mathbb{Z})$. Also, $\mathbb{H}^q$ is a $\mathbb{Z}[U]$–module; the $U$–action is given by the restriction map $r_{n+1} : H^q(S_{n+1}, \mathbb{Z}) \to H^q(S_n, \mathbb{Z})$. The same is true for $\mathbb{H}^*_{red}$. Moreover, for $q = 0$, any fixed base point $\ell_w \in S_{m_w} \subset S_n$ provides an augmentation $H^0(S_n, \mathbb{Z}) = \mathbb{Z} \oplus \widetilde{H}^0(S_n, \mathbb{Z})$, hence an augmentation of the graded $\mathbb{Z}[U]$–modules

$$\mathbb{H}^0 \simeq \mathcal{T}^+_{2m_w} \oplus \mathbb{H}^0_{red} = \Big( \bigoplus_{n \ge m_w} \mathbb{Z} \Big) \oplus \Big( \bigoplus_{n \ge m_w} \widetilde{H}^0(S_n, \mathbb{Z}) \Big) \quad \text{and} \quad \mathbb{H}^* \simeq \mathcal{T}^+_{2m_w} \oplus \mathbb{H}^*_{red}.$$

Though $\mathbb{H}^*_{red}(\mathbb{R}^r, w)$ has finite $\mathbb{Z}$–rank in any fixed homogeneous degree, in general, without certain additional properties of $w_0$, it is not finitely generated over $\mathbb{Z}$, in fact, not even over $\mathbb{Z}[U]$.

2.1.5. **Restrictions.** Assume that $T \subset \mathbb{R}^r$ is a subspace of $\mathbb{R}^r$ consisting of a union of some (closed) cubes (from $\mathcal{Q}_*$). For any $q \ge 0$ define $\mathbb{H}^q(T, w)$ as $\oplus_{n \ge \min (w_0|_T)} H^q(S_n \cap T, \mathbb{Z})$. It has a natural graded $\mathbb{Z}[U]$–module structure. The restriction map induces a natural graded $\mathbb{Z}[U]$–module homomorphism

$$r^* : \mathbb{H}^*(\mathbb{R}^r, w) \to \mathbb{H}^*(T, w) \quad \text{(of degree zero)}.$$

In some cases it can happen that the weight functions are defined only for cubes belonging to $T$.

In our applications, $T$ (besides the trivial $T = \mathbb{R}^r$ case) will be one of the following: (i) the rectangle $R(0, c) = \{x \in \mathbb{R}^r : 0 \le x \le c\}$ for some lattice point $c \ge 0$, or (ii) the first quadrant $(\mathbb{R}_{\ge 0})^r$.

2.1.6. **The Euler characteristic of $\mathbb{H}^*$ [17].** Let $T$ be as in 2.1.5 and assume that each $\mathbb{H}^q_{red}(T, w)$ has finite $\mathbb{Z}$–rank. (This happens automatically when $T$ is a finite rectangle.) We define the Euler characteristic of $\mathbb{H}^*(T, w)$ as

$$eu(\mathbb{H}^*(T, w)) := -\min\{w(\ell) : \ell \in T \cap \mathbb{Z}^r\} + \sum_q (-1)^q \mathrm{rank}_{\mathbb{Z}}(\mathbb{H}^q_{red}(T, w)).$$

## 2.2. **Graded roots and their cohomologies [14, 15]**

*Definition* 2.2.

(1) Let $\mathfrak{R}$ be an infinite tree with vertices $\mathfrak{V}$ and edges $\mathfrak{E}$. We denote by $[u,v]$ the edge with end-vertices $u$ and $v$. We say that $\mathfrak{R}$ is a *graded root* with grading $\chi:\mathfrak{V}\to\mathbb{Z}$ if

(a) $\chi(u)-\chi(v)=\pm 1$ for any $[u,v]\in\mathfrak{E}$;
(b) $\chi(u)>\min\{\chi(v),\chi(w)\}$ for any $[u,v],\ [u,w]\in\mathfrak{E}$, $v\neq w$;
(c) $\chi$ is bounded from below, $\chi^{-1}(n)$ is finite for any $n\in\mathbb{Z}$, and $\#\{\chi^{-1}(n)\}=1$ if $n\gg 0$.

(2) $v\in\mathfrak{V}$ is a *local minimum point* of the graded root $(\mathfrak{R},\chi)$ if $\chi(v)<\chi(w)$ for any edge $[v,w]$. The set of local minimum points is denoted by $\mathfrak{V}_{lm}$.

(3) An *isomorphism of graded roots* is a graph isomorphism, which preserves the gradings.

For some examples and general constructions see [14].

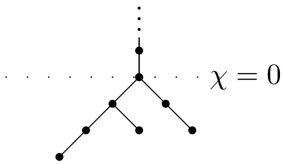
*Remark* 2.3. (1) For any vertex $v$ set $\kappa_v:=\#\{w\in\mathfrak{V}\,:\,[v,w]\in\mathfrak{E}\}$.
One can verify that the set of vertices $\{v\in\mathfrak{V}:\kappa_v=1\}$ are exactly the local minimum points $\mathfrak{V}_{lm}$ of $\chi$, and $\#\big(\mathfrak{V}_{lm}\big)<\infty$.

(2) A geodesic path connecting two vertices is *monotone* if $\chi$ restricted to the set of vertices on the path is monotone. If a vertex $v$ can be connected to another vertex $w$ by a monotone geodesic and $\chi(v)>\chi(w)$, then we say that $v$ dominates $w$, and we write $v\succ w$. $\succ$ is a partial ordering of $\mathfrak{V}$. It is easy to see that for any pair $v,w\in\mathfrak{V}$ there is a unique $\succ$–minimal vertex $\sup(v,w)$ which dominates both.

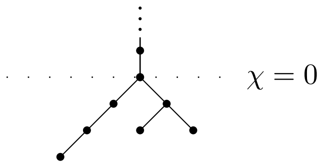
*Definition* 2.4. *(The $\mathbb{Z}[U]$–modules associated with a graded root.)* Let us identify a graded root $(\mathfrak{R},\chi)$ with its topological realization provided by vertices (0–cubes) and segments (1–cubes). Set the weight functions $w_0(v)=\chi(v)$, and $w_1([u,v])=\max\{\chi(u),\chi(v)\}$ and let $S_n$ be the union of all cubes with weight $\leq n$. Then we might set (as above) $\mathbb{H}^*(\mathfrak{R},\chi)=\bigoplus_{n\geq\min\chi}H^*(S_n,\mathbb{Z})$. However, at this time $\mathbb{H}^{\geq 1}(\mathfrak{R},\chi)=0$; so we set $\mathbb{H}(\mathfrak{R},\chi):=\mathbb{H}^0(\mathfrak{R},\chi)$. Similarly, one defines $\mathbb{H}_{red}(\mathfrak{R},\chi)$ using the reduced cohomology, hence, as usual $\mathbb{H}(\mathfrak{R},\chi)\simeq\mathcal{T}^+_{2\min\chi}\oplus\mathbb{H}_{red}(\mathfrak{R},\chi)$. The $U$–action is induced by the inclusions $S_n\hookrightarrow S_{n+1}$ similarly to paragraph 2.1.4. (For a different description of $\mathbb{H}(\mathfrak{R},\chi)$ see [14].)

*Example* 2.5. (a) For any integer $n\in\mathbb{Z}$, let $\mathfrak{R}_n$ be the tree with vertices $\mathfrak{V}=\{v^k\}_{k\geq n}$ and edges $\mathfrak{E}=\{[v^k,v^{k+1}]\}_{k\geq n}$. The grading is $\chi(v^k)=k$. Then $\mathbb{H}(\mathfrak{R}_n)=\mathcal{T}^+_{2n}$.
(b) The graded roots $\mathfrak{R}^1$ and $\mathfrak{R}^2$

$\mathfrak{R}^1$ :




$\mathfrak{R}^2$ :




are not isomorphic but the graded $\mathbb{Z}[U]$–modules associated to them are: $\mathbb{H}(\mathfrak{R}^1) = \mathbb{H}(\mathfrak{R}^2) = \mathcal{T}^+_{-6} \oplus \mathcal{T}_{-4}(1) \oplus \mathcal{T}_{-4}(2)$, see Lemma 2.6. Hence, in general, a graded root carries more information than its $\mathbb{Z}[U]$–module.

LEMMA 2.6. [14] *Let $(\mathfrak{R},\chi)$ be a graded root. We order the set $\mathfrak{V}_{lm}$ of local minimum points as follows. The first element $v_1$ is an arbitrary vertex with $\chi(v_1) = \min\{\chi(v) : v \in \mathfrak{V}_{lm}\}$. If $v_1,\dots,v_k$ is already determined, and $J := \{v_1,\dots,v_k\} \subsetneq \mathfrak{V}_{lm}$, then let $v_{k+1}$ be an arbitrary vertex in $\mathfrak{V}_{lm} \setminus J$ with $\chi(v_{k+1}) = \min\{\chi(v) : v \in \mathfrak{V}_{lm} \setminus J\}$. Let $w_{k+1} \in \mathfrak{V}$ be the unique $\succ$–minimal vertex of $\mathfrak{R}$ which dominates both $v_{k+1}$, and at least one vertex from $J$. Then one has the following isomorphism of $\mathbb{Z}[U]$–modules*

$$\mathbb{H}(\mathfrak{R},\chi) \simeq \mathcal{T}^+_{2\chi(v_1)} \oplus \bigoplus_{k\geq 2} \mathcal{T}_{2\chi(v_k)}\big(\chi(w_k)-\chi(v_k)\big).$$

*In particular, with the notations* $\min\chi := \min\{\chi(v) : v \in \mathfrak{V}_{lm}\}$ *and*

$$\mathbb{H}_{red}(\mathfrak{R},\chi) := \bigoplus_{k\geq 2} \mathcal{T}_{2\chi(v_k)}\big(\chi(w_k)-\chi(v_k)\big),$$

*one has a canonical direct sum decomposition:* $\mathbb{H}(\mathfrak{R},\chi) \simeq \mathcal{T}^+_{2\min\chi} \oplus \mathbb{H}_{red}(\mathfrak{R},\chi)$. *The $\mathbb{Z}[U]$–module $\mathbb{H}_{red}(\mathfrak{R},\chi)$ has finite $\mathbb{Z}$–rank. Moreover, $\mathbb{H}_{red}(\mathfrak{R}\chi) = 0$ if and only if $\#\big(\mathfrak{V}_{lm}\big) = 1$ and $\mathfrak{R} = \mathfrak{R}_{\min\chi}$.*

*Remark* 2.7. The $\mathbb{Z}[U]$–module splitting $\mathbb{H}(\mathfrak{R},\chi) \simeq \mathcal{T}^+_{2m_0} \oplus \mathbb{H}_{red}(\mathfrak{R},\chi)$ and the presentation of $\mathbb{H}_{red}(\mathfrak{R},\chi)$ as $\bigoplus_k \mathcal{T}_{2m_k}(n_k)$ are (non–natural but) canonical: the integers $m_0$ and $\{(m_k,n_k)\}_k$ depend only on the $\mathbb{Z}[U]$–module isomorphism type of $\mathbb{H}(\mathfrak{R},\chi)$. Indeed, the summand $\mathcal{T}^+_{2m_0}$ can be characterized as follows: $2m_0$ is the smallest degree of a homogeneous element $x \in \mathbb{H}(\mathfrak{R},\chi)$ which can be written as $U^k y$ for arbitrarily large $k$. Then, one proves the existence of a graded $\mathbb{Z}[U]$–module splitting $\mathbb{H}(\mathfrak{R},\chi) \simeq \mathcal{T}^+_{2m_0} \oplus \mathbb{H}_{red}(\mathfrak{R},\chi)$. For $\mathbb{H}(\mathfrak{R},\chi)$ associated with a graded root as above, one sees that $m_0 = \min\chi$.

Next we concentrate on $\mathbb{H}_{red}(\mathfrak{R},\chi)$. Let us consider a torsion module $\mathbb{M}$ and let $\mathbb{M}_{2k} = \{x \in \mathbb{M} \mid \deg(x) = 2k\}$. Define

$$2l := \max\{\deg(x) : x \in \mathbb{M} \text{ homogeneous}\}$$

and take the submodule generated by $\mathbb{M}_{2l}$, i.e. $\sum_k U^k\mathbb{M}_{2l} \leq \mathbb{M}$. Denote by $n := \min\{k \in \mathbb{N} \mid \ker(U^k|_{\mathbb{M}_{2l}}) \neq 0\}$. Then the submodule $\sum_k U^k\mathbb{M}_{2l}$ will have $r := \mathrm{rank}_{\mathbb{Z}}\ker(U^n|_{\mathbb{M}_{2l}})$ many direct summands of the form $\mathcal{T}_{2m}(n)$ with $m = l - n + 1$. Hence $\mathbb{M}$ splits as $\mathbb{M}' \oplus \bigoplus_{i=1}^{r} \mathcal{T}_{2m}(n)$ and we may proceed by induction.

In particular, $\mathbb{H}(\mathfrak{R}, \chi)$ can be written in the form $\mathcal{T}^+_{2m_0} \oplus \bigoplus_{k\geq 1} \mathcal{T}_{2m_k}(n_k)$ in a unique way.

2.2.1. **Degrees of the local minimum points.** Consider the multiset $\chi\big(\mathfrak{V}_{lm}\big)$ of the degrees of the local minimum points of the graded root. It can be recovered from the $U$–action on $\mathbb{H}(\mathfrak{R}, \chi)$ in the following way:

PROPOSITION 2.8. *The degree $m$ appears in the multiset $\chi\big(\mathfrak{V}_{lm}\big)$ with multiplicity $k_m$, where*

$$k_m = \mathrm{rank}_{\mathbb{Z}}\left(\ker\left(\mathbb{H}_{2m}(\mathfrak{R}, \chi) \xrightarrow{U} \mathbb{H}_{2m-2}(\mathfrak{R}, \chi)\right)\right).$$

*Proof.* Use the canonical direct sum decomposition of Lemma 2.6. □

## 2.3. The graded root of a lattice and its local minimum points

2.3.1. **The graded root associated with a lattice and a weight function.** Fix a free $\mathbb{Z}$–module, a basis $\{E_i\}_i$, and a system of compatible weight functions $\{w_q\}_q$. Consider the sequence of topological spaces (finite cubical complexes) $\{S_n\}_{n\geq m_w}$ with $S_n \subset S_{n+1}$ defined in paragraph 2.1.4. Let $\pi_0(S_n) = \{\mathcal{C}_n^1, \ldots, \mathcal{C}_n^{p_n}\}$ be the set of connected components of $S_n$.

Then we define the graded graph $(\mathfrak{R}_w, \chi_w)$ as follows. The vertex set $\mathfrak{V}(\mathfrak{R}_w)$ is $\cup_{n\in\mathbb{Z}}\pi_0(S_n)$. The grading $\chi_w : \mathfrak{V}(\mathfrak{R}_w) \to \mathbb{Z}$ is $\chi_w(\mathcal{C}_n^j) = n$, that is, $\chi_w|_{\pi_0(S_n)} \equiv n$. Furthermore, if $\mathcal{C}_n^i \subset \mathcal{C}_{n+1}^j$ for some $n$, $i$ and $j$, then we introduce an edge $[\mathcal{C}_n^i, \mathcal{C}_{n+1}^j]$. All the edges of $\mathfrak{R}_w$ are obtained in this way.

One verifies that $(\mathfrak{R}_w, \chi_w)$ satisfies all the required properties of the definition of a graded root, except maybe the last one: $|\chi_w^{-1}(n)| = 1$ whenever $n \gg 0$. However, the graded roots associated with plane curve singularities satisfy this condition as well (cf. Theorem 3.2).

This construction also works for weighted subcomplexes $(T, w)$ as in paragraph 2.1.5.

PROPOSITION 2.9. [18, Theorem 11.2.15] *If $(\mathfrak{R}_w, \chi_w)$ is a graded root associated with $(T, w)$ and $\#\{\chi_w^{-1}(n)\} = 1$ for all $n \gg 0$, then*

$$\mathbb{H}(\mathfrak{R}_w, \chi_w) = \mathbb{H}^0(T, w).$$

*Definition* 2.10. Consider a closed cubical complex $T$ as in paragraph 2.1.5 and compatible weight functions $\{w_q\}_q$ on it. Then a lattice point $\ell$ is a *local minimum point* of $w$ if

$$w_0(\ell) < w_q(\square_q) \text{ for all } q\text{–cubes } \square_q \text{ such that } \ell \text{ is a vertex of } \square_q.$$

In this case, $w_0(\ell)$ is called a *local minimum value.* Let us denote the set of local minimum points of $w_0$ by $\mathcal{W}_{lm}$.

*Remark* 2.11. If the weight functions $\{w_q\}_{q\geq 1}$ are derived from $w_0$ through formula (1), then $\ell$ is a local minimum point if and only if

$$w_0(\ell) < w_0(\ell \pm E_i) \text{ for all } i \in \mathcal{I}.$$

This definition is in fact adopted to our case of the weight functions of curve singularities. For arbitrary weight functions $w_0$ (e.g. for the topological weight function of surface singularities, cf. [16]) it can happen that $w_0$ is constant along a larger zone (cubical subcomplex). However, the weight function of curve singularities satisfies $w_0(\ell) \neq w_0(\ell \pm E_i)$ for all $\ell$ and $i$ (see Remark 3.1). This explains why we choose the above simple version in Definition 2.10. It also has the consequence that the 'local minimum zones' of $w_0$ are in fact isolated points.

The local minimum points of a graded root $(\mathfrak{R}_w, \chi_w)$ correspond to the local minimum points of the weight function $w$:

Proposition 2.12. *The map $\psi : \mathcal{W}_{lm} \to \mathfrak{V}_{lm} : \ \ell \mapsto \mathcal{C}^j_{w_0(\ell)}$, where $\ell \in \mathcal{C}^j_{w_0(\ell)}$, is a well-defined injection. If the weight functions are defined via (1) and we also know that $w_0(\ell) \neq w_0(\ell \pm E_i)$ for all $\ell$ and $i \in \mathcal{I}$ (like in the case of the lattice cohomology of curve singularities, see Remark 3.1), then $\psi$ is a bijection.*

*Proof.* By Definition 2.10 the local minimum lattice point $\ell$ must be a distinct component of $S_{w_0(\ell)}$, and thus this component $\mathcal{C}^j_{w_0(\ell)}$ cannot have any preimage under the injection $S_{w_0(\ell)-1} \hookrightarrow S_{w_0(\ell)}$ therefore $\kappa_{\mathcal{C}^j_{w_0(\ell)}} = 1$. So $\psi$ is indeed well-defined and clearly injective.

For surjectivity we have to prove that every component $\mathcal{C}^j_n$, with no preimage under the inclusion $S_{n-1} \hookrightarrow S_n$, must consist of a single lattice point $\ell$ with weight $w_0(\ell) = n$. As $\mathcal{C}^j_n \subset S_n \setminus S_{n-1}$, then all its cubes must have weight $n$, but as $w_0(\ell) \neq w_0(\ell \pm E_i)$ for all $i \in \mathcal{I}$, it cannot contain adjacent lattice points, so neither higher dimensional cubes, thus, by connectivity, $\mathcal{C}^j_n$ must just be a single lattice point. □

Corollary 2.13. *Take a closed cubical complex $T$ as in paragraph 2.1.5 and compatible weight functions $\{w_q\}_q$ on it, defined by the identities of (1), having the property that*

$$w_0(\ell) \neq w_0(\ell \pm E_i) \text{ for all } \ell \text{ and } i \in \mathcal{I}.$$

*Consider the lattice cohomology module* $\mathbb{H}(T,w)$ *and the graded root* $\mathfrak{R}(T,w)$ *associated with it. Then* $w(\mathcal{W}_{lm}) = \chi(\mathfrak{V}_{lm})$ *as multisets, and they can be recovered from the* $\mathbb{Z}[U]$*–module structure of* $\mathbb{H}(T,w)$ *as well.*

## 3. THE LATTICE COHOMOLOGY OF CURVES

### 3.1. The definition and first properties of the lattice cohomology [3]

3.1.1. **Some classical invariants of a curve via its normalization.** Let $(C,o)$ be an isolated curve singularity with local algebra $\mathcal{O}=\mathcal{O}_{C,o}$. Let $\cup_{i=1}^r(C_i,o)$ be the irreducible decomposition of $(C,o)$ and denote the local algebra of $(C_i,o)$ by $\mathcal{O}_i$. We denote the integral closure of $\mathcal{O}_i$ by $\overline{\mathcal{O}_i}=\mathbb{C}\{t_i\}$, and we consider $\mathcal{O}_i$ as a subring of $\overline{\mathcal{O}_i}$. Similarly, we denote the integral closure of $\mathcal{O}$ by $\overline{\mathcal{O}}=\oplus_i\mathbb{C}\{t_i\}$. Let $\delta=\delta(C,o)$ be the delta invariant $\dim_{\mathbb{C}}\overline{\mathcal{O}}/\mathcal{O}$ of the germ $(C,o)$.

We denote by $\mathfrak{v}_i:\overline{\mathcal{O}_i}\to\overline{\mathbb{Z}_{\geq 0}}=\mathbb{Z}_{\geq 0}\cup\{\infty\}$ the discrete valuation of $\overline{\mathcal{O}_i}$, where $\mathfrak{v}_i(0)=\infty$. This restricted to $\mathcal{O}_i$ reads as $\mathfrak{v}_i(g)=\mathrm{ord}_{t_i}(g)$ for any $g\in\mathcal{O}_i$. Let $\mathcal{S}_C=(\mathfrak{v}_1,\dots,\mathfrak{v}_r)(\mathcal{O})\cap(\mathbb{Z}_{\geq 0})^r$, or

$$\mathcal{S}_C=\{\mathfrak{v}(g):=(\mathfrak{v}_1(g),\dots,\mathfrak{v}_r(g))\,:\,g \text{ is a non-zero divisor in } \mathcal{O}\}\subset(\mathbb{Z}_{\geq 0})^r.$$

It is called the *semigroup (monoid) of values* of $(C,o)$.

Let $\mathfrak{c}=(\mathcal{O}:\overline{\mathcal{O}})$ be the *conductor ideal* of $\overline{\mathcal{O}}$, it is the largest ideal of $\mathcal{O}$ which is an ideal of $\overline{\mathcal{O}}$ too. It has the form $(t_1^{c_1},\dots,t_r^{c_r})\overline{\mathcal{O}}$. The lattice point $\mathbf{c}=(c_1,\dots,c_r)$ is called the *conductor* of $\mathcal{S}_C$. From the definitions, $\mathbf{c}+(\mathbb{Z}_{\geq 0})^r\subset\mathcal{S}_C$ and $\mathbf{c}$ is the smallest lattice point with this property. I.e.

$$\mathbf{c}+(\mathbb{Z}_{\geq 0})^r=\{\mathfrak{v}(g)\,|\,g\in\mathfrak{c}\}. \tag{2}$$

In the case of irreducible curves, the delta invariant can be computed from the semigroup: $\delta(C,o)=\#\{\mathbb{Z}_{\geq 0}\setminus\mathcal{S}_C\}$, in contrast with multiple components, where $\#\{\mathbb{Z}^r_{\geq 0}\setminus\mathcal{S}_C\}=\infty$.

3.1.2. **The valuative filtrations.** Consider the lattice $\mathbb{Z}^r$ with its natural basis $\{E_i\}_{i=1}^r$ and partial ordering defined coordinatewise (cf. paragraph 2.1.1). If $\ell=(\ell_1,\dots,\ell_r)\in\mathbb{Z}^r$ we set $|\ell|:=\sum_i\ell_i$. Then $\overline{\mathcal{O}}$ has a filtration indexed by $\ell\in\mathbb{Z}^r$ given by $\overline{\mathcal{F}}(\ell):=\{g\in\overline{\mathcal{O}}\,:\,\mathfrak{v}(g)\geq\ell\}$. Notice that $\overline{\mathcal{F}}(\ell)=\overline{\mathcal{F}}(\max\{\ell,0\})$. This induces an ideal filtration of $\mathcal{O}$ by $\mathcal{F}(\ell):=\overline{\mathcal{F}}(\ell)\cap\mathcal{O}\subset\mathcal{O}$. By (2) we have $\mathcal{F}(\mathbf{c})=\overline{\mathcal{F}(\mathbf{c})}=\mathfrak{c}$.

Set $\mathfrak{h}(\ell)=\dim\mathcal{O}/\mathcal{F}(\ell)$. Then $\mathfrak{h}$ is increasing and $\mathfrak{h}(0)=0$. It is called the Hilbert function of the filtration.

If $(C,o)$ is not smooth, $\dim(\overline{\mathcal{O}}/\mathfrak{c})=|\mathbf{c}|$ and $\dim(\mathcal{O}/\mathfrak{c})=\mathfrak{h}(\mathbf{c})$ implies that $\delta=|\mathbf{c}|-\mathfrak{h}(\mathbf{c})$. Since $\mathfrak{h}(\mathbf{c})\geq 1$, $\delta\leq|\mathbf{c}|-1$, with equality if and only if $\mathfrak{c}$ is the maximal ideal of $\mathcal{O}$. On the other hand, $\delta \geq |\mathbf{c}|/2$, with equality if and only if $(C,o)$ is Gorenstein (cf. [22, page 72]).

3.1.3. **The weight functions.** We consider the lattice $\mathbb{Z}^r$ with its fixed basis $\{E_i\}_{i=1}^r$, and the functions $\mathfrak{h}$ and $|\cdot|$ defined in paragraph 3.1.2. We also set $\mathcal{I} := \{1,\dots,r\}$. For the construction of the lattice cohomology of $(C,o)$ we consider the lattice points $\mathbb{Z}^r$ in $\mathbb{R}^r$ and the cubes from $\mathbb{R}^r$. The weight function on the lattice points $\ell \in \mathbb{Z}^r$ is defined by

$$w_0(\ell) = 2\cdot \mathfrak{h}(\ell) - |\ell|,$$

and for any $q$-cube $\square \in \mathcal{Q}_q$, $w_q(\square) := \max\{w_0(\ell) \,:\, \ell \text{ is a vertex of } \square\}$.

*Remark* 3.1. Notice that this weight function satisfies the property that

$$w_0(\ell) \neq w_0(\ell \pm E_i) \text{ for all } \ell \text{ and } i \in \mathcal{I}.$$

Indeed, the two sides of the equation have different parity. Therefore the compatible weight functions $\{w_q\}_q$ satisfy the conditions of Proposition 2.12.

In fact, if $(C,o)$ is irreducible, $w_0(\ell+E_i) = w_0(\ell)+1$ if and only if $\ell \in \mathcal{S}_C$ and $w_0(\ell+E_i) = w_0(\ell)-1$ if and only if $\ell \notin \mathcal{S}_C$.

3.1.4. **The lattice cohomology and the graded root.** The cubical decomposition of $\mathbb{R}^r$ and the weight function $\ell \mapsto w_0(\ell)$ define a lattice cohomology and a graded root. They are denoted by $\mathbb{H}^*(C,o)$ and $\mathfrak{R}(C,o)$ respectively. From the very construction we get

$$\mathbb{H}^q(C,o) = 0 \text{ for any } q \geq r.$$

THEOREM 3.2. [3]
*(a) For any $c \geq \mathbf{c}$ the inclusion $S_n \cap R(0,c) \hookrightarrow S_n$ is a homotopy equivalence. In particular, $S_n$ is contractible for $n \gg 0$.*

*(b) One has a graded $\mathbb{Z}[U]$–module isomorphism $\mathbb{H}^*(C,o) = \mathbb{H}^*(R(0,c),w)$ and a graded root isomorphism $\mathfrak{R}(C,0) = \mathfrak{R}(R(0,c),w)$ for any $c \geq \mathbf{c}$ induced by the natural inclusion map. Therefore, $\mathbb{H}^*(C,o)$ and $\mathfrak{R}(C,0)$ are determined by the weighted cubes of the rectangle $R(0,\mathbf{c})$ and $\mathbb{H}^*_{red}(C,o)$ has finite $\mathbb{Z}$–rank.*

*(c) $eu(\mathbb{H}^*(C,o)) = \delta(C,o)$, that is, $\mathbb{H}^*(C,o)$ is a 'categorification' of the delta invariant $\delta(C,o)$.*

# 4. PLANE CURVE SINGULARITIES

## 4.1. Embedded resolutions of plane curve singularities

Let $(C,0) \subset (\mathbb{C}^2,0)$ be an isolated plane curve singularity. This means that it is the zero set of a single reduced holomorphic germ $f \in \mathcal{O}_{\mathbb{C}^2,0}$.

In this section we consider certain objects associated with the embedded data and they will be compared with the abstract geometry of the germ discussed in section 3.

A good embedded resolution of $(C,0)$ is a proper map $\phi:\tilde{X}\to S$, where $\tilde{X}$ is a two dimensional complex manifold, $S\subset\mathbb{C}^2$ is a small Stein representative of the space–germ $(\mathbb{C}^2,0)$, $\phi^*f$ is a normal crossing divisor, and the restriction

$$\phi\big|_{\tilde{X}\setminus E}:\tilde{X}\setminus E\to S\setminus\{0\}$$

is an analytic isomorphism. The map $\phi$ is a minimal good embedded resolution, if any other good embedded resolution dominates it. Any good embedded resolution can be achieved by a sequence of blow–ups. For more details regarding the resolution, the dual graph, etc. see [5, 25].

We assume that $(C,0)$ is not smooth, hence $E$ is 1–dimensional.

## 4.2. **Lattices for irreducible plane curve singularities**

4.2.1. **Lattice associated to a resolution.** Let $\phi:\tilde{X}\to\mathbb{C}^2$ be the *minimal* good embedded resolution of $(C,0)\subset(\mathbb{C}^2,0)$ — fixed hereinafter — with dual embedded resolution graph $\Gamma$, whose vertices are numbered by $\{1,\dots,n\}$ in order of appearance in the blow–up procedure. E.g., $E_1$ is the exceptional curve of the first blow–up and $E_n$ of the very last one (which supports the strict transform of $(C,0)$). We also write $\mathcal{V}$ for the indexing set $\{1,\dots,n\}$. Let $\{E_v\}_{v\in\mathcal{V}}$ be the corresponding irreducible components of the exceptional set $E=\phi^{-1}(0)$. $\Gamma$ is a connected tree and all $E_v$ are rational. The link $\Sigma=\partial B^4=\partial\tilde{X}$ is $S^3$ [5, 25].

Define the lattice $L=H_2(\tilde{X},\mathbb{Z})$. It is freely generated by the classes of the exceptional divisors $E_v$, $v\in\mathcal{V}$, that is, $L=\oplus_{v\in\mathcal{V}}\mathbb{Z}\langle E_v\rangle$. $L$ carries a natural negative–definite intersection form $\langle -,-\rangle_\Gamma$. For more details regarding $L$, see [18]. Set $L':=\mathrm{Hom}(H_2(\tilde{X},\mathbb{Z}),\mathbb{Z})$. $L'$ can be identified with $H_2(\tilde{X},\Sigma,\mathbb{Z})$ via the perfect pairing $L\otimes H_2(\tilde{X},\Sigma,\mathbb{Z})\to\mathbb{Z}$.

By the homology exact sequence of the pair $(\tilde{X},\Sigma)$ one has

$$0\to L\to H_2(\tilde{X},\Sigma,\mathbb{Z})\to H_1(\Sigma,\mathbb{Z})\to 0.$$

In particular, $L'/L\cong H_1(\Sigma,\mathbb{Z})=0$. It is convenient to introduce the dual generator set for $L'$ too, namely $L'=\oplus_{v\in\mathcal{V}}\mathbb{Z}\langle E_v^*\rangle$, where $\langle E_u^*,E_v\rangle_\Gamma=-\delta_{u,v}$ (Kronecker delta) for any $u,v\in\mathcal{V}$. The elements $E_v^*$ have the following geometrical interpretation: let $D_v\subset\tilde{X}$ be a curvetta associated with $E_v$, that is, a smooth irreducible curve in $\tilde{X}$ intersecting $E_v$ transversely at a generic point. Then the divisor $D_v+E_v^*$ is numerically trivial, i.e. $(D_v+E_v^*,E_u)_{\tilde{X}}=0$ for any $u$.

4.2.2. **Divisors of functions.** If $g:(\mathbb{C}^2,0)\to(\mathbb{C},0)$ is the germ of an analytic function, then the divisor $\mathrm{div}(g\circ\phi)$ on $\tilde{X}$ is the sum of a divisor $\mathrm{div}_E(\phi^*g)=\sum_v m_vE_v$ supported on $E$ (where $m_v$ is the vanishing order of $\phi^*g:=g\circ\phi$ along $E_v$) and of the strict transform $\mathrm{str}\{g=0\}$ of $\{g=0\}$.

For example, if $f$ is irreducible and $\phi$ is the minimal good embedded resolution, then the strict transform $\mathrm{str}\{f=0\}=\mathrm{str}(C)$ intersects $E$ along $E_n$ transversely, and $\mathrm{div}_E(\phi^*f)=E^*_n$.

Note also that if $\mathfrak{v}$ is the valuation associated with the irreducible plane curve germ $f$ (cf. subsection 3.1.1), then $\mathfrak{v}(g)=i_0(f,g)$ for any germ $g$, where $i_0(f,g)$ denotes the intersection multiplicity of $f$ and $g$ at $0\in\mathbb{C}^2$.

Lemma 4.1. *(a) For any* $g\in\mathcal{O}_{\mathbb{C}^2,0}$ *one has* $i_0(g,f)\geq \mathrm{mult}_{E_n}(\mathrm{div}_E(\phi^*g))$, *with equality if and only if* $\mathrm{str}\{g=0\}\cap\mathrm{str}\{f=0\}=\emptyset$.

*(b) If additionally the strict transform of* $\{g=0\}$ *intersects* $E_n$, *then* $\mathrm{mult}_{E_n}(\mathrm{div}_E(\phi^*g))\geq\mathrm{mult}_{E_n}(\mathrm{div}_E(\phi^*f))$.

*Proof.* Set $p:=\mathrm{str}\{f=0\}\cap E_n$. Then by the projection formula $i_0(f,g)=\mathrm{mult}_{E_n}(\mathrm{div}_E(\phi^*g))+i_p(\mathrm{str}\{f=0\},\mathrm{str}\{g=0\})$, which implies *(a)*. The divisor $\mathrm{div}_E(\phi^*g)+\mathrm{str}\{g=0\}$ is linearly equivalent to zero, hence $\mathrm{div}_E(\phi^*g)=\sum_v(\mathrm{str}\{g=0\},E_v)_{\tilde{X}}E^*_v$. Since $(\mathrm{str}\{g=0\},E_n)_{\tilde{X}}\geq 1$ we get that $\mathrm{div}_E(\phi^*g)-\mathrm{div}_E(\phi^*f)$ can be written as $\sum_v n_vE^*_v$ with $n_v\geq 0$. Then use 4.2.3(a). □

4.2.3. **The Lipman cone.** For $l_1,l_2\in L$ with $l_i=\sum_v l_{iv}E_v$ $(i=1,2)$ one considers the usual partial ordering: $l_1\geq l_2$ if and only if $l_{1v}\geq l_{2v}$ for all $v\in\mathcal{V}$. The cycle $l\in L$ is called effective if $l\geq 0$ with respect to this ordering. We also set $\min\{l_1,l_2\}:=\sum_v\min\{l_{1v},l_{2v}\}E_v$.

For any fixed resolution with graph $\Gamma$ we define the Lipman cone by

$$\mathcal{S}_\Gamma:=\{l\in L\mid \langle l,E_v\rangle_\Gamma\leq 0 \text{ for all } v\in\mathcal{V}\}.$$

From the definition of the dual base elements $\{E^*_v\}_v$ one deduces that $\mathcal{S}_\Gamma$ is a cone generated over $\mathbb{Z}_{\geq 0}$ by these $\{E^*_v\}_v$. Moreover, one has the following additional properties ((a) follows from the fact that for any $v$ all the entries of $E^*_v$ are strict positive, see e.g. [18, Corollary 2.1.19]):

(a) if $s\in\mathcal{S}_\Gamma\setminus\{0\}$ then all the entries of $s$ are strict positive;

(b) $s_1,s_2\in\mathcal{S}_\Gamma$ implies $\min\{s_1,s_2\}\in\mathcal{S}_\Gamma$;

(c) for any $g\in\mathcal{O}_{\mathbb{C}^2,0}$ one has $\mathrm{div}_E(\phi^*g)\in\mathcal{S}_\Gamma$;

(d) for any $s\in\mathcal{S}_\Gamma\setminus\{0\}$ there exists an analytic function germ $g\in\mathcal{O}_{\mathbb{C}^2,0}$ such that $\mathrm{div}_E(\phi^*g)=s$, see e.g. [18, 7.1.13]. In fact, we have the following additional base-point freeness property as well: for any point $p \in E$ one can choose the germ $g$ in such a way that $p \notin \mathrm{str}\{g=0\}$.

Indeed, the cycle $s$ can be completed in $\tilde{X}$ by some curvettas $D$ such that the divisor $s+D$ is numerically trivial on $\tilde{X}$ and $p$ does not belong to the support of $D$. Then, since $\mathrm{Pic}^0(\tilde{X}) = 0$, there exists $g$ with $\mathrm{div}(\phi^* g) = s + D$ (cf. [18, Prop. 7.1.10]).

4.2.4. **The anticanonical cycle.** The *anticanonical cycle* $Z_K \in L$ is characterized uniquely by the linear system of *adjunction relations*

$$\langle -Z_K + E_v, E_v\rangle_\Gamma + 2 = 0, \text{ for all } v \in \mathcal{V}.$$

(Since the intersection matrix is unimodular, the solution is indeed an integral cycle.) One can define $Z_K$ also as the negative of the divisor of the differential form $\phi^*(dx\wedge dy)$, where $(x,y)$ are analytic local coordinates of $(\mathbb{C}^2, 0)$. E.g. the pullback of $dx \wedge dy$ after the first blow–up is $d(st)\wedge dt = tds \wedge dt$, hence $\mathrm{mult}_{E_1}(Z_K) = -1$. By induction, one verifies that $-Z_K \geq E$.

Lemma 4.2. *In fact, $-Z_K \geq E_1^*$ (and by paragraph 4.2.3, $E_1^* \geq E$).*

*Proof.* Assume that $(x, y)$ are analytic local coordinates, whith $x$ generic. Then its strict transform is a generic curvetta of $E_1$, so $\mathrm{div}_E(\phi^* x) = E_1^*$. Hence, what we have to show is that $\mathrm{div}_E(\phi^* \frac{dx\wedge dy}{x})$ is effective. This follows by induction on the number of blow–ups. □

4.2.5. **The Riemann–Roch expression.** For any non-zero effective cycle $l \in L$, the analytic Euler characteristic $\chi(\mathcal{O}_l) = h^0(\mathcal{O}_l) - h^1(\mathcal{O}_l)$ can be computed combinatorially: $\chi(\mathcal{O}_l) = -\langle l, l - Z_K\rangle_\Gamma/2$ by the Riemann–Roch theorem. This motivates the definition $\chi(l) := \chi(\mathcal{O}_l) = -\langle l, l - Z_K\rangle_\Gamma/2$ for any $l \in L$.

## 4.3. Generalities about computation sequences

*Definition* 4.3. Sequences $z_0, z_1, \ldots, z_t \in L$ with $z_{i+1} = z_i + E_{v(i)}$ (for $0 \leq i < t$), where $v(i)$ is determined by some principles fixed in each individual case, are called *computation sequences* connecting $z_0$ and $z_t$.

Lemma 4.4. [14, Lemma 7.4] *For any $l \in L$, there exists a unique minimal lattice point $s \in L$ with the properties $s \geq l$ and $s \in \mathcal{S}_\Gamma$. We call it $s(l)$. Equivalently, this is the unique minimal element of $\{s \in \mathcal{S}_\Gamma : s - l \in L_{\geq 0}\}$.*

Lemma 4.5. (Generalized Laufer's algorithm [12, 14]) *Let $l \in L$. Construct a computation sequence $z_0, \ldots, z_t$ by the following algorithm. Set*

*$z_0 = l$. Assume that $z_i$ is already constructed. If $z_i \notin \mathcal{S}_\Gamma$, i.e. $\langle z_i, E_{v(i)}\rangle_\Gamma > 0$ for some index $v(i)$, and write $z_{i+1} = z_i + E_{v(i)}$. If $z_i \in \mathcal{S}_\Gamma$, then stop and write $t = i$. The procedure necessarily stops after finitely many steps, and $z_t \in \mathcal{S}_\Gamma$ is exactly $s(l)$ considered in Lemma 4.4.*

In the next paragraphs we wish to generalize Lemmas 4.4 and 4.5. We prefer to write $\mathcal{V} = \{n\} \cup \mathcal{V}^*$, hence to distinguish the base element $E_n$ (which intersects $\mathrm{str}\{f = 0\}$).

Lemma 4.6. [14] *For any integer $\ell \geq 0$, there exists a unique cycle $x(\ell) \in L$, in the sequel called 'universal cycle', with the following properties:*

*(a)* $\mathrm{mult}_{E_n}(x(\ell)) = \ell$;
*(b)* $\langle x(\ell), E_v\rangle_\Gamma \leq 0$ *for any* $v \in \mathcal{V}^*$;
*(c)* $x(\ell)$ *is minimal with properties (a) and (b).*
*Moreover, the cycle $x(\ell)$ satisfies $x(\ell) \geq 0$.*

Lemma 4.7. (The computation sequence from $x(\ell)$ to $x(\ell+1)$ [14]) *For any integer $\ell \geq 0$ consider a computation sequence constructed as follows. Set $z_0 = x(\ell)$, $z_1 = x(\ell) + E_n$. Assume that $z_i$ $(i \geq 1)$ is already constructed. If $z_i$ does not satisfy 4.6 (b), then there exists some $v \in \mathcal{V}^*$ with $\langle z_i, E_v\rangle_\Gamma > 0$. Then choose one of these indices for $v(i)$, and write $z_{i+1} = z_i + E_{v(i)}$. If $z_i$ satisfies 4.6 (b), then stop and write $t = i$. Then $z_t$ is exactly $x(\ell+1)$. In particular, $x(\ell+1) > x(\ell)$.*

Lemma 4.5 has the following easy generalization (with the same proof):

Lemma 4.8. *Assume that $\ell \in L$ satisfies $\mathrm{mult}_{E_n}(\ell) = \ell$ and $\ell \leq x(\ell)$ for some $\ell \geq 0$. Consider a similar computation sequence as in Lemma 4.7. Namely, set $z_0 = \ell$. Assume that $z_i$ is already constructed. If for some $v \in \mathcal{V}^*$ one has $\langle z_i, E_v\rangle_\Gamma > 0$ then take $z_{i+1} = z_i + E_{v(i)}$, where $v(i)$ is such an index $v$. If $z_i$ satisfies 4.6 (b), then stop and write $t = i$. Then $z_t$ is exactly $x(\ell)$.*

Notice that, even if it is not explicitly emphasized in its notation, the cycles $\{x(\ell)\}_{\ell\geq 0}$ depend on the choice of the distinguished vertex $E_n$. For the definition and properties of the cycle $x(\ell)$ in more general situations see e.g. [9, 10, 14].

Lemma 4.9 below describes when the two universal cycles $s(\ell E_n)$ and $x(\ell)$ coincide.

Lemma 4.9. *The following facts are equivalent:*

*(a)* $\langle x(\ell), E_n\rangle_\Gamma \leq 0$,
*(b)* $x(\ell) \in \mathcal{S}_\Gamma$, *i.e.* $x(\ell)$ *belongs to the Lipman cone of* $\Gamma$,
*(c)* $x(\ell) = s(\ell E_n)$,
*(d)* $\ell \in \mathcal{S}_C$, *i.e.* $\ell$ *is an element of the numerical semigroup of* $(C, 0)$.

*Proof.* *(c)⇒(a)⇒(b)* are clear from the definitions. Assume *(b)*, Lemma 4.8 applied for $l=\ell E_n$ and Lemma 4.5 imply *(c)*. Next we prove *(b)⇒(d)*. Part (d) of subsection 4.2.3 guarantees the existence of some $g\in\mathcal{O}_{\mathbb{C}^2,0}$ such that $\mathrm{div}_E(\phi^*g)=x(\ell)$ and $\mathrm{str}\{g=0\}\cap\mathrm{str}\{f=0\}=\emptyset$. In particular (see Lemma 4.1), $i_0(f,g)=\mathrm{mult}_{E_n}(\mathrm{div}_E(\phi^*g))=\mathrm{div}_{E_n}(x(\ell))=\ell$, hence $\ell\in\mathcal{S}_C$.

Finally we prove *(d)⇒(a)*. For any $\ell\in\mathcal{S}_C$ there exists a function germ $g$ such that $\ell=i_0(f,g)$. By [23] we can choose $g$ such that the strict transform of $g$ intersects only those exceptional components $E_v$ with $\kappa_v=1$. In particular, $\mathrm{str}\{g=0\}\cap\mathrm{str}\{f=0\}=\emptyset$ and $\mathrm{mult}_{E_n}(\mathrm{div}_E(\phi^*g))=\ell$. Since $\langle\mathrm{div}_E(\phi^*g),E_v\rangle_\Gamma\leq 0$ for any $v\in\mathcal{V}$, from the minimality of $x(\ell)$ we obtain $x(\ell)\leq\mathrm{div}_E(\phi^*g)$. That is, $\mathrm{div}_E(\phi^*g)-x(\ell)$ is an effective cycle whose support does not contain $E_n$. Therefore, $\langle\mathrm{div}_E(\phi^*g)-x(\ell),E_n\rangle_\Gamma\geq 0$. Hence $\langle x(\ell),E_n\rangle_\Gamma\leq\langle\mathrm{div}_E(\phi^*g),E_n\rangle_\Gamma=0$. □

## 4.4. **Comparison of the abstract geometry with the embedded one**

We fix an irreducible plane curve singularity $(C,0)=(\{f=0\},0)$. Recall that for any $\ell\in\mathbb{Z}_{\geq 0}$ we can define the ideal $\mathcal{F}(\ell)=\{g\in\mathcal{O}_{C,0}\,:\,\mathfrak{v}(g)\geq\ell\}$, cf. paragraph 3.1.2. Let $q:\mathcal{O}_{\mathbb{C}^2,0}\to\mathcal{O}_{C,0}$ be the natural projection. This defines the ideal $q^{-1}\mathcal{F}(\ell)=\{g\in\mathcal{O}_{\mathbb{C}^2,0}\,:\,i_0(f,g)\geq\ell\}$ in $\mathcal{O}_{\mathbb{C}^2,0}$.

On the other hand, in the presence of the minimal good embedded resolution $\phi$ of $(C,0)\subset(\mathbb{C}^2,0)$ we have the effective cycle $x(\ell)$ and the ideal $(\phi_*(\mathcal{O}_{\tilde{X}}(-x(\ell))))_0$ of $\mathcal{O}_{\mathbb{C}^2,0}$ as well. Here $(\phi_*\mathcal{G})_0$ denotes the stalk at $0\in\mathbb{C}^2$ of $\phi_*\mathcal{G}$ for any sheaf $\mathcal{G}$ over $\tilde{X}$. [The fact that an element $h\in(\phi_*(\mathcal{O}_{\tilde{X}}))_0$ belongs to $\mathcal{O}_{\mathbb{C}^2,0}$ can be seen by the next well-known argument: $h$ is a holomorphic function in a neighbourhood of $E$, but since $E$ is compact, $h$ should be constant along $E$. Hence it factorizes to a continuous function in a neighbourhood of $0\in\mathbb{C}^2$, which is holomorphic in a punctured neighbourhood. Hence it is holomorphic by the normality of $\mathcal{O}_{\mathbb{C}^2,0}$.]

LEMMA 4.10.
*(a)* $(\phi_*(\mathcal{O}_{\tilde{X}}(-x(\ell))))_0\subset q^{-1}(\mathcal{F}(\ell))$ *for any* $\ell\geq 0$.
*(b)* $q^{-1}(\mathcal{F}(\ell))\subset(\phi_*(\mathcal{O}_{\tilde{X}}(-x(\ell))))_0$ *for any* $0\leq\ell\leq\mathrm{mult}_{E_n}F$, *where* $F:=\mathrm{div}_E(\phi^*f)=E_n^*$.

*Proof.* *(a)* If $g\in(\phi_*(\mathcal{O}_{\tilde{X}}(-x(\ell))))_0$ then $\mathrm{div}_E(\phi^*g)\geq x(\ell)$, hence

$$\mathrm{mult}_{E_n}(\mathrm{div}_E(\phi^*g))\geq\mathrm{mult}_{E_n}(x(\ell))=\ell.$$

Then by Lemma 4.1*(a)* $i_0(f,g)\geq\ell$, that is, $g\in q^{-1}\mathcal{F}(\ell)$.

*(b)* Take $g\in q^{-1}\mathcal{F}(\ell)$, i.e. $i_0(f,g)\geq\ell$. First we prove that

$$\mathrm{mult}_{E_n}(\mathrm{div}_E(\phi^*g))\geq\ell.$$

If $\mathrm{str}\{f = 0\} \cap \mathrm{str}\{g = 0\} = \emptyset$, then $\mathrm{mult}_{E_n}(\mathrm{div}_E(\phi^* g)) = i_0(f, g) \geq \ell$ by Lemma 4.1*(a)*.
If $\mathrm{str}\{f = 0\} \cap \mathrm{str}\{g = 0\} \neq \emptyset$, then $\mathrm{mult}_{E_n}(\mathrm{div}_E(\phi^* g)) \geq \mathrm{mult}_{E_n} F$ by Lemma 4.1*(b)*, which is greater than or equal to $\ell$ by assumption.

Finally, we have that $x(\ell) \leq x(\mathrm{mult}_{E_n}(\mathrm{div}_E(\phi^* g)))$ by the monotonicity of the $x$–operator (cf. Lemma 4.7), and $x(\mathrm{mult}_{E_n}(\mathrm{div}_E(\phi^* g))) \leq \mathrm{div}_E(\phi^* g)$ by the fact that $\mathrm{div}_E(\phi^* g) \in \mathcal{S}_\Gamma$ and by the universality of the $x$–operator. □

*Remark* 4.11. The inclusion from part *(b)* is not true in general without the assumption $\ell \leq \mathrm{mult}_{E_n} F$. Take e.g. $g \in \mathcal{O}_{\mathbb{C}^2,0}$ such that its strict transform is smooth, it intersects $E_n$ transversely at the point $p = E_n \cap \mathrm{str}\{f = 0\}$. Then $\mathrm{div}_E(\phi^* g) = F$. Choose $\ell$ as

$$i_0(f, g) = \mathrm{mult}_{E_n}(F) + i_p(\mathrm{str}\{f = 0\}, \mathrm{str}\{f = 0\}) > \mathrm{mult}_{E_n}(F),$$

hence $x(\ell) > F$ (use e.g. Example 4.12). Then we have $g \in q^{-1}\mathcal{F}(\ell)$ but $g \notin (\phi_*(\mathcal{O}_{\tilde{X}}(-x(\ell)))_0$.

4.4.1. **The universal cycle associated to the conductor.** In some cases the cycle $x(\ell)$ is a distinguished geometrical cycle closely related with the geometry of $f$, or with the cycle $F$.

*Example* 4.12. If $\ell = \mathrm{mult}_{E_n} F$ then $x(\ell) = F$. Indeed, since $\ell = \mathrm{mult}_{E_n} F$ and $F \in \mathcal{S}_\Gamma$, we get $x(\ell) \leq F$. Set $y := F - x(\ell)$. It is effective and it is supported on $\cup_{v \neq n} E_v$. Also, $\langle y, E_v \rangle_\Gamma = \langle E_n^* - x(\ell), E_v \rangle_\Gamma \geq 0$ for any $v \neq n$. Hence, by summation, $\langle y, y \rangle_\Gamma \geq 0$, which together with the negative definiteness of the intersection form implies $y = 0$.

*Example* 4.13. *(The computation of $x(\mathbf{c})$)* Let $\mathbf{c}$ be the conductor of $\mathcal{S}_C$. Note that

$$\mathrm{mult}_{E_n}(F + Z_K) = \langle -E_n^*, F + Z_K \rangle_\Gamma = \langle -F, F + Z_K \rangle_\Gamma = 2\chi(-F).$$

But, from [18, page 188] we know that $2\chi(-F) = 2\delta(C, 0)$ (this is basically A'Campo's formula for the Milnor number $\mu(C, 0) = 2\delta(C, 0)$). Moreover, in this plane curve case $2\delta(C, 0) = \mathbf{c}$ (see also subsection 5.2). It follows that $\mathrm{mult}_{E_n}(F + Z_K) = \mathbf{c}$. On the other hand, $\mathrm{mult}_{E_n}(Z_K) < 0$ (cf. Lemma 4.2). Therefore, $\mathbf{c} < \mathrm{mult}_{E_n}(F)$. Since $\mathcal{F}(\mathbf{c}) = \mathfrak{c}$, the conductor ideal, from Lemma 4.10 we deduce that

$$q^{-1}\mathfrak{c} = (\phi_*(\mathcal{O}_{\tilde{X}}(-x(\mathbf{c})))_0. \tag{3}$$

Our goal is to prove the following fact: in the minimal good embedded resolution of $(C, 0)$

$$x(\mathbf{c}) = F + Z_K. \tag{4}$$

The proof runs over several steps. Using $F \in \mathcal{S}_\Gamma$ and the adjunction formulae for $Z_K$, one verifies that $F + Z_K \in \mathcal{S}_\Gamma$ too. This, together with the minimality of $x(\mathbf{c})$ and the fact that $\mathrm{mult}_{E_n}(F + Z_K) = \mathbf{c}$, implies that $x(\mathbf{c}) \leq F + Z_K$.

In order to show the opposite inequality, we first prove that

$$q^{-1}\mathfrak{c} = (\phi_*(\mathcal{O}_{\tilde{X}}(-F - Z_K)))_0,$$

the analogue of (3). Note that $F + Z_K \in \mathcal{S}_\Gamma$ and $\mathrm{mult}_{E_n}(F + Z_K) = \mathbf{c} > 0$, hence $F + Z_K$ is non-zero and effective (cf. paragraph 4.2.3). In particular, $(\phi_*(\mathcal{O}_{\tilde{X}}(-F - Z_K)))_0$ is an ideal in $\mathcal{O}_{\mathbb{C}^2,0}$.

Now, we compare the ideals $(\phi_*(\mathcal{O}_{\tilde{X}}(-F - Z_K)))_0$ and $q^{-1}\mathfrak{c}$.

Take some $g \in (\phi_*(\mathcal{O}_{\tilde{X}}(-F - Z_K)))_0$. That is, $g \in \mathcal{O}_{\mathbb{C}^2,0}$ satisfies $\mathrm{div}_E(\phi^* g) \geq F + Z_K$. Then, from Lemma 4.1, we have

$$i_0(f, g) \geq \mathrm{mult}_{E_n}(\mathrm{div}_E(\phi^* g)) \geq \mathrm{mult}_{E_n}(F + Z_K) = \mathbf{c}.$$

Thus, via (2), $(\phi_*(\mathcal{O}_{\tilde{X}}(-F - Z_K)))_0 \subset q^{-1}\mathfrak{c}$. Next we prove that both ideals $(\phi_*(\mathcal{O}_{\tilde{X}}(-F - Z_K)))_0$ and $q^{-1}\mathfrak{c}$ of $\mathcal{O}_{\mathbb{C}^2,0}$ have the same finite codimension in $\mathcal{O}_{\mathbb{C}^2,0}$.

Indeed, consider the exact sequence of the higher direct image sheaves

$$0 \longrightarrow (\phi_*(\mathcal{O}_{\tilde{X}}(-F - Z_K)))_0 \longrightarrow (\phi_*(\mathcal{O}_{\tilde{X}}))_0 = \mathcal{O}_{\mathbb{C}^2,0} \to H^0(\mathcal{O}_{F+Z_K}) \to$$

$$\to \underbrace{(R^1\phi_*(\mathcal{O}_{\tilde{X}}(-F - Z_K)))_0}_{=0 \text{ by Grauert–Riemenschneider}} \longrightarrow \underbrace{(R^1\phi_*(\mathcal{O}_{\tilde{X}}))_0}_{=0,\text{ since } p_g(\mathbb{C}^2,0)=0} \longrightarrow \underbrace{H^1(\mathcal{O}_{F+Z_K})}_{=0 \text{ by exactness}} \to 0.$$

For the Grauert–Riemenschneider vanishing theorem see [7] (or [18, Theorem 6.4.3]). Thus

$$\dim_{\mathbb{C}} \frac{\mathcal{O}_{\mathbb{C}^2,0}}{(\phi_*(\mathcal{O}_{\tilde{X}}(-F - Z_K)))_0} = \chi(F + Z_K) = \chi(-F) = \delta(C, 0) = \dim_{\mathbb{C}} \frac{\mathcal{O}_{\mathbb{C}^2,0}}{q^{-1}\mathfrak{c}}.$$

This last equality follows from Gorenstein property [22]. Therefore, we have $q^{-1}\mathfrak{c} = (\phi_*(\mathcal{O}_{\tilde{X}}(-F - Z_K)))_0$.

Since $\mathbf{c} \in \mathcal{S}_C$, from Lemma 4.9 we get $x(\mathbf{c}) \in \mathcal{S}_\Gamma$. However, paragraph 4.2.3 (d) guarantees the existence of a holomorphic function $g \in \mathcal{O}_{\mathbb{C}^2,0}$ with $\mathrm{div}_E(\phi^* g) = x(\mathbf{c})$. Then, again, $i_0(f, g) \geq \mathrm{mult}_{E_n}(\mathrm{div}_E(\phi^* g)) = \mathbf{c}$, hence $g \in q^{-1}\mathfrak{c} \subset (\phi_*(\mathcal{O}_{\tilde{X}}(-F - Z_K)))_0$. Thus $x(\mathbf{c}) = \mathrm{div}_E(\phi^* g) \geq F + Z_K$.

4.4.2. **The weight function $w_0(\ell)$ via embedded data.** Let $(C, 0)$ be an irreducible plane curve singularity. Recall that the original definition of the analytic weight function $w(\ell)$ was given in terms of an abstract analytic invariant of $(C, 0)$, (namely its Hilbert function $\mathfrak{h}$), as $w(\ell) = 2\mathfrak{h}(\ell) - \ell$. The next theorem provides a new formula for $w(\ell)$ in terms of the embedded topology, in particular via the graph $\Gamma$ and the universal cycles $x(\ell)$ discussed in Lemma 4.6. The existence of this expression was first realized by Tamás László and the second author in a private discussion. For its generalization see [11].

THEOREM 4.14. *Let $(C,0)=(\{f=0\},0)$ be an irreducible plane curve singularity. Then*

$$w(\ell)=\langle x(\ell),F+Z_K-x(\ell)\rangle_\Gamma \quad \textit{for every } 0\le \ell\le \mathrm{mult}_{E_n}(F).$$

*Proof.* By Lemma 4.10 we have

$$\mathfrak{h}(\ell)=\dim_{\mathbb{C}}\frac{(\phi_*(\mathcal{O}_{\tilde{X}}))_0}{(\phi_*(\mathcal{O}_{\tilde{X}}(-x(\ell))))_0}.$$

From this and the long exact sequence of the higher direct image sheaves one gets the following exact sequence

$$0\to(\phi_*(\mathcal{O}_{\tilde{X}}(-x(\ell))))_0\to(\phi_*(\mathcal{O}_{\tilde{X}}))_0\to \\ \to H^0(\mathcal{O}_{x(\ell)})\to(R^1\phi_*(\mathcal{O}_{\tilde{X}}(-x(\ell))))_0\to 0$$

(via the vanishing $(R^1\phi_*(\mathcal{O}_{\tilde{X}}))_0=H^1(\mathcal{O}_{x(\ell)})=0$ from the rationality of $(\mathbb{C}^2,0)$). We get

$$\mathfrak{h}(\ell)=\chi(x(\ell))-\dim_{\mathbb{C}}(R^1\phi_*(\mathcal{O}_{\tilde{X}}(-x(\ell))))_0.$$

Next, $\langle F,x(\ell)\rangle_\Gamma=\langle E_n^*,x(\ell)\rangle_\Gamma=-\ell$, so

$$w(\ell)=2\mathfrak{h}(\ell)-\ell=2\chi(x(\ell))+\langle F,x(\ell)\rangle_\Gamma-2\cdot\dim_{\mathbb{C}}(R^1\phi_*(\mathcal{O}_{\tilde{X}}(-x(\ell))))_0.$$

Thus, what remains to prove is the next vanishing along the sequence $\{x(\ell)\}_\ell$:

CLAIM 4.15. $(R^1\phi_*(\mathcal{O}_{\tilde{X}}(-x(\ell))))_0=0$ *for all* $\ell=0,1,\dots,\mathrm{mult}_{E_n}(F)$.

We use induction on $\ell$. For $\ell=0$ the vanishing $(R^1\phi_*(\mathcal{O}_{\tilde{X}}))_0=0$ provides the statement. Next, suppose that we know the vanishing for some $\ell<\mathrm{mult}_{E_n}(F)$ and we have to prove it for $\ell+1$. We will instead prove this for every intermediary step of the generalized Laufer algorithm. First we step from $x(\ell)$ to $x(\ell)+E_n$: we have the following exact sequence:

$$0\longrightarrow(\phi_*(\mathcal{O}_{\tilde{X}}(-x(\ell)-E_n)))_0\longrightarrow(\phi_*(\mathcal{O}_{\tilde{X}}(-x(\ell))))_0\longrightarrow H^0(\mathcal{O}_{E_n}(-x(\ell)))\to \\ \to(R^1\phi_*(\mathcal{O}_{\tilde{X}}(-x(\ell)-E_n)))_0\to(R^1\phi_*(\mathcal{O}_{\tilde{X}}(-x(\ell))))_0\to H^1(\mathcal{O}_{E_n}(-x(\ell)))\longrightarrow 0.$$

**First case.** Assume that $\langle E_n,x(\ell)\rangle_\Gamma>0$. Then $H^0(\mathcal{O}_{E_n}(-x(\ell)))=0$, thus by the inductive hypothesis all terms from the next exact sequence vanish:

$$0 \to (R^1\phi_*(\mathcal{O}_{\tilde{X}}(-x(\ell)-E_n)))_0 \to (R^1\phi_*(\mathcal{O}_{\tilde{X}}(-x(\ell))))_0 \to H^1(\mathcal{O}_{E_n}(-x(\ell))) \to 0.$$

In particular, $(R^1\phi_*(\mathcal{O}_{\tilde{X}}(-x(\ell)-E_n)))_0 = 0$ and $H^1(\mathcal{O}_{E_n}(-x(\ell))) = 0$ too. From this last vanishing $\mathcal{O}_{E_n}(-x(\ell))$ equals $\mathcal{O}_{\mathbb{P}^1}(-1)$, hence, in fact, $\langle E_n, x(\ell)\rangle_\Gamma = 1$ (compare with Corollary 4.16).

**Second case.** Assume that $\langle E_n, x(\ell)\rangle_\Gamma \leq 0$. In this case, by Lemma 4.9 $x(\ell) \in \mathcal{S}_\Gamma$, hence by paragraph 4.2.3 (d), $x(\ell) = \mathrm{div}_E(\phi^* g)$ for some function $g \in \mathcal{O}_{\mathbb{C}^2,0}$. Since $\ell < \mathrm{mult}_{E_n}(F)$, by Lemma 4.1*(b)*, $\mathrm{str}\{g=0\} \cap E_n = \emptyset$, therefore $\langle E_n, x(\ell)\rangle_\Gamma = 0$. In particular, we have the following exact sequence:

$$0 \longrightarrow (\phi_*(\mathcal{O}_{\tilde{X}}(-x(\ell)-E_n)))_0 \longrightarrow (\phi_*(\mathcal{O}_{\tilde{X}}(-x(\ell))))_0 \xrightarrow{\varphi} \mathbb{C} \longrightarrow$$
$$\longrightarrow (R^1\phi_*(\mathcal{O}_{\tilde{X}}(-x(\ell)-E_n)))_0 \to (R^1\phi_*(\mathcal{O}_{\tilde{X}}(-x(\ell))))_0 = 0.$$

But, the above germ $g \in (\phi_*(\mathcal{O}_{\tilde{X}}(-x(\ell))))_0 \setminus (\phi_*(\mathcal{O}_{\tilde{X}}(-x(\ell)-E_n)))_0$, hence the mapping $\varphi$ is onto, and $(R^1\phi_*(\mathcal{O}_{\tilde{X}}(-x(\ell)-E_n)))_0 = 0$.

Finally, we consider those steps of the Laufer algorithm which connect $x(\ell)+E_n$ to $x(\ell+1)$: we have to go along $z_1 := x(\ell)+E_n$, $z_{i+1} := z_i + E_{v(i)}$, where $\langle z_i, E_{v(i)}\rangle_\Gamma > 0$. Here we can prove the vanishing $(R^1\phi_*(\mathcal{O}_{\tilde{X}}(-z_i)))_0$ inductively in a similar way as in the first case from above. □

The previous proof combined with Lemma 4.9 gives the following facts as well.

COROLLARY 4.16. *For any* $0 \leq \ell < \mathrm{mult}_{E_n}(F)$ *we have the following characterizations:*

- $\ell \in \mathcal{S}_C \iff \langle E_n, x(\ell)\rangle_\Gamma = 0$;
- $\ell \notin \mathcal{S}_C \iff \langle E_n, x(\ell)\rangle_\Gamma = 1$.

## 5. THE SEMIGROUP AND THE APÉRY SET

### 5.1. The Apéry set of a numerical semigroup [4]

The Apéry set of a numerical semigroup (monoid) with respect to one of its elements is a standard invariant commonly used in semigroup theory. It consists of the smallest elements of the semigroup from each (nonempty) residue class of the given element. For more on semigroup theory see e.g. [21].

*Definition* 5.1. The *Apéry set* of a numerical semigroup $\mathcal{S}$ with respect to an element $m \in \mathcal{S}$ of it is a set $\mathrm{Ap}(\mathcal{S}, m) = \{b_0, b_1, \ldots, b_{m-1}\}$ of semigroup elements, where $0 = b_0 < b_1 < \cdots < b_{m-1}$ form a complete residue system modulo $m$ and for each $i$ ($0 \le i \le m-1$) we have $b_i \in \mathcal{S}$ but $b_i - m \notin \mathcal{S}$.

*Example* 5.2. If $\mathcal{S}$ is minimally generated by the relatively prime numbers $m$ and $n$, then $\mathrm{Ap}(\mathcal{S}, m) = \{0, n, 2n, \ldots, (m-1)n\}$.

*Remark* 5.3. (a) The definition guarantees that $\mathcal{S} = \mathrm{Ap}(\mathcal{S}, m) + m \cdot \mathbb{Z}_{\ge 0}$. It is also clear, that $b_0 = 0$ and $b_{m-1} = \mathbf{c} + m - 1$, where $\mathbf{c}$ is the conductor of $\mathcal{S}$ (the smallest $\mathbf{c}$ such that $\mathbf{c} + \mathbb{Z}_{\ge 0} \subset \mathcal{S}$). (This last identity is one of the Selmer's formulae.)

(b) Any numerical semigroup $\mathcal{S}$ has a unique finite minimal set of generators, namely $\mathcal{S}^* \setminus (\mathcal{S}^* + \mathcal{S}^*)$, where $\mathcal{S}^* = \mathcal{S} \setminus \{0\}$. Let us denote these generators (as usually in the literature) by $\bar{\beta}_0 < \ldots < \bar{\beta}_g$. Then $\bar{\beta}_0$ is the smallest element of $\mathcal{S}^*$ and is called the multiplicity of $\mathcal{S}$.

(c) One can see that if we consider the Apéry set with respect to $\bar{\beta}_0$, then $b_1 = \bar{\beta}_1$.

### 5.2. **The semigroup of irreducible plane curve singularities**

Let $(C, 0)$ be an irreducible plane curve singularity. Below we collect some properties of $\mathcal{S}_C$, the numerical semigroup of values of $(C, 0) \subset (\mathbb{C}^2, 0)$.

(a) $\mathcal{S}_C$ satisfies the following (Gorenstein) symmetry:

$$\ell \in \mathcal{S}_C \Leftrightarrow \mathbf{c} - 1 - \ell \notin \mathcal{S}_C \quad \text{for any } \ell \in \mathbb{Z}.$$

(b) This symmetry implies that $2\delta = \mathbf{c}$.

(c) $\mathcal{S}_C$ determines the Hilbert function $\mathfrak{h}$ as follows:

$$\mathfrak{h}(\ell) = \#\{s \in \mathcal{S}_C \,:\, s < \ell\}, \quad \ell \in \mathbb{Z}.$$

(d) The weight function $w_0 : \mathbb{Z} \to \mathbb{Z}$ is defined as

$$w_0(\ell) := 2\mathfrak{h}(\ell) - \ell = \#\{s \in \mathcal{S}_C \,:\, s < \ell\} - \#\{s \notin \mathcal{S}_C \,:\, 0 \le s < \ell\}.$$

It is also symmetric with respect to the involution $\ell \leftrightarrow \mathbf{c} - \ell$. Consequently, we have $w_0(\mathbf{c}) = 0$.

(e) The local minima of the weight function $w_0$ are exactly the elements $s \in \mathcal{S}_C$ such that $s - 1 \notin \mathcal{S}_C$. Equivalently, they are exactly those elements $s \in \mathcal{S}_C$ for which $\mathbf{c} - s \in \mathcal{S}_C$ as well.

For several other properties of the semigroup $\mathcal{S}_C$ see e.g. [23].

### 5.3. **The Apéry set of an irreducible plane curve singularity**

Let $\mathbf{m}$ be the multiplicity of $(C,0)$, which in fact equals the multiplicity of the semigroup $\mathcal{S}_C$. We call the *Apéry set of the irreducible plane curve singularity* $(C,0)$ the Apéry set of $\mathcal{S}_C$ with respect to $\mathbf{m}$.

The following lemma will be used in the third proof, see section 7.

LEMMA 5.4. *If* $\mathbf{m}\geq 3$, *then* $b_{\mathbf{m}-2} > \delta = \mathbf{c}/2$.

*Proof.* By definition of the Apéry set it is clear that the $\mathbf{m}$–tuple

$$B = (b_{\mathbf{m}-2} - \mathbf{m} + 1, b_{\mathbf{m}-2} - \mathbf{m} + 2, \ldots, b_{\mathbf{m}-2})$$

contains exactly $\mathbf{m}-1$ semigroup elements. Even more, for every nonnegative integer $n \in \mathbb{N}$

$$(b_{\mathbf{m}-2} - \mathbf{m} + 1 + n, b_{\mathbf{m}-2} - \mathbf{m} + 2 + n, \ldots, b_{\mathbf{m}-2} + n)$$

contains at least $\mathbf{m}-1$ semigroup elements. Consider the $\mathbf{m}$–tuple

$$(\mathbf{c} - 1 - b_{\mathbf{m}-2}, \mathbf{c} - b_{\mathbf{m}-2}, \ldots, \mathbf{c} - b_{\mathbf{m}-2} + \mathbf{m} - 2),$$

the symmetrical of $B$. By the Gorenstein symmetry it contains exactly 1 semigroup element. Since $1 < \mathbf{m}-1$, we conclude that

$$\mathbf{c} - 1 - b_{\mathbf{m}-2} < b_{\mathbf{m}-2} - \mathbf{m} + 1$$

hence $2b_{\mathbf{m}-2} > \mathbf{c} + \mathbf{m} - 2 > \mathbf{c}$. $\square$

5.3.1. **Apéry theorem.** Next we recall a theorem of Apéry, which connects the Apéry sets of $(C,0)$ and the germ obtained from $(C,0)$ as the strict transform of a single blow–up.

Let $\mathcal{S}'_C$ denote the semigroup of the strict transform under blow–up of the singularity $(C,0)$. Now $\mathbf{m} \in \mathcal{S}'_C$. Indeed, the strict transform of the singular curve after the blow–up and the reduced exceptional divisor have intersection multiplicity $\mathbf{m}$. Let $\mathrm{Ap}(\mathcal{S}'_C, \mathbf{m}) = \{b'_0, b'_1, \ldots, b'_{\mathbf{m}-1}\}$ denote the Apéry set of $\mathcal{S}'_C$ with respect to $\mathbf{m}$.

THEOREM 5.5. (APÉRY THEOREM [4]) *The Apéry sets of* $\mathcal{S}_C$ *and* $\mathcal{S}'_C$ *(with respect to the multiplicity* $\mathbf{m}$ *of* $\mathcal{S}_C$*) are related by the following formulae:*

$$b_j = b'_j + j\mathbf{m} \ (j = 0, \ldots, \mathbf{m}-1).$$

## 6. **FIRST TWO PROOFS**

Let $(C,0) = \big(\{f=0\},0\big) \subset (\mathbb{C}^2,0)$ be an irreducible plane curve singularity with multiplicity $\mathbf{m}$ and semigroup $\mathcal{S}_C$. Recall that our goal is to prove the next Main Theorem.

Theorem 6.1. *Let $s \notin \{0, \mathbf{c}\}$ be a local minimum of the weight function $w$. Then*

$$w(s) \leq 2 - \mathbf{m},$$

*and equality holds (at least) for the local minimum point $s = \mathbf{m}$.*

We start with the following observation of Tamás Ágoston and the second author, which motivated our approach:

Proposition 6.2. *If for some nonnegative integer $\ell \in \mathbb{N}$ the weight $w(\ell) \geq 2$, then $\ell > \mathbf{c}$. Furthermore, the following facts also hold:*

*(a) the local minimum values of the weight function are no higher then $0$;*
*(b) the local minimum points are realized for $0 \leq s \leq \mathbf{c}$;*
*(c) and, finally, the $S_n$ spaces for $n \geq 1$ are all contractible.*

*Remark* 6.3. A more general treatment of this last statement regarding the contractibility of the spaces $S_n$, $n \geq 1$, for (non-necessarily irreducible) curve singularities will appear in [8].

Next we provide a short possible proof of Proposition 6.2, which inspired the constructions of the second proof presented in subsection 6.0.1.

*Proof of Proposition 6.2.* Suppose indirectly that $\ell$ has the property that $w(\ell) \geq 2$. By the definition of the analytic weight function this means that

$$\#\big([0, \ell) \cap \mathcal{S}_C\big) - \#\big([0, \ell) \cap (\mathbb{Z}_{\geq 0} \setminus \mathcal{S}_C)\big) \geq 2. \tag{5}$$

First notice, that it is enough to prove that $\ell \in \mathcal{S}_C$. Indeed, that would imply that $w(\ell + 1) = w(\ell) + 1 \geq 3 \geq 2$, thus by induction $\ell + 1 \in \mathcal{S}_C$, and so on, for every $n \geq \ell$ we would have $n \in \mathcal{S}_C$ with $w(n) \geq 2$. Therefore, we obtain that $\ell \geq \mathbf{c}$, even more — as $w(\mathbf{c}) = 0$ —, $\ell > \mathbf{c}$.

In order to prove that $\ell \in \mathcal{S}_C$ we argue as follows: inequality (5) guarantees that the set $\{1, 2, \ldots, \ell - 1\}$ contains strictly more semigroup elements than non–semigroup elements. Therefore, there exists a pair of a natural numbers $(k, \ell - k)$, $1 \leq k \leq \lfloor \ell/2 \rfloor$, with $k, \ell - k \in \mathcal{S}_C$, hence $\ell = k + (\ell - k) \in \mathcal{S}_C$ as well. □

Another proof for Proposition 6.2 can be given through the following lemma as well (cf. Remark 6.5), which will also be used in the proof of the Main Theorem 6.1.

Lemma 6.4. *Let $\mathcal{S} \subset \mathbb{N}$ be a symmetric numerical semigroup with conductor $\mathbf{c}$ and let $a < b \leq \mathbf{c}/2$ be natural numbers such that $b - a \in \mathcal{S}$. Then $w(a) \geq w(b)$.*

*Proof.* We observe that given $k \in \mathbb{N}$ and $\ell \in \mathcal{S}$,

$$\#\big(\mathcal{S} \cap [k, k+\ell)\big) \leq \#\big(\mathcal{S} \cap [k+1, k+\ell+1)\big),$$

since the failure of this inequality would imply that $k \in \mathcal{S}$ but $k+\ell \notin \mathcal{S}$.

Assume by contradiction that $w(a) < w(b)$. This is equivalent to the inequality $\#\big(\mathcal{S} \cap [a,b)\big) > \frac{b-a}{2}$, cf. subsection 5.2 (d). Applying the above observation inductively and using the symmetry of $\mathcal{S}$ we get

$$\begin{aligned}\frac{b-a}{2} &< \#\big(\mathcal{S} \cap [a,b)\big) \leq \#\big(\mathcal{S} \cap [a+1, b+1)\big) \leq \cdots \\ &\leq \#\big(\mathcal{S} \cap [\mathbf{c}-b, \mathbf{c}-a)\big) = (b-a) - \#\big(\mathcal{S} \cap [a,b)\big) \\ &< (b-a) - \frac{b-a}{2} = \frac{b-a}{2},\end{aligned}$$

a contradiction. $\square$

*Remark* 6.5. Lemma 6.4 implies Proposition 6.2 as follows. Assume that $\ell \in \mathcal{S}_C$ and $\ell \leq \mathbf{c}/2$. Then by the above Lemma 6.4 $w(\ell) \leq w(0) = 0$. This means that $w(k) \leq 1$ for any $k \leq \mathbf{c}/2$ (hence, by symmetry, for any $0 \leq k \leq \mathbf{c})$). Indeed, let $n$ be the smallest integer such that $w(n) = 2$. Then $\ell = n-1$ should belong to $\mathcal{S}_C$ with $w(\ell) = 1$ (cf. Remark 3.1), a contradiction.

*First proof of Theorem 6.1.* We will use some well–known properties of the generators of $\mathcal{S}_C$ (consult e.g. paragraph 9.1.1). If $\mathbf{m} = 2$ or $3$ then $(C, 0)$ has exactly one Puiseux pair and $\mathcal{S}_C = \langle \mathbf{m}, n \rangle$. In fact, in the case of $\mathbf{m} = 2$ all the local minimum values are 0, and for $\mathbf{m} = 3$ via Lemma 5.4 one has $n > \delta$, and the local minima have the form $3k$ with weight $-k$.

If $\mathbf{m}, \mathbf{m}+1 \in \mathcal{S}_C$, then $(C, o)$ has exactly one Puiseux pair $(\mathbf{m}, \mathbf{m}+1)$ and $\mathcal{S}_C$ is generated only by $\mathbf{m}$ and $\mathbf{m}+1$. In this case one checks that the local minima of $w$ occur at the points $\{k\mathbf{m}\}_{k=0,\dots,\mathbf{m}-1}$ with $w(k\mathbf{m}) = k(k+1-\mathbf{m})$, so $w(s) \leq 2 - \mathbf{m}$ if $s \neq 0$ or $\mathbf{c}$.

For all remaining cases, set $\overline{\mathcal{S}_C} := \mathbb{Z}_{\geq 0} \setminus \mathcal{S}_C$. Assume by contradiction that there exists a local minimum $s \in \mathcal{S}_C$ with $w(\mathbf{m}) < w(s)$. This is equivalent to

$$\#\big(\mathcal{S}_C \cap [\mathbf{m}, s)\big) > \#\big(\overline{\mathcal{S}_C} \cap [\mathbf{m}, s)\big).$$

Let $\mathbf{m} = \overline{\beta}_0 < \overline{\beta}_1 < \cdots < \overline{\beta}_g$ denote the set of minimal generators of $\mathcal{S}_C$. It is known that $2\overline{\beta}_0 < \overline{\beta}_2$ (this can be proved using the 'edge determinant positivity' and the expressions of the semigroup generators in paragraph 9.1.1; but the reader might also consult [23, 24]), whence

$$\#\big(\mathcal{S}_C \cap (\mathbf{m}, 2\mathbf{m})\big) \leq 1.$$

Together with the assumption that $\mathbf{m} \geq 4$ and $\mathbf{m}+1 \notin \mathcal{S}_C$, this implies that $s > 2\mathbf{m}$, since $w(s) \leq w(\mathbf{m})$ for any possible local minimum $s \in \mathcal{S}_C \cap [\mathbf{m}, 2\mathbf{m}]$.

Consider the set $I$ of the intervals

$$[\mathbf{m}, s) \supset [\mathbf{m}, s-1) \supset [\mathbf{m}+1, s-1) \supset [\mathbf{m}+2, s-1) \supset \cdots \supset [2\mathbf{m}-2, s-1).$$

We have the following sequence of inequalities:

$$\begin{aligned}
\#\big(\mathcal{S}_C \cap [\mathbf{m}, s)\big) &\geq \#\big(\overline{\mathcal{S}_C} \cap [\mathbf{m}, s)\big) + 1 && \text{by assumption}\\
\#\big(\mathcal{S}_C \cap [\mathbf{m}, s-1)\big) &\geq \#\big(\overline{\mathcal{S}_C} \cap [\mathbf{m}, s-1)\big) + 2 && \text{since } s-1 \notin \mathcal{S}_C\\
\#\big(\mathcal{S}_C \cap [\mathbf{m}+1, s-1)\big) &\geq \#\big(\overline{\mathcal{S}_C} \cap [\mathbf{m}+1, s-1)\big) + 1 && \text{since } \mathbf{m} \in \mathcal{S}_C\\
\#\big(\mathcal{S}_C \cap [\mathbf{m}+2, s-1)\big) &\geq \#\big(\overline{\mathcal{S}_C} \cap [\mathbf{m}+2, s-1)\big) + 2 && \text{since } \mathbf{m}+1 \notin \mathcal{S}_C\\
\#\big(\mathcal{S}_C \cap [\mathbf{m}+3, s-1)\big) &\geq \#\big(\overline{\mathcal{S}_C} \cap [\mathbf{m}+3, s-1)\big) + 1 &&\\
&\vdots&&\\
\#\big(\mathcal{S}_C \cap [2\mathbf{m}-2, s-1)\big) &\geq \#\big(\overline{\mathcal{S}_C} \cap [2\mathbf{m}-2, s-1)\big) + 1, &&
\end{aligned}$$

where the inequalities concerning the intervals $[\mathbf{m}+3, s-1), \ldots, [2\mathbf{m}-2, s-1)$ follow from the fact that $\#\big(\mathcal{S}_C \cap (\mathbf{m}, 2\mathbf{m})\big) \leq 1$. This means that all intervals $[a, b) \in I$ satisfy $w(a) < w(b)$. The lengths of the intervals in $I$ attain all possible $\mathbf{m}$ values between $s - \mathbf{2m} + 1$ and $s - \mathbf{m}$. Hence there must exist an interval $[a, b) \in I$ with $b - a \in \mathcal{S}_C$, contradicting Lemma 6.4. □

6.0.1. **Second proof.** We give here another combinatorial proof of the Main Theorem 6.1 using techniques from the proofs of Proposition 6.2 and Lemma 6.4.

*Second proof of Theorem 6.1.* Assume by contradiction that there exists a local minimum point $s$ with value $w(s) > 2 - \mathbf{m}$. We can also assume, due to the properties of $\mathcal{S}_C$ (e.g. Gorenstein symmetry), that $\mathbf{m} < s < \delta = \mathbf{c}/2$.

Then, by Lemma 6.4 (for $a = s - \mathbf{m}$ and $b = s$) we have $w(s-\mathbf{m}) \geq w(s)$, i.e. (see 5.2(d))

$$\#\big(\mathcal{S}_C \cap [s-\mathbf{m}, s)\big) \leq \#\big(\mathbb{Z}_{\geq 0} \setminus \mathcal{S}_C \cap [s-\mathbf{m}, s)\big).$$

Therefore $\#\big(\mathcal{S}_C \cap [s-\mathbf{m}, s)\big) \leq \lfloor \mathbf{m}/2 \rfloor$. On the other hand, the indirect assumption asking for $w(s) = 2\#\big(\mathcal{S}_C \cap [0, s)\big) - s > 2 - \mathbf{m}$ implies the inequality $\#\big(\mathcal{S}_C \cap [0, s)\big) > \lfloor \frac{s+2-\mathbf{m}}{2} \rfloor$. Now these two combined give that

$$\#\big(\mathcal{S}_C \cap [\mathbf{m}, s-\mathbf{m})\big) > \left\lfloor \frac{s+2-\mathbf{m}}{2} \right\rfloor - \lfloor \mathbf{m}/2 \rfloor - 1 \geq \left\lfloor \frac{s-2\mathbf{m}}{2} \right\rfloor.$$

Similarly to the proof of Proposition 6.2, this inequality implies that there exists a pair of integers $(k, s-1-k)$, $\mathbf{m} \leq k \leq \lfloor \frac{s-1}{2} \rfloor$, with $k, s-1-k \in \mathcal{S}_C$. Hence $s - 1 = k + s - 1 + k \in \mathcal{S}_C$, which contradicts the local minimality of $s$, characterized in subsection 5.2 (e). □

## 7. THIRD PROOF

This section contains the third proof of the Main Theorem 1.4. We will use the notations of subsection 5.3.

LEMMA 7.1. *The intervals* $(b_{j-1}, b_j]$, $j = 1, \dots, \mathbf{m}-1$, *have the following property:*

*for any* $s \in (b_{j-1}, b_j] \cap \mathbb{Z}$ *one has the equivalence:* $s \in \mathcal{S}_C \Leftrightarrow s - j\mathbf{m} \in \mathcal{S}'_C$.

*Thus, the weight functions corresponding to* $\mathcal{S}_C$ *and* $\mathcal{S}'_C$ *in the regions* $(b_{j-1}, b_j]$ *and* $(b'_{j-1} - \mathbf{m}, b'_j]$ *make the same 'up and down' movements. More precisely:*

$$\text{(6)} \qquad \text{if } s \in (b_{j-1}, b_j] \cap \mathbb{Z}, \text{ then } w(s) = w'(s - j\mathbf{m}) - \Big(\sum_{i=1}^{j} (\mathbf{m} - 2i)\Big);$$

*where* $w'$ *denotes the weight function of the blown–up curve.*

*Example* 7.2. For $s = \mathbf{m} \in (b_0, b_1]$: $w(\mathbf{m}) = w'(0) - (\mathbf{m} - 2) = 2 - \mathbf{m}$.

*Proof of Lemma 7.1.* By definition, for any $s \in (b_{j-1}, b_j] :\ s \in \mathcal{S}_C$ if and only if $s \equiv b_i \pmod{\mathbf{m}}$ for some $i = 0, 1, \dots, j-1$ or $s = b_j$. Similarly, for any $s - j\mathbf{m} \in (b'_{j-1} - \mathbf{m}, b'_j] :\ s - j\mathbf{m} \in \mathcal{S}'_C \Longleftrightarrow s - j\mathbf{m} \equiv b'_i \pmod{\mathbf{m}}$ for some $i = 0, 1, \dots, j-1$ or $s - j\mathbf{m} = b'_j = b_j - j\mathbf{m}$. But $s \equiv s - j\mathbf{m} \pmod{\mathbf{m}}$ and $b'_i \equiv b_i \pmod{\mathbf{m}}$, so the two conditions agree.

As $w(s) = 2 \cdot \#\big([0, s) \cap \mathcal{S}_C\big) - s$, in view of the previous paragraph, it is enough to prove (6) for every $b_i \in \mathrm{Ap}(\mathcal{S}_C, \mathbf{m})$. We use induction on $i$. For $b_0 = 0 :\ w(0) = w'(0) = 0$. Now, suppose that (6) holds for $b_j$, then we have to prove it for $b_{j+1}$:

$$\begin{aligned} w(b_{j+1}) &= w(b_j) + 2 \cdot \#\big([b_j, b_{j+1}) \cap \mathcal{S}_C\big) - (b_{j+1} - b_j) = \\ &= w(b_j) + 2 \cdot \#\big((b_j, b_{j+1}] \cap \mathcal{S}_C\big) - (b_{j+1} - b_j) = \\ &= w'(b_j - j\mathbf{m}) - \sum_{i=1}^{j} (\mathbf{m} - 2i) + 2 \cdot \big(j + 1 + \#\big((b'_j, b'_{j+1}] \cap \mathcal{S}'_C\big)\big) \\ &\qquad\qquad - \mathbf{m} - (b'_{j+1} - b'_j) = \\ &= w'(b'_j) + 2 \cdot \#\big((b'_j, b'_{j+1}] \cap \mathcal{S}'_C\big) - (b'_{j+1} - b'_j) - \sum_{i=1}^{j} (\mathbf{m} - 2i) \\ &\qquad\qquad - (\mathbf{m} - 2(j+1)) = \\ &= w'(b'_{j+1}) - \sum_{i=1}^{j+1} (\mathbf{m} - 2i) = w'(b_{j+1} - (j+1)\mathbf{m}) - \sum_{i=1}^{j+1} (\mathbf{m} - 2i). \end{aligned}$$

Therefore, by induction, the lemma follows. $\square$

*Remark* 7.3. The statement of (6) is also true for $s \leq b_0 = 0$ with $j = 0$ and $s > b_{\mathbf{m}-1}$ with $j = \mathbf{m} - 1$, though one can easily see that in these regions the weight function behaves trivially.

*Remark* 7.4. Notice that

$$\sum_{i=1}^{j}(\mathbf{m} - 2j) = \begin{cases} 0 & \text{for } j = 0; \\ \geq \mathbf{m} - 2 & \text{for } j = 1, \dots, \mathbf{m} - 2; \\ 0 & \text{for } j = \mathbf{m} - 1. \end{cases}$$

Therefore, the weight function of the irreducible plane curve singularity is never higher then the weight function of the blown–up curve at the corresponding place.

*Remark* 7.5. As the integers $b_j$ for $j = 0, \dots, \mathbf{m} - 1$ are semigroup elements, the lattice points of the form $b_j + 1$, $j = 0, \dots, \mathbf{m} - 1$, are not local minimum points of the weight function $w$, see subsection 5.2(e). Therefore, if some $s \in [b_{j-1} + 2, b_j]$ is a local minimal point of $w$, then so is $s - j\mathbf{m} \in [b'_{j-1} + 2, b'_j]$ in the weight distribution $w'$ of the blown–up curve.

The previous remarks yield the Main Theorem 1.4:

Corollary 7.6. *If $s \in (0, \mathbf{c})$ is a local minimum point of the weight function $w$, then $w(s) \leq 2 - \mathbf{m}$. Example 7.2 then shows, that for non-smooth germs $2 - \mathbf{m}$ is indeed the highest local minimum value apart from the two obvious ones at level $0$ ($\mathbf{m} - 1 \notin \mathcal{S}_C, \mathbf{m} \in \mathcal{S}_C$).*

*Proof.* For smooth germs this statement is empty. In the non-smooth setting, due to the Gorenstein symmetry, it is enough to prove the statement in the $s < \delta$ case.

Let us compare the weight function $w$ of the irreducible plane curve singularity $(C, 0)$ with $w'$ of its blow–up $(C', 0)$. From Remark 7.5 we get that the local minimum points of $w$ correspond to those of $w'$, i.e. if $s \in (b_{j-1}, b_j]$ is a local minimum point, then so is $s - j\mathbf{m}$. If $\mathbf{m} \geq 3$, then by Lemma 5.4 we see that for $s < \delta$ the relevant cases of equations (6) are that of $j = 1, \dots, \mathbf{m} - 2$. Then by Remark 7.4 we have

$$w(s) \leq w'(s - j\mathbf{m}) + 2 - \mathbf{m} \leq 2 - \mathbf{m},$$

where the last inequality comes from Proposition 6.2 using the local minimality of $s - j\mathbf{m}$. If $\mathbf{m} = 2$, then $2 - \mathbf{m} = 0$, so the assertion is clear again due to Proposition 6.2. $\square$

# 8. FOURTH PROOF

## 8.1. Idea of the proof

Let $(C,0)=\big(\{f=0\},0\big)\subset(\mathbb{C}^2,0)$ be an irreducible plane curve singularity with multiplicity $\mathbf{m}$ and semigroup $\mathcal{S}_C$. Again, recall that our goal is to prove the Main Theorem 1.4, i.e. that for any local minimum $s\notin\{0,\mathbf{c}\}$ of the weight function $w$,

$$w(s)\leq 2-\mathbf{m},$$

and equality holds (at least) for the local minimum point $s=\mathbf{m}$. Note that by the Gorenstein symmetry of the semigroup (and hence of the weight function $w$) it is enough to prove this for local minima $s$ with $s\leq\delta$.

A sufficient condition for this statement is given in Proposition 8.1. For ease of reading, several technical lemmas used in its proof have been moved to subsection 8.2.

Fix the minimal good embedded resolution $\phi:\tilde{X}\to S$. We adopt all the notations of previous sections. Additionally, we set $m_1(\ell):=\mathrm{mult}_{E_1}(x(\ell))$ for every $\ell\geq 0$.

Following Theorem 4.14, we want $\langle x(s),F+Z_K-x(s)\rangle_\Gamma\leq 2-\mathbf{m}$ for any local minimum point $s$. Notice that it is sufficient to show that

$$\langle x(s),F+Z_K-x(s)\rangle_\Gamma\leq -m_1(s)(\mathbf{m}-1-m_1(s)),$$

as by Lemma 8.3 we have that $-m_1(s)(\mathbf{m}-1-m_1(s))\leq 2-\mathbf{m}$.

Proposition 8.1. *$\langle x(s),F+Z_K-x(s)\rangle_\Gamma\leq -m_1(s)(\mathbf{m}-1-m_1(s))$ for any $s\in\mathcal{S}_C$.*

*Proof.* Let $X_1$ be the partial resolution obtained from $S\subset\mathbb{C}^2$ via a single blow–up and denote by $\pi:\tilde{X}\to X_1$ the composition of the other ones. Denote by $\Gamma_1=\{E_1\}$ the dual graph of this first blow–up and by $\Gamma'$ the dual graph of the resolution $\pi$, with irreducible exceptional divisors $\{E_v\}_{v\in\mathcal{V}\setminus\{1\}}$. We remark that $\pi$ is an embedded resolution of the once blown–up curve $(C',0)$ (considered above in the Apéry theorem 5.5 and in the third proof).

Notice that

$$\begin{aligned}-m_1(s)(\mathbf{m}-1-m_1(s))=&\big\langle\pi_*x(s),\pi_*(F+Z_K-x(s))\big\rangle_{\Gamma_1}\\ =&\big\langle\pi^*\pi_*x(s),\pi^*\pi_*(F+Z_K-x(s))\big\rangle_{\Gamma}\ ,\end{aligned}$$

so the inequality to be shown becomes

$$\langle x(s),F+Z_K-x(s)\rangle_\Gamma\leq\big\langle\pi^*\pi_*x(s),\pi^*\pi_*(F+Z_K-x(s))\big\rangle_\Gamma\ .$$

This can be rewritten as

$$(7)\qquad \big\langle x(s)-\pi^*\pi_*x(s),F+Z_K-x(s)-\pi^*\pi_*(F+Z_K-x(s))\big\rangle_\Gamma\leq 0$$

through a computation using the projection formula. Since

$$\langle x(s) - \pi^*\pi_* x(s), E_v\rangle_\Gamma = \langle x(s), E_v\rangle_\Gamma - \langle \pi_* x(s), \pi_* E_v\rangle_{\Gamma_1} \leq 0$$

for all $v \neq 1$ (use $\pi_* E_v = 0$ and $\langle x(s), E_v\rangle_\Gamma \leq 0$ due to $x(s) \in \mathcal{S}_\Gamma$) and $F + Z_K - x(s) - \pi^*\pi_*(F + Z_K - x(s))$ is supported on $E' := \cup_{v\neq 1} E_v$, it is enough to show that $F + Z_K - x(s) - \pi^*\pi_*(F + Z_K - x(s))$ is an effective divisor, i.e. that

$$F + Z_K - \pi^*\pi_*(F + Z_K) \geq x(s) - \pi^*\pi_* x(s). \tag{8}$$

Both sides of (8) are supported on the divisor $E'$ with corresponding graph $\Gamma' = \Gamma \setminus \{E_1\}$. Let $F' := (E_n)^*_{\Gamma'}$ be the $\Gamma'$–dual of $E_n$ and let $Z'_K$ be the anticanonical cycle of the graph $\Gamma'$. A computation shows that

$$F - \pi^*\pi_* F\big|_{\Gamma'} = F' \qquad \text{and} \qquad Z_K - \pi^*\pi_* Z_K\big|_{\Gamma'} = Z'_K.$$

As for the right-hand side of (8), $x(s) - \pi^*\pi_* x(s)\big|_{\Gamma'} = x'(s - \mathbf{m}\cdot m_1(s))$ by Lemma 8.4, where $x' : \mathbb{Z}_{\geq 0} \to \mathbb{Z}^{n-1}$ denotes the sequence of universal cycles of the graph $\Gamma'$ with respect to $E_n$. Hence, after restricting to $\Gamma'$, the desired identity (8) becomes

$$F' + Z'_K \geq x'(s - \mathbf{m}\cdot m_1(s)).$$

Finally, by Example 4.13, $F' + Z'_K = x'(\mathbf{c}')$, where $\mathbf{c}'$ is the conductor of the semigroup $\mathcal{S}'_C$. The Proposition hence follows from Lemma 8.5 (stating that $s - \mathbf{m}\cdot m_1(s) \leq \mathbf{c}'$) and the monotoneity of the $x'$ operator (see 4.7). □

*Remark* 8.2. Notice that in Lemma 7.1 we have shown that if $s \in (b_{j-1}, b_j]$, then

$$w(s) = w'(s - j\mathbf{m}) - j(\mathbf{m} - 1 - j).$$

On the other hand the left-hand side of (7) is $w'(s - m_1(s)\mathbf{m})$, thus one can derive the formula

$$w(s) = w'(s - \mathbf{m}\cdot m_1(s)) - m_1(s)(\mathbf{m} - 1 - m_1(s)).$$

This suggests (but does not prove) that $j = m_1(s)$. This equality is in fact true and we give a proof of this fact in section 9. This has a beautiful geometric interpretation: notice that the function $m_1 : \mathcal{S}_C \to \mathbb{Z}_{\geq 0}$ can be defined equivalently as

$$m_1(s) = \min\{\mathrm{mult}(g) \,:\, g \in \mathcal{O}_{C,0}, \text{ with } i_0(g, f) = s\}.$$

Then $m_1$ is an increasing function on $\mathcal{S}_C$, and $j = m_1$ means that $m_1$ jumps exactly at the elements of the Apéry set $\mathrm{Ap}(\mathcal{S}_C, \mathbf{m})$.

## 8.2. **The Lemmas used in the proof of Proposition 8.1**

LEMMA 8.3. *In the case of any non-zero semigroup element* $s \leq \delta$ *we have that* $1 \leq m_1(s) \leq \mathbf{m} - 2$.

*Proof.* It is sufficient to construct a cycle $D \in \mathcal{S}_\Gamma$ with the properties that $1 \leq \mathrm{mult}_{E_1}(D) \leq \mathbf{m} - 2$ and $\mathrm{mult}_{E_n}(D) \geq \delta \geq s$, because in this case $x(s) \leq x(\mathrm{mult}_{E_n}(D)) \leq D$. If $(C, 0)$ has $r$ Puiseux pairs, set $D = kE_v^*$, where $v$ is the end–vertex of the leg corresponding to the last Puiseux pair, and $k$ equals $p_r - 1$ if $r \geq 2$ or $p_1 - 2$ if $r = 1$, for notations see paragraph 9.1.1. A simple computation on the resolution graph (or the corresponding splice diagram) now shows that $D$ satisfies the desired properties. □

LEMMA 8.4. *Let* $\pi$, $\Gamma$ *and* $\Gamma'$ *denote the same resolution and resolution graphs as in the proof of Proposition 8.1. Let* $x$ *and* $x'$ *denote the universal cycle operators associated with the graphs* $\Gamma$ *and* $\Gamma'$ *and the exceptional divisor* $E_n$. *Then for any integer* $\ell \geq 0$:

$$x(\ell) - \pi^*\pi_*x(\ell)\big|_{\Gamma'} = x'(\ell - \mathbf{m} \cdot m_1(\ell)),$$

*where* $m_1(\ell) = \mathrm{mult}_{E_1}(x(\ell))$.

Notice that this combined with Remark 8.2 can be thought of as an inductive formula for the computation of $x(\ell)$.

*Proof.* First we prove that

$$\mathrm{mult}_{E_n}\big(x(\ell) - \pi^*\pi_*x(\ell)\big|_{\Gamma'}\big) = \ell - \mathbf{m} \cdot m_1(\ell). \tag{9}$$

Since $\pi_*x(\ell) = m_1(\ell)E_1$, it is sufficient to show that $\mathrm{mult}_{E_n}(\pi^*E_1) = \mathbf{m}$. Observe that $\pi^*E_1 = E_1^*$, since $\langle \pi^*E_1, E_v\rangle_\Gamma = \langle E_1, \pi_*E_v\rangle_{\Gamma_1} = -\delta_{1,v}$ by the projection formula. We compute

$$\begin{aligned} \mathrm{mult}_{E_n}(\pi^*E_1) &= -\langle E_n^*, \pi^*E_1\rangle_\Gamma = -\langle F, \pi^*E_1\rangle_\Gamma = \\ &= -\langle F, E_1^*\rangle_\Gamma = \mathrm{mult}_{E_1}(F) = \mathbf{m}. \end{aligned} \tag{10}$$

Next, we show that $x(\ell) - \pi^*\pi_*x(\ell)\big|_{\Gamma'} \geq x'(\ell - \mathbf{m} \cdot m_1(\ell))$. For every $v \in \mathcal{V} \setminus \{1, n\}$:

$$\begin{aligned} &\langle x(\ell) - \pi^*\pi_*x(\ell)\big|_{\Gamma'}, E_v\rangle_{\Gamma'} = \\ &\quad = \langle x(\ell) - \pi^*\pi_*x(\ell), E_v\rangle_\Gamma && \text{since } \mathrm{mult}_{E_1}\big(x(\ell) - \pi^*\pi_*x(\ell)\big) = 0 \\ &\quad = \langle x(\ell), E_v\rangle_\Gamma - \langle \pi_*x(\ell), \pi_*E_v\rangle_{\Gamma_1} && \text{by the projection formula} \\ &\quad = \langle x(\ell), E_v\rangle_\Gamma \leq 0 && \text{by definition of the } x \text{ operator.} \end{aligned}$$

Through equation (9), the universality property (4.6*(c)*) of the $x'$ operator gives indeed

$$x'(\ell - \mathbf{m} \cdot m_1(\ell)) \leq x(\ell) - \pi^*\pi_* x(\ell)\big|_{\Gamma'} . \tag{11}$$

Finally, we prove that $x(\ell) \leq x'(\ell - \mathbf{m} \cdot m_1(\ell)) + \pi^*\pi_* x(\ell)$, where we extend $x'(\ell - \mathbf{m} \cdot m_1(\ell))$ to $\Gamma$ by setting its value on $E_1$ to zero (we will use the same notation for the extended version). Now we want to use the universality property of the operator $x$. If $v \in \mathcal{V} \setminus \{1, n\}$, then

$$\begin{aligned} &\langle x'(\ell - \mathbf{m} \cdot m_1(\ell)) + \pi^*\pi_* x(\ell), E_v\rangle_\Gamma = \\ &\qquad = \underbrace{\langle x'(\ell - \mathbf{m} \cdot m_1(\ell)), E_v\rangle_{\Gamma'}}_{\leq 0} + \langle \pi_* x(\ell), \underbrace{\pi_* E_v}_{0}\rangle_{\Gamma_1} \leq 0. \end{aligned}$$

For the $v = 1$ case, observe that $\text{mult}_{E_1}\big(x(\ell) - \pi^*\pi_* x(\ell) - x'(\ell - \mathbf{m} \cdot m_1(\ell))\big) = 0$ and by inequality (11) we have $x(\ell) - \pi^*\pi_* x(\ell) - x'(\ell - \mathbf{m} \cdot m_1(\ell)) \geq 0$. Therefore

$$\begin{aligned} &\langle x(\ell) - \pi^*\pi_* x(\ell) - x'(\ell - \mathbf{m} \cdot m_1(\ell)), E_1\rangle_\Gamma \geq 0 \Leftrightarrow \\ &\qquad \Leftrightarrow \langle x'(\ell - \mathbf{m} \cdot m_1(\ell)) + \pi^*\pi_* x(\ell), E_1\rangle_\Gamma \leq \langle x(\ell), E_1\rangle_\Gamma \leq 0. \end{aligned}$$

Identity (10) implies that $\text{mult}_{E_n}(x'(\ell - \mathbf{m} \cdot m_1(\ell)) + \pi^*\pi_* x(\ell)) = \ell$. So, by the universality property of the $x$ operator

$$x(\ell) \leq x'(\ell - \mathbf{m} \cdot m_1(\ell)) + \pi^*\pi_* x(\ell).$$

This concludes the proof. □

LEMMA 8.5. *Let $s \in \mathbb{Z}$ be a local minimum of the weight function $w$ of the irreducible plane curve singularity $(C, 0)$. Then $s - \mathbf{m} \cdot m_1(s) \leq \mathbf{c}'$, where $\mathbf{c}'$ is the conductor of the once blown–up curve $(C', 0)$.*

*Proof.* First note that by Remark 5.3 and the Apéry theorem 5.5 the conductor $\mathbf{c}' = \mathbf{c} - \mathbf{m}(\mathbf{m} - 1)$, therefore it is enough to prove that

$$\mathbf{m}(\mathbf{m} - 1 - m_1(s)) \leq \mathbf{c} - s.$$

Notice that by identity (4), we have $\mathbf{m} - 1 = m_1(\mathbf{c})$. It is also useful to observe, that by the minimality property *(c)* from Lemma 4.6 of the $x$ operator we have $x(s) + x(\mathbf{c} - s) \geq x(\mathbf{c})$, thus $m_1(\mathbf{c}) - m_1(s) \leq m_1(\mathbf{c} - s)$. So it is enough to prove the inequality

$$\mathbf{m} \cdot m_1(\mathbf{c} - s) \leq \mathbf{c} - s.$$

Here we use the fact that $s$ is a local minimum, and hence both $s$ and $\mathbf{c} - s \in \mathcal{S}_C$. Then, by Lemma 4.9, we know that $x(\mathbf{c} - s) \in \mathcal{S}_\Gamma$ and by Lemma 4.2.3 (d) there exists $g \in \mathcal{O}_{\mathbb{C}^2,0}$ with $\text{div}_E(\phi^* g) = x(\mathbf{c} - s)$ and $\text{str}\{g = 0\} \cap \text{str}\{f = 0\} = \emptyset$. Therefore, by Lemma 4.1*(a)* we have $i_0(g, f) = \text{mult}_{E_n}(\text{div}_E(\phi^* g)) = \mathbf{c} - s$.

Now we can find a generic linear function $l \in \mathcal{O}_{\mathbb{C}^2,0}$, with $i_0(l,f) = \mathbf{m}$ and $i_0(l,g) = \mathrm{mult}_{E_1}(\mathrm{div}_E(\phi^* g)) = m_1(\mathbf{c}-s)$. Thus, in conclusion we have that $m_1(\mathbf{c}-s)\cdot \mathbf{m} = i_0(l,g)\cdot i_0(l,f) \le i_0(f,g) = \mathbf{c}-s$. $\square$

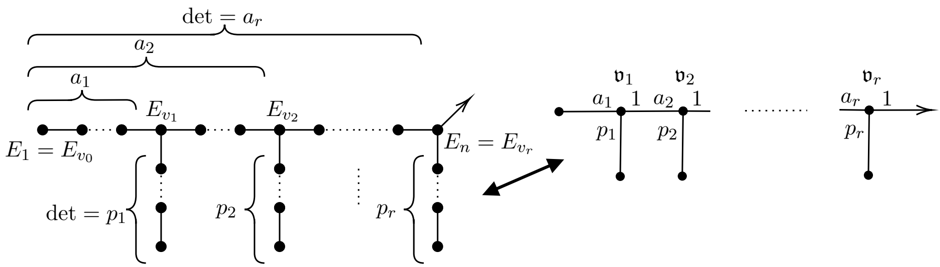
## 9. APÉRY SET AND JUMPING LOCI

### 9.1. Semigroup via the splice diagram

The main result of this section, Theorem 9.1, connects the third and the fourth proofs of the Main Theorem, via the comments of Remark 8.2.

Furthermore, this theorem provides a *new characterization of the Apéry set* of an irreducible plane curve singularity and might be the starting point of generalizations.

9.1.1. **Splice diagrams.** The simplest presentation of certain parts of the proof can be realized using the combinatorial properties of the decorations of the splice diagram $\mathfrak{S}$ associated with the minimal embedded resolution graph $\Gamma$ of $(C,0)$ [6], with which we assume some familiarity. The procedure how one constructs $\mathfrak{S}$ from $\Gamma$ is shown in the next diagram.




Here the integers $a_i, p_i$ ($i = 1,\dots,r$; $\gcd(a_i,p_i) = 1$ for all $i$) are the absolute values of the determinants of the corresponding subgraphs, and they appear as the decorations on the edges of the splice diagram. Using these decorations $\{a_i,p_i\}_i$ one also reads directly the entries of the inverse of the intersection matrix $-\langle E_v^*, E_w^*\rangle_\Gamma = \mathrm{mult}_{E_v}(E_w^*)$ as the product of all splice diagram decorations along but not on the shortest path connecting $v$ and $w$ (the corresponding vertices of the splice diagram), cf. [6]. Furthermore, they satisfy the *edge determinant positivity*, namely $a_i > a_{i-1}p_{i-1}p_i$ for any $1 < i \le r$, cf. [6]. In terms of $\{a_i,p_i\}_i$ the minimal set of generators of the numerical semigroup $\mathcal{S}_C$ are the following (cf. [23]):

- $\overline{\beta}_0 = \mathbf{m} = p_1p_2\dots p_r$;

- $\overline{\beta}_l = a_l p_{l+1} p_{l+2} \dots p_r$ for all $1 \leq l \leq r-1$;
- $\overline{\beta}_r = a_r$.

These generators are represented by divisors of functions with one smooth and transverse strict transform (arrow) at the end of the corresponding leg of the embedded resolution graph (or at the corresponding end–vertex of the splice diagram).

Furthermore, by a theorem of Teissier–Zariski [23], every semigroup element $s \in \mathcal{S}_C$ can be written uniquely in the form:

$$s = k_0\overline{\beta}_0 + \sum_{i=1}^{r} k_i\overline{\beta}_i \text{ with } k_0 \geq 0 \text{ and } \ 0 \leq k_i < p_i \text{ for all } i = 1, \dots, r.$$

## 9.2. **A new characterization of the Apéry set**

Recall the notation $m_1(\ell) := \mathrm{mult}_{E_1}(x(\ell))$, and let $\mathcal{J}$ denote the set of the 'jumping loci' of the sequence $\ell \mapsto x(\ell)$, namely:

$$\mathcal{J} = \{\ell \in \mathbb{Z} \,:\, 0 \leq \ell < \mathrm{mult}_{E_n}(F),\, m_1(\ell) < m_1(\ell+1)\}.$$

As $\mathrm{mult}_{E_1}(x(0)) = 0$ and $\mathrm{mult}_{E_1}(F) = \mathbf{m}$ (cf. 4.12), it follows that the cardinality $\#(\mathcal{J}) \leq \mathbf{m}$.

We claim the following:

Theorem 9.1. $\mathcal{J} = \mathrm{Ap}(\mathcal{S}_C, \mathbf{m})$ *(and as such, $\#(\mathcal{J}) = \mathbf{m}$).*

It is enough to prove that $\mathrm{Ap}(\mathcal{S}_C, \mathbf{m}) \subset \mathcal{J}$. Let us start with the following observation.

The largest Apéry element $b_{\mathbf{m}-1}$ satisfies $b_{\mathbf{m}-1} = \mathbf{c} + \mathbf{m} - 1$ (cf. Remark 5.3), by Example 4.13 $\mathrm{mult}_{E_n}(F + Z_K) = \mathbf{c}$, and by Lemma 4.2 and (10) $\mathrm{mult}_{E_n}(-Z_K) \geq \mathbf{m}$. Hence

$$\mathrm{mult}_{E_n}(F) \geq \mathbf{c} + \mathbf{m}, \tag{12}$$

and

$$\mathrm{Ap}(\mathcal{S}_C, \mathbf{m}) \subset \{0, 1, \dots, \mathrm{mult}_{E_n}(F) - 1\}. \tag{13}$$

In the proof we will use the following terminology.

*Definition* 9.2. The *spine* of the minimal embedded resolution graph $\Gamma$ is the unique path from $E_1$ to $E_n$ with $E_n$ removed from the end at the right.

We say that a cycle $l \in L$ *has no arrow* on the spine, if $\langle l, E_v\rangle_\Gamma = 0$ for any vertex on the spine.

We start the proof of Theorem 9.1 with the following lemma.

Lemma 9.3. *If for some* $\ell \in \{0, \ldots, \mathrm{mult}_{E_n}(F)-1\}$ *the cycle* $x(\ell)$ *has no arrows on the spine of the minimal embedded resolution graph, then* $\ell \in \mathcal{J}$.

*Proof.* To get $x(\ell+1)$ from $x(\ell)$ we apply the Laufer algorithm to the cycle $z_1 = x(\ell) + E_n$, cf. Lemma 4.7. Now, if $E_{w_1}, E_{w_2}, \ldots, E_{w_k}$ are the exceptional divisors corresponding to the vertices on the spine of the embedded resolution graph Γ 'from left to right' ($E_{w_1} = E_1$, and $E_{w_k}$ is adjacent to $E_n$), then we can make the following choices in the Laufer algorithm:

- $\langle x(\ell), E_{w_k}\rangle_\Gamma = 0 \Rightarrow \langle z_1, E_{w_k}\rangle_\Gamma = 1 > 0 \ \Rightarrow\ z_2 := z_1 + E_{w_k}$;
- $\langle x(\ell), E_{w_{k-1}}\rangle_\Gamma = 0 \Rightarrow \langle z_2, E_{w_{k-1}}\rangle_\Gamma = 1 > 0 \ \Rightarrow\ z_3 := z_2 + E_{w_{k-1}}$;
- and so on: $\langle x(\ell), E_{w_{k-j+1}}\rangle_\Gamma = 0 \Rightarrow \langle z_j, E_{w_{k-j+1}}\rangle_\Gamma = 1 > 0 \Rightarrow$
$\Rightarrow\ z_{j+1} := z_j + E_{w_{k-j+1}}$;

therefore $z_{k+1} = z_k + E_1$, and $x(\ell+1) = z_t \geq z_{k+1}$, so indeed, $\ell \in \mathcal{J}$. □

Therefore, via (13) and the previous lemma it is enough to prove that

$$\mathrm{Ap}(\mathcal{S}_C, \mathbf{m}) \subset \{\ell : 0 \leq \ell < \mathrm{mult}_{E_n}(F),\ x(\ell) \text{ has no arrows on the spine}\}.$$

Choose any element $s \in \mathrm{Ap}(\mathcal{S}_C, \mathbf{m}) \subset \mathcal{S}_C$. Then the universal cycle $x(s)$ belongs to the Lipman cone (cf. Lemma 4.9), therefore it can be given as the divisor of some (non-unique) function $g \in \mathcal{O}_{\mathbb{C}^2,0}$. Hence, it is sufficient to show that if a function $g \in \mathcal{O}_{\mathbb{C}^2,0}$ satisfies $\mathrm{mult}_{E_n}(\mathrm{div}_E(\phi^* g)) \in \mathrm{Ap}(\mathcal{S}_C, \mathbf{m})$, then $\mathrm{div}_E(\phi^* g)$ cannot have any arrows on the spine of the embedded resolution graph.

Let us rewrite this statement as follows.

Lemma 9.4. *If for some* $g \in \mathcal{O}_{\mathbb{C}^2,0}$ *the cycle* $\mathrm{div}_E(\phi^* g)$ *has some arrows on the spine of* Γ, *then* $\mathrm{mult}_{E_n}(\mathrm{div}_E(\phi^* g)) \notin \mathrm{Ap}(\mathcal{S}_C, \mathbf{m})$.

It is enough to prove this for a divisor with a single arrow on the spine of the embedded resolution graph.

We will denote the exceptional divisors corresponding to the nodes of the resolution graph by $E_{v_1}, E_{v_2}, \ldots, E_{v_{r-1}}$ and in this context let $E_{v_0} := E_1$ and $E_{v_r} := E_n$.

Suppose now that the single arrow is supported on $E_v$ between $E_{v_l}$ and $E_{v_{l+1}}$ (including $E_{v_l}$ but not $E_{v_{l+1}}$). Then we prove the following technical version of Lemma 9.4.

Claim 9.5. *Let* $C_l = \{g_l = 0\}$ *be a generic plane curve singularity whose dual resolution graph is the first part of* Γ *truncated after the vertex corresponding to* $E_{v_l}$ *(this corresponds to a transverse smooth curvetta through some* *generic point of* $E_{v_l}$ *in* $\tilde{X}$*). Then its multiplicity is* $\mathbf{m}_l = p_1p_2\dots p_l$ *and let us denote its conductor by* $\mathbf{c}_l$*. Then*

*1.* $\text{mult}_{E_{v_l}}(E^*_v) \geq \text{mult}_{E_{v_l}}(E^*_{v_l}) = \text{mult}_{E_{v_l}}(\text{div}_E(\phi^* g_l)) \geq \mathbf{c}_l + \mathbf{m}_l$*;*

*2.* $\text{mult}_{E_{v_{l+1}}}(E^*_v) \geq \text{mult}_{E_{v_{l+1}}}(E^*_{v_l}) =$
$= \text{mult}_{E_{v_{l+1}}}(\text{div}_E(\phi^* g_l)) = \text{mult}_{E_{v_l}}(\text{div}_E(\phi^* g_l)) \cdot p_{l+1}$*;*

*3.* $\text{mult}_{E_{v_{l+1}}}(E^*_v) = k_0p_1p_2\dots p_{l+1} + \sum_{i=1}^{l} k_ia_ip_{i+1}\dots p_{l+1}$ *with* $k_0 \geq 0$ *and* $0 \leq k_i < p_i$*;*

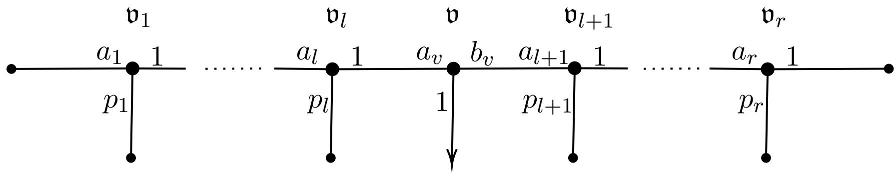
*4. The previous three combined give the assertion, that* $k_0$ *cannot be* $0$*. Thus there exists a system of positive arrows/curvettas (the strict transform of a not necessarily irreducible curve* $h$*) on the end vertices of the legs of* $\Gamma$ *with determinants* $a_1, p_1, \dots, p_l$*, whose corresponding multiplicity system satisfies the following equality:*

$$\text{mult}_{E_{v_{l+1}}}(\text{div}_E(\phi^* h)) = \text{mult}_{E_{v_{l+1}}}(E^*_v) - p_1p_2\dots p_{l+1};$$

*5.* $i_0(f,h) = \text{mult}_{E_{v_r}}(\text{div}_E(\phi^* h)) = \text{mult}_{E_{v_r}}(E^*_v) - p_1p_2\dots p_{l+1}p_{l+2}\dots p_r =$
$= \text{mult}_{E_{v_r}}(E^*_v) - \mathbf{m}$*, therefore* $\text{mult}_{E_{v_r}}(E^*_v)$ *cannot be an element of the Apéry set.*

*Proof.*

1. If we add an arrow at vertex $v$ to the embedded resolution graph we get the following (non-minimal) splice diagram:




Now, $\text{mult}_{E_{v_l}}(E^*_{v_l}) = a_l \cdot p_l$ and $\text{mult}_{E_{v_l}}(E^*_v) = a_l \cdot p_l \cdot b_v$, which shows the first inequality. For the second one use (12) for the singularity $(C_l, 0)$.

2. One has the following:

$$\text{mult}_{E_{v_{l+1}}}(E^*_v) = a_v \cdot p_{l+1};$$

$$\text{mult}_{E_{v_{l+1}}}(E^*_{v_l}) = a_l \cdot p_l \cdot p_{l+1} = p_{l+1}\text{mult}_{E_{v_l}}(\text{div}_E(\phi^* g_l)).$$

Then the edge–determinant positivity applied for the edge $(\mathfrak{v}_l, \mathfrak{v})$ gives $a_v > a_l \cdot p_l \cdot b_v$ (cf. [6]).

3. We use the Teissier–Zariski theorem for $(C_{l+1},0)$, a generic plane curve singularity whose dual resolution graph is the first part of $\Gamma$ truncated after the vertex corresponding to $E_{v_{l+1}}$. (Its splice diagram is similar to the diagram of $(C,0)$ but now the nodes are $\mathfrak{v}_1,\dots,\mathfrak{v}_{l+1}$, and this last node supports the arrow.) Its numerical semigroup is generated by $p_1\dots p_{l+1}$ , $a_1p_2\dots p_{l+1}$ , $\dots$ , $a_lp_{l+1}$ , $a_{l+1}$, so
$$\mathrm{mult}_{E_{v_{l+1}}}(E_v^*) = k_0p_1p_2\dots p_{l+1} + \sum_{i=1}^{l+1} k_ia_ip_{i+1}\dots p_{l+1}$$
with $k_0 \geq 0$ and $0 \leq k_i < p_i$. We only need to prove that $k_{l+1} = 0$. Indeed, this is true, because $\mathrm{mult}_{E_{v_{l+1}}}(E_v^*) = a_v \cdot p_{l+1}$ is divisible by $p_{l+1}$ just as all the other generators of this semigroup apart from $a_{l+1}$.

4. The fact that $\mathbf{c}_l$ is the conductor of
$$\mathcal{S}_{C_l} = \mathbb{Z}_{\geq 0}\langle p_1\dots p_l, a_1p_2\dots p_l,\dots,a_{l-1}p_l,a_l\rangle$$
means that every integer $n \geq \mathbf{c}_l$ is contained in the semigroup. Therefore, every integer not less than $\mathbf{c}_lp_{l+1}$ and divisible by $p_{l+1}$ is contained in the semigroup
$$p_{l+1}\cdot\mathcal{S}_{C_l} = \mathbb{Z}_{\geq 0}\langle\, p_1\dots p_lp_{l+1}\,,\, a_1p_2\dots p_lp_{l+1}\,,\,\dots\,,\, a_{l-1}p_lp_{l+1}\,,\, a_lp_{l+1}\,\rangle$$
and $p_{l+1}\cdot\mathcal{S}_{C_l} \subset \mathcal{S}_{C_{l+1}}$. Now, by parts 1. and 2.
$$\mathrm{mult}_{E_{v_{l+1}}}(E_v^*) - p_1p_2\dots p_lp_{l+1} \geq \mathbf{c}_lp_{l+1}$$
and, by part 3., it is divisible by $p_{l+1}$, thus $\mathrm{mult}_{E_{v_{l+1}}}(E_v^*) - p_1p_2\dots p_lp_{l+1}$ is contained in $p_{l+1}\cdot\mathcal{S}_{C_l}$. So by the uniqueness of the Teissier–Zariski decomposition of $\mathrm{mult}_{E_{v_{l+1}}}(E_v^*)$ in $\mathcal{S}_{C_{l+1}}$ we get that $k_0 \neq 0$ and there exists an arrow system as in the statement, represented by a germ $h$, with
$$\mathrm{mult}_{E_{v_{l+1}}}(\mathrm{div}_E(\phi^*h)) = \mathrm{mult}_{E_{v_{l+1}}}(E_v^*) - p_1p_2\dots p_{l+1}.$$

5. As $h$ does not have any arrows on the vertices starting from $v_{l+1}$, the intersection calculations on the splice diagram show that
$$\begin{aligned} i_0(f,h) &= \mathrm{mult}_{E_{v_r}}(\mathrm{div}_E(\phi^*h)) = && \text{by Lemma 4.1}\textit{(a)}\\ &= \mathrm{mult}_{E_{v_{l+1}}}(\mathrm{div}_E(\phi^*h))p_{l+2}\dots p_r = \\ &= \mathrm{mult}_{E_{v_{l+1}}}(E_v^*)p_{l+2}\dots p_r - p_1p_2\dots p_{l+1}p_{l+2}\dots p_r = \\ &= \mathrm{mult}_{E_{v_r}}(E_v^*) - \mathbf{m}. \end{aligned}$$

This concludes the proof. □

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