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LATTICE COHOMOLOGY OF PARTIALLY ORDERED SETS

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ABSTRACT

In this paper we introduce a construction for a weighted CW complex (and the associated lattice cohomology) corresponding to partially ordered sets with some additional structure. This is a generalization of the construction seen in [4] where we started from a system of subspaces of a given vector space. We then proceed to prove some basic properties of this construction that are in many ways analogous to those seen in the case of subspaces, but some aspects of the construction result in complexities not present in that scenario.

KEYWORDS

Lattice cohomology, posets, local singularities

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1. INTRODUCTION

The lattice cohomology of a singularity was originally defined in the context of a complex normal surface singularity (X, o), whenever the link is a rational homology sphere. The link of such a singularity is a plumbed (graph) 3-manifold, where the plumbing graph can be chosen as one of the resolution graphs of the germ. In particular, it is connected and negative definite. In such a situation, using the combinatorics of the graph, a lattice cohomology $\mathbb{H}^*_{top}(X, o)$ was constructed, with an extra grading (the spin^c-structures of the link), with the Euler characteristic of the cohomology being the Seiberg–Witten invariant of the link [16]. In [16] it was conjectured (and in some cases verified) and in [25] proved that it coincides with the Heegaard Floer cohomology of the link. For some more connections with topology (e.g. applications of the graded roots) see [5, 9, 10, 11].

In order to define a lattice cohomology, one usually needs a lattice $\mathbb{Z}^s$, or more generally a CW complex $\mathcal{X}$ and a suitable integer-valued weight function on the set of cells (see Section 2.1). In various geometric contexts the lattice itself arises rather clearly and naturally, however how to retrieve a useful weight function from the geometric structure can often be much less obvious. E.g. in the classical case, the lattice is generated by the vertices of the graph, while the weight function is obtained from a Riemann–Roch expression.

The analytic version $\mathbb{H}^*_{an}(X, o)$ of the lattice cohomology associated with a normal surface singularity was defined in [1, 2], and even a graded $\mathbb{Z}[U]$-module morphism $\mathbb{H}^*_{an}(X, o) \to \mathbb{H}^*_{top}(X, o)$ was provided. In this analytic case the Euler characteristic equals the geometric genus of the germ.

Later, in [3], for any n-dimensional complex isolated singularity ($n \geq 2$) an analytic lattice cohomology was constructed — though its topological analogue is not currently known (conjecturally it is the embedded contact homology of the link). Also, an invariant applicable to isolated curve singularities was constructed in [4], which is a categorification of the δ-invariant, and presents a variety of useful properties on top of that, like a connection with flat deformations of the singularities.

In the case of all these analytic constructions, the weight functions arise from particular $\mathbb{Z}^s$-filtrations connected to the analytic structure (e.g. the multivariable divisorial filtration in the higher dimensional case), with a more general derivation of the weights and the cohomology induced by a filtration described in [4].

When proving functoriality under deformations, it was necessary to further abstract away the combinatorial structure to which we associate a lattice cohomology. Filtrations were replaced with systems of subspaces of a given vector space (the set of subspaces that comprise the filtration), and a CW complex with a weight function was defined purely based on the properties of this set.

In the present paper, we will even further generalize this construction to partially ordered sets equipped with additional structure corresponding to the codimensions of pairs of subspaces.

The original aim of this generalization was to be used when proving the aforementioned functoriality in [4]. Though this more general construction turned out not to be ultimately necessary for proving it on the $\mathbb{H}^0$ level — which is what ended up being included in the final version of that paper — we expect it to still be useful further down the line, e.g. when dealing with the higher cohomologies. For further details about that application, see Theorem 4.4. Also, due to the generality of the setting, we hope that the construction of the CW complex and its cohomology could be applicable in a variety of topics when these kinds of partially ordered sets naturally occur.

We now provide a quick summary of the concepts involved in this construction (for the details, see Section 3).

DEFINITION 1.1. A(n *integrally*) *metrized poset* is a poset $(P,<)$ together with a function

$$d:\ \{(x,y)\in P\times P \mid x\geq y \text{ or } x\leq y\}\to \mathbb{Z}_{\geq 0}\cup\{\infty\}$$

such that

(M1) $d(x,y)=0$ if and only if $x=y$;
(M2) $d(x,y)=d(y,x)$;
(M3) $\max\big(d(x,y),d(y,z)\big)\leq d(x,z)\leq d(x,y)+d(y,z)$ for all $x\geq y\geq z$.

We call such a function a *partial metric* on $(P,<)$ (as it is only defined between comparable pairs of elements).

DEFINITION 1.2. Consider an integrally metrized poset $(P,<,d)$ as above. We call a subset $C\subseteq P$ a combinatorial $(P,<,d)$-cell (or (P,d)-cell for short) if it is of the form

$$C=\left\{\bigwedge_{G\in\mathcal{H}} G \mid \emptyset\neq\mathcal{H}\subseteq\mathcal{G}\right\}\cup\{x\} \tag{$*$}$$

for some $x\in P$ and a finite set $\mathcal{G}\subseteq P$ with $d(x,G)=1$ for all $G\in\mathcal{G}$, where $\wedge$ denotes the meet operation — thus, in particular, the meet needs to exist for all non-empty sets $\mathcal{H}\subseteq\mathcal{G}$.

We define the *root*, *tip* and *generators* of a (P,d)-cell C as

$$\begin{aligned} R(C)&=\vee C,\\ T(C)&=\wedge C,\\ \mathcal{G}(C)&=\big\{\,y\in C \mid d\big(R(C),y\big)=1\,\big\}\end{aligned}$$

respectively. Note that the join and meet above always exist: for the set C of the form $(*)$, they would be x and $\wedge\mathcal{G}$ respectively.

Furthermore, we define the *dimension* and *height* of the (P,d)-cell C as

$$\dim C=\big|\mathcal{G}(C)\big|,\qquad \operatorname{ht} C=d\big(R(C),T(C)\big).$$

Using this purely combinatorial definition, we proceed to construct a CW complex where to each of these cells C, we associate a topological cell $\square(C)$ in that CW complex, and the above defined "dimension" of a combinatorial cell will correspond to the dimension of the topological one. These topological cells will be glued together in a way that respects the following (combinatorial) notion of faces:

DEFINITION 1.3. For a given combinatorial (P,d)-cell C, its *faces* are the elements of

$$\big\{\,D\subseteq C \mid D \text{ is a combinatorial } (P,d)\text{-cell}\,\big\}.$$

The construction gives us the CW complex $\mathcal{X}(P,<,d)$. Since we intend to consider a lattice cohomology on this complex (for the definition, see Section 2.1), we add weights to the original poset that are compatible with the metrized structure, as well as to cells of the associated CW complex. This is also where the *height* of a cell comes into play: it is used to determine the weights of the higher dimensional cells.

DEFINITION 1.4. We call $(P,<,d,w)$ a *weighted (integrally) metrized poset* (WIMP) if $(P,<,d)$ is an integrally metrized poset, and $w: P \to \mathbb{Z}$ is a function bounded from below, where for any $x, y \in P$ with $x \geq y$, we have

$$w(y) \leq w(x) + d(x,y). \tag{$**$}$$

The *weighted CW complex associated to* $(P,<,d,w)$ is the complex $\mathcal{X}(P,d)$ together with the weight function

$$\tilde{w}\big(\square(C)\big) = w\big(R(C)\big) + \operatorname{ht} C.$$

This definition of the weights is inspired by the construction in [4]. The condition $(**)$ is required to ensure the compatibility of the weight function on the CW complex (cf. Definition 2.2). Putting all of this together:

DEFINITION 1.5. Given a WIMP $(P,<,d,w)$, we define its lattice cohomology $\mathbb{H}^*(P,<,d,w)$ as that of the weighted CW complex $\big(\mathcal{X}(P,<,d),w\big)$ (cf. Definition 2.3).

2. PRELIMINARY

2.1. The lattice cohomology associated with a system of weights [16]

The construction of the *lattice cohomology* associates a graded $\mathbb{Z}[U]$-module to a CW complex $\mathcal{X}$ endowed with a set of weights. For a more detailed description, refer to [4]. The original definition can be seen in [16, 17, 18], where $\mathcal{X}$ is a cubical complex generated by a free $\mathbb{Z}$-module, i.e., a lattice.

Now let us summarize the definitions, first setting some notations regarding $\mathbb{Z}[U]$-modules.

NOTATION 2.1. Let $\mathcal{X}$ be a CW complex (for definitions and properties see e.g. [24]). Let $\{\operatorname{sk}_q \mathcal{X}\}_{q\geq 0}$ be the skeleton decomposition of $\mathcal{X}$. The 'q-dimensional cells' of $\mathcal{X}$, i.e., the images of the characteristic maps $\kappa_{q,\alpha}: D^q \to \mathcal{X}$, constitute the set $\mathcal{Q}_q = \mathcal{Q}_q(\mathcal{X})$, and this forms a basis in $\mathcal{C}_q = \mathcal{C}_q(\mathcal{X}) = \mathbb{Z}\langle \mathcal{Q}_q\rangle$, the free $\mathbb{Z}$-module generated by them.

The elements of $\mathcal{Q}_q$ will also be referred to as 'closed cells in $\mathcal{X}$' to differentiate them from their relative interiors $\square_q^\circ := \square_q \setminus \operatorname{sk}_{q-1}\mathcal{X}$, which we call the 'open cells'.

Using the above setting, in order to define an 'interesting' cohomology theory, we consider a set of compatible *weight functions* $\{w_q\}_q$.

DEFINITION 2.2. A set of functions $w_q: \mathcal{Q}_q \to \mathbb{Z}$ is called a *set of compatible weight functions* if

(a) w_0 is bounded from below;
(b) for any $\square_q \in \mathcal{Q}_q$ and any point $p \in \square_q \setminus \square_q^\circ$, consider $r < q$ such that $p \in \operatorname{sk}_r \mathcal{X} \setminus \operatorname{sk}_{r-1}\mathcal{X}$, and $\square_r^\circ$, the unique open cell of $\operatorname{sk}_r \mathcal{X}$ with $p \in \square_r^\circ$. Then in any such case we require $w_q(\square_q) \geq w_r(\square_r)$.

The index q of w_q may be omitted henceforth if it causes no confusion, i.e., we set $w = \cup_q w_q$. Such a pair $(\mathcal{X}, w)$ is called a *weighted CW complex*.

Let $\mathcal{S}_n = \mathcal{S}_n(w) \subseteq \mathcal{Q}_*$ denote the set of closed cells $\square_q$ (of any dimension) with $w(\square_q) \leq n$. By property (b), these form a subcomplex of $\mathcal{X}$, as such the weighted structure can also be viewed as a $\mathbb{Z}$-filtration $\mathcal{S}_{m_w} \subseteq \mathcal{S}_{m_w+1} \subseteq \cdots$ where $m_w = \min w_0$ ($\mathcal{S}_n = \emptyset$ for all $n < m_w$).

To a CW complex $\mathcal{X}$ and compatible weight function w, we associate the lattice cohomology $\mathbb{H}^*(\mathcal{X}, w)$ (or when the pair $(\mathcal{X}, w)$ is clear from the context, $\mathbb{H}^*$ for short). Here we summarize a geometric realization of this cohomology. For a more complete picture and another construction as the homology of a cochain complex, see [16, 4].

DEFINITION 2.3. For each $n \in \mathbb{Z}$ define $S_n = S_n(w) = \bigcup_{\square \in \mathcal{S}_n} \square \subseteq \mathcal{X}$. We have $S_n = \emptyset$ whenever $n < m_w = \min w_0$, and $S_{m_w} \subseteq S_{m_w+1} \subseteq \cdots$ form an increasing chain of subsets of $\mathcal{X}$.

For any $q \geq 0$, set

$$\mathbb{H}^q(\mathcal{X}, w) := \bigoplus_{n \geq m_w} H^q(S_n, \mathbb{Z}).$$

Then $\mathbb{H}^q$ is $\mathbb{Z}$ (in fact, conventionally $2\mathbb{Z}$)-graded: the $d = 2n$-homogeneous elements $\mathbb{H}^q_d$ consist of $H^q(S_n, \mathbb{Z})$.

Furthermore, $\mathbb{H}^q$ has a $\mathbb{Z}[U]$-module structure: the U-action is given by the restriction map $r_{n+1} : H^q(S_{n+1}, \mathbb{Z}) \to H^q(S_n, \mathbb{Z})$. Namely, $U * (\alpha_n)_n = (r_{n+1}\alpha_{n+1})_n$. Thus, we obtain a doubly graded $\mathbb{Z}[U]$-module

$$\mathbb{H}^*(\mathcal{X}, w) = \bigoplus_{q \geq 0} \mathbb{H}^q(\mathcal{X}, w).$$

Moreover, for $q = 0$, the choice of a fixed base point $x_w \in S_{m_w} \subseteq S_n$ provides an augmentation (splitting) $H^0(S_n, \mathbb{Z}) = \mathbb{Z} \oplus \widetilde{H}^0(S_n, \mathbb{Z})$, hence an augmentation of the graded $\mathbb{Z}[U]$-modules

$$\mathbb{H}^0 = \Big(\bigoplus_{n \geq m_w} \mathbb{Z}\Big) \oplus \Big(\bigoplus_{n \geq m_w} \widetilde{H}^0(S_n, \mathbb{Z})\Big) = \Big(\bigoplus_{n \geq m_w} \mathbb{Z}\Big) \oplus \mathbb{H}^0_{red}.$$

2.2. Weighted systems of subspaces [4]

In [4, Section 6.1], we introduced the concept of a (weighted) system of subspaces, and constructed an associated (weighted) CW complex. First, we recall the definitions essentially as-is, but we will then immediately note where some of the restrictions may be omitted or relaxed.

DEFINITION 2.4. Given a fixed vector space $\mathcal{O}$, we say that a set $\mathcal{A}$ is a 'system of subspaces' (of $\mathcal{O}$) if $\mathcal{O} \in \mathcal{A}$, $\mathcal{A}$ is closed under intersection (i.e., if $V_1, V_2 \in \mathcal{A}$, then $V_1 \cap V_2 \in \mathcal{A}$), and $\operatorname{codim}(V \subset \mathcal{O}) < \infty$ for any $V \in \mathcal{A}$.

DEFINITION 2.5. Given a system of subspaces $\mathcal{A}$ in a vector space $\mathcal{O}$, we call a subset $C \subseteq \mathcal{A}$ a combinatorial $\mathcal{A}$-cell if it is of the form

$$C = \left\{ \bigcap_{G \in \mathcal{H}} G \,\middle|\, \emptyset \neq \mathcal{H} \subseteq \mathcal{G} \right\} \cup \{V\}$$

for some $V \in \mathcal{A}$ and a finite set $\mathcal{G} \subset \mathcal{A}$ with $\operatorname{codim}(G \subseteq V) = 1$ for all $G \in \mathcal{G}$. We emphasize that *all* partial intersections must be contained within $\mathcal{A}$.

We define the *dimension* of the combinatorial $\mathcal{A}$-cell as $\dim C = |\mathcal{G}|$. We call the spaces V and $G \in \mathcal{G}$ the *root* and the *generators* respectively, while $\cap_{G \in \mathcal{G}} G$ (or V if $\mathcal{G} = \emptyset$) will be referred to as the *tip* of C. The root, tip and the set of generators are respectively denoted as $R(C)$, $T(C)$, and $\mathcal{G}(C)$, while the combinatorial $\mathcal{A}$-cell obtained from a root R and set of generators $\mathcal{G}$ as $C(R, \mathcal{G})$.

We further define the *height* of the $\mathcal{A}$-cell C as $\operatorname{ht} C = \operatorname{codim}\big(T(C) \subseteq R(C)\big)$.

REMARK 2.6. (a) Note that the root, tip, and generators are uniquely determined by the set C, so the above Definition 2.5 makes sense. Indeed,

$$R(C) = \textstyle\sum_{V \in C} V, \quad T(C) = \bigcap_{V \in C} V, \quad \mathcal{G}(C) = \{W \in C \mid \operatorname{codim}\big(W \subseteq R(C)\big) = 1\}.$$

The last identity holds since all intersections of some distinct $G, G' \in \mathcal{G}$ have codimension ≥ 2.

(b) We always have $\operatorname{ht} C \leq \dim C$. The inequality can be strict e.g. in the case when all the generators are distinct subspaces of some $V = R(C)$ containing a fixed codimension-2 subspace $W = T(C)$, and their number is $\dim C \geq 3$.

DEFINITION 2.7. For a given combinatorial $\mathcal{A}$-cell C, its *faces* are the elements of

$$\{D \subseteq C \mid D \text{ is a combinatorial } \mathcal{A}\text{-cell}\}.$$

We then proceed to construct a CW complex $\mathcal{X}(\mathcal{A})$ where to each of the above defined combinatorial cells C we associate a cube $\square(C)$ of the same dimension, and glue these together in such a way that the geometry of the CW complex corresponds to the above described, purely combinatorial notion of one cell being the face of another. Since most of the construction will be done later in a more general form (cf. Section 3.2), we omit it from here.

Note instead that merely for Definition 2.5 to work, basically all the conditions on $\mathcal{A}$ that we introduced in Definition 2.4 are superfluous: neither $\mathcal{O} \in \mathcal{A}$, nor $\mathcal{A}$ being closed under intersection, or elements of $\mathcal{A}$ having finite codimension is required. Omitting the intersection-closed property means that we need to be generally more careful about what is and is not in $\mathcal{A}$ though. In particular, in this case the existence of some $V \in \mathcal{A}$ and a finite set $\mathcal{G} \subset \mathcal{A}$ with $\operatorname{codim}(G \subseteq V) = 1$ for all $G \in \mathcal{G}$ does not guarantee that a corresponding combinatorial cell with root V and set of generators $\mathcal{G}$ exists.

In the generalization defined in Section 3, we will indeed make no analogous assumptions.

REMARK 2.8. If we do keep the assumptions in Definition 2.4, then the homotopy type of $\mathcal{X}(\mathcal{A})$ is quite simple: all connected components will be contractible (for a proof, see [4, Lemma 6.1.12]).

EXAMPLE 2.9. Otherwise this is not the case: assume that $V = \langle e_1, e_2, e_3 \rangle$ is 3-dimensional, and (the non-intersection-closed) $\mathcal{A}$ consists of the subspaces 0, $\langle e_1 \rangle$, $\langle e_1, e_3 \rangle$, V, $\langle e_2, e_3 \rangle$, $\langle e_2 \rangle$. Then $\mathcal{X}(\mathcal{A}) \sim S^1$.

Next, we add weights to the picture:

DEFINITION 2.10. We call $(\mathcal{A}, w)$ a *weighted system of vector spaces* (WSS) if $\mathcal{A}$ is a system of subspaces as in Definition 2.4, and $w : \mathcal{A} \to \mathbb{Z}$ is a function bounded from below. Additionally, for any $V, W \in \mathcal{A}$ with $W \subseteq V$, we require

$$w(W) \leq w(V) + \operatorname{codim}(W \subseteq V). \tag{2.1}$$

DEFINITION 2.11. Let $(\mathcal{A}, w)$ be a weighted system of vector spaces, and consider the associated complex $\mathcal{X} = \mathcal{X}(\mathcal{A})$. For an $\mathcal{A}$-cell C, let

$$\tilde{w}\big(\square(C)\big) = w\big(R(C)\big) + \operatorname{ht} C.$$

This forms a set of compatible weight functions on $\mathcal{X}$, and the resulting pair $(\mathcal{X}, \tilde{w})$ is called the *weighted CW complex associated to* $(\mathcal{A}, w)$.

In the original case of (intersection-closed) systems of subspaces, the weight function will give us most of the interesting structure, see Remark 2.8. In particular, we get a lattice cohomology:

DEFINITION 2.12. The lattice cohomology of a WSS $(\mathcal{A}, w)$ is the lattice cohomology of its associated weighted CW complex, and is denoted by $\mathbb{H}^*(\mathcal{A}, w)$.

3. METRIZED POSETS AND THEIR CW COMPLEXES

Observe that for the concepts in Section 2.2, what we need is essentially a set with a partial ordering (containment), and a way to measure how far away two comparable elements are (codimension). In the following pages, we will abstract away the particular setup used in 2.2, and then proceed to adapt the construction of the CW complex accordingly.

3.1. Combinatorial structure

To begin with, we introduce the following concepts, describing the kind of structures to which we aim to associate a CW complex, as well the class of subsets within such a structure that will give us the cells in that complex.

DEFINITION 3.1. A(n *integrally) metrized poset* is a poset $(P, <)$ together with a function

$$d : \{(x, y) \in P \times P \mid x \geq y \text{ or } x \leq y\} \to \mathbb{Z}_{\geq 0} \cup \{\infty\}$$

such that

(M1) $d(x, y) = 0$ if and only if $x = y$;
(M2) $d(x, y) = d(y, x)$;
(M3) $\max\big(d(x, y), d(y, z)\big) \leq d(x, z) \leq d(x, y) + d(y, z)$ for all $x \geq y \geq z$.

We call such a function a *partial metric* on $(P, <)$ (as it is only defined between comparable pairs of elements).

A *graded poset* is a poset $(P, <)$ together with a function $g : P \to \mathbb{Z}$ such that

(G1) if $x \geq y$, then $g(x) \leq g(y)$;
(G2) if $x \geq y$ and $g(x) = g(y)$, then $x = y$.

In particular, this grading naturally induces a metrized structure as well, with the choice $d(x,y) = |g(x) - g(y)|$.

For the sake of simplicity, we sometimes omit the ordering $<$ from the notation for the poset (and objects constructed from it) when it is obvious from the context.

DEFINITION 3.2. Consider an integrally metrized poset $(P, <, d)$ as above. We call a subset $C \subseteq P$ a combinatorial $(P, <, d)$-cell (or (P, d)-cell for short) if it is of the form

$$C = \left\{ \bigwedge_{G \in \mathcal{H}} G \mid \emptyset \neq \mathcal{H} \subseteq \mathcal{G} \right\} \cup \{x\} \tag{$*$}$$

for some $x \in P$ and a finite set $\mathcal{G} \subseteq P$ with $d(x, G) = 1$ for all $G \in \mathcal{G}$, where $\wedge$ denotes the meet operation — thus, in particular, the meet needs to exist for all non-empty sets $\mathcal{H} \subseteq \mathcal{G}$.

We define the *root*, *tip* and *generators* of a (P, d)-cell C as

$$R(C) = \vee C,$$
$$T(C) = \wedge C,$$
$$\mathcal{G}(C) = \{y \in C \mid d\big(R(C), y\big) = 1\}$$

respectively. Note that the join and meet above always exist: for the set C of the form $(*)$, they would be x and $\wedge\mathcal{G}$ respectively. The set $\mathcal{G}(C)$ may in fact be larger than $\mathcal{G}$ though, but starting from the entire $\mathcal{G}(C)$ instead would still ultimately give us the same set C (cf. Remark 3.3). The (P, d)-cell obtained from a given root R and a set $\mathcal{G}$ in the manner described in $(*)$ is denoted as $C(R, \mathcal{G})$.

Furthermore, we define the *dimension* and *height* of the (P, d)-cell C as

$$\dim C = \big|\mathcal{G}(C)\big|, \quad \operatorname{ht} C = d\big(R(C), T(C)\big).$$

REMARK 3.3. The generators of C needed to be explicitly defined as those elements $y \in C$ with $d(x, y) = 1$ where x is the root. That is because unlike with the original case of a system of subspaces $\mathcal{A}$ and $d = \operatorname{codim}$ (see Section 2.2, in particular part (a) of Remark 2.6), here potentially not all elements of $\mathcal{G}(C)$ are needed to "generate" the entire C as meets: $d(x, y \wedge y') = d(x, y) = d(x, y') = 1$ can occur for some $y \neq y'$. Nonetheless, we do still want to include all of these in $\mathcal{G}(C)$; and starting from this (potentially larger) set to begin with in place of $\mathcal{G}$ in $(*)$ will still yield the same set C since the extra elements would be obtained as meets anyway. Consequently, we can mostly dispense with this seeming ambiguity and declare the *generators of a cell* C to mean $\mathcal{G}(C)$ as defined above.

A simple application of the property (M3) from Definition 3.1 yields the following statement, similar to how it works for a system of subspaces:

PROPOSITION 3.4. *We always have* $\operatorname{ht} C \le \dim C$ *for a combinatorial* (P,d)*-cell* C.

For constructing a CW complex, we introduce the following, still purely combinatorial concept:

DEFINITION 3.5. For a given combinatorial (P,d)-cell C, its *faces* are the elements of

$$\{D \subseteq C \mid D \text{ is a combinatorial } (P,d)\text{-cell}\}.$$

3.2. Construction of the CW complex

Now, we start the recursive construction of a CW complex $\mathcal{X}$ associated with (P,d) whose cells will correspond to the combinatorial (P,d)-cells of appropriate dimensions. The gluing of these cells will be compatible with the above combinatorial notion of faces.

NOTATION 3.6. For any combinatorial (P,d)-cell C, let

$$\square^\circ(C) = \left\{R(C)\right\} \times (0,1)^{\mathcal{G}(C)}, \quad \text{and} \quad \square(C) = \left\{R(C)\right\} \times [0,1]^{\mathcal{G}(C)}.$$

We will glue the boundaries of the cubes $\square(C)$ to the lower dimensional skeleta when constructing the CW complex. Hence, the sets $\square^\circ(C)$ will serve as the open cells of $\mathcal{X}$. We call $\square^\circ(C)$ the *topological open* (P,d)*-cell* associated to the combinatorial cell C. We may view this as just the open cube $(0,1)^{\mathcal{G}(C)}$. Or in the case of $\dim C = 0$, this vertex is simply representing $R(C) \in P$. However, in the above notation of $\square^\circ(C)$ we wished to insert the root of the cube as well.

The closed cubes $\square(C)$ will be called *topological closed* (P,d)*-cells.*

The exact method of gluing will be defined in the sequel. The general principle is that in $[0,1]^{\mathcal{G}(C)}$, each vertex $u\colon \mathcal{G}(C) \to \{0,1\}$ will be glued to the 0-cell representing $\bigcap\{G \mid G \in \mathcal{G}(C),\ u(G) = 1\}$ for $u \ne 0$, and $R(C)$ for $u = 0$.

For the 0-dimensional skeleton, set

$$\operatorname{sk}_0 \mathcal{X} = P \times \{\emptyset\},$$

i.e., the vertices of $\operatorname{sk}_0 \mathcal{X}$ are essentially the elements of P.

Assume that we have already defined the $(q-1)$-dimensional skeleton $\operatorname{sk}_{q-1} \mathcal{X}$. For any $\square(C)$ of dimension less than q let $\kappa_C\colon \square(C) \to \operatorname{sk}_{q-1} \mathcal{X}$ be the characteristic map of $\square(C)$ into $\operatorname{sk}_{q-1} \mathcal{X}$ with attaching map $\kappa_C|_{\partial\square(C)}$.

Next, we wish to define the attaching map of a fixed topological (P,d)-cell C of dimension q (i.e., with $|\mathcal{G}(C)| = q$). We need to give a continuous map

$$\partial\kappa_C\colon \partial\square(C) = \left\{R(C)\right\} \times \partial[0,1]^{\mathcal{G}(C)} \to \operatorname{sk}_{q-1} \mathcal{X}.$$

Let us fix $u \in \partial[0,1]^{\mathcal{G}(C)} = \{u\colon \mathcal{G}(C) \to [0,1] \mid \exists\, G \in \mathcal{G}(C)\colon u(G) \in \{0,1\}\}$, and set

$$\mathcal{G}_0(u) = u^{-1}(0), \quad \mathcal{G}_1(u) = u^{-1}(1), \quad \mathcal{G}_*(u) = u^{-1}\big((0,1)\big).$$

Define also $R_u = \cap_{W \in \mathcal{G}_1(u) \cup \{R(C)\}} W$. (If $\mathcal{G}_1(u) \ne \emptyset$, then $R_u = \cap_{W \in \mathcal{G}_1(u)} W$; otherwise $R_u = R(C)$.) Finally, set

$$\widetilde{\mathcal{G}}_*(u) = \{R_u \cap G \mid G \in \mathcal{G}_*(u)\} \setminus \{R_u\} \subset P.$$

Since $\mathcal{G}_0(u) \cup \mathcal{G}_1(u) \ne \emptyset$, $|\mathcal{G}_*(u)| < q$. Furthermore, the map $G \mapsto R_u \cap G$ ($G \in \mathcal{G}_*(u)$) is usually not injective, and the intersection $R_u \cap G$ can also be equal to R_u. In particular, $|\widetilde{\mathcal{G}}_*(u)| \le |\mathcal{G}_*(u)|$, but the inequality in some cases can be strict. The element u defines the proper face C_u of C, with root R_u and generator set $\widetilde{\mathcal{G}}_*(u)$. Furthermore, the point $(R(C), u)$ from the boundary of the topological cell $\square(C)$ is sent to the topological cell $\square(C_u) = \square(R_u, \widetilde{\mathcal{G}}_*(u))$ by the map

$$(R(C), u) \mapsto (R_u, \tilde{u}) \in \{R_u\} \times (0,1)^{\tilde{\mathcal{G}}_*(u)} = \square^\circ\big(R_u, \tilde{\mathcal{G}}_*(u)\big),$$

where $\tilde{u} : \widetilde{\mathcal{G}}_*(u) \to (0,1)$ is defined as

$$\tilde{u}(\widetilde{G}) = \max\{u(G) \mid G \in \mathcal{G}_*(u), R_u \cap G = \widetilde{G}\}, \quad \widetilde{G} \in \widetilde{\mathcal{G}}_*(u). \tag{3.1}$$

Finally, the value $\partial\kappa_C((R(C), u))$ via the attaching map of $\partial\square(C)$ is given by

$$\partial\kappa_C\big((R(C), u)\big) = \kappa_{C_u}((R_u, \tilde{u})) \in \mathrm{sk}_{q-1}\mathcal{X}. \tag{3.2}$$

This, in the case $\widetilde{\mathcal{G}}_*(u) = \emptyset$, says that $\partial\kappa_C$ sends $(R(C), u)$ to the vertex $R_u \in \mathrm{sk}_0\mathcal{X}$.

REMARK 3.7. Let us summarize and refine some details. The element $u \in \partial[0,1]^{\mathcal{G}(C)}$ defines two topological cubes at two different levels and also a cell of $\mathrm{sk}_{q-1}\mathcal{X}$.

The first cube is a $|\mathcal{G}_*(u)|$-face sitting in the boundary of $\square(C) = [0,1]^{\mathcal{G}(C)}$. Its interior is

$$\widetilde{D}^\circ(u) := \{v : \mathcal{G}(C) \to [0,1] \mid \mathcal{G}_0(v) = \mathcal{G}_0(u), \mathcal{G}_1(v) = \mathcal{G}_1(u)\},$$

while its closure is

$$\widetilde{D}(u) = \{v : \mathcal{G}(C) \to [0,1] \mid \mathcal{G}_0(v) \supseteq \mathcal{G}_0(u), \mathcal{G}_1(v) \supseteq \mathcal{G}_1(u)\}.$$

The second cube is the topological cube/cell $\square(R_u, \widetilde{\mathcal{G}}_*(u))$ associated with a face of C, namely with the combinatorial $|\widetilde{\mathcal{G}}_*(u)|$-cube $C_u = (R_u, \widetilde{\mathcal{G}}_*(u))$. Its interior is $\square^\circ(R_u, \widetilde{\mathcal{G}}_*(u))$.

The map from (3.1) (with the substitutions $u \rightsquigarrow v$) defines a map $v \in \widetilde{D}^\circ(u) \mapsto \tilde{v} \in \square^\circ(R_u, \widetilde{\mathcal{G}}_*(u))$. This extends naturally to a continuous map $\delta_{\widetilde{D}(u)} : \widetilde{D}(u) \to \square(R_u, \widetilde{\mathcal{G}}_*(u))$ by the very same formula (3.1) (we simply allow v to take values of 0 or 1 for elements in $\mathcal{G}(C) \setminus (\mathcal{G}_0(u) \cup \mathcal{G}_1(u))$ too).

If $\widetilde{\mathcal{G}}_*(u) = \mathcal{G}_*(u)$, then $\delta_{\widetilde{D}(u)}$ basically is the identity, however, if $|\widetilde{\mathcal{G}}_*(u)| < |\mathcal{G}_*(u)|$, then $\delta_{\widetilde{D}(u)}$ is a non-linear topological surjective contraction with all fibers contractible. For a concrete case see Example 3.10.

The third object is the image of $\square(R_u, \widetilde{\mathcal{G}}_*(u))$ via κ_{C_u} in $\mathrm{sk}_{q-1}\mathcal{X}$, it is a $|\widetilde{\mathcal{G}}_*(u)|$-cell of this CW complex. Its interior is homeomorphic with $\square^\circ(R_u, \widetilde{\mathcal{G}}_*(u))$, but the boundary of $\square(R_u, \widetilde{\mathcal{G}}_*(u))$ might be contracted under the gluing procedure.

Regarding the gluing map $\partial\kappa_C$, for any fixed u, the gluing principle is the following: among the coordinates of u, the 1's determine the root R_u of the open cell we are gluing into, the 0's determine which codimension-1 subspaces we omit from the old generators, and the values in $(0,1)$ identify the exact point in the face $\square^\circ(R_u, \widetilde{\mathcal{G}}_*(u))$ to which we attach the point $\big(R(C), u\big)$. Finally, the map $\mathcal{G}_*(u) \to \widetilde{\mathcal{G}}_*(u)$, $G \mapsto R_u \cap G$, guides the cube-contraction $\delta_{\widetilde{D}(u)}$ via $v \mapsto \tilde{v}$ (cf. (3.1)).

REMARK 3.8. Once the continuity of $\partial\kappa_C : \partial\square_q(C) \to \operatorname{sk}_{q-1}\mathcal{X}$ is verified (cf. Proposition 3.9), we can also consider the inclusion $\iota_C : \partial\square_q(C) \to \square_q(C)$ too. The fibered coproduct of these two maps gives the following commutative diagram (which defines/identifies κ_C as well)

$$\begin{array}{ccc}
\partial\square_q(C) & \xrightarrow{\ \partial\kappa_C\ } & \operatorname{sk}_{q-1}\mathcal{X} \\
\downarrow{\scriptstyle \iota_C} & & \downarrow \\
\square_q(C) & \xrightarrow{\ \kappa_C\ } & \operatorname{sk}_{q-1}\mathcal{X} \sqcup \square_q(C)
\end{array}$$

From the continuity of $\partial\kappa_C$ the continuity of κ_C follows automatically.

Regarding the map κ_C note the following fact as well. The formula (3.1) can be considered even if $u \in \square^\circ(C)$, i.e., for u not on the boundary. In that case, we simply have $\mathcal{G}_0(u) = \mathcal{G}_1(u) = \emptyset$, and $R_u = R(C)$, $\widetilde{\mathcal{G}}_*(u) = \mathcal{G}(C)$, with the map leaving such points fixed. This extension of $\partial\kappa_C$ to $\square_q(C)$ is the map κ_C in the diagram.

Once the inductive construction of $\mathcal{X}$ is completed, we can consider the composition

$$\square_q(C) \xrightarrow{\kappa_C} \operatorname{sk}_{q-1}\mathcal{X} \sqcup \square_q(C) \hookrightarrow \operatorname{sk}_q\mathcal{X} \hookrightarrow \mathcal{X}.$$

This is the characteristic map of the cell $\square_q(C)$ in $\mathcal{X} = \mathcal{X}(P, d)$, also denoted by κ_C.

In order to finish the inductive construction of the CW complex $\mathcal{X}$, we verify the following:

PROPOSITION 3.9. *Assume that $\operatorname{sk}_{q-1}\widetilde{\mathcal{X}}$ was already constructed (hence all the maps $\partial\kappa_C$ and κ_C are continuous for any (P, d)-cell C of dimension $< q$). Then, for any C with $|\mathcal{G}(C)| = q$, $\partial\kappa_C$ is continuous.*

Proof. Let us write $R = R(C)$ and $\mathcal{G} = \mathcal{G}(C)$.

In order to prove the statement, it is enough to prove that the restriction of $\partial\kappa_C$ to any closed face of $\square(C)$ is continuous. The interior of such a face can be defined as

$$\widetilde{D}^\circ = \widetilde{D}^\circ_{\mathcal{G}_0,\mathcal{G}_1} = \{u \in [0,1]^{\mathcal{G}} \mid \mathcal{G}_0(u) = \mathcal{G}_0,\ \mathcal{G}_1(u) = \mathcal{G}_1\} \subseteq \partial[0,1]^{\mathcal{G}},$$

for some fixed $\mathcal{G}_0, \mathcal{G}_1 \subseteq \mathcal{G}$, not both of them empty. Let $\widetilde{D}$ be its closure in the cube $[0,1]^{\mathcal{G}}$. They can be identified with $\widetilde{D}^\circ(u)$ and $\widetilde{D}(u)$ for a certain u considered in Remark 3.7.

Also, let $\mathcal{G}_* = \mathcal{G} \setminus (\mathcal{G}_0 \cup \mathcal{G}_1)$.

If one only considers $\partial\kappa_C|_{\widetilde{D}^\circ}$, we see that R_u and $\widetilde{\mathcal{G}}_*(u)$ are fixed, and thus the continuity can be seen directly from the continuity of the max-function in (3.1). In other words, $\partial\kappa_C$ would be continuous if the topology on $\operatorname{sk}_{q-1}\mathcal{X}$ was obtained simply via taking the disjoint union of its open cells. Now we need to check that $\partial\kappa_C$ is in fact compatible with the attaching maps corresponding to the cells in $\operatorname{sk}_{q-1}\mathcal{X}$. That is to say, for any given such open face $\widetilde{D}^\circ = \widetilde{D}^\circ_{\mathcal{G}_0,\mathcal{G}_1}$, let $R_{\widetilde{D}}$ and $\widetilde{\mathcal{G}}_*(\widetilde{D})$ be the values of R_u and $\widetilde{\mathcal{G}}_*(u)$ for all $u \in \widetilde{D}^\circ$, $C_{\widetilde{D}} = C\big(R_{\widetilde{D}}, \widetilde{\mathcal{G}}_*(\widetilde{D})\big)$, and extend continuously the formula in (3.1) to a map

$$\delta_{\widetilde{D}} : \widetilde{D} \to \square(C_{\widetilde{D}})$$

as in Remark 3.7 (we simply allow v to take values of 0 or 1 for elements in $\mathcal{G} \setminus (\mathcal{G}_0 \cup \mathcal{G}_1)$ too as in the definition of $\delta_{\widetilde{D}(u)}$). Then, we claim that this map satisfies the identity

$$\kappa_{C_{\widetilde{D}}} \circ \delta_{\widetilde{D}} = \kappa_C|_{\widetilde{D}}. \tag{3.3}$$

This can be checked directly: for any element $(R, v) \in \widetilde{D} = \{R\} \times \{v \in [0,1]^{\mathcal{G}} \mid v|_{\mathcal{G}_0} = 0, v|_{\mathcal{G}_1} = 1\}$,

$$\delta_{\widetilde{D}}\big((R,v)\big) = (R', v') \in \{R'\} \times [0,1]^{\mathcal{G}'}, \ \ R' = R_{\widetilde{D}}, \ \ \mathcal{G}' = \widetilde{\mathcal{G}}_*(\widetilde{D}), \ \ v'(G') = \max\{v(G) \mid R' \cap G = G'\}.$$

That point (R', v') is in turn taken by $\kappa_{C_{\widetilde{D}}}$ to some $(R'', v'') \in [0,1]^{\mathcal{G}''}$, where

$$\begin{aligned} R'' &= \cap\big(\{R'\} \cup \{G' \mid v'(G') = 1\}\big) = \cap\big(\{R'\} \cup \{G' \in \mathcal{G}' \mid \exists G : v(G) = 1, R' \cap G = G', G \in \mathcal{G}_*\}\big) = \\ &= \cap\big(\{R'\} \cup \{R' \cap G \mid G \in \mathcal{G}_*, v(G) = 1, G \not\supseteq R'\}\big) = \cap\big(\{R\} \cup \mathcal{G}_1 \cup \{G \in \mathcal{G}_* \mid v(G) = 1\}\big) = R_v, \\ \mathcal{G}'' &= \{R'' \cap G' \mid G' \in \mathcal{G}', v'(G') \in (0,1)\} \setminus \{R''\} = \\ &= \{R_v \cap (R_{\widetilde{D}} \cap G) \mid G \in \mathcal{G}_*, G \not\supseteq R_{\widetilde{D}}, v(G) \in (0,1)\} \setminus \{R_v\} = \{R_v \cap G \mid G \in \mathcal{G}_*(v)\} \setminus \{R_v\} = \\ &= \widetilde{\mathcal{G}}_*(v), \end{aligned}$$

and for any $G'' \in \mathcal{G}'' = \widetilde{\mathcal{G}}_*(v)$, we have

$$\begin{aligned} u''(G'') &= \max\{v'(G') \mid G' \in \mathcal{G}', R'' \cap G' = G''\} = \max\{v(G) \mid G \in \mathcal{G}, R'' \cap G = G''\} = \\ &= \max\{v(G) \mid G \in \mathcal{G}, R_v \cap G = G''\}. \end{aligned}$$

So, the equality (3.3) is indeed true.

From (3.3), the continuity of $\kappa_C|_{\widetilde{D}}$ follows because both maps on the left hand side are continuous: $\kappa_{C_{\widetilde{D}}}$ because of the induction hypothesis (since $C_{\widetilde{D}}$ is lower dimensional), $\delta_{\widetilde{D}}$ by its definition (see above). Thus, κ_C is continuous when restricted to any closed face, and as these sets cover $\partial\square(C)$, the map κ_C is continuous on the whole. □

EXAMPLE 3.10. For a better intuition about the above picture, it is worth considering an example (this is the same as the one provided in [4]). Let P be a system of subspaces of a 2-dimensional space, with $\{e_1, e_2\}$ the canonical basis, and $R = \langle e_1, e_2\rangle$, $G_1 = \langle e_1\rangle$, $G_2 = \langle e_2\rangle$, $G_3 = \langle e_1 + e_2\rangle$, $T = \{0\}$ are all elements of P (and set $d = \operatorname{codim}$).

These vertices together form a 3-dimensional combinatorial cell C of height 2: the names of vertices reflect that. Between them we also have 6 edges in total: one connecting each (1-dimensional) generator to both R and T. There will also be 3 different 2-dimensional cells.

Since $\operatorname{ht} C < \dim C$, we can see that some of the contraction maps onto the boundary topological cells will be nontrivial: some nontrivial configurations of points on the boundary of $\square(C) = [0,1]^3$ will be mapped to the same point. Figure 1 illustrates this.

The black dots and the set of 3 thick black edges on the first diagram represent what will collapse into a single vertex each upon the gluing: the latter ones will be attached to the vertex T. The 3 blue edges and 3 blue faces (one in the back) will collapse onto a single edge each. The

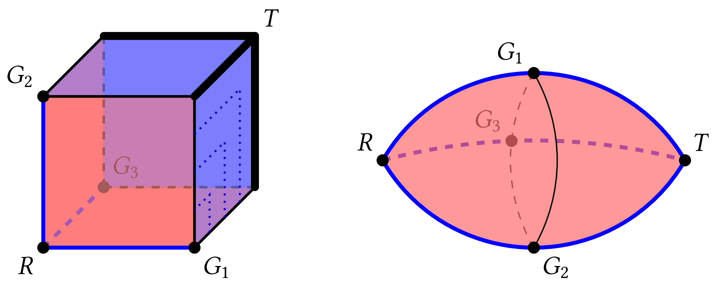

FIGURE 1.

dotted lines on the rightmost blue face show some points that will be glued to the same point on that edge. Lastly, the red faces will become the 2-cells.

Thus, we end up with a single 3-cell looking like what is seen on the second diagram. This is the image of $\square(\widetilde{C})$ in $\mathcal{X}$. (The black edges G_1G_2, G_2G_3, G_3G_1 in the middle are not cells in $\mathcal{X}$, they serve only to illustrate the spatial arrangement of the faces.)

DEFINITION 3.11. For a metrized poset $(P, <, d)$, its associated CW complex $\mathcal{X}(P, <, d)$ (or $\mathcal{X}(P, d)$ for short) is that obtained by repeating the above described gluing for each integer $q \geq 0$ (in increasing order, taking the limit if the set of dimensions is not bounded).

3.3. Adding the weights

Finally, we introduce weights to this complex as well.

DEFINITION 3.12. We call $(P, <, d, w)$ a *weighted (integrally) metrized poset* (WIMP) if $(P, <, d)$ is an integrally metrized poset, and $w : P \to \mathbb{Z}$ is a function bounded from below, where for any $x, y \in P$ with $x \geq y$, we have

$$w(y) \leq w(x) + d(x, y). \tag{3.4}$$

For any WIMP $(P, <, d, w)$, we introduce a weight function (cf. Definition 2.2) on its associated CW complex from Definition 3.11.

DEFINITION 3.13. Let $(P, <, d, w)$ be a weighted metrized poset, and consider the associated complex $\mathcal{X} = \mathcal{X}(P, d)$. For a (P, d)-cell C, let

$$\tilde{w}\big(\square(C)\big) = w\big(R(C)\big) + \operatorname{ht} C.$$

The resulting pair $(\mathcal{X}, \tilde{w})$ is called the *weighted CW complex associated to* $(P, <, d, w)$.

REMARK 3.14. This definition of the weights on the higher dimensional cells is inspired by the construction in [4]. There it arises naturally upon collapsing a cubical lattice with weights into a CW complex of this kind: the weights of the cells in the resulting complex in that scenario are the respective minimuma of the weights of cells collapsing onto them.

We can also now see that the condition (3.4) is needed to ensure the compatibility of the weight function on the CW complex (cf. Definition 2.2):

PROPOSITION 3.15. *The above $\tilde{w}$ forms a set of compatible weight functions on $\mathcal{X}(P,d)$.*

Proof. This is a consequence of Definition 3.12, since for any face D of a (P,d)-cell C,

$$\tilde{w}\big(\square(D)\big) = w\big(R(D)\big) + \operatorname{ht} D \le w\big(R(C)\big) + d\big(R(C),R(D)\big) + d\big(R(D),T(D)\big) \le$$
$$\le w\big(R(C)\big) + d\big(R(C),T(C)\big) = \tilde{w}\big(\square(C)\big).$$ □

DEFINITION 3.16. Given a WIMP $(P,<,d,w)$, we define its lattice cohomology $\mathbb{H}^*(P,<,d,w)$ as that of the weighted CW complex $\big(\mathcal{X}(P,<,d),w\big)$ (cf. Definition 2.3).

4. EXAMPLES

We illustrate how the presented the concept behaves through some simple examples.

EXAMPLE 4.1. Given a metrized poset $(P,<,d)$ with the weight function w being identically 0, we have

$$\operatorname{rk} \mathbb{H}^0_{2n}(P,d,0) = \begin{cases} |P|, & \text{if } n = 0 \\ \operatorname{rk} H^0\big(\mathcal{X}(P,d)\big), & \text{if } n > 0. \end{cases}$$

Indeed, $\tilde{w}\big(\square(C)\big) = 0$ for $\dim C = 0$ but $\tilde{w}\big(\square(C)\big) = \operatorname{ht} C > 0$ otherwise, since then $R(C) > T(C)$, so the level set S_0 just contains the vertices. But 1-cells can only have height 1, therefore two vertices (elements of the poset P) are in the same component of S_1 if and only if they are in the same component of $\mathcal{X}(P,d)$, i.e., one can connect them with down- or upward steps where the distance (given by d) is 1. In particular, for connected $\mathcal{X}(P,d)$, we have $\mathbb{H}^0_{\text{red}} = \mathbb{H}^0_{\text{red},0}$.

By a similar reasoning,

$$\mathbb{H}^{q-1}_{2n}(P,d,0) = H^{q-1}\big(\mathcal{X}(P,d)\big) \quad \text{if } n \ge q.$$

This follows from the fact that all $(\le q)$-cells have height at most q, as per Proposition 3.4.

Note that this is drastically different from the case we are used to, i.e., defining the weights of higher dimensional cells to be the respective maxima of the weights of their vertices: setting $w = 0$ everywhere on P does not imply the contractibility of all level sets. That is because in this construction, we explicitly set the weights of higher dimensional cells to always be bigger than that of their vertices, as motivated by the construction in [4]. The vertices here do not correspond to the vertices in the original, cubical lattice, rather the one where "superfluous" cells have been already collapsed, as described in section 6. The vertices that remain are only those where something interesting is happening (the filtration jumps), and the cohomology reflects that.

EXAMPLE 4.2. Let $(P,<,g)$ be a graded poset (together with the induced partial metric $d_g(x,y) = |g(x) - g(y)|$), and set the weights to be $w = g$. Then

$$\operatorname{rk} \mathbb{H}^q_{2n}(P,d_g,g) \le \operatorname{rk} H^q\big(\mathcal{X}(P,d_g)\big)$$

for all $q \ge 0$ and $n \in \mathbb{Z}$. Furthermore, if w is bounded from above, then we have equality for all sufficiently large n.

We can observe that

$$\tilde{w}\big(\square(C)\big) = w\big(R(C)\big) + \operatorname{ht} C = g\big(R(C)\big) + g\big(T(C)\big) - g\big(R(C)\big) = w\big(T(C)\big),$$

which is exactly the maximum of the weight function on the vertices (since the weight function $w = g$ is increasing on P). Thus, we can essentially "push down" the level sets within themselves (i.e., construct a suitable deformation retract) whenever we can do so within $\mathcal{X}(P, d_g)$. A rigorous proof of this fact is left to the reader.

After seeing these two examples, we can make the following basic observation:

PROPOSITION 4.3. *Let* $(P, <, d, w)$ *be a WIMP with* w *bounded from above. Then*

$$\mathbb{H}^{q-1}_{2n}(P, d, w) = H^{q-1}\big(\mathcal{X}(P, d)\big)$$

whenever $n \geq \max w + q$.

This simply follows from the fact that all $(\leq q)$-cells have height at most q, thus their weight is bounded by $n \geq \max w + q$, and gluing higher dimensional cells does not affect the lower singular cohomologies.

We will also mention a case that arises when studying the lattice cohomologies of systems of subspaces, as in [4]; in particular when aiming to prove the functoriality of $\mathbb{H}^*$ with respect to deformations of curve singularities. This was, in fact, the original motivation for introducing the generalized construction described in the present paper. Though neither the present generalization to metrized posets nor the following example are included in the final version of [4] as they were ultimately not needed for the statement proved (the functoriality of $\mathbb{H}^0$), it could still be useful for e.g. dealing with the higher cohomologies.

EXAMPLE 4.4. When aiming to associate a map between lattice cohomologies to any deformation of an isolated curve singularity $X_{t\neq 0} \to X_0$, one observes that the limits of the vector spaces in the filtration $\mathcal{F}$ we consider (that form the vertices of the associated CW complex $\mathcal{X}_t$) can end up being nontrivial subspaces of the corresponding vertices in $\mathcal{X}_0$.

To define a map, we might wish to extend the system $\mathcal{A} = \operatorname{Im} \mathcal{F}$ by incorporating the subspaces below each element. One obvious option would be to consider the set

$$\overline{\mathcal{A}} = \{V \oplus W \subseteq \mathcal{O} \oplus \mathcal{O} \mid V \in \mathcal{A}, W \subset V\}.$$

We would then find a copy of the original $\mathcal{X}(\mathcal{A})$ within $\mathcal{X}(\overline{\mathcal{A}})$ by looking at the subcomplex spanned by the diagonal elements (V, V) — except that this is not the case. By Definition 2.5, there will be no higher dimensional cells at all in this subcomplex, since the codimensions are always at least 2 (or 0) for diagonal elements.

To fix that, we need to modify the definition. Consider the function

$$\operatorname{cd}(V \oplus W, V' \oplus W') = \operatorname{codim}(V' \subseteq V) + \operatorname{codim}(W' \subseteq W \cap V')$$

instead of

$$\operatorname{codim}(V \oplus W, V' \oplus W') = \operatorname{codim}(V' \subseteq V) + \operatorname{codim}(W' \subseteq W)$$

for some $V \oplus W, V' \oplus W' \in \overline{\mathcal{A}}$ with $V' \oplus W' \subseteq V \oplus W$. Intuitively, since we are only considering pairs (V, W) where $W \subseteq V$, when measuring the "distance" between $V \oplus W$ and $V' \oplus W'$

(previously done simply via taking the codimension), we force the second component to be a subspace of V' before seeing how much further away W' is.

For the weight function, we set $\overline{w}(V,W) = w(V) + \mathrm{codim}(V,W)$.

One can verify that $(\overline{\mathcal{A}}, \subset, \mathrm{cd}, \overline{w})$ is indeed a WIMP. Also, since

$$\mathrm{cd}\big((V,V),(W,W)\big) = \mathrm{codim}(V,W),$$

we will naturally get an embedding

$$\mathcal{X}(\overline{\mathcal{A}}, \mathrm{cd}) \to \mathcal{X}(\mathcal{A}, \mathrm{codim})$$

via a kind of diagonal map.

We now state this in a more precise manner.

DEFINITION 4.5. Given a weighted system of subspaces $(\mathcal{A}, w)$ with all $V \in \mathcal{A}$ being subspaces of $\mathcal{O}$, consider the set

$$\overline{\mathcal{A}} = \{\iota_1(V) + \iota_2(W) \leq \mathcal{O} \oplus \mathcal{O} \mid V \in \mathcal{A}, W \leq V\}$$

where $\iota_1, \iota_2 : \mathcal{O} \to \mathcal{O} \oplus \mathcal{O}$ are the standard embeddings of the two components in the direct sum. For simplicity, we write $\iota(V,W) = \iota_1(V) + \iota_2(W)$. Also, let

$$d\big(\iota(V,W), \iota(V',W')\big) = \mathrm{codim}(V' \subseteq V) + \mathrm{codim}(W' \subseteq W \cap V')$$

and

$$\overline{w} : \overline{\mathcal{A}} \to \mathbb{Z}, \quad \overline{w}\big(\iota(V,W)\big) = w(V) + \mathrm{codim}(V,W).$$

These form a WIMP $(\overline{\mathcal{A}}, d, \overline{w})$.

Let us further consider the weighted CW complexes $(\mathcal{X}, w)$ and $(\overline{\mathcal{X}}, \overline{w})$ associated to $(\mathcal{A}, w)$ and $(\overline{\mathcal{A}}, d, \overline{w})$ respectively. Then the diagonal map $\mathcal{A} \to \overline{\mathcal{A}}$, $V \mapsto \iota(V,V)$ gives us an embedding

$$\delta : \mathcal{X} \to \overline{\mathcal{X}}, \quad \square(V, \mathcal{G}) \ni (V,u) \overset{\delta}{\mapsto} \Big(\iota(V,V), \big(\iota(G,G) \overset{\overline{u}}{\mapsto} u(G)\big)\Big) \in \square\big(\iota(V,V), \{\iota(G,G) \mid G \in \mathcal{G}\}\big).$$

In fact, one can show that this diagonal map is a (weighted) homotopy equivalence, i.e., it respects the filtration of the CW complex induced by the weight and induces a homotopy equivalence on each level set. Indeed, this is exactly the reason it is worth considering: from a weighted cellular map into such an "extended" CW complex, one can also naturally get a morphism into the lattice cohomology of the original.

Do note, however, that $\overline{\mathcal{X}}$ as defined here is usually extremely large (infinite dimensional, and has uncountably many cells). It *is* still a CW complex: the definition of the attaching maps ensures that the closure of each open cell is covered by a finite number of open cells, those associated to the faces of the corresponding combinatorial cell. Still, this is not a complex that is practical for any kind of computation. One can observe that this is rather similar to how singular homology is useful when dealing with maps between topological spaces, but for concrete calculations, one tends to use, say, cellular or simplicial homology. We also see why it can be advisable to relax some of the finiteness conditions imposed on weight functions in the classical construction of lattice cohomology — which we indeed did as a side effect of generalizing to arbitrary CW complexes.

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