Chapter 1
Lattice Cohomology

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This note is a gentle introduction to the lattice cohomology of isolated complex analytic germs. The analytic version is sensitive to the analytic type of the germs. It can be defined in any positive dimension, in particular for reduced curve singularities as well. Here we will treat mostly the curve case. The topological version is defined for the topological type of an isolated surface singularity with a rational homology sphere link.

These cohomology theories are categorifications of famous numerical invariants. E.g., the Euler characteristic of the lattice cohomology of a reduced curve singularity is its delta invariant.

In section 1 we review some notations and elementary properties of singular analytic germs. In the case of isolated plane curve singularities we compare numerical invariants read from the embedded topological type with invariants read from the abstract analytic type.

In section 2 we treat the lattice cohomology. We provide some combinatorial statements and also several examples both in the curve and surface cases.

1.1 Isolated singular germs

1.1.1 Preliminary

Definition 1.1.1 Let $f_i\colon (\mathbb{C}^n,0)\to(\mathbb{C},0)$ be holomorphic germs $(1\le i\le k)$, and set $f=(f_1,\dots,f_k)\colon(\mathbb{C}^n,0)\to(\mathbb{C}^k,0)$. Then if $(V(f),0)$ is defined as

$$(V(f),0) := \{\mathbf{x}\in(\mathbb{C}^n,0)\mid f(\mathbf{x})=0\}\subset(\mathbb{C}^n,0)$$

we call $(V(f),0)$ the germ of an analytic set. If $k=1$ then we call $(V(f),0)$ a *hypersurface* germ.

At each point $\mathbf{x}\in V(f)$, we can consider the Jacobian

$$\mathrm{d}f(\mathbf{x})=\left(\frac{\mathrm{d}f_i}{\mathrm{d}x_j}\right)_{i,j}\in\mathbb{C}^{k\times n}.$$

The *dimension* of $(V(f),0)$ is defined as

$$\dim(V(f),0)=n-\max_{\mathbf{x}\in V(f)}\operatorname{rk}\mathrm{d}f(\mathbf{x})=n-\operatorname{rk}\mathrm{d}f(\mathbf{x}_{gen}),$$

where the second equality also shows that for a generic choice of $\mathbf{x}$, the rank of the Jacobian will be maximal. By definition, points where the rank is *not* maximal form the *singular locus* of $(V(f),0)$:

$$\mathrm{Sing}(V(f),0)=\{\mathbf{x}\in(V(f),0)\mid\operatorname{rk}\mathrm{d}f(\mathbf{x})<n-\dim(V(f),0)\}.$$

Remark 1.1.2 In the above setting, $\operatorname{codim}\big((V(f),0)\subset(\mathbb{C}^n,0)\big)=n-\dim V(f)\leq k$. In particular, for the hypersurface case — i.e. when $k=1$ and $f\neq 0$ — the codimension is 1.

Remark 1.1.3 $\mathrm{Sing}(V(f),0)$ is also the germ of an analytic set: it is described by the simultaneous vanishing of f and the determinants of all $(\operatorname{codim}V)\times(\operatorname{codim}V)$ minors of $\mathrm{d}f$.

For hypersurfaces, i.e. $f\colon(\mathbb{C}^n,0)\to(\mathbb{C},0)$, the point $\mathbf{x}\in(V(f),0)$ is singular if and only if $\mathrm{d}f(\mathbf{x})=0$.

Example 1.1.4 If $f=(xy,xz,yz)\colon\mathbb{C}^3\to\mathbb{C}^3$, then $V(f)\subset\mathbb{C}^3$ is the union of the coordinate axes. The Jacobian is

$$\mathrm{d}f=\begin{pmatrix}y&x&0\\z&0&x\\0&z&y\end{pmatrix},$$

and one can verify that the rank of $\mathrm{d}f$ is 2 for all $(x,y,z)\in V(f)$ except 0 ($\det(\mathrm{d}f)=-2xyz\equiv 0$).

Definition 1.1.5 The analytic germ $(f,0)$ is said to define an *isolated singularity* when 0 is an isolated point in $\mathrm{Sing}(V(f),0)$.

Example 1.1.6

- If $f=x_1^2+\cdots+x_n^2$ then $\mathrm{Sing}(V(f),0)=\{0\}$, hence $(f,0)$ is an isolated singularity;

- if $f = x_1^2(x_1^2 + x_2^3)$ then $\mathrm{Sing}(V(f), 0) = \{x_1 = 0\}$, hence $(f, 0)$ is not an isolated singularity.

For a given germ $(V(f), 0) \subset (\mathbb{C}^n, 0)$ with $f = (f_1, \ldots, f_k)$, we can consider its topological and analytic types. The analytic type is characterized by the $\mathbb{C}$-algebra isomorphism type of its coordinate ring $\mathscr{O}_{V,0} = \mathscr{O}_{\mathbb{C}^n,0} \,/\, (f_1, \ldots, f_k)$, where $\mathscr{O}_{\mathbb{C}^n,0} = \mathbb{C}\{z_1, \ldots, z_n\}$ is the (local) ring of holomorphic functions defined near 0 (its unique maximal ideal consists of those germs which vanish at 0) and $(f_1, \ldots, f_k)$ denotes the ideal generated by $f_1, \ldots, f_k$.

Example 1.1.7 Let $f = x^2 - y^3$, $f' = (x^2 - y^3, z - x^{100})$ and consider them at the point 0. Then the two local algebras, $\mathbb{C}\{x, y\}/(x^2 - y^3)$ and $\mathbb{C}\{x, y, z\}/(x^2 - y^3, z - x^{100})$ are isomorphic, hence f and f' define analytically equivalent curve singularities at 0.

Regarding the topological type, we will distinguish two notions. The abstract topological type is the topology of $V(f)$ in some sufficiently small neighborhood of 0, while the embedded topology is the topology of the pair of $V(f) \subset \mathbb{C}^n$ within some sufficiently small neighborhood of 0. In order to formulate these statements more precisely, the following facts are crucial:

Theorem 1.1.8 (see [16, 23] and the references therein.) *Let $(f, 0)$ be an analytic germ and assume that $(V, 0) = (V(f), 0) \subset (\mathbb{C}^n, 0)$ defines an isolated singularity. Let also S_r^{2n-1} and B_r^{2n} denote the sphere and ball in $\mathbb{C}^n$ centered at 0 with radius r. Then for some sufficiently small ε_0:*

(1) $S_\varepsilon^{2n-1} \pitchfork V$ for every $0 < \varepsilon \leq \varepsilon_0$.
(2) There is a homeomorphism $\Phi\colon (B_{\varepsilon_0}^{2n}, V \cap B_{\varepsilon_0}^{2n}) \to \big(C(S_{\varepsilon_0}^{2n-1}), C(V \cap S_{\varepsilon_0}^{2n-1})\big)$ where $C(X)$ denotes the real cone over X. This map is a diffeomorphism away from 0, and takes the sections $V \cap S_\varepsilon^{2n-1}$ to sections of the cone $C(V \cap S_{\varepsilon_0}^{2n-1})$ parallel to the base.

Definition 1.1.9 In the above setting, the intersections $V \cap S_\varepsilon^{2n-1}$ are all diffeomorphic for sufficiently small ε. We call this the *(abstract) link* L_V of $(V, 0)$.

Likewise, the diffeomorphism type of the pair $(S_\varepsilon^{2n-1}, V \cap S_\varepsilon^{2n-1})$ is constant for sufficiently small ε. We call this the *embedded link* of $(V, 0)$ into S_ε^{2n-1}.

Remark 1.1.10 Because of the cone structure, the local topological type of $(V, 0)$ is given by the homeomorphism type of the abstract link, while the embedded topological type of $(\mathbb{C}^n, V, 0)$ is equivalent to the topological type of the embedded link respectively. In particular, when talking about the (abstract) topological and embedded topological types, in fact we might consider the respective links.

The analytic type (or the embedded topological type) determines the abstract topological type.

Remark 1.1.11 If $\dim(V(f), 0) = d$, then V has dimension is $\dim_{\mathbb{R}}(V(f), 0) = 2d$ over $\mathbb{R}$. Hence, by the transversality property from Theorem 1.1.8, the link L_V is a compact, oriented, smooth real manifold of dimension $2d - 1$, embedded in the sphere S^{2n-1}.

Example 1.1.12 Let $(V, 0)$ be a plane curve singularity: $(V, 0) \subset (\mathbb{C}^2, 0)$ and $\dim(V, 0) = 1$. Then the abstract link L_V is a compact real 1-manifold, i.e. a disjoint union of circles. The number of such circles agrees with the number of irreducible components of the curve germ $(V, 0)$ (for more details see below), and we obtain the embedded link as well:

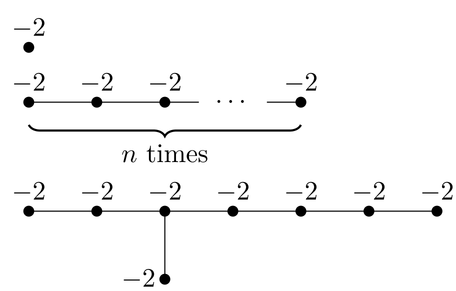
$$L_V = \bigsqcup_{i=1}^{r} S^1 \hookrightarrow S^3.$$

Example 1.1.13 If $\dim(V, 0) = 2$ then $(V, 0)$ is called a surface singularity. Then L_V is a compact, oriented real 3-manifold. In fact, it is a plumbed 3-manifold associated with a plumbing graph whose intersection form is negative definite. If $(V, 0)$ is irreducible then the graph is connected.

Theorem 1.1.14 (Grauert [14]) *Every oriented plumbed 3-manifold with a negative definite intersection form is the link of a surface singularity.*

Example 1.1.15 For the following singularities, the plumbing graphs are the following:

- $\{x^2 + y^2 + z^2 = 0\}$ $\longrightarrow$
- $\{x^2 + y^2 + z^{n+1} = 0\}$ $\longrightarrow$
- $\{x^2 + y^3 + z^5 = 0\}$ $\longrightarrow$



1.1.2 Curves in $(\mathbb{C}^2, 0)$

In this section, we assume $f\colon (\mathbb{C}^2,0) \to (\mathbb{C},0)$. For such an $f \in \mathscr{O}_{\mathbb{C}^2,0} = \mathbb{C}\{x,y\}$, we have a decomposition $f_1^{\alpha_1} \cdots f_r^{\alpha_r}$, where each f_i is an irreducible germ and $f_i \neq f_j$ $(i \neq j)$ up to invertible elements of $\mathbb{C}\{x,y\}$. This induces the irreducible decomposition $(V(f),0) = \bigcup_{i=1}^{r} (V(f_i),0)$. Correspondingly, we get the embedded link

$$L_f = \bigsqcup_{i=1}^{r} L_{f_i} \approx \bigsqcup_{i=1}^{r} S^1 \hookrightarrow S^3$$

where L_f and L_{f_i} denote the links of $(V(f),0)$ and $(V(f_i),0)$ respectively. The germ defines an isolated singularity if and only if $\alpha_i = 1$ for all i. Such isolated singularities are also called *plane curve singularities.* fact env

Fact 1.1.16 ([35]) *The embedded topological type of an isolated plane curve singularity is completely classified by the following data:*

- *The embedded topological types of each component L_{f_i} of the link. This is an iterative torus knot, encoded by the Puiseux pairs of f_i.*
- *The linking numbers between the components L_{f_i} and L_{f_j} $(i \neq j)$. Algebraically, the linking number of L_{f_i} and L_{f_j} $(i \neq j)$ equals the (local) intersection multiplicity at the origin of f_i and f_j.*

A notable property of the embedded topological type is the following:

Theorem 1.1.17 (Milnor [16]) *For a sufficiently small $\varepsilon > 0$, the map $\Phi\colon S^3_\varepsilon \setminus V(f) \to S^1$, $\Phi(z) = \frac{f(z)}{|f(z)|}$ $(z = (x,y))$ defines a C^∞ locally trivial fibration (called the Milnor fibration):*

$$\begin{array}{ccc} F & \longrightarrow & S^3_\varepsilon \setminus L_f \\ & & \big\downarrow \Phi \\ & & S^1 \end{array}$$

In particular, each $L_f \subset S^3_\varepsilon$ is a fibered link. The Milnor fiber F is homotopically equivalent to a bouquet of S^1's, and its first Betti number is equal to the Milnor number $\mu(f) = \mu(f,0)$:

$$F \cong \bigvee_{i=1}^{\mu(f,0)} S^1 \quad \textit{where} \quad \mu(f,0) = \dim_{\mathbb{C}} \frac{\mathscr{O}_{\mathbb{C}^2,0}}{\left(\frac{\mathrm{d}f}{\mathrm{d}x}, \frac{\mathrm{d}f}{\mathrm{d}y}\right)}.$$

Remark 1.1.18 A similar statement holds for isolated hypersurface singularities in any dimension ($F = \vee S^{n-1}$, $b_{n-1}(F) = \mu(f,0) = \dim_{\mathbb{C}}(\mathscr{O}_{\mathbb{C}^n,0}/(\mathrm{d}f))$), see [16].

As for understanding certain numerical invariants of the analytic type of a plane curve singularity, consider first the irreducible case.

1.1.2.1 Irreducible plane curve singularities

Proposition 1.1.19 (see e.g. [7, 34].) *Let f define an irreducible plane curve singularity $(V(f),0) \hookrightarrow (\mathbb{C}^2,0)$. Then there exists an analytic homeomorphism $n\colon (\mathbb{C},0) \to (V(f),0)$, also called the Puiseux parametrization, or the normalization of $\big(V(f),0\big)$. Furthermore, this parametrization $n\colon t \mapsto \big(x(t),y(t)\big)$ can be written in the form*

$$x(t) = t^m,$$
$$y(t) = a_n t^n + a_{n+1}t^{n+1} + \cdots, \quad (a_i \in \mathbb{C},\ a_n \neq 0)$$

such that $f\big(x(t),y(t)\big) \equiv 0$.

Example 1.1.20 For $f = x^a - y^b$ with a, b coprime (which implies that f is irreducible), we have

$$t \mapsto \begin{pmatrix} x(t) = t^b \\ y(t) = t^a \end{pmatrix}.$$

For any analytic map $g \in \mathscr{O}_{\mathbb{C}^2,0}$, we can then take the pullback $n^*g \in \mathscr{O}_{\mathbb{C},0} = \mathbb{C}\{t\}$: the bijection n induces $n^*\colon \mathscr{O}_{V,0} \hookrightarrow \mathscr{O}_{\mathbb{C},0} = \mathbb{C}\{t\}$, and we can apply that to $g|_V$. Using diagrams:

$$\begin{array}{ccccc} (\mathbb{C},0) & \xrightarrow{n} & \big(V(f),0\big) & \hookrightarrow & (\mathbb{C}^2,0) \\ & {\scriptstyle n^*g} \searrow & \downarrow {\scriptstyle g|_V} & \swarrow {\scriptstyle g} & \\ & & (\mathbb{C},0) & & \end{array}$$

Definition 1.1.21 Define the *δ-invariant* of $(f,0)$ as

$$\dim_{\mathbb{C}} \mathscr{O}_{\mathbb{C},0}/\operatorname{Im} n^* = \dim_{\mathbb{C}} \mathbb{C}\{t\}/\mathscr{O}_{V,0} < \infty,$$

and its *semigroup* (or *monoid*) *of values* as

$$\mathscr{S} = \{\operatorname{ord}_t n^*g \mid g \in \mathscr{O}_{V,0}\} \subset \mathbb{Z}_{\geq 0}.$$

Note that this satisfies

$$\delta = \dim \frac{\mathbb{C}\{t\}}{\mathscr{O}_{V,0}} = |\mathbb{Z}_{\geq 0} \setminus \mathscr{S}| < \infty.$$

In particular, there exists a smallest element $c \in \mathbb{Z}_{\geq 0}$ such that $c + \mathbb{Z}_{\geq 0} \subset \mathscr{S}$, which is called the *conductor* element of the semigroup $\mathscr{S}$.

Example 1.1.22 Let $f = x^3 - y^4$. Then, via 1.1.20, we get

$$n^* g(t) = g\big(x(t), y(t)\big) = g(t^4, t^3)$$

for any $g \in \mathbb{C}\{x, y\}$. In particular, for $g = x^k y^\ell$, we get $n^* g(t) = t^{4k+3\ell}$, so $\mathscr{S} \supseteq \langle 3, 4 \rangle$. One can also verify that this is in fact an equality.

Example 1.1.23 More generally, for $f = x^a - y^b$ with a, b coprime, we get

$$\mathscr{S} = \langle a, b \rangle, \quad \delta = \frac{(a-1)(b-1)}{2}, \quad c = (a-1)(b-1).$$

As for the topological type, in this case the abstract link is S^1 (since f is irreducible), and it is embedded in S^3 as the torus knot $T_{a,b}$. The Milnor number is $\mu = (a-1)(b-1)$ too.

Example 1.1.24 Consider the singularity parametrised as $n\colon t \mapsto (t^4, t^6 + t^7)$. Then $n^* g(t) = g(t^4, t^6 + t^7)$, and one can show that

$$\mathscr{S} = \langle 4, 6, 13 \rangle, \quad \delta = 8, \quad c = 16.$$

Exercise 1.1.25 Find a polynomial f that corresponds to this parametrization, and compute the Milnor number as well.

One can observe that in all these examples, we always have $c = 2\delta = \mu$. This is indeed the case when f is an irreducible plane curve germ. For the general statement see Theorem 1.1.39.

It is a notable fact that the semigroup can be computed from the embedded topological type. In fact, the semigroup is a complete embedded topological invariant, i.e. it always distinguishes singularities with different embedded topological types (i.e. different embedded algebraic links).

Another complete invariant of irreducible isolated plane curve singularities is the Alexander polynomial $\Delta(t)$ of the embedded link $S^1 \subset S^3_\varepsilon$. The connection between the two can be expressed as follows:

Proposition 1.1.26 (Campillo, Delgado, Gusein-Zade [8]) *For an irreducible plane curve singularity* $(V,0) \subset (\mathbb{C}^2,0)$ *with an associated semigroup* $\mathscr{S}$*, we have*

$$\sum_{s\in\mathscr{S}} t^s = \frac{\Delta(t)}{1-t}.$$

Remark 1.1.27 The Alexander polynomial is not a complete invariant of knots in S^3 in general (for not necessarily algebraic knots). It does, however distinguish between all links of irreducible plane curve singularities.

It can be immediately seen then that $\mathscr{S}$ and Δ can indeed be computed from each other. Another complete invariant is the Hilbert series $H(t)$. Its equivalence with the semigroup can be seen from its very definition.

Definition 1.1.28 For an irreducible plane curve singularity $(V,0)$ with semigroup $\mathscr{S}$, we can define its *Hilbert function* $\ell \mapsto \mathfrak{h}(\ell)$ and *Hilbert series* $H(t)$ respectively as

$$\mathfrak{h}(\ell) = |\{s \in \mathscr{S} \mid s < \ell\}| \quad (\ell \in \mathbb{Z}_{\geq 0}) \quad \text{and} \quad H(t) = \sum_{\ell\geq 0} \mathfrak{h}(\ell)t^\ell.$$

Proposition 1.1.29 *With the above notations,*

$$H(t) = \frac{t}{1-t}\sum_{s\in\mathscr{S}} t^s = \frac{t\Delta(t)}{(1-t)^2}.$$

Remark 1.1.30 The function $\mathfrak{h}$ *is* actually the Hilbert function of a filtration. We will talk about that in detail later in this section.

Example 1.1.31 Consider the plane curve singularity defined by $f = x^2-y^3$. Recall from Example 1.1.23 that its embedded link is the torus knot $T_{2,3}$ (the trefoil), and the semigroup is $\mathscr{S} = \langle 2,3\rangle = \mathbb{Z}_{\geq 0} \setminus \{1\}$ (with $\delta = 1$ and $c = 2$). Then

$$\sum_{s\in\mathscr{S}} t^s = 1 + t^2 + t^3 + t^4 + \cdots = 1 + \frac{t^2}{1-t} = \frac{1-t+t^2}{1-t},$$

and $1-t+t^2$ is indeed the Alexander polynomial of the knot $T_{2,3}$. Also, we get

$$H(t) = \sum_{\ell\geq 0} \mathfrak{h}(\ell)t^\ell = t+t^2+2t^3+3t^4+\cdots = t+\frac{t^2}{(1-t)^2} = \frac{t-t^2+t^3}{(1-t)^2} = \frac{t\Delta(t)}{(1-t)^2}.$$

1.1.2.2 General curves

Next, we define the delta invariant and the Hilbert function for non-irreducible curve germs as well. In fact, in this subsection we will consider an arbitrary isolated curve singularity (which is not necessarily embedded in $(\mathbb{C}^2, 0)$). In particular the next definition will not use any embedded data. (On the contrary, for the definition of the Milnor number μ one needs an embedding in $(\mathbb{C}^2, 0)$), or at least a smoothing.)

Now, write $(V, 0) = \bigcup_{i=1}^{r} (V_i, 0)$, where $(V_i, 0) \subset (\mathbb{C}^n, 0)$ are irreducible curve germs.

Consider the normalizations $n_i\colon (\mathbb{C}_{t_i}, 0) \to (V_i, 0)$ to get the mapping $\mathscr{O}_{\mathbb{C}^n,0} \ni g \mapsto n_i^* g \in \mathbb{C}\{t_i\}$ induced via the diagram

$$\begin{array}{ccccccc} (\mathbb{C}_{t_i}, 0) & \xrightarrow{n_i} & (V_i, 0) & \hookrightarrow & (V, 0) & \hookrightarrow & (\mathbb{C}^n, 0) \\ & \searrow^{n_i^* g} & & & \downarrow{\scriptstyle g|_V} & \swarrow_{g} & \\ & & & & (\mathbb{C}, 0) & & \end{array}$$

and the valuations $(\mathrm{ord}_{t_1} n_1^* g, \ldots, \mathrm{ord}_{t_r} n_r^* g)$ on $\mathscr{O}_{\mathbb{C}^n,0}$. Since here $n_i^* g$ only depends on $g|_V$, we might as well equivalently define $n_i^* g$ on $\mathscr{O}_{V,0}$ via the left part of the diagram. Group the valuations as

$$\mathfrak{v}(g) = (\mathrm{ord}_{t_1} n_1^* g, \ldots, \mathrm{ord}_{t_r} n_r^* g) \quad (g \in \mathscr{O}_{V,0}).$$

This induces a $(\mathbb{Z}_{\geq 0})^r$-filtration on $\mathscr{O}_{V,0}$, which in turn gives rise to a Hilbert function (and series):

$$\begin{aligned} \mathscr{F}(\ell) &= \{g \in \mathscr{O}_{V,0} \mid \forall i : \mathrm{ord}_{t_i} n_i^* g \geq \ell_i\} \subset \mathscr{O}_{V,0}, \qquad (\ell = (\ell_1, \ldots, \ell_r)), \\ \mathfrak{h}(\ell) &= \dim \mathscr{O}_{V,0}/\mathscr{F}(\ell), \\ H(\mathbf{t}) &= \sum_{\ell \geq 0} \mathfrak{h}(\ell) \mathbf{t}^\ell \qquad (\mathbf{t}^\ell = t_1^{\ell_1} \cdots t_r^{\ell_r}), \end{aligned}$$

and provides an embedding $n^* : \mathscr{O}_{V,0} \hookrightarrow \bigoplus_{i=1}^{r} \mathbb{C}\{t_i\}$.

Remark 1.1.32 Observe also that the definition of $\ell \mapsto \mathfrak{h}(\ell)$ does not depend at all on the embedding (realization) of $(V, 0)$ in some $(\mathbb{C}^n, 0)$, it depends only the local algebra $\mathscr{O}_{V,0}$.

Definition 1.1.33 We define the *δ-invariant* of $(V, 0)$ as

$$\dim_{\mathbb{C}} \bigoplus_{i=1}^{r} \mathbb{C}\{t_i\}/\mathrm{Im}\, n^* < \infty,$$

and its *semigroup of values* as

$$\mathscr{S} = \{\mathfrak{v}(g) \mid g \in \mathscr{O}_{V,0}\} = \{(\mathrm{ord}_{t_1} n_1^* g, \dots, \mathrm{ord}_{t_r} n_r^* g) \mid g \in \mathscr{O}_{V,0}\} \subset (\mathbb{Z}_{\geq 0})^r.$$

There exists a unique smallest element $c \in (\mathbb{Z}_{\geq 0})^r$ such that $c + (\mathbb{Z}_{\geq 0})^r \subset \mathscr{S}$, which is called the *conductor* element of $\mathscr{S}$.

Example 1.1.34 Let $(V,0) = \{xy = 0\} \subset (\mathbb{C}^2, 0)$. Then $n_1(t_1) = (t_1, 0)$ and $n_2(t_2) = (0, t_2)$, so for some $g(x,y)$, we get

$$n_1^* g = g(t_1, 0) \in \mathbb{C}\{t_1\}, \quad n_2^* g = g(0, t_2) \in \mathbb{C}\{t_2\}.$$

One can then verify that $\delta = 1$, $\mu = 1$, $c = (1,1)$, and $\mathscr{S} = (0,0) \cup ((1,1) + (\mathbb{Z}_{\geq 0})^2)$.

Example 1.1.35 Let $(V,0) = \{xy = yz = zx = 0\} \subset (\mathbb{C}^3, 0)$, i.e. the union of the 3 coordinate axes. Then we get the maps $n_1(t_1) = (t_1, 0, 0)$, $n_2(t_2) = (0, t_2, 0)$, $n_3(t_3) = (0, 0, t_3)$. We can then compute $\delta = 2$.

Example 1.1.36 Let $(V,0) = \{xy(x-y) = 0\} \subset (\mathbb{C}^2, 0)$. We leave as an exercise to the reader to show that $\delta = 3$ in this case.

Note however, that while this singularity has the same abstract topologically type as the germ from the previous example (the link is $S^1 \sqcup S^1 \sqcup S^1$) and even 'geometrically' they are very similar (3 lines, each pair of them intersecting in the same point), the δ-invariants differs. Indeed the two are *not* analytically isomorphic. E.g., $\{xy = yz = zx = 0\}$ cannot be embedded into $(\mathbb{C}^2, 0)$.

Example 1.1.37 Let $(V,0) = \left\{x^2 + y^4 = 0\right\} \subset (\mathbb{C}^2, 0)$. Again, we leave as an exercise to check that $\mu = 3$ and $\delta = 2$ and $r = 2$.

The semigroup $\mathscr{S}$ and the Hilbert function determine each other:

Lemma 1.1.38 (see e.g. [13].) *The semigroup can be deduced from the Hilbert function as follows:*

$$\mathscr{S} = \{\ell \in (\mathbb{Z}_{\geq 0})^r \mid \mathfrak{h}(\ell + E_i) > \mathfrak{h}(\ell) \ \text{ for every } \ i = 1, \dots, r\}.$$

On the other hand, $\mathfrak{h}(\ell + E_i) - \mathfrak{h}(\ell) \in \{0, 1\}$ *for any* $\ell \geq 0$ *and* $i \in \{1, \dots, r\}$. *Moreover,* $\mathfrak{h}(\ell + E_i) = \mathfrak{h}(\ell) + 1$ *if there is an element* $s \in \mathscr{S}$ *such that* $s_i = \ell_i$ *and* $s_j \geq \ell_j$ *for* $j \neq i$. *Otherwise* $\mathfrak{h}(\ell + E_i) = \mathfrak{h}(\ell)$. *(Here* $\{E_i\}_{i=1}^r$ *denotes the standard basis of* $\mathbb{Z}^r$*.)*

1.1.2.3 The case of plane curve singularities revisited

We have the following two statements valid specifically for plane curve singularities $(V(f),0) \subset (\mathbb{C}^2,0)$. The first one relates the delta invariant with the Milnor number.

Theorem 1.1.39 (Jung, Milnor, see [16].) *For a plane curve singularity* $(f,0)$*, we have* $\mu(f) = 2\delta(f) - (r-1)$ *where* r *is the number of irreducible components of* $(V(f),0)$*.*

The second group of statements relates the Hilbert function (or, equivalently, the semigroup) with the multivariable Alexander polynomial $\Delta(\mathbf{t})$ of the link (here $(\mathbf{t} = (t_1,\dots,t_r))$).

The key intermediate bridge is the multivariable Poincaré series. First define the *extended* Hilbert series as

$$H^{ext}(\mathbf{t}) = \sum_{\ell\in\mathbb{Z}^r} \mathfrak{h}(\ell)\mathbf{t}^\ell \in \mathbb{Z}[[t_1,t_1^{-1},\dots,t_r,t_r^{-1}]],$$

where for an arbitrary $\ell \in \mathbb{Z}^r$ we set $\mathfrak{h}(l) = \mathfrak{h}(\max\{l,0\})$. Note that $H^{ext}|_{(\mathbb{Z}_{\geq 0})^r} = H$.

Then we define the multivariable Poincaré series $P(\mathbf{t})$ as follows. If $r=1$, then the Poincaré series of the graded ring $\oplus_l \mathscr{F}(l)/\mathscr{F}(l+E_1)$ is $P(t) = -H(t)(1-t^{-1})$. For general r, one defines the Poincaré series by

$$P(t_1,\dots,t_r) := -H(t_1,\dots,t_r)\cdot\prod_i(1-t_i^{-1}).$$

This means that the coefficient $\mathfrak{p}(\ell)$ of $P(\mathbf{t}) = \sum_\ell \mathfrak{p}(\ell)\cdot t_1^{\ell_1}\dots t_r^{\ell_r}$ satisfies

$$\mathfrak{p}(\ell) = \sum_{I\subset\{1,\dots,r\}} (-1)^{|I|-1}\mathfrak{h}(l+E_I), \qquad (E_I = \sum_{i\in I} E_i).$$

$\mathbb{Z}[[t_1,t_1^{-1},\dots,t_r,t_r^{-1}]]$ is a module over the ring of Laurent power series, hence the multiplication by $\prod_i(1-t_i^{-1})$ is well-defined. One can check (using the property $\mathfrak{h}(l) = \mathfrak{h}(\max\{l,0\})$) that $P(\mathbf{t})$ is a power series involving only nonnegative powers of t_i. In fact, the support of $P(\mathbf{t})$ is included in $\mathscr{S}$, that is, $\mathfrak{p}(\ell) = 0$ whenever $\ell \notin \mathscr{S}$.

If $r=1$, then by Lemma 1.1.38 $P(t) = \sum_{s\in\mathcal{S}} t^s = -\sum_{s\notin\mathcal{S}} t^s + 1/(1-t)$, where $-\sum_{s\notin\mathcal{S}} t^s$ is a polynomial. Furthermore, $P(\mathbf{t})$ is a polynomial for $r>1$ (see Theorem 1.1.40 below).

The following result motivated the introduction of $P(\mathbf{t})$, it creates the bridge between the abstract analytic invariant $H(\mathbf{t})$ and the embedded topological invariant $\Delta(\mathbf{t})$.

Theorem 1.1.40 (Campillo, Delgado, Gusein-Zade, see [9, 10, 11].) *The following identity holds:*

$$P(\mathbf{t}) = \begin{cases} \Delta(\mathbf{t}), & \text{if } r > 1, \\ \Delta(\mathbf{t})/(1-t), & \text{if } r = 1. \end{cases}$$

Multiplication by $\prod_i (1 - t_i^{-1})$ of series with $\mathfrak{h}(0) = 0$ is injective if $r = 1$, however it is not if $r > 1$. In particular, in such cases it can happen that for two different series $H(\mathbf{t})$ we obtain the very same $P(\mathbf{t})$. (This really can happen for concrete isolated curve singularities, which have higher embedded dimensions, that is for non-plane curve singularities, see [11]). Nevertheless, for plane curve singularities even for $r > 1$ one can recover $H(\mathbf{t})$ as follows.

Using this, we write the multivariate analogue of 1.1.29:

Theorem 1.1.41 (Gorsky–Némethi [13]) *For a plane curve singularity* $(V, 0) = \bigcup_{i=1}^{r} (V_i, 0)$*, with the above notations we have*

$$H(\mathbf{t}) = \frac{1}{\prod_{i=1}^{r}(1-t_i)} \left(\sum_{\emptyset \neq I \subseteq \{1,\dots,r\}} (-1)^{|I|+1} \Big(\prod_{i \in I} t_i \Big) P_{V_I}(\mathbf{t}_I) \right)$$

where $\mathbf{t} = (t_1, \dots, t_r)$ *and for* $\emptyset \neq I = \{i_1, \dots, i_s\} \subseteq \{1, \dots, r\}$*, we denote* $(V_I, 0) = \bigcup_{i \in I} (V_i, 0)$*,* $\mathbf{t}_I = (t_{i_1}, \dots, t_{i_s})$*, and* P_{V_I} *is the Poincaré series of* $(V_I, 0)$ *in the variables* $t_{i_1}, \dots, t_{i_s}$*.*

The multivariable Alexander polynomials (hence the multivariable Poincaré series/polynomials) can be directly computed from the splice diagrams of the pair $L_V \subset S^3$ (or from the embedded resolution graph of $V, 0) \subset (\mathbb{C}^2, 0)$), see [12]. In this way we can compute $H(\mathbf{t})$ as well.

We emphasize again:

Proposition 1.1.42 *For any isolated plane curve isolated singularity* $(V, 0)$*, its semigroup, Hilbert and Poincaré series, as well as the multivariate Alexander polynomial of the link are all complete embedded topological invariants.*

1.2 Lattice cohomology

1.2.1 Combinatorial setup

The lattice cohomology was originally introduced as a way to construct a particular topological invariant for normal surface singularities. In recent years, however, the method was generalized and used to study the analytic structure of singularities in any dimension as well.

Let us take a look at the general, abstract setup first. We follow [20, 23], see also [1].

Goal 1.2.1 *Given some sort of combinatorial data — that will, ultimately, come from some topological or analytic information — we aim to construct a bigraded* $\mathbb{Z}[U]$*-module* $\mathbb{H}^* = \bigoplus_{q\geq 0} \mathbb{H}^q$ *where*

- $\mathbb{H}^q$ *is a* $\mathbb{Z}$*-graded* $\mathbb{Z}[U]$*-module,*
- *for every* $x \in \mathbb{H}^q$*, there is some* $n(x) \in \mathbb{Z}_{\geq 0}$ *such that* $U^{n(x)} \cdot x = 0$*.*

Here $\mathbb{Z}[U]$ *is the polynomial ring, with* U *denoting the free variable.*

Example 1.2.2 Some particular $\mathbb{Z}$-graded $\mathbb{Z}[U]$-modules:

- $\mathcal{T}_0^+ = \mathbb{Z}[U,U^{-1}] \,/\, U\mathbb{Z}[U] = \mathbb{Z}\langle 1, U^{-1}, U^{-2}, \ldots\rangle$ with $\deg U^{-k} = 2k$,
- $\mathcal{T}_{2n}^+ = \mathbb{Z}[U,U^{-1}] \,/\, U^{-n+1}\mathbb{Z}[U] = \mathbb{Z}\langle U^{-n}, U^{-n-1}, U^{-n-2}, \ldots\rangle$ (with the same grading),
- $\mathcal{T}_{2n}^+(m) = U^{-n-m+1}\mathbb{Z}[U] \,/\, U^{-n+1}\mathbb{Z}[U] = \mathbb{Z}\langle U^{-n}, U^{-n-1}, \ldots, U^{-n-m+1}\rangle$ (with the same grading and with $\mathbb{Z}$-rank m).

In each of these cases, multiplication with U is a degree-(-2) morphism, which together with the grading being bounded from below implies in particular that the second condition above is satisfied.

We should also introduce some further notations that we will use extensively. Assume that $\{E_i\}_{i=1}^r$ is a basis of $\mathbb{R}^r$.

Definition 1.2.3 (a) For a set $I \subseteq \{1, \ldots, r\}$, let $E_I = \sum_{i\in I} E_i$.

(b) For $a, b \in \mathbb{R}^r$ with $a \leq b$, let $R(a,b) = \{x \in \mathbb{R}^r \mid a \leq x \leq b\}$ denote the (hyper)rectangle spanned by a and b with edges parallel to the coordinate axes. We will also allow $a, b \in (\mathbb{R} \cup \{\pm\infty\})^r$.

Next, we present the construction.

Definition 1.2.4 Consider the lattice $L = \mathbb{Z}^r = \mathbb{Z}\langle E_i \rangle_{i=1}^r \hookrightarrow \mathbb{Z}^r \otimes_{\mathbb{Z}} \mathbb{R} = \mathbb{R}^r$, which induces a cubical cellular decomposition of $\mathbb{R}^r$:

- 0-cubes: $\ell \in L$ lattice points (technically the singleton sets $\{\ell\}$, but for convenience we identify these with the lattice points themselves when it causes no confusion),
- 1-cubes: $R(\ell, \ell + E_i)$ segments, where $\ell \in L$ and $i \in \{1, \ldots, r\}$,
- q-cubes: $R(\ell, \ell + E_I)$ cubes, where $l \in L$ and $I \subset \{1, \ldots, r\}$, $|I| = q$.

We denote the set of q-cubes by $\mathcal{Q}_q$.

On this cubical complex we introduce a set of *compatible weight functions*:

- $w_0 \colon \mathcal{Q}_0 = \mathbb{Z}^r \to \mathbb{Z}$ such that $\forall n : \left|w_0^{-1}\big((-\infty, n)\big)\right| < \infty$ (in particular there exists some $m_w = \min w_0 \in \mathbb{Z}$),
- $w_q \colon \mathcal{Q}_q \to \mathbb{Z}$ for any q-cell $\square$, where $w_q(\square) = \max\limits_{\ell \in L \cap \square} w_0(\ell)$,
- $w = \bigcup\limits_{q \geq 0} w_q$.

(We could allow more general configurations here, but this will suffice for now; the interested reader is invited to check e.g. [23].)

For every $n \in \mathbb{Z}$, consider the 'level set'

$$S_n = \bigcup_{w(\square) \leq n} \square,$$

which gives us a chain of inclusions of finite cubical spaces

$$\emptyset \hookrightarrow S_{m_w} \hookrightarrow S_{m_w+1} \hookrightarrow \cdots .$$

Due to the way the weights were defined, we can view the weights as a $\mathbb{Z}$-filtration on the cubical complex: cubes $\square$ with $w(\square) \leq n$ form the subcomplex S_n.

Definition 1.2.5 Given the above setup, we define the graded $\mathbb{Z}$-module

$$\mathbb{H}^q(L, w) = \bigoplus_{n \geq m_w} H^q(S_n, \mathbb{Z}),$$

where $\mathbb{H}^q(L, w)$ is $\mathbb{Z}$-garded in such a way that $H^q(S_n, \mathbb{Z})$ is the homogeneous component of degree $2n$. Together with a U-action

$$0 \xleftarrow{\cdot U} H^q(S_{m_w}, \mathbb{Z}) \xleftarrow{\cdot U} H^q(S_{m_w+1}, \mathbb{Z}) \xleftarrow{\cdot U} \cdots ,$$

induced by the inclusion maps, we get a graded $\mathbb{Z}[U]$-module structure as well, just as we wanted. Finally, set $\mathbb{H}^*(L, w) = \bigoplus\limits_{q \geq 0} \mathbb{H}^q(L, w)$. It is called the *lattice cohomology* associated with the pair (L, w). (We will often simply write $\mathbb{H}^q$ and $\mathbb{H}^*$ if the meaning is clear from the context.)

Definition 1.2.6 We define the *reduced lattice cohomology* as well. For $q = 0$ we set

$$\bigoplus_{n \geq m_w} H^0(S_n, \mathbb{Z}) = \bigoplus_{n \geq m_w} \big(\mathbb{Z} \oplus \widetilde{H}^0(S_n, \mathbb{Z})\big)$$

with its natural $\mathbb{Z}$-graded $\mathbb{Z}[U]$-module structure. This gives us a splitting of graded $\mathbb{Z}[U]$-modules

$$\mathbb{H}^0 = \mathcal{T}^+_{2m_w} \oplus \mathbb{H}^0_{red}, \quad \text{where} \quad \mathbb{H}^0_{red} := \bigoplus_{n \geq m_w} \widetilde{H}^0(S_n, \mathbb{Z}).$$

We then set $\mathbb{H}^*_{red} = \mathbb{H}^0_{red} \oplus \bigoplus_{q>0} \mathbb{H}^q$.

Remark 1.2.7 For all $q \geq r$, we have $\mathbb{H}^q = 0$.

Definition 1.2.8 The same construction can be done for just a smaller part of the lattice, indeed, for any subcomplex of the cubical complex. The most oft-occurring cases are the restrictions to a (possibly infinite) rectangle $R(a, b)$, in particular to $(\mathbb{R}_{\geq 0})^r$.

If we restrict an already given weight function w to a smaller subcomplex R, we will use the notation $\mathbb{H}^*(L, w)|_R = \mathbb{H}^*(L \cap R, w|_R)$. In fact, there also always exists a graded $\mathbb{Z}[U]$-module morphism $\mathbb{H}^*(L, w) \to \mathbb{H}^*(L, w)|_R$.

Example 1.2.9 Let $r = 1$, and consider just $L_{\geq 0} = \mathbb{Z}_{\geq 0} \subset \mathbb{R}_{\geq 0}$. If $w_0 \colon \mathbb{Z}_{\geq 0} \to \mathbb{Z}$ is an increasing function then all S_n are of the form $R(0, a)$, in particular contractible. Hence, $\mathbb{H}^*_{red} = 0$.

Example 1.2.10 Again, let $L_{\geq 0} = \mathbb{Z}_{\geq 0} \subset \mathbb{R}_{\geq 0}$. Now consider a monoid $\mathscr{S} \subset \mathbb{Z}_{\geq 0}$ (a semigroup with $0 \in \mathscr{S}$) such that $\delta = |\mathbb{Z}_{\geq 0} \setminus \mathscr{S}| < \infty$ (cf. 1.1.21), and define

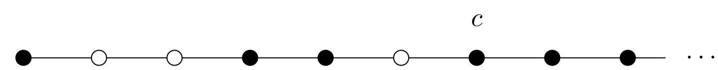
$$\mathfrak{h}(\ell) = |\{a \in \mathscr{S} \mid a < \ell\}|,$$

$$\overline{\mathfrak{h}}(\ell) = |\{a \in \mathbb{Z}_{\geq 0} \setminus \mathscr{S} \mid a < \ell\}| = \ell - \mathfrak{h}(\ell),$$

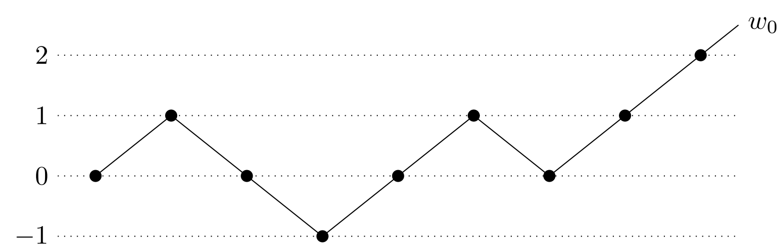
$$w_0(\ell) = \mathfrak{h}(\ell) - \overline{\mathfrak{h}}(\ell) = 2\mathfrak{h}(\ell) - \ell.$$

Note that for every ℓ, we have $w_0(\ell + 1) - w_0(\ell) = \pm 1$ depending on whether $\ell \in \mathscr{S}$ or not, and due to $|\mathbb{Z}_{\geq 0} \setminus \mathscr{S}| < \infty$, the weight function w_0 becomes strictly increasing after a point. Hence in particular w_0 satisfies the conditions from 1.2.4, and it indues a lattice cohomology $\mathbb{H}^* = \mathbb{H}^0$.

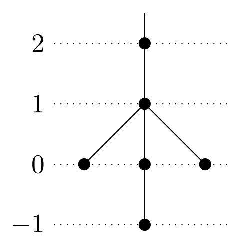
Now, set $\mathscr{S} = \langle 3, 4 \rangle$ as in Example 1.1.22. We have $\delta = |\mathbb{Z}_{\geq 0} \setminus \mathscr{S}| = 3$, and $c = 6$ is the conductor, the element above which every integer is in $\mathscr{S}$: this is the point where w_0 becomes increasing. Computing the functions in question (filled dots in the top row denote elements of $\mathscr{S}$):



ℓ	0	1	2	3	4	5	6	7	8
$\mathfrak{h}(\ell)$	0	1	1	1	2	3	3	4	5
$\overline{\mathfrak{h}}(\ell)$	0	0	1	2	2	2	3	3	3
$w_0(\ell)$	0	1	0	-1	0	1	0	1	2

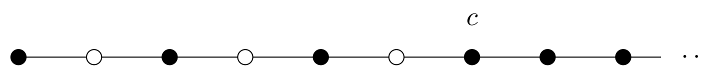


We see that in the induced $\mathbb{H}^0$, there is one generator at level -1, three at level 0, and for all $n > 0$, the set S_n is contractible. (The part of the lattice above $c = 6$ does not matter since there w_0 is increasing, so for any such n the subset $S_n \cap R(0, 6)$ is a strong deformation retract of S_n.) Drawing the so-called *graded root* which encodes this information:

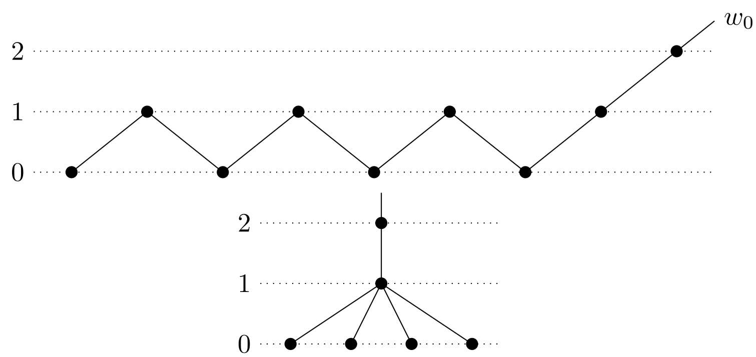


Here the integers on the left are the w_0-weights, the black dots at weight/level k correspond to the connected components of S_k, with the edges denoting the inclusion relations. Algebraically, we can write $\mathbb{H}^0 = \mathcal{T}^+_{-2} \oplus \mathcal{T}_0(1) \oplus \mathcal{T}_0(1)$, and $\mathbb{H}^{\geq 1} = 0$.

Example 1.2.11 For $\mathscr{S} = \langle 2, 7 \rangle$ we get $\delta = 3$, $c = 6$, and



ℓ	0	1	2	3	4	5	6	7	8
$\mathfrak{h}(\ell)$	0	1	1	2	2	3	3	4	5
$\overline{\mathfrak{h}}(\ell)$	0	0	1	1	2	2	3	3	3
$w_0(\ell)$	0	1	0	1	0	1	0	1	2



Before we proceed further, we introduce one more general concept for lattice cohomology:

Definition 1.2.12 Assume that $\text{rank}_{\mathbb{Z}}\mathbb{H}^*_{red} < \infty$. Then let the *Euler characteristic* of $\mathbb{H}^*$ be

$$\text{eu}(\mathbb{H}^*) = -m_w + \sum_{q\geq 0}(-1)^q \,\text{rank}_{\mathbb{Z}}\mathbb{H}^q_{red}.$$

For a finite rectangle we can also write this in another way:

Proposition 1.2.13 *In the case of $R = R(a,b)$ for finite $a, b \in \mathbb{Z}^r$:*

$$\text{eu}(\mathbb{H}^*)(R(a,b),w) = \sum_{q\geq 0}\;\sum_{\square\in\mathcal{Q}_q\cap R}(-1)^{q+1}w(\square).$$

Looking back on the previous examples, we notice that the Euler characteristic is always equal to δ. This is not a coincidence, though it does rely on some conditions being satisfied:

Theorem 1.2.14 (Ágoston–Némethi [1]) *Consider the finite rectangle $R(0,c)$ for some $c \in (\mathbb{Z}_{>0})^r$, and let the weight function w_0 be obtained in the following way:*

- $\mathfrak{h}\colon L \to \mathbb{Z}$ *is increasing (with respect to the partial ordering) and* $h(0) = 0$,
- $\mathfrak{h}^\circ\colon L \to \mathbb{Z}$ *is decreasing (with respect to the partial ordering),*
- $\mathfrak{h}$ *satisfies the following 'matroid inequality' (MAT):*

$$\mathfrak{h}(\ell_1) + \mathfrak{h}(\ell_2) \geq \mathfrak{h}\big(\max(\ell_1, \ell_2)\big) + \mathfrak{h}\big(\min(\ell_1, \ell_2)\big),$$

- *the pair* $(\mathfrak{h}, \mathfrak{h}^\circ)$ *satisfies the following 'combinatorial duality property' (CDP):*

$$\mathfrak{h}(\ell + E_i) - \mathfrak{h}(\ell) \text{ and } \mathfrak{h}^\circ(\ell + E_i) - \mathfrak{h}^\circ(\ell) \text{ cannot be simultaneously } \neq 0,$$

- $w_0(\ell) = \mathfrak{h}(\ell) + \mathfrak{h}^\circ(\ell) - \mathfrak{h}^\circ(0)$.

Then for the induced lattice cohomology $\mathbb{H}^*(R(0, c), w)$*, we have* $\operatorname{eu}\mathbb{H}^* = \mathfrak{h}^\circ(0) - \mathfrak{h}^\circ(c)$.

One can easily verify that for the examples before (when $r = 1$), setting $\mathfrak{h}^\circ(\ell) = \mathfrak{h}(\ell) - \ell$ gives us the weight function we were computing, and also that $\mathfrak{h}$ and $\mathfrak{h}^\circ$ then satisfy both the MAT and CDP properties.

This is the method we will use to construct the weight function for $\mathbb{H}^*$ in various contexts (but in some cases the weight function can be more complicated).

Remark 1.2.15 The MAT property is automatically satisfied whenever $\mathfrak{h}$ is obtained as the Hilbert function of a filtration induced by a valuation: given a decreasing $\mathbb{Z}^r$-filtration $\mathscr{F}$ of a vector space M where $\mathscr{F}(0) = M$ and $\mathscr{F}(c) = 0$ for some $c \in (\mathbb{Z}_{\geq 0})^r$, we get an increasing function $\mathfrak{h}(\ell) = \dim M/\mathscr{F}(\ell)$. If $\mathscr{F}$ is obtained from some valuation $\mathfrak{v}\colon M \to (\mathbb{Z}_{\geq 0} \cup \{\infty\})^r$ via $\mathscr{F}(\ell) = \{f \in M \mid \mathfrak{v}(f) \geq \ell\}$ then MAT holds.

As a result, there is a plethora of situations in which the above theorem may be applicable. The CDP condition is more particular, but certain duality properties can often be used to ensure it being satisfied, like it was in the previous examples — and the general construction for curve singularities, to be discussed later.

1.2.2 Overview and motivation

The introduction and study of lattice cohomology associated with various situations of singularity theory is motivated by two packages of results and principles. Firstly, by the appearances of different (co)homology theories in low-dimensional topology (Heegaard Floer, Khovanov, contact, etc.), see e.g. [28, 29, 32, 33]. The other motivation is based on several results of surface singularities, which aims to understand certain analytic invariants in terms of the topology of the link, and in this way compares invariants of singularities — both topological and analytical — and studies their connections to each other. While going over a part of this larger picture, we will also give a bit of a historical overview on the topic of lattice cohomologies.

As mentioned at the beginning of section 1.2, lattice cohomology was first introduced in [20] as a topological invariant of normal surface singularities. We will talk about it in detail in section 1.2.4. The construction of a bi-graded $\mathbb{Z}[U]$-module is a cohomology paralleled of the *Heegaard–Floer homology* HF^- of the link (a smooth, compact 3-manifold) — and in fact they were proved to be equivalent by Zemke in 2021 [36]. This cohomology $\mathbb{H}^*_{top}$ also has connections to other notable topological properties, e.g. its Euler characteristic is the *Seiberg–Witten invariant* of the link corresponding to the canonical $spin^c$-structure. (This means that $\mathbb{H}^*_{top}$ is a categorification of the Seiberg–Witten invariant.) Its weight function is constructed from the Riemann-Roch expression χ.

Analogously, it seemed natural to try and construct an analytic version: one whose Euler characteristic is the so-called *geometric genus* of a surface singularity $(V, 0)$, which can be considered an analytic analogue to the Seiberg–Witten invariant (though we will not go into details now). For detailed discussions and parallel statements between the topological and analytical invariants (in particular, between the Seiberg-Witten invariant and the geometric genus) see [23]. In this sense, $\mathbb{H}^*_{an}(V, 0)$ appears as a 'categorification' of the geometric genus.

Such an invariant was constructed by Ágoston and Némethi in [2] (see also [3] for the equivariant case which covers the cohomology modules corresponding to all the $spin^c$-structures of the link). The construction (and the weigth function) relies an the divisorial filtration associated with a good resolution. The analytic lattice cohomology $\mathbb{H}^*_{an}(V, 0)$ also has a direct connection to $\mathbb{H}^*_{top}$ in the form of graded $\mathbb{Z}[U]$-module morphism $\mathbb{H}^*_{an}(V, 0) \to \mathbb{H}^*_{top}(V, 0)$.

This particular construction valid for surfaces inspired the more general combinatorial setup described e.g. in Theorem 1.2.14 and used to construct other lattice cohomologies: not just for surfaces, but for isolated singularities of any dimension $n \geq 2$ [4] (using again the divisorial filtration of a resolution), as well as for curves [1] (using the Hilbert function associated with the valuative filtration of the normalization). The curve case — which covers the case of all reduced curves, with arbitrary embedded dimension, i.e. not necessarily plane curves — will be discussed in the next section 1.2.3.

Unfortunately, at this moment, no direct topological analogue is known in the cases $n > 2$ yet. (Conjecturally we have to find connection with some version of Embedded Contact Homology.) The construction of $\mathbb{H}^*_{top}$ for surfaces does not easily lend itself to generalizations in the same manner. In fact, for $n > 2$, the abstract link itself holds too little information to be sufficient. (This is why we need to introduce some additional information, e.g. the contact structure.) The same is true in the case $n = 1$, in that case too the abstract link has very little information (it gives only the number of local irreducible components). However, in the case of plane curve singularities $(V, 0)$ specifically, the knot/link Floer homology HFL^- of the embedded link (and other equivalent embedded invariants like the multivariate Alexander polynomial Δ, or the motivic multivariable Poincaré series) can be related with a filtered version of the analytic lattice (co)homology, see [24]. (For the filtered version in the surface case see also [25].)

Some of these cohomology theories are summarized in the table below:

dim	topological	analytic
$\dim_{\mathbb{C}}(V,0)=2$ $\dim_{\mathbb{R}} L_V = 3$	$HF^- = \mathbb{H}^*_{top}$ (from χ of a resolution) $eu(\mathbb{H}^*_{top})$ =Seiberg-Witten invariant	$\mathbb{H}^*_{an}$ (div. filtration of a resolution) $eu(\mathbb{H}^*_{an})$ =geometric genus
$\dim_{\mathbb{C}}(V,0)=d>2$ $\dim_{\mathbb{R}} L_V = 2d-1$	?	$\mathbb{H}^*_{an}$ (div. filtration of a resolution) $eu(\mathbb{H}^*_{an})$ =geometric genus
$\dim_{\mathbb{C}}(V,0)=1$ $L_V = \sqcup_{i=1}^r S^1$ .... $(V,0)\subset(\mathbb{C}^2,0)$ $L_V=\sqcup S^1 \subset S^3$	? .... (embedded topological) HFL^-	$\mathbb{H}^*_{an}$ (val. filtration of normalization) $eu(\mathbb{H}^*_{an})$ =delta invariant .... (filtered version) spectral sequence $\Rightarrow\ \mathrm{Gr}^F_*\mathbb{H}_{*,an}$

1.2.3 The case of isolated curve singularities [1]

Fix an isolated curve singularity $(V,0)$ with irreducible decomposition $\bigcup_{i=1}^{r}(V_i,0)$ (which is not necessarily a plane curve singularity). We consider the lattice $\mathbb{Z}^r$ and the first quadrant of $\mathbb{Z}^r\otimes_{\mathbb{Z}}\mathbb{R}$, namely $(\mathbb{R}_{\geq 0})^r$, together with its cubical decomposition. Additionally we define the weight function as follows.

Definition 1.2.16

$$w_0(\ell)=\mathfrak{h}(\ell)+\mathfrak{h}^\circ(\ell)-\mathfrak{h}^\circ(0)=2\mathfrak{h}(\ell)-|\ell|$$

where

$$|\ell|=\sum_{i=1}^{r}\ell_i,\quad \mathfrak{h}^\circ(\ell)=\mathfrak{h}(\ell)-|\ell|.$$

This then induces the lattice cohomology $\mathbb{H}^*((\mathbb{R}_{\geq 0})^r,w)$.

Proposition 1.2.17 *(1) For any $c'\geq c$ (where c is the conductor), one has a graded $\mathbb{Z}[U]$-module isomorphism $\mathbb{H}^*((\mathbb{R}_{\geq 0})^r,w)=\mathbb{H}^*(R(0,c'),w)$.*

(2) For any $c' \geq c$ the pair $(\mathfrak{h}, \mathfrak{h}^\circ)$ satisfies the conditions in 1.2.14. In particular,

$$\mathrm{eu}\,\mathbb{H}^*((R(0,c), w) = \mathfrak{h}(c) - |c| = \delta.$$

The graded $\mathbb{Z}[U]$-module $\mathbb{H}^*((\mathbb{R}_{\geq 0})^r, w) = \mathbb{H}^*(R(0,c'), w)$ is called the lattice cohomology of $(V,0)$, and it is denoted by $\mathbb{H}^*(V,0)$. It is a categorification of the delta invariant of $(V,0)$.

$\mathbb{H}^*_{red}(C,0) = 0$ if and only if $(C,0)$ is smooth. Examples 1.2.10 and 1.2.11 provide the lattice cohomology for irreducible plane curves $\{x^3+y^4=0\}$ and $\{x^2+y^7=0\}$ respectively. The next examples has two irreducible components.

Example 1.2.18 Let $(V,0) = \{x^2 - y^4 = 0\}$. Here $x^2 - y^4 = (x-y^2)(x+y^2)$, so $(V,0)$ has two components (two isomorphic components, in fact, so we can expect the Hilbert function etc. to have an additional symmetry).

Both components $(V_1,0)$ and $(V_2,0)$ are regular, so their respective links are the unknots; hence, their individual Alexander polynomials are trivial. Their linking number is the intersection multiplicity (cf. Fact 1.1.16), which is 2. From this, one can get the Poincaré series (cf. Theorems 1.1.40 and 1.1.41) for all V_I where $\emptyset \neq I \subset \{1,2\}$:

$$P_{V_1}(t_1) = \frac{1}{1-t_1}, \quad P_{V_2}(t_2) = \frac{1}{1-t_2}, \quad P_{V_{1,2}} = 1 + t_1t_2.$$

Finally, using Theorem 1.1.41, we get

$$H(t_1,t_2) = \frac{1}{(1-t_1)(1-t_2)}\Big(\frac{t_1}{1-t_1} + \frac{t_2}{1-t_2} - t_1t_2(1+t_1t_2)\Big).$$

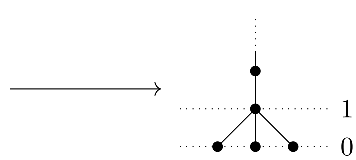
Computing the coefficients $\mathfrak{h}(\ell)$ and the weight function:

$\mathfrak{h}$

4	4	4	5	6
3	3	3	4	5
2	2	2	3	4
1	1	2	3	4
0	1	2	3	4

w

4	3	2	3	4
3	2	1	2	3
2	1	0	1	2
1	0	1	2	3
0	1	2	3	4



Note that, compatibly with the theory, $\mathrm{eu}\,\mathbb{H}^*(V,0) = \delta = 2$.

1.2.4 Surface singularities: the topological cohomology [20, 23]

Moving on to another context in which we can apply the lattice cohomology construction, consider a normal surface singularity $(V, 0) \hookrightarrow (\mathbb{C}^n, 0)$. We will aim to understand the topological lattice cohomology: the one that was originally defined, in 2000.

1.2.4.1 Plumbed 3-manifolds and topological lattice cohomology

Now we will want to rely on solely (abstract) topological invariants for obtaining a suitable weight function on some lattice. As it so happens, both a lattice and some combinatorial data on that lattice naturally arises from the topology.

Fact 1.2.19

1) For a normal (in particular, isolated) surface singularity $(V, 0)$*, the link* L_V *is a connected, smooth, oriented real 3-manifold.*
2) L_V *is plumbed* 3*-manifold associated with a connected negative definite plumbing graph* Γ*.*

Assuming the plumbing construction as known, let us just quickly recall the basic notations and concepts and how they apply in our current context:

- A *plumbing graph* is a graph $\Gamma = \big(\mathcal{V}, \mathcal{E}, \{e_v\}_{v\in\mathcal{V}}, \{g_v\}_{v\in\mathcal{V}}\big)$ where the decorations e_v and g_v are integers assigned to each vertex $v \in \mathcal{V}$. We will not allow loops now (edges from a vertex to itself), but multiple edges are possible. For simplicity we will assume that Γ is connected, cf. Fact 1.2.19(2).
- Through the *plumbing construction*, we get the plumbed 4-manifold $P = P(\Gamma)$, with its boundary being the plumbed 3-manifold $M = M(\Gamma)$. Here we have $H_2(P, \mathbb{Z}) \simeq \mathbb{Z}\langle E_v\rangle_{v\in\mathcal{V}}$: to each vertex corresponds a compact oriented submanifold E_v of real dimension two, all of which together freely generate $H_2(P, \mathbb{Z})$. Then the genus of E_v is the decoration $g_v \geq 0$.
- The intersection form $(\bullet, \bullet)$ on $\mathbb{Z}\langle E_v\rangle_{v\in\mathcal{V}}$ is also given by the plumbing graph:

$$(E_u, E_v) = \begin{cases} e_v, & \text{if } u = v, \\ \#\{\text{edges between } u \text{ and } v\}, & \text{if } u \neq v. \end{cases}$$

(Hence, e_v is the Euler numbers for the normal bundles of $E_v \subset P(\Gamma)$.) A plumbing graph being *negative definite* means that this intersection form is such.

In particular, we indeed have a lattice $L = \mathbb{Z}\langle E_v \rangle_{v \in \mathcal{V}} = H_2(P, \mathbb{Z})$ and the intersection form — both of which are topological. Well, almost: the plumbing graph Γ of a 3-manifold is not unique, so if we rely on it to construct a lattice cohomology, we will need to show that different choices of Γ result in the same $\mathbb{H}^*$. The *set* of plumbing graphs for the link is a topological invariant, so if the previous statement is true then $\mathbb{H}^*$ would indeed be topological as well.

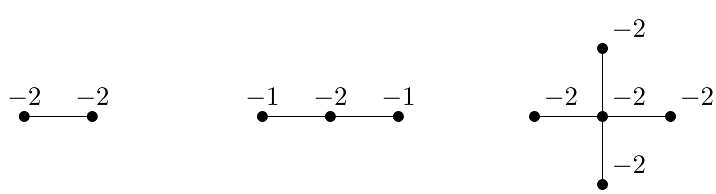
We will further restrict our attention to singularities $(V, 0)$ whose link is a $\mathbb{Q}HS^3$ (rational homology sphere), i.e. all homologies over $\mathbb{Q}$ are isomorphic to that of S^3. This means that the Betti numbers satisfy $b_0(L_V) = b_3(L_V) = 1$ and $b_1(L_V) = b_2(L_V) = 0$. In terms of the plumbing graph Γ, this means:

Proposition 1.2.20 *A connected plumbed* 3*-manifold with plumbing graph* $\Gamma = \big(\mathcal{V}, \mathcal{E}, \{e_v\}, \{g_v\}\big)$ *is a* $\mathbb{Q}HS^3$ *if and only if all* $g_v = 0$*, and* Γ *is a tree.*

As a result, we will omit the genus decorations altogether from now on and just assume $g_v = 0$, and also no multiple edges (and circles in the graph) will be present.

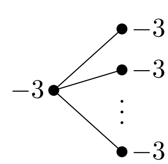
Aside from these restrictions though, it is not immediately obvious what the graph being negative definite (ND) means, how we can identify such a graph.

Example 1.2.21 Some examples and non-examples of negative definite graphs.



ND not ND not ND

In the following graph



if the number of (-3) vertices on the right (i.e. the number of end vertices) is < 9 then the graph is negative definite, otherwise it is not.

In general, negative definiteness can be tested by Sylvester's Criterion. Another criterion is the following:

Proposition 1.2.22 *Assume that some* $D = \sum_{v\in\mathcal{V}} d_v E_v$ *satisfies:*

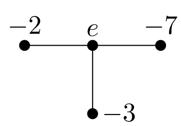
- $\forall v : d_v > 0$,
- $\forall u : (D, E_u) \leq 0$,
- $\exists u : (D, E_u) < 0$.

Then $(\bullet, \bullet)$ *is negative definite.*

Example 1.2.23 Let us test the above property for the effective cycle $D = \sum_{v\in\mathcal{V}} E_v$. Let let κ_v denote the degree of the vertex $v \in \mathcal{V}$. Then $D = \sum_{v\in\mathcal{V}} E_v$ satisfies the properties from Proposition 1.2.22 if and only if

- $\forall v : e_v + \kappa_v \leq 0$,
- $\exists v : e_v + \kappa_v < 0$.

In particular, these two conditions imply negative definiteness. Graphs with such properties are very special, they are called *minimal rational*.

Example 1.2.24 The graph 

is *minimal rational* (hence negative definite) for $e \leq -3$ as per the above condition. However, this is not a *necessary* condition: both $e = -2$ and $e = -1$ result in negative definite graphs as well.

Also important to note:

Proposition 1.2.25

- *If* Γ *is negative definite and* $\Gamma' \subseteq \Gamma$ *is a (full) subgraph then* Γ' *is negative definite too.*
- *If* $\Gamma = \big(\mathcal{V}, \mathcal{E}, \{e_v\}\big)$ *is negative definite and* $e'_v \leq e_v$ *for all* v *then* $\Gamma' = \big(\mathcal{V}, \mathcal{E}, \{e'_v\}\big)$ *is negative definite too.*

Getting back to constructing the lattice cohomology $\mathbb{H}^*_{top}(V, 0)$, the question is what the weight function w should be on the lattice $L = \mathbb{Z}\langle E_v\rangle_{v\in\mathcal{V}}$.

Definition 1.2.26 Let $Z_K \in L \otimes \mathbb{Q}$ denote the anticanonical cycle.

Remark 1.2.27 The adjunction formula implies $(Z_K, E_i) = (E_i, E_i) + 2$ for all i, so Z_K can be computed from the intersection form (which is nondegenerate). In particular, Z_K is topological.

Definition 1.2.28 Let us define

$$\chi\colon L \to \mathbb{Z}, \quad \chi(\ell) = \frac{1}{2}(\ell, Z_K - \ell).$$

From the adjunction formula, we get that indeed $\chi(\ell) \in \mathbb{Z}$, in particular $\chi(0) = 0$ and $\chi(E_v) = 1$ for all $v \in \mathcal{V}$.

Remark 1.2.29 Note that

$$\chi(\ell) = \frac{1}{8}(Z_K, Z_K) - \frac{1}{2}(\ell - Z_K/2, \ell - Z_K/2).$$

Given that $(\bullet, \bullet)$ is negative definite, this implies that χ is bounded from below, and even more, for any $k \in \mathbb{Z}$, the set $\chi^{-1}\big((-\infty, k]\big)$ is finite. (Also, χ is symmetric with respect to the involution $\ell \mapsto Z_K - \ell$ if $Z_K \in L$.)

As per the above observation, we can set χ as the weight function:

Definition 1.2.30 For a negative definite plumbing graph Γ (when Γ is a tree and with vanishing g_v's), define the weight function

$$w_0\colon L = \mathbb{Z}\langle E_v \rangle_{v \in \mathcal{V}} \to \mathbb{Z}, \quad w_0(\ell) = \chi(\ell)$$

on the lattice L. We denote the resulting cohomology by $\mathbb{H}^*(\Gamma)$.

The following statement shows that it is independent of the choice of the plumbing graph of the (fixed) link.

Proposition 1.2.31 *If two connected negative definite plumbing graphs Γ and Γ' give rise to diffeomorphic 3-manifolds $M(\Gamma)$ and $M(\Gamma')$ through the plumbing construction then $\mathbb{H}^*(\Gamma) \simeq \mathbb{H}^*(\Gamma')$.*

Definition 1.2.32 Let $(V, 0)$ be a normal surface singularity with $\mathbb{Q}HS^3$ link. We can define $\mathbb{H}^*_{top}(V, 0) = \mathbb{H}^*(L_V)$ as $\mathbb{H}^*(\Gamma)$ for a connected negative definite plumbing graph of the link L_V, and the resulting object is a topological invariant of $(V, 0)$.

As for the Euler characteristic, one can show that

Proposition 1.2.33 *(1) If the plumbing graph Γ gives rise to a $\mathbb{Q}HS^3$ link, then* $\mathrm{rank}_{\mathbb{Z}}\mathbb{H}^*_{red}(\Gamma) < \infty$*. Hence in this case the Euler characteristic is well-defined.*

(2) For a negative definite plumbed $\mathbb{Q}HS^3$ M, we have $\mathrm{eu}\,\mathbb{H}^*(M) = \mathfrak{sw}_{can}(M)$*, the Seiberg–Witten invariant for the canonical spin^c-structure of M.*

It has also been recently proved that:

Theorem 1.2.34 (Zemke [36]) *The topological invariant $\mathbb{H}^*(M)$ is equivalent to the Heegaard–Floer homology $HF^-(M)$.*

1.2.4.2 The topological lattice cohomology and other singularity invariants

Now, we want to relate $\mathbb{H}^*_{top}(V,0)$ to some other properties of the normal surface singularity $(V,0)$.

Definition 1.2.35 The *geometric genus* of a germ $(V,0)$ is

$$p_g = \dim_{\mathbb{C}} \frac{\{\text{holomorphic 2-forms on } V\setminus\{0\}\}}{\{\text{holomorphic 2-forms on } V\setminus\{0\} \text{ ``that can be extended to 0''}\}}.$$

Technically this means the following. Let $\pi : \widetilde{V} \to V$ be a resolution of the singularity of $(V,0)$, that is, $\widetilde{V}$ is smooth, π is proper and surjective. Set $E := \pi^{-1}(0) \subset \widetilde{V}$, the exceptional set of π. Then

$$p_g = \dim_{\mathbb{C}} \frac{\Big\{\text{holomorphic 2-forms on } \widetilde{V}\setminus E\Big\}}{\Big\{\text{holomorphic 2-forms on } \widetilde{V}\setminus E \text{ that can be extended to } \widetilde{V}\Big\}}.$$

(The right hand side of the identity is independent of the choice of the resolution π.)

Definition 1.2.36 The germ $(V,0)$ is called *rational* if $p_g(V,0) = 0$.

It turns out that rationality is expressible in terms of the plumbing graph of the link. (Below all the plumbing graphs are considered connected and negative definite.)

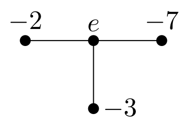
Theorem 1.2.37 (Artin [5, 6]) *A normal surface singularity $(V,0)$ has $p_g = 0$ if and only if the plumbing graph Γ of the link satisfies $\chi(\ell) > 0$ for all nonzero integral effective cycles $\ell \gneqq 0$.*

In particular, Artin's Criterion replaces the a priori analytic property of the vanishing of the geometric genus p_g with a combinatorial property of the plumbing graph Γ of the link. This motivated to formulate this property for plumbing graphs as well.

Definition 1.2.38 The plumbing graph Γ is *rational* if $\chi(\ell) > 0$ for all nonzero integral effective cycles $\ell \gneqq 0$.

Proposition 1.2.39

(1) if Γ is rational then any full connected subgraph $\Gamma' \subset \Gamma$ is rational,
(2) if Γ is rational and $M(\Gamma) = M(\Gamma')$ then Γ' is rational too,
(3) if $\Gamma = (\mathcal{V}, \mathcal{E}, \{e_v\})$ is rational, and $e'_v \leq e_v$, then $\Gamma' = (\mathcal{V}, \mathcal{E}, \{e'_v\})$ is rational too,
(4) if Γ is negative definite and $E = \sum_{i \in \mathcal{V}} E_i$ satisfies $(E, E_i) \leq 0$ for all $i \in \mathcal{V}$ $\implies$ then Γ is rational (that is, minimal rational graphs are rational),
(5) for fixed $(\mathcal{V}, \mathcal{E})$, the plumbing graph $\Gamma = (\mathcal{V}, \mathcal{E}, \{e_i\}_i)$ is always rational if all (e_i) are sufficiently small.

Example 1.2.40 The graph 

is negative definite for all $e \leq -1$ but rational only for $e \leq -2$.

The condition in Theorem 1.2.37 translates directly to the lattice cohomology of the graph:

Theorem 1.2.41 ([18, 20, 23]) *Γ is rational if and only if for the associated lattice cohomology $\mathbb{H}^*$,*

- *$\mathbb{H}^0 = \mathcal{T}_0^+$ and $\mathbb{H}^q = 0$ for $q > 1$, i.e. $\mathbb{H}^*_{red} = 0$;*
- *or equivalently: $S_n = \emptyset$ for all $n < 0$ and S_n is contractible for all $n \geq 0$.*

To prove this statement, first we observe that when computing $\mathbb{H}^*$, one can restrict the lattice to $(\mathbb{R}_{\geq 0})^{|\mathcal{V}|}$ and not lose anything:

Theorem 1.2.42 *Let S_n be the level sets corresponding to $\mathbb{H}^*(\Gamma) = \mathbb{H}^*(\mathbb{R}^{|\mathcal{V}|}, w)$. Then S_n has the same homotopy type as $S_n \cap (\mathbb{R}_{\geq 0})^{|\mathcal{V}|}$. In fact, $S_n \cap (\mathbb{R}_{\geq 0})^{|\mathcal{V}|}$ is a strong deformation retract of S_n.*

Proof (sketch) First we can verify directly that

Lemma 1.2.43 *$\chi(A + B) = \chi(A) + \chi(B) - (A, B)$ for all $A, B \in L$.* □

Then, let us fix some $x = \sum_i n_i E_i \leq 0$, $x \neq 0$. We intend to show that there is an $i \in \{1, \dots, r\}$ with $n_i > 0$ such that $\chi(x + E_i) \leq \chi(x)$. This would essentially mean that the 'negative quadrant' can be retracted to the origin using several inductive steps by a χ-nonincreasing combinatorial flow.

Note that for any $y \geq 0$, $y \neq 0$, it must exists some E_i in the support of y with $(y, E_i) < 0$, because otherwise we would get $(y, y) \geq 0$, which would contradict the negative definiteness. Multiplying y with -1, we get that for the above x, a vertex $i \in \mathcal{V}$ exists with $(x, E_i) > 0$. Then, by the above lemma,

$$\chi(x + E_i) = \chi(x) + \chi(E_i) - (x, E_i) = \chi(x) + 1 - (x, E_i) \leq \chi(x).$$

Next, if $x \ngeq 0$, then we write x as $x_1 - x_2$, where $x_1 \geq 0$, $x_2 > 0$ and they have different supports. Then we can repeat the above argument for x_2: we find E_i in the support of x_2 such that $\chi(x + E_i) \leq \chi(x)$. This allows us to contract the parts of S_n outside the positive quadrant back inside it. (Technically, one would need to check that the choice of i can be made consistently to give us a retraction on the entire S_n, but we omit that now. For details see e.g. [23].) □

Corollary 1.2.44 *For a plumbing graph Γ, we have an isomorphism $\mathbb{H}^*(\Gamma) \simeq \mathbb{H}^*((\mathbb{R}_{\geq 0})^{|\mathcal{V}|}, w)$.*

In this way we are a little bit closer to Artin's Criterion, Theorem 1.2.37, since both are tests in the first quadrant. Hence, let us return back to the rationality of Γ:

Proof (of Theorem 1.2.41) By the above discussion it is enough to consider the cubical decomposition of $(\mathbb{R}_{\geq 0})^{|\mathcal{V}|}$. On the other hand, By Artin's Criterion 1.2.37 we can replace rationality with the positivity condition on χ for the lattice points in the positive quadrant (aside from 0).

Assume then that $\chi(\ell) > 0$ for $\ell \gneq 0$. We of course automatically get $S_n = \emptyset$ for $n < 0$, so we need to check that S_n is contractible for $n \geq 0$. We apply the same method as when proving Theorem 1.2.42: we intend to show that

$$\text{for any } x \gneq 0 \text{ there exists } E_i \text{ in the support of } x \text{ such that } \chi(x - E_i) \leq \chi(x).$$

Assume by contradiction that this is not the case, i.e. $\chi(x - E_i) \geq \chi(x) + 1$ for all E_i in the support of x. Then

$$\chi(x-E_i)-1 \geq \chi(x)=\chi(x-E_i+E_i)=\chi(x-E_i)+\chi(E_i)-(x-E_i,E_i)=$$
$$=\chi(x-E_v)+1-(x-E_i,E_i),$$
$$(x-E_i,E_i)\geq 2,$$
$$(x,E_i)\geq (E_i,E_i)+2=(Z_K,E_i)$$

for all E_i in the support of x. Summing up we obtain that for all $x \geq 0$, we have $(x,x)\geq (Z_K,x)$, i.e.

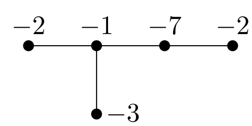
$$\chi(x)=\frac{1}{2}(x,Z_K-x)\leq 0,$$

which contradicts Artin's Criterion of rationality.

Now once again, we can use this to retract $S_n \cap (\mathbb{R}_{\geq 0})^{|\mathcal{V}|}$ by a χ-nonincreasing flow this time down to a single lattice point 0.

Conversely, assume S_n is contractible for $n \geq 0$; in particular, S_0 is connected. Since $\chi(0)=0$ and $\chi(E_i)=1$ for all i, this means S_0 can not contain any point $x \in (\mathbb{Z}_{\geq 0})^{|\mathcal{V}|}$ other than 0 as otherwise we would have at least 2 connected components in S_0. This finishes the proof. □

Example 1.2.45 [23] Though $p_g=0$ is determined by the link, p_g itself is *not* topological. Indeed, e.g. for the link defined by the graph



one can find an analytic structure with $p_g=2$ (let $V=\{x^2+y^3+z^{13}=0\}$), while the generic analytic structure has $p_g=1$.

In general, for any Γ we have

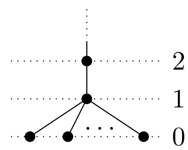
Proposition 1.2.46 ([18, 23]) *S_n is connected for $n \geq 1$.*

Also, we can analyze a slightly weaker condition than Artin's Rationality Criterion:

Definition 1.2.47 ([18, 23]) The graph Γ is *elliptic* if $\chi(\ell)\geq 0$ for all $\ell \geq 0$, but Γ is not rational.

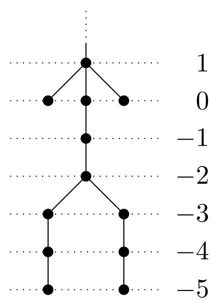
Proposition 1.2.48 *Γ is elliptic if and only if $\mathbb{H}^0_{red}(\Gamma)=\mathcal{T}_0(1)^k$ for some integer $k>0$. For elliptic graphs $\mathbb{H}^{>0}=0$.*

Example 1.2.49 In case of an elliptic graph Γ, the graded root looks like this:



Note that if $Z_K \in L$ then χ is symmetric (cf. Remark 1.2.29), and in particular $\chi(Z_K) = 0$, so 0 and Z_K correspond to two different "legs" of the above graded root.

Example 1.2.50 Let $(V, 0) = \{x^5 + y^5 + z^6 = 0\} \subset (\mathbb{C}^3, 0)$. Then the graded root is



Problem 1.2.51 *Characterize the graded* $\mathbb{Z}[U]$*-modules* $\mathbb{H}^0$ *(or the graded roots) that can be realized analytically, i.e. as the* $\mathbb{H}^0_{top}$ *of some surface singularity* $(V, 0)$.

1.2.4.3 Bad vertices and the reduction theorem

As a closing note, one can observe that in regards to $\mathbb{H}^*(\Gamma)$, some vertices of Γ 'matter more than others'. We can identify a set of 'bad vertices' in a graph, and focus only on those in a sense. More concretely,

Definition 1.2.52 ([18, 23]) Assume $(\mathcal{V}, \mathcal{E}, \{e_i\}_i)$ is a fixed plumbing graph Γ with Euler numbers $\{e_i\}_i$. We say that $\{i_1, \ldots, i_k\} \subseteq \mathcal{V}$ is a *set of bad vertices of* Γ if there exist $e'_{i_j} \leq e_{i_j}$ such that by replacing all e_{i_j} with e'_{i_j}, we get a rational graph $\Gamma' = (\mathcal{V}, \mathcal{E}, \{e'_i\}_i)$.

Remark 1.2.53 By Proposition 1.2.39, we know that at least $\mathcal{V}$ itself is such a set — usually a much smaller set also suffices though.

Example 1.2.54 When $M(\Gamma)$ is a $\mathbb{Q}HS^3$ then the set of vertices with degree > 2 (the 'nodes') is a suitable set of bad vertices.

Example 1.2.55 In particular, given a 'star-shaped' graph Γ, i.e. a tree with a single node (and $M(\Gamma)$ a $\mathbb{Q}HS^3$), then the singleton set containing that node is a set of bad vertices.

Definition 1.2.56 A graph Γ with $M(\Gamma)$ a $\mathbb{Q}HS^3$ is called *almost rational* if admits a set of bad vertices of cardinality one.

As for how to use these 'bad vertices' exactly:

Theorem 1.2.57 (Reduction Theorem; László–Némethi [15]) *Let Γ be a plumbing graph for a $\mathbb{Q}HS^3$, with a set of bad vertices $\overline{\mathcal{V}} = \{i_1, \ldots, i_k\}$. Then*

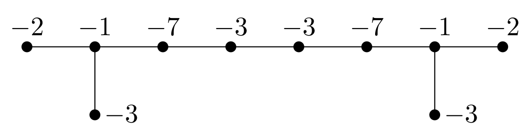
$$\mathbb{H}^*(\Gamma) \simeq \mathbb{H}^*((\mathbb{R}_{\geq 0})^{|\overline{\mathcal{V}}|}, \overline{w})$$

for a suitable choice of weight function $\overline{w}\colon (\mathbb{Z}_{\geq 0})^{|\overline{\mathcal{V}}|} = (\mathbb{Z}_{\geq 0})^k \to \mathbb{Z}$ (which we will not explicitly construct here).

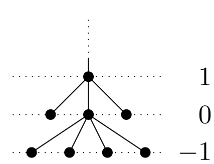
This means that with a modification of the weight function (that is combinatorially and algorithmically computable from χ), we can often drastically reduce the dimension of the lattice we need to consider. This is, of course, very important for any kind of computation, but also, as an immediate consequence we obtain:

Corollary 1.2.58 *If $\overline{\mathcal{V}}$ is a set of bad vertices in Γ then $\mathbb{H}^q_{red}(\Gamma) = 0$ for all $q \geq |\overline{\mathcal{V}}|$. (In particular, $\mathbb{H}^* = \mathbb{H}^0$ for almost rational graphs.)*

Example 1.2.59 Consider the following graph:



Here we have 2 nodes so $\mathbb{H}^q = 0$ for all $q \geq 2$. By a computation $\mathrm{rank}_{\mathbb{Z}}\mathbb{H}^1 = 1$ (supported in degree zero), and the graded root is the following (which provides $\mathbb{H}^0$ as well):



Remark 1.2.60 For any particular $q > 0$ one can also construct examples for graphs with non-vanishing $\mathbb{H}^q$. (Of course, such a graph needs to have at least $q + 1$ nodes.)

References

1. Ágoston, T. and Némethi, A.: The analytic lattice cohomology of isolated curve singularities, arXiv 2301.08981, 2023.
2. Ágoston, T. and Némethi, A.: The analytic lattice cohomology of surface singularities, arXiv 2108.12294, 2021.
3. Ágoston, T. and Némethi, A.: Analytic lattice cohomology of surface singularities, II (the equivariant case), arXiv 2108.12429, 2021.
4. Ágoston, T. and Némethi, A.: The analytic lattice cohomology of isolated singularities, arXiv 2109.11266, 2021.
5. Artin, M.: Some numerical criteria for contractibility of curves on algebraic surfaces. *Amer. J. of Math.*, **84**, 485-496, 1962.
6. Artin, M.: On isolated rational singularities of surfaces. *Amer. J. of Math.*, **88**, 129-136, 1966.
7. Brieskorn, E., Knörrer, H.: Plane Algebraic Curves, Birkhäuser, Boston, 1986.
8. Campillo, A., Delgado de la Mata, F. and Gusein-Zade, S. M.: On the monodromy of a plane curve singularity and the Poincaré series of the ring of functions on the curve, Functional Analysis and Its Applications 33, 56–57 (1999).
9. Campillo, A., Delgado de la Mata, F. and Gusein-Zade, S. M.: Integrals with respect to the Euler characteristic over spaces of functions and the Alexander polynomial, Proc. of the Steklov Inst. of Math. 238, 134–147 (2002).
10. Campillo, A., Delgado de la Mata, F. and Gusein-Zade, S. M.: The Alexander polynomial of a plane curve singularity via the ring of functions on it, Duke Math. Journal 117 (1), 125–156 (2003).
11. Campillo, A., Delgado de la Mata, F. and Gusein-Zade, S. M.: Multi-index Filtrations and Generalized Poincaré Series, Monatshefte für Mathematik 150, 193–209 (2007).
12. Eisenbud, D. and Neumann. W. D.: Three-Dimensional Link Theory and Invariants of Plane Curve Singularities, Annals of Math. Studies 110, Princeton University Press 1985.
13. Gorsky, E. and Némethi, A.: Lattice and Heegaard Floer homologies of algebraic links, Int. Math. Research Notices 2015 (23), 12737–12780 (2015).
14. Grauert, H.: Über die Deformation isolierter Singularitäten analytischer Mengen, Inventiones mathematicae, 15 (3), 171–198 (1971).

15. László, T. and Némethi, A.: Reduction theorem for lattice cohomology, Int. Math. Research Notices **2015** (1) (2015), 2938–2985.
16. Milnor, J.: Singular Points of Complex Hypersurfaces, Annals of Math. Studies 61, 1969, Princeton Univ. Press.
17. Moyano-Ferdnández, J. J.: Poincaré series associated with curves defined over perfect fields, PhD Thesis, University of Valladolid, 2008.
18. Némethi, A.: On the Ozsváth–Szabó invariant of negative definite plumbed 3-manifolds, Geometry & Topology 9(2), 991–1042 (2005).
19. Némethi, A.: Graded roots and singularities, in Singularities in Geometry and Topology, World Scientific 394–463 (2007).
20. Némethi, A.: Lattice cohomology of normal surface singularities, Publ. of the Res. Inst. for Math. Sci. 44(2), 507–543 (2008).
21. Némethi, A.: Poincaré series associated with surface singularities, in Singularities I: Algebraic and Analytic Aspects, Contemp. Math. 474, AMS 271–297 (2008).
22. Némethi, A.: The Seiberg–Witten invariants of negative definite plumbed 3-manifolds, J. Eur. Math. Soc. 13(4), 959–974 (2011).
23. Némethi, A.: Normal surface singularities, Ergebnisse der Math. und ihrer Grenzgebiete, Springer 2022.
24. Némethi, A.: Filtered lattice homology of curve singularities, arXiv:2306.13889 .
25. Némethi, A.: Filtered lattice homology of surface singularities, arXiv:2307.16581.
26. Ozsváth, P. S. and Szabó, Z.: On the Floer homology of plumbed three-manifolds, Geometry & Topology 7(1), 185–224 (2003).
27. Ozsváth, P. S. and Szabó, Z.: Knot Floer homology and the four-ball genus, Geometry & Topology 7, 615–639 (2003).
28. Ozsváth, P. and Szabó, Z.: Holomorphic disks and three-manifold invariants: properties and applications, Ann. of Math. (2), 159(2004), no. 3, 1159–1245.
29. Ozsváth, P. and Szabó, Z.: Holomorphic discs and topological invariants for closed three-manifolds. Ann. of Math. (2) 159 (2004), no. 3, 1027–1158.
30. Ozsváth, P., Szabó, Z. and Stipsicz, A.: A spectral sequence on lattice homology, Quantum Topol. 5 (2014), 487–521.
31. Ozsváth, P., Szabó, Z. and Stipsicz, A.: Knot lattice homology in L-spaces, Journal of Knot Theory and Its Ramifications, Vol. 25, No. 01, 1650003 (2016).
32. J. Rasmussen: Knot polynomials and knot homologies, in *Geometry and Topology of Manifolds*, Boden et. al. eds., Fields Institute Communications 47, 261–280 (2005).
33. J. Rasmussen: Khovanov homology and the slice genus, Invent. math. 182, 419–447 (2010).
34. Wall, C.T.C.: Singular points of plane curves, London Math. Soc. Student Texts **63**, Cambridge University Press 2004.
35. Zariski, O.: General theory of saturation and of saturated local rings II: Saturated local rings of dimension 1, Am. J. Math 93 (4), 872–964 (1971).
36. Zemke, I.: The equivalence of lattice and Heegaard Floer homology, arXiv:2111.14962.