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Categorification of the plurigenera of Gorenstein normal surface singularities

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Abstract

Consider a complex normal surface singularity and its three plurigenera, the m-th L^2–plurigenus of Watanabe, the m-th plurigenus of Knöller and the m-th log-plurigenus of Morales. For any of these invariants we construct a double graded $\mathbb{Z}[U]$–module, whose Euler characteristic is the chosen plurigenus. The three outputs are compared with the analytic lattice cohomology of the germ, whose Euler characteristic is the classical geometric genus.

Keywords Normal surface singularities · Geometric genus · Plurigenus of Watanabe · Plurigenus of Knöller · Plurigenus of Morales · Categorification · Lattice cohomology

Mathematics Subject Classification Primary: 14B05 · 14Fxx · 32S05; Secondary: 32S10 · 32S25

1 Introduction

1.1 In mathematical classification procedures we use invariants. If a certain invariant is not sufficiently ‘strong’, then we try to endow it with some additional structure. A *categorification* of an invariant is a (co)homology theory whose Euler characteristic is the invariant considered.

In the last decades several famous categorifications were introduced. E.g., in knot theory, the Khovanov invariant was introduced as the categorification of the Jones polynomial [10], the Link Heegaard Floer homology as the categorification of the Alexander polynomial [26]. Or, in the 3–manifold theory, the Heegaard Floer homology of Ozsváth and Szabó is the categorification of the Seiberg–Witten invariant [24, 25].

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Motivated by the theory of complex normal surface singularities, in [18] an another categorification was introduced for the Seiberg–Witten invariant, namely the (topological) lattice cohomology $\mathbb{H}^*_{top}$ associated with links of such singularities, or equivalently, with negative definite graph 3–manifolds [19]. Later, in [1, 2] its analytic version $\mathbb{H}^*_{an}$ was also introduced, in this way we obtain for any analytic type of normal surface singularity a categorification of the geometric genus.

Once we have such a construction for the geometric genus of a complex normal surface singularity, it is natural to ask: is there any analogous construction for the plurigenera? In this note we provide a positive answer.

Recall that in the literarure there are different versions of the plurigenera: the m-th L^2–plurigenus of Watanabe [29], the m-th plurigenus of Knöller [11] and the m-th log-plurigenus of Morales [14] (see 2.1.2). In this note we construct categorifications for all these numerical invariants in the case when the singularity is Gorenstein and the link is a rational homology sphere (see Sect. 7 for δ_m and γ_m and Sect. 8 for λ_m).

The corresponding cohomologies $\mathbb{H}^*$ have the following structure: $\mathbb{H}^*$ decomposes into a direct sum (of graded $\mathbb{Z}[U]$–modules) $\oplus_{q\geq 0}\mathbb{H}^q$, and each $\mathbb{H}^q$ is a $\mathbb{Z}$–graded $\mathbb{Z}[U]$–module. (They have the same structure as the Heegaard Floer homology HF^+ with an additional q–grading, or the topological/analytical lattice cohomologies mentioned above.)

In fact, we succeed to compare our new $\mathbb{Z}[U]$–modules with the analytic lattice cohomology $\mathbb{H}^*_{an}$ (categorification of the classical geometric genus), see Corollary 6.3.3, Theorems 7.1.1, 7.2.1, 8.1.1 and 8.5.3 . In this way their concrete computations are strongly linked with the modules $\mathbb{H}^*_{an}$. We find out that the weight tables (hence the cohomologies themselves) differ only by translations or only by well-understood modifications. This will be exemplified in Sect. 9 as well. On the other hand, for several concrete computations of $\mathbb{H}^*_{an}$ see e.g. [1, 2], or [20].

1.2 In the body of the paper we always assume that the normal surface singularity in hand is Gorenstein and has rational homology sphere link. For technical simplicity we always use the minimal good resolution.

We also exclude the cases when the singularity is of type A, D or E. Indeed, in these cases all the plurigenera (including the geometric genus) are 0, and even the weight function w_{an} and the $\mathbb{Z}[U]$–module $\mathbb{H}^*_{an}$ of the analytic lattice cohomology is the simplest possible: $\mathbb{H}^*_{an,red} = 0$. Since in these cases we do not get any extra information we prefer to completely omit them.

1.3 The structure of the article is the following. In Sect. 2 we introduce several notations and facts regarding complex normal surface singularities and their resolutions. In Sect. 3 we discuss the general definition and first properties of the lattice cohomology associated with a system of weights. In order to define a lattice cohomology we need a lattice $\mathbb{Z}^s$ (with fixed basis) and a weight function $w : \mathbb{Z}^s \to \mathbb{Z}$. Sect. 4 treats properties of lattice cohomology associated with special weight functions of type $w(l) = h(l) + h^\circ(l) - h^\circ(0)$ sometimes written also as $w(l) = h(l) - h'(l)$, where $h'(l) := h^\circ(0) - h^\circ(l)$. If the pair (h, h°) (or, equivalently (h, h')) satisfies certain combinatorial properties ('stability' and 'combinatorial duality property'), then the Euler characteristic of the lattice cohomology can be computed from h° (respectively h') .

In Sect. 5 we recall the definitions and some needed properties of the 'analytic lattice cohomology of the singularity' $\mathbb{H}^*_{an}$. Its weight function is defined via a pair $(\mathfrak{h}, \mathfrak{h}^\circ)$ determined via dimensions of sheaf theoretical cohomology vector spaces. Its Euler characteristic is the geometric genus.

Then in Sect. 6 we consider a common generalization $\pi_{m,n}$ of all three versions of the plurigenera and provide a categorification of it for all m and n (satisfying some inequality). The model is the construction of $\mathbb{H}^*_{an}$. We also prove some properties of these lattice cohomologies at this level of generality.

Finally, in Sect. 7 we collect all the results specified for the δ_m, γ_m cases and in Sect. 8 we discuss λ_m. In all the cases we link the respective lattice cohomologies to $\mathbb{H}^*_{an}$. The last Sect. 9 contains some concrete examples of plurigenera computations.

2 Notations

2.1 The combinatorics of the resolutions [15–17, 20]

2.1.1. Let (X, o) be the germ of a complex analytic normal surface singularity. We call a proper analytic map $\phi : \widetilde{X} \to X$ a *resolution* of (X, o) if $\widetilde{X}$ is smooth and

$$\phi\big|_{\phi^{-1}(X\setminus o)} : \phi^{-1}(X \setminus o) \to X \setminus o$$

is an isomorphism. Let $\cup_{v\in\mathcal{V}} E_v$ be the irreducible decomposition of the *exceptional curve* $E := \phi^{-1}(o)$ (with reduced structure). A famous theorem of Hironaka claims that such a resolution always exists [8].

A resolution is called *minimal* if it does not dominate any other resolution. Through Castelnuovo's Contractibility Criterion this is equivalent to the fact that there is no irreducible exceptional component E_v such that $E_v = \mathbb{P}^1$ with self-intersection $E_v^2 = -1$.

A resolution is called *good* if the exceptional curve E is a normal crossing divisor and all the irreducible components E_v are smooth. A resolution is called *minimal good* if it does not dominate any other good resolution. For every normal surface singularity (X, o) there exists a unique minimal good resolution (cf. Sect. 2.4 in [20]).

Let Γ denote the dual resolution graph of ϕ.

2.1.2. The lattice $L := H_2(\widetilde{X}, \mathbb{Z})$ is endowed with the natural negative definite intersection form $(\, , \,)$. It is a free $\mathbb{Z}$–module generated by the fundamental classes of $\{E_v\}_{v\in\mathcal{V}}$. The dual lattice is $L' = \mathrm{Hom}_{\mathbb{Z}}(L, \mathbb{Z}) \simeq \{l' \in L \otimes \mathbb{Q} \, : \, (l', L) \in \mathbb{Z}\}$. L is embedded in L' with $L'/L \simeq \mathrm{Tors}(H_1(M, \mathbb{Z}))$, where M denotes the link of the singularity (cf. 2.1.3).

There is a natural partial ordering of L' and L: we write $l'_1 \geq l'_2$ if $l'_1 - l'_2 = \sum_v r_v E_v$ with every $r_v \geq 0$. We set $L_{\geq 0} = \{l \in L : l \geq 0\}$ and $L_{>0} = L_{\geq 0} \setminus \{0\}$. The elements of $L_{\geq 0}$ are called effective cycles. We define the minima and the maxima of two cycles $l_1 = \sum_v l_{1,v} E_v$ and $l_2 = \sum_v l_{2,v} E_v$ as $\min\{l_1, l_2\} = \sum_v \min\{l_{1,v}, l_{2,v}\} E_v$ and $\max\{l_1, l_2\} = \sum_v \max\{l_{1,v}, l_{2,v}\} E_v$. The support of a cycle $l = \sum l_v E_v$ is defined as $|l| = \cup_{l_v \neq 0} E_v$.

We define the *Lipman cone* as $\mathcal{S}' := \{l' \in L' \, : \, (l', E_v) \leq 0$ for all $v\}$, and we also set $\mathcal{S} := \mathcal{S}' \cap L$. If $s' \in \mathcal{S}' \setminus \{0\}$, then all its E_v–coordinates are strictly positive. Thus, if $s \in \mathcal{S} \setminus \{0\}$, then $s \geq E$.

2.1.3. Assume that (X, o) is embedded in some $(\mathbb{C}^N, 0)$. The *link of* (X, o) is the intersection of X with a sphere S^{2N-1}_ϵ centered at the origin of small enough radius. It is a smooth, oriented, closed 3-manifold independent of the concrete embedding and ϵ. It is a rational homology sphere (i.e. its first Betti number is zero) if and only if each E_v is rational and the dual graph of the (any) resolution is a tree.

2.1.4. For a good resolution $\phi : \widetilde{X} \to X$ the *(anti)canonical cycle* $Z_K \in L'$ is defined by the *adjunction formulae* $(Z_K, E_v) = (E_v, E_v) + 2 - 2g_v$ for all $v \in \mathcal{V}$, where g_v denotes the genus of E_v. In particular, if the link is a rational homology sphere, then $g_v = 0$ for all

$v \in \mathcal{V}$. In fact, the cycle $-Z_K$ is the first Chern class of the line bundle $\Omega^2_{\widetilde{X}}$ (the sheaf of holomorphic 2–forms).

The singularity (or, its topological type) is called *numerically Gorenstein* if $Z_K \in L$. (Since $Z_K \in L$ if and only if the line bundle $\Omega^2_{X\setminus\{o\}}$ of holomorphic 2–forms on $X \setminus \{o\}$ is topologically trivial, see e.g. [6], the $Z_K \in L$ property is independent of the resolution). (X, o) is called *Gorenstein* if $Z_K \in L$ and $\Omega^2_{\widetilde{X}}$ is isomorphic to $\mathcal{O}_{\widetilde{X}}(-Z_K)$ (or, equivalently, if the line bundle $\Omega^2_{X\setminus\{o\}}$ is holomorphically trivial).

As usual, the *canonical divisor* $K_{\widetilde{X}}$ in $\mathrm{Div}(\widetilde{X})$ is defined (up to linear equivalence) via the identity $\Omega^2_{\widetilde{X}} \cong \mathcal{O}_{\widetilde{X}}(K_{\widetilde{X}})$, hence, in the Gorenstein case:

$$\mathcal{O}_{\widetilde{X}}(-Z_K) \cong \Omega^2_{\widetilde{X}} \cong \mathcal{O}_{\widetilde{X}}(K_{\widetilde{X}}),$$

and $K_{\widetilde{X}}$ can be chosen as $-Z_K$.

2.1.5. If $\widetilde{X}$ is a minimal resolution then (by the adjunction formulae) $Z_K \in \mathcal{S}'$, and thus $Z_K \geq 0$. Note that if the resolution graph is of type A, D or E, then $Z_K = 0$. For any other Gorenstein singularity with minimal resolution $Z_K \geq E$. Even more, by Example 6.3.4 of [20], $Z_K \geq E$ is also true for the minimal good resolution of any non–ADE Gorenstein singularity.

This paper only deals with the case of (X, o) being a non–ADE Gorenstein singularity with rational homology sphere link and $\phi : \widetilde{X} \to X$ the minimal good resolution.

2.2 The geometric genus and other plurigenera [20, 23]

2.2.1 Let $\phi : \widetilde{X} \to X$ be a good resolution of a normal surface singularity (X, o), when X is a small Stein representative of the singularity germ (X, o), e.g. for any embedding $(X, o) \hookrightarrow (\mathbb{C}^N, 0)$ one can choose X as the intersection of the analytic set with a small enough ball B_ε. The *geometric genus* $p_g = p_g(X, o)$ is defined by

$$p_g(X, o) = \dim(R^1\phi_*\mathcal{O}_{\widetilde{X}})_o = \dim H^1(\widetilde{X}, \mathcal{O}_{\widetilde{X}}) =: h^1(\widetilde{X}, \mathcal{O}_{\widetilde{X}}).$$

The singularity (X, o) is *rational* if $p_g = 0$. Though the geometric genus is not topological (usually it cannot be determined from the graph), its vanishing (i.e. rationality) is topological [5].

The *plurigenera* of a normal surface singularity are defined for each $m \in \mathbb{Z}_{>0}$ as follows:

- the m-th L^2-plurigenus (Watanabe [29]): $\delta_m(X, o) = \dim \dfrac{H^0(\widetilde{X} \setminus E, \mathcal{O}_{\widetilde{X}}(mK_{\widetilde{X}}))}{H^0(\widetilde{X}, \mathcal{O}_{\widetilde{X}}(mK_{\widetilde{X}} + (m-1)E))}$;
- the m-th plurigenus (Knöller [11]): $\gamma_m(X, o) = \dim \dfrac{H^0(\widetilde{X} \setminus E, \mathcal{O}_{\widetilde{X}}(mK_{\widetilde{X}}))}{H^0(\widetilde{X}, \mathcal{O}_{\widetilde{X}}(mK_{\widetilde{X}}))}$;
- the m-th log-plurigenus (Morales [14]): $\lambda_m(X, o) = \dim \dfrac{H^0(\widetilde{X} \setminus E, \mathcal{O}_{\widetilde{X}}(mK_{\widetilde{X}}))}{H^0(\widetilde{X}, \mathcal{O}_{\widetilde{X}}(mK_{\widetilde{X}} + mE))}$.

By the ramification formula and the log-ramification formula (cf. [9, Lemma 1.6]) these definitions are independent of resolutions.

Remark 2.2.2 By the definitions, for each $m \in \mathbb{Z}_{>0}$, we have $\lambda_m(X, o) \leq \delta_m(X, o) \leq \gamma_m(X, o)$. Following Laufer ([12], 2.3.10):

$$p_g(X, o) = \delta_1(X, o) = \gamma_1(X, o).$$

In our case (Gorenstein singularity with $\mathbb{Q}HS^3$ link) these also agree with $\lambda_1(X, o)$ (cf. Corollary 8.3.4).

2.3 Review of some analytic properties

2.3.1 Let (X, o) be a normal surface singularity and we fix any (not necessarily good) resolution $\phi : \widetilde{X} \to X$. In this subsection we present some statements that will help in the later discussion and proofs. First we state two versions of Serre duality (see Theorem 1.40 and 2.2 in [23] and references therein, or [20]):

Theorem 2.3.2 Serre duality for surfaces. *For a locally free $\mathcal{O}_{\widetilde{X}}$–module $\mathcal{G}$ the following duality homomorphism is an isomorphism:*

$$H^1_c(\widetilde{X}, \mathcal{G}) \to H^1(\widetilde{X}, \mathcal{G}^\vee \otimes \Omega^2_{\widetilde{X}})^*,$$

where $\mathcal{G}^\vee$ denotes the dual sheaf $\mathrm{Hom}_{\mathcal{O}_{\widetilde{X}}}(\mathcal{G}, \mathcal{O}_{\widetilde{X}})$ *and H^1_c denotes cohomology with compact support.*

Theorem 2.3.3 Serre duality for effective cycles. *Let $l \in L$ be a nonzero effective cycle on $\widetilde{X}$. Then for a locally free $\mathcal{O}_l$–module $\mathcal{G}$ the following duality homomorphism is an isomoprhism:*

$$H^0(l, \mathcal{G}) \to H^1(l, \mathcal{G}^\vee \otimes \mathcal{O}_l(K_{\widetilde{X}} + l))^*,$$

where $\mathcal{G}^\vee$ denotes the dual sheaf $\mathrm{Hom}_{\mathcal{O}_l}(\mathcal{G}, \mathcal{O}_l)$.

Theorem 2.3.4 h^0-vanishing Theorem. *[20, Theorem 6.4.2] Let $l \in L_{>0}$ be a positive cycle and $\widetilde{\mathcal{L}} \in$* $\mathrm{Pic}(\widetilde{X})$ *a line bundle such that $(c_1(\widetilde{\mathcal{L}}), E_v) \le 0$ for all $E_v \subset |l|$. Then $h^0(l, \mathcal{O}_l(l) \otimes \widetilde{\mathcal{L}}) = \dim H^0(l, \mathcal{O}_l(l) \otimes \widetilde{\mathcal{L}}) = 0$.*

Corollary 2.3.5 *For any effective cycle $l \ge 0$ one has $H^0(\mathcal{O}_{\widetilde{X}}(l)) \cong H^0(\mathcal{O}_{\widetilde{X}})$.*

Proof In the cohomological exact sequence associated with $0 \to \mathcal{O}_{\widetilde{X}} \to \mathcal{O}_{\widetilde{X}}(l) \to \mathcal{O}_l(l) \to 0$, we have the vanishing $H^0(\mathcal{O}_l(l)) \cong 0$ by applying the h^0–vanishing Theorem to $\widetilde{\mathcal{L}} = \mathcal{O}_{\widetilde{X}}$. □

By Serre duality for cycles the h^0–vanishing Theorem is equivalent with the following:

Theorem 2.3.6 Grauert–Riemenschneider Vanishing Theorem. [7, 12, 27] [20, Theorem 6.4.3] *Consider a line bundle $\mathcal{L} \in$* $\mathrm{Pic}(\widetilde{X})$ *such that $c_1(\mathcal{L}(Z_K)) \in -\mathcal{S}'$. Then $h^1(l, \mathcal{L}|_l) = 0$ for any $l \in L_{>0}$. In particular, $h^1(\widetilde{X}, \mathcal{L}) = 0$ too.*

Corollary 2.3.7 *If $\mathcal{L} \in$* $\mathrm{Pic}(\widetilde{X})$ *and $l \in L_{>0}$ satisfies $l \in c_1(\mathcal{L}) + Z_K + \mathcal{S}'$, then $H^1(\widetilde{X}, \mathcal{L}) = H^1(l, \mathcal{L}|_l)$.*

Proof Theorem 2.3.6 applied to $\mathcal{L}(-l)$ gives $h^1(\widetilde{X}, \mathcal{L}(-l)) = 0$, then use the cohomological exact sequence associated with the sheaf exact sequence $0 \to \mathcal{L}(-l) \to \mathcal{L} \to \mathcal{L}|_l \to 0$. □

Corollary 2.3.8 *Let (X, o) be a singularity and ϕ the minimal good resolution. Then the above statements imply the following:*

- *if $Z_K = 0$, then $p_g = 0$;*
- *if $Z_K > 0$, then for any $Z \ge Z_K$, $Z \in L$, $p_g = h^1(Z, \mathcal{O}_Z)$.*

Remark 2.3.9 [28, 4.8] [20, Remark 6.4.21] Let $l_1, l_2 \in L_{>0}$ be effective cycles, set $l = \min\{l_1, l_2\}$ and $\bar{l} = \max\{l_1, l_2\}$. Then

$$h^1(\mathcal{O}_{\bar{l}}) + h^1(\mathcal{O}_l) \ge h^1(\mathcal{O}_{l_1}) + h^1(\mathcal{O}_{l_2}).$$

We will refer to this inequality as the *'opposite' matroid rank inequality* of h^1.

In particular, in the numerically Gorenstein case for any $l \in L_{>0}$ we have $h^1(\mathcal{O}_l) = h^1(\mathcal{O}_{\min\{l, Z_K\}})$.

2.3.10 [12], [13, p. 1281] Following Laufer we can identify the dual space $H^1(\widetilde{X}, \mathcal{O}_{\widetilde{X}})^*$ with the space of global holomorphic 2-forms on $\widetilde{X}\setminus E$ up to the subspace of those forms which can be extended holomorphically over $\widetilde{X}$. That is, $H^1(\widetilde{X}, \mathcal{O}_{\widetilde{X}})^* \cong H^0(\widetilde{X}\setminus E, \Omega^2_{\widetilde{X}})/H^0(\widetilde{X}, \Omega^2_{\widetilde{X}})$. Here $H^0(\widetilde{X}\setminus E, \Omega^2_{\widetilde{X}})$ can be replaced by $H^0(\widetilde{X}, \Omega^2_{\widetilde{X}}(Z))$ for any $Z > 0$ with $h^1(\mathcal{O}_Z) = p_g$. Indeed, for any $Z > 0$, from the exact sequence of sheaves $0 \to \Omega^2_{\widetilde{X}} \to \Omega^2_{\widetilde{X}}(Z) \to \mathcal{O}_Z(Z + K_{\widetilde{X}}) \to 0$ (where $\Omega^2_{\widetilde{X}} \cong \mathcal{O}_{\widetilde{X}}(K_{\widetilde{X}})$), the Grauert–Riemenschneider vanishing $h^1(\Omega^2_{\widetilde{X}}) = 0$ and Serre duality

$$H^0(\Omega^2_{\widetilde{X}}(Z))/H^0(\Omega^2_{\widetilde{X}}) \cong H^0(\mathcal{O}_Z(Z + K_{\widetilde{X}})) \cong H^1(\mathcal{O}_Z)^*. \tag{2.3.11}$$

Therefore, if $H^1(\mathcal{O}_Z) \cong H^1(\mathcal{O}_{\widetilde{X}})$, the inclusion $H^0(\Omega^2_{\widetilde{X}}(Z))/H^0(\Omega^2_{\widetilde{X}}) \hookrightarrow H^0(\widetilde{X}\setminus E, \Omega^2_{\widetilde{X}})/H^0(\Omega^2_{\widetilde{X}})$ is an isomorphism. In particular, in the Gorenstein case

$$p_g = \dim H^0(\mathcal{O}_{\widetilde{X}})/H^0(\mathcal{O}_{\widetilde{X}}(-Z_K)). \tag{2.3.12}$$

3 The lattice cohomology associated with a system of weights

In this section we follow [18, 20].

3.1 General construction

3.1.1 We consider a free $\mathbb{Z}$–module, with a fixed basis $\{E_v\}_{v\in\mathcal{V}}$, denoted by $\mathbb{Z}^s$. It is also convenient to fix a total ordering of the index set $\mathcal{V}$, which in the sequel will be denoted by $\{1, \dots, s\}$. The next construction associates a graded $\mathbb{Z}[U]$–module to the pair $(\mathbb{Z}^s, \{E_v\}_v)$ and a set of weights.

We will use the following notation. Consider the graded $\mathbb{Z}[U]$–module $\mathbb{Z}[U, U^{-1}]$ and denote by $\mathcal{T}_0^+$ its quotient by the submodule $U \cdot \mathbb{Z}[U]$. This has a grading in such a way that $\deg(U^{-d}) = 2d$ $(d \ge 0)$. More generally, for any graded $\mathbb{Z}[U]$–module P with d-homogeneous elements P_d and for any $k \in \mathbb{Z}$ we denote by $P[k]$ the same module graded in such a way that $P[k]_{d+k} = P_d$. Then set $\mathcal{T}_k^+ := \mathcal{T}_0^+[k]$. Hence, for $m \in \mathbb{Z}$, $\mathcal{T}_{2m}^+ = \mathbb{Z}\langle U^{-m}, U^{-m-1}, \dots\rangle$ as a $\mathbb{Z}$–module.

3.1.2 The cochain complex. $\mathbb{Z}^s \otimes \mathbb{R}$ has a natural decomposition into cubes. The set of zero-dimensional cubes consists of the lattice points $\mathbb{Z}^s$. Any $l \in \mathbb{Z}^s$ and subset $I \subset \mathcal{V}$ of cardinality q defines a q-dimensional cube, which has its vertices in the lattice points $(l + \sum_{v\in I'} E_v)_{I'\subset I}$. On each such cube we fix an orientation. This can be determined, e.g., by the order $(E_{v_1}, \dots, E_{v_q})$, where the indices of the involved base elements $\{E_v\}_{v\in I}$ fulfill $v_1 < \cdots < v_q$. This orientation remains fixed throughout the constructions. The set of oriented q-dimensional cubes defined in this way is denoted by $\mathcal{Q}_q$ $(0 \le q \le s)$.

Let $\mathcal{C}_q$ be the free $\mathbb{Z}$–module generated by oriented cubes $\square_q \in \mathcal{Q}_q$. Clearly, for each $\square_q \in \mathcal{Q}_q$, the oriented boundary $\partial\square_q$ (of 'classical' cubical homology) has the form $\sum_k \varepsilon_k \square^k_{q-1}$ for some $\varepsilon_k \in \{-1, +1\}$. These $\square^k_{q-1}$-s appearing in the boundary are the faces of $\square_q$.

Clearly, the homology of the chain complex $(\mathcal{C}_*, \partial)$ is trivial: it is the homology of $\mathbb{R}^s$. The (co)homology what we will consider is constructed via a set of compatible *weight functions* $\{w_q\}_q$.

Definition 3.1.3 A set of functions $w_q : \mathcal{Q}_q \to \mathbb{Z}$ $(0 \le q \le s)$ is called a *set of compatible weight functions* if the following hold:

(a) For any integer $k \in \mathbb{Z}$ the set $w_0^{-1}((-\infty, k])$ is finite;
(b) $w_q(\square_q) \geq w_{q-1}(\square_{q-1})$ for any $\square_q \in \mathcal{Q}_q$ and any of its faces $\square_{q-1} \in \mathcal{Q}_{q-1}$.

Example 3.1.4 Once a function $w_0 : L \to \mathbb{Z}$ (with property (a)) is fixed, one can define w_q as:

$$w_q(\square_q) := \max\{w_0(l) \,:\, l \text{ is a vertex of } \square_q\}. \tag{3.1.5}$$

The set $\{w_q\}_{q\geq 0}$ obtained this way clearly satisfies property (b) of Definition 3.1.3 too.

3.1.6 In the presence of any fixed set of compatible weight functions $\{w_q\}_q$ we define $\mathcal{F}^q$ as the set of morphisms $\mathrm{Hom}_{\mathbb{Z}}(\mathcal{C}_q, \mathcal{T}_0^+)$ with finite support on $\mathcal{Q}_q$.

Then $\mathcal{F}^q$ is a $\mathbb{Z}[U]$–module by $(p * \phi)(\square_q) := p(\phi(\square_q))$ $(p \in \mathbb{Z}[U])$. Moreover, $\mathcal{F}^q$ has the following $\mathbb{Z}$-grading: $\phi \in \mathcal{F}^q$ is homogeneous of degree $\deg(\phi) = d \in \mathbb{Z}$ if for each $\square_q \in \mathcal{Q}_q$ with $\phi(\square_q) \neq 0$, $\phi(\square_q)$ is a homogeneous element of $\mathcal{T}_0^+$ of degree $d - 2 \cdot w(\square_q)$. (Hence, in fact, we obtain a $2\mathbb{Z}$-grading; this is motivated by the fact that we wish to keep a certain similarity with the Heegaard Floer homology of the link.)

Next, we define the differential $\delta_w : \mathcal{F}^q \to \mathcal{F}^{q+1}$. For this, fix $\phi \in \mathcal{F}^q$ and we indicate how $\delta_w \phi$ acts on a cube $\square_{q+1} \in \mathcal{Q}_{q+1}$. First write $\partial \square_{q+1} = \sum_k \varepsilon_k \square_q^k$, then set

$$(\delta_w \phi)(\square_{q+1}) := \sum_k \varepsilon_k \, U^{w(\square_{q+1}) - w(\square_q^k)} \, \phi(\square_q^k).$$

One verifies that $\delta_w \circ \delta_w = 0$, hence $(\mathcal{F}^*, \delta_w)$ is a cochain complex.
3.1.7 The complex $(\mathcal{F}^*, \delta_w)$ has a natural *augmentation*. Indeed, set $m_w := \min_{l \in \mathbb{Z}^s} w_0(l)$ and define the $\mathbb{Z}[U]$-linear map

$$\epsilon_w : \mathcal{T}_{2m_w}^+ \longrightarrow \mathcal{F}^0$$

such that $\epsilon_w(U^{-m_w - s})(l)$ is the class of $U^{-m_w + w_0(l) - s}$ in $\mathcal{T}_0^+$ for any $l \in L$ and $s \geq 0$. Then ϵ_w is injective and $\delta_w \circ \epsilon_w = 0$. Furthermore, ϵ_w and δ_w are morphisms of $\mathbb{Z}[U]$–modules, and are homogeneous of degree zero.

Definition 3.1.8 The homology of the cochain complex $(\mathcal{F}^*, \delta_w)$ is called the *lattice cohomology* of the pair $(\mathbb{R}^s, w)$, and it is denoted by $\mathbb{H}^*(\mathbb{R}^s, w)$. The homology of the augmented cochain complex

$$0 \longrightarrow \mathcal{T}_{2m_w}^+ \xrightarrow{\epsilon_w} \mathcal{F}^0 \xrightarrow{\delta_w} \mathcal{F}^1 \xrightarrow{\delta_w} \cdots$$

is called the *reduced lattice cohomology* of the pair $(\mathbb{R}^s, w)$, and it is denoted by $\mathbb{H}^*_{red}(\mathbb{R}^s, w)$.

For any $q \geq 0$ fixed, the $\mathbb{Z}$-grading of $\mathcal{F}^q$ induces a $\mathbb{Z}$-grading on $\mathbb{H}^q$ and $\mathbb{H}^q_{red}$. Moreover, both $\mathbb{H}^q$ and $\mathbb{H}^q_{red}$ admit an induced graded $\mathbb{Z}[U]$–module structure and $\mathbb{H}^q \cong \mathbb{H}^q_{red}$ for $q > 0$. Furthermore, one has a graded $\mathbb{Z}[U]$–module isomorphism $\mathbb{H}^0 \cong \mathcal{T}_{2m_w}^+ \oplus \mathbb{H}^0_{red}$.

Remark 3.1.9 The condition 3.1.3(a) guarantees that the U–action is nilpotent on the lattice cohomology modules $\mathbb{H}^*$ and $\mathbb{H}^*_{red}$ defined above. On the other hand, the U–action is not (necessarily) nilpontent on the lattice homology modules $\mathbb{H}_*$ and $\mathbb{H}_{*,red}$ (these are defined for example in [21, 22]). In this sense, lattice cohomology is analogous to HF^+, while lattice homology is analogous to HF^-.

3.1.10 Restrictions. Assume that $T \subset \mathbb{R}^s$ is a subspace of $\mathbb{R}^s$ consisting of a union of some closed cubes (from $\mathcal{Q}_*$). Let $\mathcal{C}_q(T)$ be the free $\mathbb{Z}$–module generated by q-cubes of T, $\mathcal{F}^q(T)$ be the restriction of $\mathcal{F}^q$ to $\mathcal{C}_q(T)$. Then $(\mathcal{F}^*(T), \delta_w)$ is a complex, whose homology will be denoted by $\mathbb{H}^*(T, w)$. It has a natural graded $\mathbb{Z}[U]$–module structure. Again, $\mathbb{H}^0(T, w) \cong \mathcal{T}^+_{2\min\{w|_T\}} \oplus \mathbb{H}^0_{red}(T, w)$.

Remark 3.1.11 Though $\mathbb{H}^*_{red}(\mathbb{R}^s, w)$ has finite $\mathbb{Z}$-rank in any fixed homogeneous degree, in general, it is not finitely generated over $\mathbb{Z}$, in fact, not even over $\mathbb{Z}[U]$.

Definition 3.1.12 Fix $T \subset \mathbb{R}^s$ as above, and assume that $\mathbb{H}^*_{red}(T, w)$ has finite $\mathbb{Z}$–rank. Then we define the *Euler characteristic* of $\mathbb{H}^*(T, w)$ as

$$eu(\mathbb{H}^*(T, w)) := -\min\{w_0|_T\} + \sum_q (-1)^q \operatorname{rank}_{\mathbb{Z}} \mathbb{H}^q_{red}(T, w).$$

In this article we will use the following sets T: either $T = (\mathbb{R}_{\geq 0})^s$ or it is a rectangle $R = R(0, c) := \{l \in \mathbb{R}^s : 0 \leq l \leq c\}$ for a certain $c \in L_{\geq 0}$. Furthermore, in all cases considered in this note the weight system will be determined by w_0 by $w_q(\square_q) = \max\{w_0(l),\ l \text{ is a vertex of } \square_q\}$ for all $1 \leq q \leq s$ and $\square_q \in \mathcal{Q}_q$.

3.2 Example: topological lattice cohomology of a normal surface singularity [16, 18, 20]

We consider a good resolution $\phi : \widetilde{X} \to X$ and we assume that the link M is a rational homology sphere. We write $s := |\mathcal{V}|$. Then we automatically have a free $\mathbb{Z}$-module $L = H_2(\widetilde{X}, \mathbb{Z}) = \mathbb{Z}^s$ with a fixed bases $\{E_v\}_v$. The Riemann–Roch expression $\chi : l \mapsto -(l, l - Z_K)/2$ defines a weight function $w_{top,0}(l) = \chi(l)$ on the lattice points, hence a set of compatible weight functions by $w_{top,q}(\square_q) = \max\{\chi(l) : l \text{ is a vertex of } \square_q\}$.

The $\mathbb{Z}[U]$-modules $\mathbb{H}^*(\mathbb{R}^s, w_{top})$ and $\mathbb{H}^*_{red}(\mathbb{R}^s, w_{top})$ obtained in this way are called the *topological lattice cohomologies* associated with the canonical spin^c–structure. They are denoted by $\mathbb{H}^*(\Gamma, -Z_K)$, respectively $\mathbb{H}^*_{red}(\Gamma, -Z_K)$, where Γ denotes the dual resolution graph. One can prove, that the graded $\mathbb{Z}[U]$–module $\mathbb{H}^*(\Gamma, -Z_K)$ depends only on the (diffeomorphism type of the) link M and thus it is independent of the choice of the resolution ϕ (see Proposition 3.4.2 in [18] or Proposition 11.1.24 in [20]).

As $\mathbb{H}^*_{red}(\Gamma, -Z_K)$ is finitely generated over $\mathbb{Z}$, its Euler characteristic is well defined. By [19], $eu(\mathbb{H}^*(\Gamma, -Z_K)) = \mathfrak{sw}_{\sigma_{can}}(M) - (Z_K^2 + |\mathcal{V}|)/8$, where $\mathfrak{sw}$ denotes the Seiberg–Witten invariant $\mathrm{Spin}^c(M) \to \mathbb{Q}$, which associates a rational number $\mathfrak{sw}_\sigma(M)$ to each spin^c–structure σ of the link. In other words, $\mathbb{H}^*(\Gamma, -Z_K)$ *is the categorification of* $\mathfrak{sw}_{\sigma_{can}}(M)$ (normalized by $(Z_K^2 + |\mathcal{V}|)/8$).

As the name in the definition suggests, there are more variants of the topological lattice cohomology according to different spin^c–structures. These correspond to different choices of the characteristic cohomology cycle $k \in Char$ used in the weight function: $\chi_k(l) := -(l, l - k)/2$; and they provide invariant $\mathbb{Z}[U]$–modules with similar properties.

4 Combinatorial lattice cohomology with special weight functions

In this and the next section we follow [1, 20].

4.1 The combinatorial setup

4.1.1 Fix $\mathbb{Z}^s$ with a fixed basis $\{E_v\}_{v \in \mathcal{V}}$, $|\mathcal{V}| = s$. Fix also an element $c \in \mathbb{Z}^s$, $c \geq 0$. Consider the real rectangle $R = R(0, c) := \{l \in \mathbb{R}^s : 0 \leq l \leq c\}$. Here we can consider the case $c = \infty$ too, in such a case $R = (\mathbb{R}_{\geq 0})^s$. Furthermore, assume that to each $l \in R \cap \mathbb{Z}^s$ we assign

(i) an integer $h(l)$ such that $h(0)=0$ and $h(l+E_v)\geq h(l)$ for any $v\in\mathcal{V}, l, l+E_v\in R\cap\mathbb{Z}^s$,
(ii) an integer $h^\circ(l)$ such that $h^\circ(l+E_v)\leq h^\circ(l)$ for any $v\in\mathcal{V}$ and $l, l+E_v\in R\cap\mathbb{Z}^s$.

Once h and c are fixed with (i), a possible choice for h° is h^{sym}, where $h^{sym}(l):=h(c-l)$. Clearly $h^\circ=h^{sym}$ defined in this way depends on c.

We consider the set of cubes $\{\mathcal{Q}_q\}_{q\geq 0}$ of R as in 3.1.2 and the weight function

$$w_0:\mathcal{Q}_0\to\mathbb{Z}\ \text{ by }\ w_0(l):=h(l)+h^\circ(l)-h^\circ(0). \tag{4.1.2}$$

Clearly $w_0(0)=0$. Moreover, we define the other $w_q:\mathcal{Q}_q\to\mathbb{Z}$ by Eq. (3.1.5): $w_q(\square_q)=\max\{w_0(l)\ :\ l$ is a vertex of $\square_q\}$. We will use the symbol w for the system $\{w_q\}_q$. These compatible weight functions define the lattice cohomology $\mathbb{H}^*(R,w)$. In particular, if $\mathbb{H}^*_{red}(R,w)$ has finite rank (which in the $c<\infty$ case is automatic), we obtain the Euler characteristic $eu(\mathbb{H}^*(R,w))$ as well.

4.1.3 Notice that if we replace h° by one of its translations (that is, $l\mapsto h^\circ(l)+m$ for a constant m) then the weight function w_0 stays stable.

In geometric applications sometimes it is more natural to replace h° by another function (which together with h reflects better certain dualities). That is, we pair the function h (with property (i)) in 4.1.1 with another function h' assigning to each $l\in R\cap\mathbb{Z}^s$

(ii)' an integer $h'(l)$ such that $h'(0)=0$ and $h'(l+E_v)\geq h'(l)$ for any $v\in\mathcal{V}$ and $l, l+E_v\in R\cap\mathbb{Z}^s$.

In this case we consider the weight function

$$w_0:\mathcal{Q}_0\to\mathbb{Z}\ \text{ by }\ w_0(l):=h(l)-h'(l) \tag{4.1.4}$$

and $w_q:\mathcal{Q}_q\to\mathbb{Z}$ by $w_q(\square_q):=\max\{w_0(l)\ :\ l$ is a vertex of $\square_q\}$.

The transition between the two notations is realised through the identity:

$$h'(l)=h^\circ(0)-h^\circ(l),\quad l\in R\cap\mathbb{Z}^s \tag{4.1.5}$$

where h° is well-defined from h' up to a translation. Once c is fixed (and $c<\infty$), then we can assume that $h^\circ(c)=0$, and in that case we can choose $h^\circ(l)=h'(c)-h'(l)$.

4.1.6 We will focus on pairs (h,h°) (or (h,h')) which satisfy certain additional properties.

Definition 4.1.7 We say that h satisfies the *'matroid rank inequality'* if

$$h(l_1)+h(l_2)\geq h(\min\{l_1,l_2\})+h(\max\{l_1,l_2\}),\text{ for all } l_1,l_2\in R\cap\mathbb{Z}^s. \tag{4.1.8}$$

Note that the 'matroid rank inequality' implies the *'stability property'*

$$h(l)=h(l+E_v)\ \Rightarrow\ h(l+\bar{l})=h(l+\bar{l}+E_v); \tag{4.1.9}$$

valid for any l and $\bar{l}$ such that $\bar{l}\geq 0$, $|\bar{l}|\not\supset E_v$ and $l+\bar{l}+E_v\in R\cap\mathbb{Z}^s$.

Example 4.1.10 (1) Fix $\mathbb{Z}^s$ and c as in 4.1.1. Let M be a finite dimensional vector space with a $\mathbb{Z}^s$-grading $\{M_{\mathbf{a}}\}_{\mathbf{a}}$ such that $M_{\mathbf{a}}=0$ whenever either $\mathbf{a}\not\geq 0$ or $\mathbf{a}\geq c$. Let $h: R(0,c)\cap\mathbb{Z}^s\to\mathbb{Z}$ be the function $l\mapsto\sum_{\mathbf{a}\not\geq l}\dim M_{\mathbf{a}}$. Then $h(0)=0$, $h(c)=\dim M$, and h satisfies the matroid rank inequality.

(2) Assume that M is a finite dimensional vector space endowed with a decreasing $\mathbb{Z}^s$-filtration such that $F(0)=M$ and $F(c)=0$ and define $h(l)=\dim(M/F(l))$ for any $l\in R\cap\mathbb{Z}^s$. Then usually the matroid rank inequality is not satisfied.

(3) Suppose M is a vector space (not necessarily finite dimensional) and let $\{F(\mathbf{a})\}_{\mathbf{a}}$ be a $\mathbb{Z}^s$–filtration on it. If this filtration satisfies the identity

$$F(\max\{\mathbf{a},\mathbf{b}\})=F(\mathbf{a})\cap F(\mathbf{b}),\quad\text{for all }\mathbf{a},\mathbf{b}\in\mathbb{Z}^s, \tag{4.1.11}$$

and if the codimension of $F(\mathbf{a})$ in M is finite for all $\mathbf{a} \in \mathbb{Z}^s$, then the function $\mathbf{a} \mapsto \dim M/F(\mathbf{a})$ satisfies the matroid rank inequality.

(4) Let $\{F(\mathbf{a})\}_{\mathbf{a}}$ be a $\mathbb{Z}^s$–filtration of the local ring $\mathcal{O}_{X,o}$ of the germ at o of an analytic space X. If the filtration is provided by a collection of order functions (or valuations), then $\{F(\mathbf{a})\}_{\mathbf{a}}$ is a $\mathbb{Z}^s$–filtration of ideals and $F(\mathbf{a}) \cap F(\mathbf{b}) = F(\max\{\mathbf{a}, \mathbf{b}\})$. In particular, if the codimension of $F(\mathbf{a})$ in $\mathcal{O}_{\widetilde{X},o}$ is finite, then the function $\mathbf{a} \mapsto \dim \mathcal{O}_{X,o}/F(\mathbf{a})$ satisfies the matroid rank inequality. However, for more general filtrations the matroid rank inequality does not necessarily hold.

(5) Let (X, o) be a normal surface singularity and $\widetilde{X}$ a fixed resolution. Then the height function $l \in L_{\geq 0}, l \mapsto -h^1(\mathcal{O}_l)$ satisfies the matroid rank inequality (cf. Remark 2.3.9 or [20, Remark 6.4.21], [28, 4.8]).

Definition 4.1.12 We say that the pair h and h° (respectively h and h') satisfies the 'Combinatorial Duality Property' (CDP) if $h(l+E_v) - h(l)$ and $h^\circ(l+E_v) - h^\circ(l)$ (respectively $h(l+E_v) - h(l)$ and $h'(l) - h'(l+E_v)$) simultaneously cannot be nonzero for any $l, l+E_v \in R \cap \mathbb{Z}^s$. Furthermore, we say that h satisfies the CDP if the pair (h, h^{sym}) satisfies it.

Examples of pairs (h, h°) satisfying the CDP can be found in [3, 4] or in the following sections.

Theorem 4.1.13 [1, Theorem 5.2.1] *Assume that* $c < \infty$ *and* h *satisfies the stability property, and the pair* (h, h°) *(respectively* (h, h')*) satisfies the Combinatorial Duality Property. Then* $eu(\mathbb{H}^*(R, w)) = h^\circ(0) - h^\circ(c)$ $(= h'(c))$.

5 Analytic lattice cohomology of normal surface singularities

5.1 General construction

Let (X, o) be a normal surface singularity, and we fix a good resolution $\phi : \widetilde{X} \to X$. For the definition of analytic lattice cohomology we do not need explicitly the minimal good resolution, neither the assumptions that (X, o) has to be Gorenstein with rational homology sphere link ([1, 20]).

5.1.1 For any $c \in L$, $c \geq Z_K$, we consider the rectangle $R(0, c)$. Here we might consider the $c = \infty$ case too, in such a case $R(0, c) = (\mathbb{R}_{\geq 0})^s$. Then we consider the multivariable Hilbert function $\mathfrak{h} : R(0, c) \cap \mathbb{Z}^s \to \mathbb{Z}$, $\mathfrak{h}(l) = \dim H^0(\mathcal{O}_{\widetilde{X}})/H^0(\mathcal{O}_{\widetilde{X}}(-l))$ associated with the divisorial filtration of $\mathcal{O}_{X,o}$ and the resolution ϕ, cf. [1, 4.2], or [20].

Clearly, $\mathfrak{h}$ is increasing (that is, $\mathfrak{h}(l_1) \geq \mathfrak{h}(l_2)$ whenever $l_1 \geq l_2$) and $\mathfrak{h}(0) = 0$.

Next, set $\mathfrak{h}^\circ(l) = p_g - h^1(\mathcal{O}_l)$ too (where, by definition, $h^1(\mathcal{O}_{l=0}) = 0$). Then $\mathfrak{h}^\circ$ is decreasing, $\mathfrak{h}^\circ(0) = p_g$ and $\mathfrak{h}^\circ(c) = 0$ (by Corollary 2.3.8), cf. [1, 4.2] or [20]. Finally we define the weight function

$$w_{an,0} : \mathcal{Q}_0 \to \mathbb{Z}, \quad w_{an,0}(l) = \mathfrak{h}(l) + \mathfrak{h}^\circ(l) - \mathfrak{h}^\circ(0) = \mathfrak{h}(l) - h^1(\mathcal{O}_l). \tag{5.1.2}$$

If we reorganize $\mathfrak{h}^\circ(0) - \mathfrak{h}^\circ$ as $\mathfrak{h}'$ then $w_{an,0}(l) = \mathfrak{h}(l) - \mathfrak{h}'(l)$, where $\mathfrak{h}'(l) = h^1(\mathcal{O}_l)$ and thus $\mathfrak{h}'(0) = 0$ and $\mathfrak{h}'(c) = p_g$. Clearly, $w_{an,0}(0) = 0$.

We consider next the natural cube–decomposition of $R(0, c)$ as in 3.1.2 and we define $w_{an,q} : \mathcal{Q}_q \to \mathbb{Z}$ by $w_{an,q}(\square_q) = \max\{w_{an,0}(l) : l$ is any vertex of $\square_q\}$. This system defines the lattice cohomology $\mathbb{H}^*(R(0, c), w_{an})$.

5.1.3 This weight function has several useful properties:

First of all, note that $0 \leq \mathfrak{h}^\circ(l) \leq p_g$ for every l, hence when $c = \infty$ then $\mathfrak{h}$ and $w_{an,0}$ have comparable asymptotic behaviour for $l \gg 0$. A computation shows that $w_{an,0}$ satisfies the requirement 3.1.3(a), namely, $w_{an,0}^{-1}((\infty, n])$ is finite for any $n \in \mathbb{Z}$.

Since $\mathfrak{h}$ is induced by a filtration given by valuations, it satisfies the matroid rank inequality (cf. Example 4.1.10). On the other hand, h^1 satisfies the 'opposite' matroid rank inequality (cf. Remark 2.3.9). Therefore, $w_{an,0}$ itself satisfies the matroid rank inequality.

Theorem 5.1.4 [1, 20]

(a) **(Independence of** *c)* *$\mathbb{H}^*(R(0,c), w_{an})$ is independent of the choice of $c \geq Z_K$.*
(b) **(Independence of** *ϕ)* *Assume that the resolution graph is a tree (a property independent of the resolution). Then $\mathbb{H}^*(R(0,c), w_{an})$ $(c \geq Z_K)$ is independent of the choice of the resolution ϕ.*
(c) **(CDP)** *Assume that $g_v = 0$ for any $v \in \mathcal{V}$. Then there exists no $l \in L_{\geq 0}$ and $v \in \mathcal{V}$ such that the differences $\mathfrak{h}(l + E_v) - \mathfrak{h}(l)$ and $\mathfrak{h}^\circ(l) - \mathfrak{h}^\circ(l + E_v)$ are simultaneously strictly positive.*
(d) **(Finiteness)** *The module $\mathbb{H}^*_{red}(R(0,c), w_{an})$ has finite $\mathbb{Z}$–rank, hence the Euler characteristic $eu(\mathbb{H}^*(R(0,c), w_{an}))$ is well-defined.*
(e) **(The Euler characteristic** *$eu(\mathbb{H}^*(R(0,c), w_{an}))$)* *If the link is a rational homology sphere, then $eu(\mathbb{H}^*(R(0,c), w_{an})) = p_g(X,o)$. In particular, $\mathbb{H}^*(R(0,c), w_{an})$ is a categorification of the geometric genus.*

Definition 5.1.5 Assume that the link is a rational homology sphere. The graded $\mathbb{Z}[U]$–module $\mathbb{H}^*(R(0,c), w_{an})$ $(c \geq Z_K)$ will be denoted by $\mathbb{H}^*_{an,0}(X,o)$. It is called the *analytic lattice cohomology of* (X,o) associated with the canonical spin^c–structure.

Remark 5.1.6 Similarly to the topological case (see Example 3.2), the analytic lattice cohomology has different versions for different spin^c–structures, too. For more information on these see [2].

5.2 Reinterpretation of $\mathfrak{h}^\circ$ in the Gorenstein case

Suppose that the normal surface singularity (X,o) is Gorenstein, not of type A, D or E, and $\phi : \widetilde{X} \to X$ is the minimal good resolution . Thus $Z_K \in L$, $Z_K \geq E$, and $\mathcal{O}_{\widetilde{X}}(K_{\widetilde{X}}) \cong \Omega^2_{\widetilde{X}} \cong \mathcal{O}_{\widetilde{X}}(-Z_K)$, therefore in the sequel we will only use the $-Z_K$ notation.

Proposition 5.2.1 *If we choose $c = Z_K$, then $\mathfrak{h}^\circ(l) = \mathfrak{h}(Z_K - l)$, i.e. $\mathfrak{h}^\circ$ is obtained as the symmetrization of $\mathfrak{h}$. In particular $w_{an,0}(l) = \mathfrak{h}(l) + \mathfrak{h}(Z_K - l) - p_g = w_{an,0}(Z_K - l)$.*

Proof From (2.3.10) we have that $\dim H^0(\mathcal{O}_{\widetilde{X}}(-Z_K + Z))/H^0(\mathcal{O}_{\widetilde{X}}(-Z_K)) = h^1(\mathcal{O}_Z) = \mathfrak{h}'(Z)$ for any $Z > 0$. On the other hand, by the opposite matroid rank inequality (cf. Remark 2.3.9) we also have $h^1(\mathcal{O}_Z) = h^1(\mathcal{O}_{\min\{Z, Z_K\}})$.
For a general $c \geq Z_K$, from (4.1.5) we have $\mathfrak{h}^\circ(l) = \mathfrak{h}'(c) - \mathfrak{h}'(l)$, so by Corollary 2.3.5

$$\mathfrak{h}^\circ(l) := \dim \frac{H^0(\widetilde{X}, \mathcal{O}_{\widetilde{X}}(-Z_K + c))}{H^0(\widetilde{X}, \mathcal{O}_{\widetilde{X}}(-Z_K + l))} = \dim \frac{H^0(\widetilde{X}, \mathcal{O}_{\widetilde{X}})}{H^0(\widetilde{X}, \mathcal{O}_{\widetilde{X}}(-Z_K + \min\{l, Z_K\}))}. \quad (5.2.2)$$

Specifically for $c = Z_K$ and $0 \leq l \leq Z_K$:

$$\mathfrak{h}^\circ(l) = \dim H^0(\mathcal{O}_{\widetilde{X}})/H^0(\mathcal{O}_{\widetilde{X}}(-Z_K + l)) = \mathfrak{h}(Z_K - l); \quad (5.2.3)$$

and also $\mathfrak{h}'(l) = \mathfrak{h}^\circ(0) - \mathfrak{h}^\circ(l) = \dim H^0(\mathcal{O}_{\widetilde{X}}(-Z_K + l))/H^0(\mathcal{O}_{\widetilde{X}}(-Z_K))$. (5.2.4)

□

Remark 5.2.5 Note that this symmetry is true for the topological weight function too: $\chi(l) = \chi(Z_K - l)$ (for the topological lattice cohomology see Example 3.2 or [18, 20]). However, in the analytic case, the symmetry might fail for non-Gorenstein germs (even if we consider a numerically Gorenstein topological type).

5.2.6 As it will be the model for our approach later, we present here a proof of the fact, that if (X, o) is a Gorenstein normal surface singularity, not of type ADE, with rational homology sphere link, then the pair $(\mathfrak{h}, \mathfrak{h}^{sym})$ satisfies the Combinatorial Duality property. We choose $c = Z_K$.

For each holomorphic function germ $f \in \mathcal{O}_{X,o}$ consider $\mathrm{div}_E(f) \in L_{\geq 0}$. One can easily prove, that $\mathrm{div}_E(f) \in \mathcal{S}$. Then let us define $\mathcal{S}_{an}$ as $\{\mathrm{div}_E(f) : f \in \mathcal{O}_{X,o}\} \subset \mathcal{S}$.

Now assume by contradiction that the pair $(\mathfrak{h}, \mathfrak{h}^{sym})$ does not satisfy the CDP. Then there exists a lattice point l, a vertex v and $s', s'' \in \mathcal{S}_{an}$ such that $\mathrm{div}_E(s') \geq l$, $s'_v = l_v$, $\mathrm{div}_E(s'') \geq Z_K - l - E_v$, $s''_v = (Z_K - l)_v - 1$. Hence, $\mathrm{div}_E(s's'') = \mathrm{div}_E(s') + \mathrm{div}_E(s'') \geq Z_K - E_v$ with equality at E_v–coordinate. But this contradicts Theorem 2.3.3, which states that $H^0(\mathcal{O}_{E_v}(-Z_K + E_v)) \cong H^1(\mathcal{O}_{E_v})^* \cong 0$.

6 The general construction of $\pi_{m,n}$ and its properties

Let (X, o) be a Gorenstein normal surface singularity, not of type ADE, and we fix the minimal good resolution $\phi : \widetilde{X} \to X$. As above, $Z_K \in L$ and $Z_K \geq E$ (see 2.1.5). We also assume that the link is a rational homology sphere.

Recall that in the case of Gorenstein singularities $\mathcal{O}_{\widetilde{X}}(-Z_K) \cong \Omega^2_{\widetilde{X}} \cong \mathcal{O}_{\widetilde{X}}(K_{\widetilde{X}})$, therefore $K_{\widetilde{X}}$ can be chosen as $-Z_K$. In the sequel we will only use the $-Z_K$ notation.

In this section we construct a lattice cohomology module, which will be a categorification of a common generalization $\pi_{m,n}$ of the numerical invariants δ_m, γ_m and λ_m.

6.1 δ_m, γ_m and λ_m in the Gorenstein case

The definitions of the δ_m, γ_m and λ_m plurigenera in the Gorenstein case transform into the following. For every $m \geq 1$, $m \in \mathbb{Z}$:

- $\delta_m = \dim H^0(\widetilde{X} \setminus E, \mathcal{O}_{\widetilde{X}}(-mZ_K))/H^0(\widetilde{X}, \mathcal{O}_{\widetilde{X}}(-mZ_K + (m-1)E))$;
- $\gamma_m = \dim\ H^0(\widetilde{X} \setminus E, \mathcal{O}_{\widetilde{X}}(-mZ_K))/H^0(\widetilde{X}, \mathcal{O}_{\widetilde{X}}(-mZ_K))$;
- $\lambda_m = \dim\ H^0(\widetilde{X} \setminus E, \mathcal{O}_{\widetilde{X}}(-mZ_K))/H^0(\widetilde{X}, \mathcal{O}_{\widetilde{X}}(-mZ_K + mE))$.

Note that $\delta_1 = \gamma_1 = p_g$ (see Remark 2.2.2).

Lemma 6.1.1 $H^0(\widetilde{X} \setminus E, \mathcal{O}_{\widetilde{X}}(-mZ_K)) \cong H^0(\widetilde{X}, \mathcal{O}_{\widetilde{X}})$.

Proof Since $-mZ_K$ has support in E, it follows that $H^0(\widetilde{X} \setminus E, \mathcal{O}_{\widetilde{X}}(-mZ_K)) \cong H^0(\widetilde{X} \setminus E, \mathcal{O}_{\widetilde{X}})$. Next, take the following exact sequence of sheaf cohomology with compact support (cf. Theorem 1.39 in [23]):

$$0 \to H^0\big(\widetilde{X}, \mathcal{O}_{\widetilde{X}}\big) \to H^0\big(\widetilde{X} \setminus E, \mathcal{O}_{\widetilde{X}}\big) \to H^1_c\big(\widetilde{X}, \mathcal{O}_{\widetilde{X}}\big) \to H^1\big(\widetilde{X}, \mathcal{O}_{\widetilde{X}}\big) \to H^1\big(\widetilde{X} \setminus E, \mathcal{O}_{\widetilde{X}}\big) \to \cdots$$

Here the middle term is isomorphic to $H^1(\widetilde{X}, \Omega^2_{\widetilde{X}})^*$ via Theorem 2.3.2, which vanishes by Theorem 2.3.6. □

Corollary 6.1.2 *For every $m \geq 1$, $m \in \mathbb{Z}$:*

- $\delta_m = \dim H^0(\widetilde{X}, \mathcal{O}_{\widetilde{X}})/H^0(\widetilde{X}, \mathcal{O}_{\widetilde{X}}(-mZ_K + (m-1)E))$;
- $\gamma_m = \dim\ H^0(\widetilde{X}, \mathcal{O}_{\widetilde{X}})/H^0(\widetilde{X}, \mathcal{O}_{\widetilde{X}}(-mZ_K))$;
- $\lambda_m = \dim\ H^0(\widetilde{X}, \mathcal{O}_{\widetilde{X}})/H^0(\widetilde{X}, \mathcal{O}_{\widetilde{X}}(-mZ_K + mE))$.

6.2 General construction

Suppose that the singularity (X, o) is not of type ADE, in particular $Z_K \geq E$. Then we will construct a categorification of $\pi_{m,n} := \dim H^0(\widetilde{X}, \mathcal{O}_{\widetilde{X}})/H^0(\widetilde{X}, \mathcal{O}_{\widetilde{X}}(-mZ_K + nE))$ for every pair $m \geq 1, n \geq 0,\ m, n \in \mathbb{Z}$ such that $mZ_K - nE \geq 0$.

Clearly, $\delta_m = \pi_{m,m-1}$, $\gamma_m = \pi_{m,0}$ and $\lambda_m = \pi_{m,m}$ satisfy the required condition. (These $\pi_{m,n}$ were considered already in a more general setting in [20, Sect. 6.8.D].)

Definition 6.2.1 In the lattice L, with fixed basis $\{E_v\}_{v\in\mathcal{V}}$, consider the rectangle $R(0, mZ_K - nE)$ (here $mZ_K - nE \geq 0$). Let

$$h_{\pi_{m,n}}(l) := \dim \frac{H^0(\mathcal{O}_{\widetilde{X}}((m-1)Z_K - nE))}{H^0(\mathcal{O}_{\widetilde{X}}((m-1)Z_K - nE - l))}$$

be the 'Hilbert function' of the divisorial filtration on $H^0(\mathcal{O}_{\widetilde{X}}((m-1)Z_K - nE))$; and let

$$h'_{\pi_{m,n}}(l) := \dim \frac{H^0(\mathcal{O}_{\widetilde{X}}(-mZ_K + nE + l))}{H^0(\mathcal{O}_{\widetilde{X}}(-mZ_K + nE))}.$$

We define the weight function $w_{\pi_{m,n},0} : R(0, mZ_K - nE) \cap \mathbb{Z}^s \to \mathbb{Z}$ as in (4.1.4):

$$w_{\pi_{m,n},0}(l) = h_{\pi_{m,n}}(l) - h'_{\pi_{m,n}}(l),$$

and extend it to get $w_{\pi_{m,n},q}$ as in (3.1.5). Associated with this lattice and weight function we get a lattice cohomology $\mathbb{H}^*(R(0, mZ_K - nE), w_{\pi_{m,n}})$.

Remark 6.2.2 $h_{\pi_{m,n}}$ and $h'_{\pi_{m,n}}$ are both increasing functions with $h_{\pi_{m,n}}(0) = h'_{\pi_{m,n}}(0) = 0$, so they give the h and $h' = h^\circ(0) - h^\circ$ functions of a combinatorial lattice cohomology on $R(0, mZ_K - nE)$. In this analogy $h^\circ_{\pi_{m,n}}$ is (compare with (4.1.5))

$$\begin{aligned} h^\circ_{\pi_{m,n}}(l) &= h'_{\pi_{m,n}}(mZ_K - nE) - h'_{\pi_{m,n}}(l) = \dim \frac{H^0(\mathcal{O}_{\widetilde{X}})}{H^0(\mathcal{O}_{\widetilde{X}}(-mZ_K + nE + l))} \\ &= \mathfrak{h}(mZ_K - nE - l). \end{aligned}$$

Lemma 6.2.3 *$h_{\pi_{m,n}}$ satisfies the matroid rank inequality.*

Proof The divisorial filtration $L \ni l \mapsto H^0(\mathcal{O}_{\widetilde{X}}((m-1)Z_K - nE - l)) \subset H^0(\mathcal{O}_{\widetilde{X}}((m-1)Z_K - nE))$ satisfies Eq. (4.1.11) and has finite codimension, hence 4.1.10 can be applied. □

Lemma 6.2.4 *The function $l \mapsto -h'_{\pi_{m,n}}(l)$ satisfies the matroid rank inequality and it stabilizes:*

$$h'_{\pi_{m,n}}(l) = \dim \frac{H^0(\mathcal{O}_{\widetilde{X}})}{H^0(\mathcal{O}_{\widetilde{X}}(-mZ_K + nE))} = \pi_{m,n} \quad \textit{for all } \geq mZ_K - nE.$$

Proof For the first part, notice that the inequality (4.1.8) remains the same after translating the height function by a constant. Therefore, in our case it is enough to prove that

$$h^{\circ}_{\pi_{m,n}}(l_1) + h^{\circ}_{\pi_{m,n}}(l_2) \geq h^{\circ}_{\pi_{m,n}}(\min\{l_1, l_2\}) + h^{\circ}_{\pi_{m,n}}(\max\{l_1, l_2\}).$$

However, by Remark 6.2.2 this translates as

$$\begin{aligned} &\mathfrak{h}(mZ_K - nE - l_1) + \mathfrak{h}(mZ_K - nE - l_2) \\ &\quad \geq \mathfrak{h}(\max\{mZ_K - nE - l_1, mZ_K - nE - l_2\}) \\ &\quad\quad + \mathfrak{h}(\min\{mZ_K - nE - l_1, mZ_K - nE - l_2\}), \end{aligned}$$

which indeed holds, because the Hilbert function $\mathfrak{h}$ already satisfies the matroid rank inequality (cf. 5.1.3 or 4.1.10 applied to the divisorial filtration of $H^0(\mathcal{O}_{\widetilde{X}})$).

The second part follows from Corollary 2.3.5. □

Proposition 6.2.5 *The pair* $(h_{\pi_{m,n}}, h'_{\pi_{m,n}})$ *(or, equivalently, the pair* $(h_{\pi_{m,n}}, h^{\circ}_{\pi_{m,n}})$*) satisfies the Combinatorial Duality Property.*

Proof Let us consider the step from l to $l + E_v$. If $h_{\pi_{m,n}}(l + E_v) > h_{\pi_{m,n}}(l)$ then there exists some $f \in H^0(\mathcal{O}_{\widetilde{X}}((m-1)Z_K - nE))$ with $\mathrm{div}_E(f) \geq l$, $\mathrm{div}_{E_v}(f) = l_v$. Similarly, if $h'_{\pi_{m,n}}(l + E_v) > h'_{\pi_{m,n}}(l)$, then there exists a meromorphic section s of $\mathcal{O}_{\widetilde{X}}(-mZ_K + nE)$ with $\mathrm{div}_E(s) \geq -l - E_v$, $\mathrm{div}_{E_v}(s) = -l_v - 1$. In particular, if both functions jump, then fs, as a meromorphic section of $\mathcal{O}_{\widetilde{X}}(-Z_K)$, has divisor $\mathrm{div}_E(fs) \geq -E_v$, $\mathrm{div}_{E_v}(fs) = -1$.

Hence $fs \neq 0$ in $H^0(\mathcal{O}_{\widetilde{X}}(-Z_K + E_v))/H^0(\mathcal{O}_{\widetilde{X}}(-Z_K)) \cong H^0(\mathcal{O}_{E_v}(-Z_K + E_v)) \cong H^1(\mathcal{O}_{E_v})^* = 0$ (where we have used Serre duality on the rational curve E_v, see Theorem 2.3.3). This leads to contradiction. □

Corollary 6.2.6 *The lattice cohomology* $\mathbb{H}^*(R(0, mZ_K - nE), w_{\pi_{m,n}})$ *defined on* $R(0, mZ_K - nE)$ *with the weight function* $w_{\pi_{m,n},0}(l) = h_{\pi_{m,n}}(l) - h'_{\pi_{m,n}}(l)$ *has Euler characteristic* $h'_{\pi_{m,n}}(mZ_K - nE) = \pi_{m,n}$.

In particular, as a graded $\mathbb{Z}[U]$*–module,* $\mathbb{H}^*(R(0, mZ_K - nE), w_{\pi_{m,n}})$ *is a categorification of* $\pi_{m,n}$.

Proof From Remark 6.2.2, Lemma 6.2.3 and Proposition 6.2.5 we see that the functions $h_{\pi_{m,n}}$ and $h'_{\pi_{m,n}}$ satisfy the requirements of Theorem 4.1.13. The value of the Euler characteristic comes from Lemma 6.2.4. □

Corollary 6.2.7 $\mathbb{H}^*(R(0, c), w_{\pi_{m,n}})$ *is independent of the choice of* $c \geq mZ_K - nE$.

Proof One can just copy the argument from the proof of Theorem 5.1.4*(a)*, see also Lemma 11.9.2 in [20]. Indeed, that proof only uses the fact that (with our notation) $w_{\pi_{m,n},0}$ satisfies the matroid rank inequality (see Lemmas 6.2.3 and 6.2.4) and is increasing in the rectangle $R(mZ_K - nE, c)$ (by the stabilization $h'_{\pi_{m,n}}(l) = h'_{\pi_{m,n}}(mZ_K - nE)$ for all $l \geq mZ_K - nE$, cf. Lemma 6.2.4). □

Definition 6.2.8 The lattice cohomology (computed from the minimal good resolution $\phi : \widetilde{X} \to X$ of a non–ADE Gorenstein normal surface singularity (X, o) with rational homology sphere link) associated with the weight function $w_{\pi_{m,n},0}$ and lattice $R(0, c)$ with c large enough will be denoted by $\mathbb{H}^*((X, o), w_{\pi_{m,n}})$ (it is a double graded $\mathbb{Z}[U]$–module, which by the above discussion, is independent of the choice of $c \geq mZ_K + nE$).

6.3 Comparison with the analytic lattice cohomology

We compare the 'new' weight function $w_{\pi_{m,n},0}$ with the 'old one', $w_{an,0}(l) = \mathfrak{h}(l) - h^1(\mathcal{O}_l) = \mathfrak{h}(l) - \mathfrak{h}'(l)$ (corresponding to $m = 1,\ n = 0$, or to the case of the analytic lattice cohomology as categorification of p_g). In this discussion we need to use a stronger assumption, namely

$$(m-1)Z_K - nE \geq 0,$$

which is fulfilled in the case of δ_m and γ_m but not for λ_m (at least not for any Gorenstein normal surface singularity with rational homology sphere link). The specific cases not covered by this inequality will be treated in Sect. 8.

Proposition 6.3.1 *Suppose that $(m-1)Z_K - nE \geq 0$. If $l \leq (m-1)Z_K - nE$, then $h_{\pi_{m,n}}(l) = 0$. If $l \geq (m-1)Z_K - nE$, then we have $h_{\pi_{m,n}}(l) = \mathfrak{h}(l - (m-1)Z_K + nE)$ and $h'_{\pi_{m,n}}(l) = \mathfrak{h}'(l - (m-1)Z_K + nE) + d_{\pi_{m,n}}$, where $d_{\pi_{m,n}}$ is the constant $\pi_{m,n} - p_g$.*

Proof Assume that $l \leq (m-1)Z_K - nE$. Then the first statement follows from Corollary 2.3.5:

$$H^0(\mathcal{O}_{\widetilde{X}}((m-1)Z_K - nE - l)) \cong H^0(\mathcal{O}_{\widetilde{X}}((m-1)Z_K - nE)) \cong H^0(\mathcal{O}_{\widetilde{X}}). \tag{6.3.2}$$

If $l \geq (m-1)Z_K - nE$, then (via the second identity of (6.3.2))

$$h_{\pi_{m,n}}(l) = \dim \frac{H^0(\widetilde{X}, \mathcal{O}_{\widetilde{X}})}{H^0(\widetilde{X}, \mathcal{O}_{\widetilde{X}}(-(l-(m-1)Z_K + nE)))} = \mathfrak{h}(l - (m-1)Z_K + nE).$$

Now let us consider $h'_{\pi_{m,n}}$ in the case $l \geq (m-1)Z_K - nE$. By Eq. (5.2.4)

$$\begin{aligned}
\mathfrak{h}'(l - (m-1)Z_K + nE) &= \dim \frac{H^0(\widetilde{X}, \mathcal{O}_{\widetilde{X}}(-Z_K + l - (m-1)Z_K + nE))}{H^0(\widetilde{X}, \mathcal{O}_{\widetilde{X}}(-Z_K))} \\
&= \dim \frac{H^0(\widetilde{X}, \mathcal{O}_{\widetilde{X}}(-mZ_K + nE + l))}{H^0(\widetilde{X}, \mathcal{O}_{\widetilde{X}}(-mZ_K + nE))} \\
&\quad - \dim \frac{H^0(\widetilde{X}, \mathcal{O}_{\widetilde{X}}(-Z_K))}{H^0(\widetilde{X}, \mathcal{O}_{\widetilde{X}}(-mZ_K + nE))} = h'_{\pi_{m,n}}(l) - d_{\pi_{m,n}},
\end{aligned}$$

where we set $d_{\pi_{m,n}} := \dim\ H^0(\widetilde{X}, \mathcal{O}_{\widetilde{X}}(-Z_K))/H^0(\widetilde{X}, \mathcal{O}_{\widetilde{X}}(-mZ_K + nE))$.

Finally, the identity $d_{\gamma_m} = \gamma_m - p_g$ follows from Eq. (2.3.12) and the definition of the number $\pi_{m,n}$ (see notation in 6.2). □

Corollary 6.3.3 *If $(m-1)Z_K - nE \geq 0$, then $\mathbb{H}^*((X, o), w_{\pi_{m,n}}) \cong \mathbb{H}^*_{an,0}(X, o)[-2d_{\pi_{m,n}}]$.*

Proof By Proposition 6.3.1 the weight function $l \mapsto w_{\pi_{m,n},0}(l)$ is decreasing in the rectangle $R(0, (m-1)Z_K - nE)$. As $w_{\pi_{m,n},0}$ also satisfies the matroid rank inequality, similarly to the proof of Corollary 6.2.7 (see also Lemma 7.3.9 from [20]), we get that the map

$$\mathbb{H}^*(R(0, mZ_K - nE), w_{\pi_{m,n}}) \to \mathbb{H}^*(R((m-1)Z_K - nE, mZ_K - nE), w_{\pi_{m,n}})$$

induced by the natural inclusion is a graded $\mathbb{Z}[U]$–module isomorphism. However, the affine translation $\tau : l \mapsto l - (m-1)Z_K + nE$ identifies the rectangles $R((m-1)Z_K - nE, mZ_K - nE)$ and $R(0, Z_K)$ and by Proposition 6.3.1 we have that for any $l \in R((m-1)Z_K - nE, mZ_K - nE) \cap \mathbb{Z}^s$: $w_{\pi_{m,n}}(l) = w_{an,0}(\tau(l)) - d_{\pi_{m,n}}$. Therefore, τ induces a graded $\mathbb{Z}[U]$–module cochain complex isomorphism

$$\tau^* : \mathcal{F}^*(R(0, c), w_{an})[-2d_{\pi_{m,n}}] \xrightarrow{\cong} \mathcal{F}^*(R((m-1)Z_K - nE, mZ_K - nE), w_{\pi_{m,n}}).$$

This induces the graded $\mathbb{Z}[U]$–module isomorphism on the cohomologies stated in the lemma. □

7 The cases δ_m, γ_m revisited

Let (X, o) be a Gorenstein normal surface singularity, not of type ADE, and we fix the minimal good resolution $\phi : \widetilde{X} \to X$. We also assume that the link is a rational homology sphere.

In this section we collect all the results from the previous sections specified to the δ_m and γ_m cases.

7.1 The categorification of δ_m

Recall that $\delta_m = \pi_{m,m-1} = \dim H^0(\widetilde{X}, \mathcal{O}_{\widetilde{X}})/H^0(\widetilde{X}, \mathcal{O}_{\widetilde{X}}(-mZ_K + (m-1)E))$ for any $m \geq 1$. In the lattice L, with fixed basis $\{E_v\}_{v\in\mathcal{V}}$, consider the rectangle $R(0, mZ_K - (m-1)E)$. Let

$$h_{\delta_m}(l) := \dim \frac{H^0(\mathcal{O}_{\widetilde{X}}((m-1)(Z_K - E)))}{H^0(\mathcal{O}_{\widetilde{X}}((m-1)(Z_K - E) - l))} \text{ and}$$
$$h'_{\delta_m}(l) := \dim \frac{H^0(\mathcal{O}_{\widetilde{X}}(-mZ_K + (m-1)E + l))}{H^0(\mathcal{O}_{\widetilde{X}}(-mZ_K + (m-1)E))}$$

be increasing functions (with $h^\circ_{\delta_m}(l) = \dim H^0(\mathcal{O}_{\widetilde{X}})/H^0(\mathcal{O}_{\widetilde{X}}(-mZ_K + (m-1)E + l))$). We define the weight function $w_{\delta_m,0} : R(0, mZ_K - (m-1)E) \cap \mathbb{Z}^s \to \mathbb{Z}$ as

$$w_{\delta_m,0}(l) = h_{\delta_m}(l) - h'_{\delta_m}(l) = h_{\delta_m}(l) + h^\circ_{\delta_m}(l) - \delta_m,$$

and $w_{\delta_m,q}$ as in (3.1.5). Associated with this lattice and weight function we get a lattice cohomology $\mathbb{H}^*(R(0, mZ_K - (m-1)E), w_{\delta_m})$.

Theorem 7.1.1 *(a)* **(Matroid rank inequality)** *both h_{δ_m} and $-h'_{\delta_m}$ satisfy the matroid rank inequality;*

(b) **(Lattice cohomology)** *the lattice cohomology associated with the weight function $w_{\delta_m,0} = h_{\delta_m} - h'_{\delta_m}$ in the rectangle $R(0, c)$, whenever $c \geq mZ_K - (m-1)E$, stabilizes with respect to c. It will be denoted by $\mathbb{H}^*((X, o), w_{\delta_m})$;*

(c) **(CDP)** *the pair $(h_{\delta_m}, h'_{\delta_m})$ satisfies the CDP;*

(d) **(Euler characteristic)** *the lattice cohomology defined on $R(0, mZ_K - (m-1)E)$ with the weight function $w_{\delta_m,0}(l) = h_{\delta_m}(l) - h'_{\delta_m}(l) = h_{\delta_m}(l) + h^\circ_{\delta_m}(l) - h^\circ_{\delta_m}(0)$ has Euler characteristic $h'_{\delta_m}(mZ_K - (m-1)E) - h'_{\delta_m}(0) = \delta_m$. In particular, as a graded $\mathbb{Z}[U]$–module, $\mathbb{H}^*((X, o), w_{\delta_m})$ is a categorification of δ_m;*

(e) **(Relations with the original Hilbert functions)** *if $l \leq (m-1)(Z_K - E)$, then $h_{\delta_m}(l) = 0$, if $l \geq (m-1)(Z_K - E)$, then $h_{\delta_m}(l) = \mathfrak{h}(l - (m-1)(Z_K - E))$ and $h'_{\delta_m}(l) = \mathfrak{h}'(l - (m-1)(Z_K - E)) + d_{\delta_m}$, where $d_{\delta_m} = \delta_m - p_g$ is the non–negative constant*

$$d_{\delta_m} = \dim\ H^0(\mathcal{O}_{\widetilde{X}}(-Z_K))/H^0(\mathcal{O}_{\widetilde{X}}(-mZ_k + (m-1)E).$$

(f) **(Relation with the analytic lattice cohomology)** *there exists a natural double graded $\mathbb{Z}[U]$–module isomorphism:*

$$\mathbb{H}^*((X, o), w_{\delta_m}) \cong \mathbb{H}^*_{an,0}(X, o)[-2d_{\delta_m}].$$

7.2 The categorification of γ_m

8.1 In this case $\gamma_m = \pi_{m,0} = \dim H^0(\widetilde{X}, \mathcal{O}_{\widetilde{X}})/H^0(\widetilde{X}, \mathcal{O}_{\widetilde{X}}(-mZ_K))$ for any $m \geq 1$, and in the rectangle $R(0, mZ_K)$ we set the increasing functions

$$h_{\gamma_m}(l) := \dim \frac{H^0(\mathcal{O}_{\widetilde{X}}((m-1)Z_K))}{H^0(\mathcal{O}_{\widetilde{X}}((m-1)Z_K - l))} \quad \text{and}$$
$$h'_{\gamma_m}(l) := \dim \frac{H^0(\mathcal{O}_{\widetilde{X}}(-mZ_K + l))}{H^0(\mathcal{O}_{\widetilde{X}}(-mZ_K)}$$

corresponding to $h^\circ_{\gamma_m}(l) = \dim H^0(\mathcal{O}_{\widetilde{X}})/H^0(\mathcal{O}_{\widetilde{X}}(-mZ_K + l))$. We define the weight function $w_{\gamma_m,0} : R(0, mZ_K) \cap \mathbb{Z}^s \to \mathbb{Z}$ as

$$w_{\gamma_m,0}(l) = h_{\gamma_m}(l) - h'_{\gamma_m}(l) = h_{\gamma_m}(l) + h^\circ_{\gamma_m}(l) - \gamma_m,$$

and $w_{\gamma_m,q}$ as in (3.1.5). In this way we obtain the lattice cohomology $\mathbb{H}^*(R(0, mZ_K), w_{\gamma_m})$.

Theorem 7.2.1 *(a)* **(Matroid rank inequality)** *both h_{γ_m} and $-h'_{\gamma_m}$ satisfy the matroid rank inequality;*

(b) **(Lattice cohomology)** *the lattice cohomology associated with the weight function $w_{\gamma_m,0} = h_{\gamma_m} - h'_{\gamma_m}$ in the rectangle $R(0, c)$, whenever $c \geq mZ_K$, stabilizes with respect to c. It will be denoted by $\mathbb{H}^*((X, o), w_{\gamma_m})$;*

(c) **(CDP)** *the pair $(h_{\gamma_m}, h'_{\gamma_m})$ satisfies the CDP;*

(d) **(Euler characteristic)** *the lattice cohomology defined on $R(0, mZ_K)$ with the weight function $w_{\gamma_m,0}(l) = h_{\gamma_m}(l) - h'_{\gamma_m}(l) = h_{\gamma_m}(l) + h^\circ_{\gamma_m}(l) - h^\circ_{\gamma_m}(0)$ has Euler characteristic $h'_{\gamma_m}(mZ_K) - h'_{\gamma_m}(0) = \gamma_m$. In particular, as a graded $\mathbb{Z}[U]$–module, $\mathbb{H}^*((X, o), w_{\gamma_m})$ is a categorification of γ_m;*

(e) **(Relations with the original Hilbert functions)** *if $l \leq (m-1)Z_K$, then $h_{\gamma_m}(l) = 0$, if $l \geq (m-1)Z_K$, then $h_{\gamma_m}(l) = \mathfrak{h}(l - (m-1)Z_K)$ and $h'_{\gamma_m}(l) = \mathfrak{h}'(l - (m-1)Z_K) + d_{\gamma_m}$, where $\mathfrak{h}$ and $\mathfrak{h}'$ denote the standard Hilbert $d_{\gamma_m} = \gamma_m - p_g$ is the non–negative constant*

$$d_{\gamma_m} = \dim \ H^0(\mathcal{O}_{\widetilde{X}}(-Z_K))/H^0(\mathcal{O}_{\widetilde{X}}(-mZ_K).$$

(f) **(Relation with the analytic lattice cohomology)** *there exists a natural graded $\mathbb{Z}[U]$–module isomorphism:*

$$\mathbb{H}^*((X, o), w_{\gamma_m}) \cong \mathbb{H}^*_{an,0}(X, o)[-2d_{\gamma_m}].$$

Remark 7.2.2 Assume that the minimal good resolution ϕ is already a minimal resolution. Then several cohomological invariants considered above have additional equivalent descriptions.

(i) We claim that

$$\gamma_m = h^1\big(\mathcal{O}_{\widetilde{X}}((m-1)Z_K)\big) \text{ for every } m \geq 1. \tag{7.2.3}$$

Indeed, since $Z_K \in \mathcal{S}$, by Theorem 2.3.6 the module $H^1(\widetilde{X}, \mathcal{O}_{\widetilde{X}}(-mZ_K)) \cong 0$ and the following sequence is exact:

$$0 \to H^0\big(\widetilde{X}, \mathcal{O}_{\widetilde{X}}(-mZ_K)\big) \to H^0\big(\widetilde{X} \setminus E, \mathcal{O}_{\widetilde{X}}(-mZ_K)\big) \to H^1_c\big(\widetilde{X}, \mathcal{O}_{\widetilde{X}}(-mZ_K)\big) \to 0.$$

Furthermore, by Theorem 2.3.2 $H^1_c(\widetilde{X}, \mathcal{O}_{\widetilde{X}}(-mZ_K)) \cong H^1(\widetilde{X}, \mathcal{O}_{\widetilde{X}}((m-1)Z_K))^*$.

(ii) Furthermore, we can also give another equivalent description of h'_{γ_m} :

$$h'_{\gamma_m}(l) = \dim \frac{H^0(\widetilde{X}, \mathcal{O}_{\widetilde{X}}(-mZ_K + l))}{H^0(\widetilde{X}, \mathcal{O}_{\widetilde{X}}(-mZ_K))} = h^1(l, \mathcal{O}_l((m-1)Z_K), \tag{7.2.4}$$

which better resembles the original $\mathfrak{h}'$ function of the analytic lattice cohomology (cf. 5.1.5).

Indeed, as $Z_K \in \mathcal{S}$, from the vanishing $H^1(\mathcal{O}_{\widetilde{X}}(-mZ_K)) = 0$ (cf. Theorem 2.3.6) we have the exact sequence:

$$0 \to H^0(\mathcal{O}_{\widetilde{X}}(-mZ_K)) \to H^0(\mathcal{O}_{\widetilde{X}}(-mZ_K + l)) \to H^0(\mathcal{O}_l(-mZ_K + l)) \to 0;$$

and then by Theorem 2.3.3

$$H^0(\mathcal{O}_l(-mZ_K + l))^* \cong H^1(\mathcal{O}_l(-Z_K + l + mZ_K - l)) \cong H^1(\mathcal{O}_l((m-1)Z_K)).$$

(iii) We can reprove the stabilization of the h'_{γ_m} function for $l \geq mZ_K$ in this new setup as follows. Consider the sheaf exact sequence:

$$0 \to \mathcal{O}_{\widetilde{X}}(-Z_K) \to \mathcal{O}_{\widetilde{X}}((m-1)Z_K) \to \mathcal{O}_{mZ_K}((m-1)Z_K) \to 0.$$

The exact sequence of sheaf cohomologies and the vanishing $H^1(\mathcal{O}_{\widetilde{X}}(-Z_K)) = 0$ (via Theorem 2.3.6) give

$$H^1\big(\widetilde{X}, \mathcal{O}_{\widetilde{X}}((m-1)Z_K)\big) \cong H^1\big(mZ_K, \mathcal{O}_{mZ_K}((m-1)Z_K)\big).$$

Since in general we have surjections $H^1(\widetilde{X}, \mathcal{L}) \to H^1(l_2, \mathcal{L}|_{l_2}) \to H^1(l_1, \mathcal{L}|_{l_1})$ for any $l_2 \geq l_1$, we obtain that

$$H^1(l, \mathcal{O}_l((m-1)Z_K)) = H^1(\mathcal{O}_{\widetilde{X}}((m-1)Z_K)) \ \text{ for any} l \geq mZ_K. \tag{7.2.5}$$

Hence from (7.2.3) we then have that $h_{\gamma_m}(l) = \gamma_m$ for any $l \geq mZ_K$.

8 The categorification of λ_m

In this case of $n = m$ we have to separate two cases. First, all the results from Subsect. 6.2 (proved under the assumption $mZ_K - nE \geq 0$) are valid for any singularity (non-ADE Gorenstein normal surface singularity, with rational homology sphere link and minimal good resolution). On the other hand, the results of Subsect. 6.3 are valid under the stronger assumption $(m-1)Z_K - nE \geq 0$. The remaining case will be analysed independently and will produce a different structure theorem.

In all cases $\lambda_m = \pi_{m,m} = \dim H^0(\widetilde{X}, \mathcal{O}_{\widetilde{X}})/H^0(\widetilde{X}, \mathcal{O}_{\widetilde{X}}(-mZ_K + mE))$ for any $m \geq 1$. Using Subsect. 6.2 we have the following facts. Define in the rectangle $R(0, mZ_K - mE)$

$$h_{\lambda_m}(l) := \dim \frac{H^0(\mathcal{O}_{\widetilde{X}}((m-1)Z_K - mE))}{H^0(\mathcal{O}_{\widetilde{X}}((m-1)Z_K - mE - l))} \quad \text{and}$$

$$h'_{\lambda_m}(l) := \dim \frac{H^0(\mathcal{O}_{\widetilde{X}}(-mZ_K + mE + l))}{H^0(\mathcal{O}_{\widetilde{X}}(-mZ_K + mE))},$$

the two increasing functions (with $h^\circ_{\lambda_m}(l) = \dim H^0(\mathcal{O}_{\widetilde{X}})/H^0(\mathcal{O}_{\widetilde{X}}(-mZ_K + mE + l))$). Define also the weight function $w_{\lambda_m,0} : R(0, mZ_K - mE) \cap \mathbb{Z}^s \to \mathbb{Z}$ by

$$w_{\lambda_m,0}(l) = h_{\lambda_m}(l) - h'_{\lambda_m}(l) = h_{\lambda_m}(l) + h^\circ_{\lambda_m}(l) - \lambda_m,$$

and $w_{\lambda_m,q}$ as in (3.1.5). They define the lattice cohomology $\mathbb{H}^*(R(0, mZ_K - mE), w_{\lambda_m})$.

Theorem 8.1.1 *(a)* **(Matroid rank inequality)** *both h_{λ_m} and $-h'_{\lambda_m}$ satisfy the matroid rank in equality;*

(b) **(Lattice cohomology)** *the lattice cohomology associated with the weight function $w_{\lambda_m,0} = h_{\lambda_m} - h'_{\lambda_m}$ in the rectangle $R(0, c)$, whenever $c \geq mZ_K - mE$, stabilizes with respect to c. It will be denoted by $\mathbb{H}^*((X, o), w_{\lambda_m})$;*

(c) **(CDP)** *the pair $(h_{\lambda_m}, h'_{\lambda_m})$ satisfies the CDP;*

(d) **(Euler characteristic)** *the lattice cohomology defined on $R(0, mZ_K - mE)$ with the weight function $w_{\lambda_m,0}(l) = h_{\lambda_m}(l) - h'_{\lambda_m}(l) = h_{\lambda_m}(l) + h^\circ_{\lambda_m}(l) - h^\circ_{\lambda_m}(0)$ has Euler characteristic $h'_{\lambda_m}(mZ_K - mE) - h'_{\lambda_m}(0) = \lambda_m$. Hence, as a graded $\mathbb{Z}[U]$–module, $\mathbb{H}^*((X, o), w_{\lambda_m})$ is a categorification of λ_m.*

In the comparison of $(h_{\lambda_m}, h'_{\lambda_m})$ with $(\mathfrak{h}, \mathfrak{h}')$ we need to divide the discussions in different cases.

8.1.2 Assume that $mZ_K - mE \geq Z_K$. Under this assumption we can apply the results of Subsect. 6.3. Then the statements of Theorem 8.1.1 can be complemented by the following facts:

(e) **(Relations with the original Hilbert functions)** If $l \leq (m-1)Z_K - mE$, then $h_{\lambda_m}(l) = 0$, if $l \geq (m-1)Z_K - mE$, then $h_{\lambda_m}(l) = \mathfrak{h}(l - (m-1)Z_K + mE)$ and $\mathfrak{h}'(l - (m-1)Z_K + mE) = h'_{\lambda_m}(l) - d_{\lambda_m}$, where $d_{\lambda_m} = \lambda_m - p_g$. (From definition, $\lambda_m - p_g$ equals the non-negative integer dim $H^0(\mathcal{O}_{\widetilde{X}}(-Z_K))/H^0(\mathcal{O}_{\widetilde{X}}(-mZ_K + mE))$ as well.)

(f) **(Relation with the analytic lattice cohomology)** in this case there exists a natural graded $\mathbb{Z}[U]$–module isomorphism:

$$\mathbb{H}^*((X, o), w_{\lambda_m}) \cong \mathbb{H}^*_{an,0}(X, o)[-2d_{\lambda_m}].$$

8.2 In the remaining part of this section we assume that $mZ_K - mE \ngeq Z_K$.

In this case (the first version of) the structure theorem — as a comparison of $\mathbb{H}^*((X, o), w_{\lambda_m})$ with the analytic lattice cohomology — is the following.

Theorem 8.2.1 *Set $d_{\lambda_m} := \lambda_m - p_g$ (similarly as in the previous case 8.1.2). Then*

$$\mathbb{H}^*((X, o), w_{\lambda_m}) \cong \mathbb{H}^*(R(E, Z_K), w_{an})[-2d_{\lambda_m} - 2]. \tag{8.2.2}$$

The proof runs differently for $m = 1$ and $m \geq 2$. In both cases we will also provide the specific cohomological descriptions of h_{λ_m} and h'_{λ_m}, and we also prove that $d_{\lambda_m} \geq 0$ as well. Finally, in Theorem 8.5.3 we rewrite the right hand side of (8.2.2) in terms of $\mathbb{H}^*_{an,0}(X, o)$.

8.3. Assume $m = 1$.

Clearly $Z_K - E \ngeq Z_K$, so the discussion 8.1.2 does not hold. However, in this case all the involved terms can be computed explicitly. Indeed, for $l \geq 0$,

$$\begin{aligned} h_{\lambda_1}(l) &= \dim \frac{H^0(\mathcal{O}_{\widetilde{X}}(-E))}{H^0(\mathcal{O}_{\widetilde{X}}(-E-l))} = \dim \frac{H^0(\mathcal{O}_{\widetilde{X}})}{H^0(\mathcal{O}_{\widetilde{X}}(-E-l))} - \dim \frac{H^0(\mathcal{O}_{\widetilde{X}})}{H^0(\mathcal{O}_{\widetilde{X}}(-E))} \\ &= \mathfrak{h}(l+E) - \mathfrak{h}(E). \end{aligned}$$

But $H^0(\mathcal{O}_{\widetilde{X}})/H^0(\mathcal{O}_{\widetilde{X}}(-E)) \simeq \mathcal{O}_{X,o}/\mathfrak{m}_{X,o} \simeq \mathbb{C}$, where $\mathfrak{m}_{X,o}$ is the maximal ideal of $\mathcal{O}_{X,o}$. Hence

$$\mathfrak{h}(E) = \dim H^0(\mathcal{O}_{\widetilde{X}})/H^0(\mathcal{O}_{\widetilde{X}}(-E)) = 1. \tag{8.3.1}$$

These two facts combined give $h_{\lambda_1}(l) = \mathfrak{h}(l+E) - 1$. Furthermore,

$$h'_{\lambda_1}(l) = \dim \frac{H^0(\mathcal{O}_{\widetilde{X}}(-Z_K + E + l))}{H^0(\mathcal{O}_{\widetilde{X}}(-Z_K + E))} = \dim \frac{H^0(\mathcal{O}_{\widetilde{X}}(-Z_K + E + l))}{H^0(\mathcal{O}_{\widetilde{X}}(-Z_K))}$$

$$- \dim \frac{H^0(\mathcal{O}_{\widetilde{X}}(-Z_K + E))}{H^0(\mathcal{O}_{\widetilde{X}}(-Z_K))}.$$

Hence, $h'_{\lambda_1}(l) = \mathfrak{h}'(l + E) - \mathfrak{h}'(E)$. But $\mathfrak{h}'(E) = h^1(\mathcal{O}_E) = 0$, hence $h'_{\lambda_1}(l) = \mathfrak{h}'(l + E)$. Thus

$$w_{\lambda_1,0} : R(0, Z_K - E) \cap \mathbb{Z}^s \to \mathbb{Z} : \ w_{\lambda_1,0}(l) = h_{\lambda_1}(l) - h'_{\lambda_1}(l) = w_{an,0}(l + E) - 1. \quad (8.3.2)$$

Corollary 8.3.3 $\mathbb{H}^*((X, o), w_{\lambda_1}) \cong \mathbb{H}^*(R[E, Z_K], w_{an})[-2]$.

Proof Note that in (8.3.2) l runs in the rectangle $R(0, Z_K - E)$, see also Theorem 8.1.1*(b)*, hence $l + E \in R(E, Z_K)$. Then the affine translation $\tau : R(E, Z_K) \to R(0, Z_K - E)$, $l \mapsto l - E$ satisfies $w_{an,0}(l) - 1 = w_{\lambda_1,0}(\tau(l))$, hence it induces a graded isomorphism on the cochain complexes and, therefore, on the cohomology modules, just as in the case of Corollary 6.3.3. □

This together with the next fact prove (8.2.2) for $m = 1$.

Corollary 8.3.4 *$\lambda_1 = p_g$ in the Gorenstein, rational homology sphere case.*

Proof By Corollary 6.1.2 and identities (2.3.12) and (5.2.4) we have

$$\lambda_1 - p_g = \dim \frac{H^0(\mathcal{O}_{\widetilde{X}}(-Z_K + E))}{H^0(\mathcal{O}_{\widetilde{X}}(-Z_K))} = \mathfrak{h}'(E) = h^1(\mathcal{O}_E) = 0.$$

□

8.4. Assume that $m \geq 2$ and $mZ_K - mE \not\geq Z_K$.

Let $\widetilde{E}$ be the sum of those irreducible exceptional divisors E_v, where the multiplicity of Z_K is 1. From assumption $\widetilde{E} > 0$. Let $\widetilde{Z_K} := Z_K - \widetilde{E}$. Clearly, for any $m \geq 2$ we have $m(Z_K - E) \geq \widetilde{Z_K}$.

Notation 8.4.1 Note that if $l \geq 0$ and $l \geq (m-1)Z_K - mE$, then automatically $l \geq (m-1)Z_K - mE + \widetilde{E}$. Let us denote this latter effective cycle with $Z := (m-1)Z_K - mE + \widetilde{E}$.

Lemma 8.4.2 *The natural inclusion $H^0(\mathcal{O}_{\widetilde{X}}(-Z_K)) \to H^0(\mathcal{O}_{\widetilde{X}}(-\widetilde{Z_K}))$ is an isomorphism.*

Proof This follows from the vanishing

$$h^0(\mathcal{O}_{\widetilde{E}}(-\widetilde{Z_K})) = h^1\big(\mathcal{O}_{\widetilde{E}}(\widetilde{Z_K} + \widetilde{E} - Z_K)\big) = h^1(\mathcal{O}_{\widetilde{E}}) = 0.$$

□

Corollary 8.4.3 *In the case of a non-ADE Gorenstein singularity with rational homology sphere link $p_g = \dim H^0(\mathcal{O}_{\widetilde{X}})/H^0(\mathcal{O}_{\widetilde{X}}(-\widetilde{Z_K}))$*

Proof Combine (2.3.12) and Lemma 8.4.2. □

Proposition 8.4.4 *If $l \leq Z$, then $h_{\lambda_m}(l) = 0$. If $l \geq Z$, then $h_{\lambda_m}(l) = \mathfrak{h}(l - ((m-1)Z_K - mE)) - 1$ and $h'_{\gamma_m}(l) = \mathfrak{h}'(l - ((m-1)Z_K - mE)) + d_{\lambda_m}$, where d_{λ_m} is the constant $\lambda_m - p_g$. (For a homological expression of d_{λ_m} valid in this case see (8.4.6) too.)*

Proof As h_{λ_m} is increasing, for the first part it is enough to prove that $h_{\lambda_m}(Z) = 0$. Let us consider the following exact sequence of sheaves:

$$0 \to \mathcal{O}_{\widetilde{X}}(-\widetilde{E}) \to \mathcal{O}_{\widetilde{X}}(Z - \widetilde{E}) \to \mathcal{O}_Z(Z - \widetilde{E}) \to 0.$$

The respective cohomological exact sequence is the following:

$$0 \to H^0\big(\mathcal{O}_{\widetilde{X}}(-\widetilde{E})\big) \to H^0\big(\mathcal{O}_{\widetilde{X}}((m-1)Z_K - mE)\big) \to H^0\big(\mathcal{O}_Z(Z - \widetilde{E})\big) \to \cdots$$

According to Theorem 2.3.4 the last term $H^0(\mathcal{O}_Z(Z - \widetilde{E}))$ vanishes, so

$$H^0(\mathcal{O}_{\widetilde{X}}(-\widetilde{E})) \cong H^0(\mathcal{O}_{\widetilde{X}}((m-1)Z_K - mE)), \tag{8.4.5}$$

hence $h_{\lambda_m}(Z) = 0$ by definition.
Now, for $l \geq Z$

$$\begin{aligned}
h_{\lambda_m}(l) &= \dim \frac{H^0(\mathcal{O}_{\widetilde{X}}((m-1)Z_K - mE))}{H^0(\mathcal{O}_{\widetilde{X}}((m-1)Z_K - mE - l))} \\
&= \dim \frac{H^0(\mathcal{O}_{\widetilde{X}}((m-1)Z_K - mE))}{H^0(\mathcal{O}_{\widetilde{X}}(-\widetilde{E}))} + \dim \frac{H^0(\mathcal{O}_{\widetilde{X}}(-\widetilde{E}))}{H^0(\mathcal{O}_{\widetilde{X}}((m-1)Z_K - mE - l))} \\
&= \dim \frac{H^0(\mathcal{O}_{\widetilde{X}})}{H^0(\mathcal{O}_{\widetilde{X}}(-(l - (m-1)Z_K + mE)))} - \dim \frac{H^0(\mathcal{O}_{\widetilde{X}})}{H^0(\mathcal{O}_{\widetilde{X}}(-\widetilde{E}))} \\
&= \mathfrak{h}(l - (m-1)Z_K + mE) - 1
\end{aligned}$$

by (8.3.1) and (8.4.5).

Next, for the statement regarding h'_{λ_1} take $l \geq Z$. Then

$$\begin{aligned}
\mathfrak{h}'(l - (m-1)Z_K + mE) &= \dim\ H^0(\mathcal{O}_{\widetilde{X}}(-mZ_K + mE + l))/H^0(\mathcal{O}_{\widetilde{X}}(-\widetilde{Z_K})) \\
&= \dim \frac{H^0(\mathcal{O}_{\widetilde{X}}(-mZ_K + mE + l))}{H^0(\mathcal{O}_{\widetilde{X}}(-mZ_K + mE))} \\
&\quad - \dim \frac{H^0(\mathcal{O}_{\widetilde{X}}(-\widetilde{Z_K}))}{H^0(\mathcal{O}_{\widetilde{X}}(-mZ_K + mE))} \\
&= h'_{\lambda_m}(l) - d_{\lambda_m},
\end{aligned}$$

where d_{λ_m} is the non-negative integer

$$d_{\lambda_m} = \dim\ H^0(\mathcal{O}_{\widetilde{X}}(-\widetilde{Z_K})/H^0(\mathcal{O}_{\widetilde{X}}(-mZ_K + mE)). \tag{8.4.6}$$

From Corollary 8.4.3 we indeed get that $d_{\lambda_m} = \lambda_m - p_g$. □

Corollary 8.4.7

$$\begin{aligned}
\mathbb{H}^*(R(0, mZ_K - mE), w_{\lambda_m}) &\cong \mathbb{H}^*(R(Z, mZ_K - mE), w_{\lambda_m}) \\
&\cong \mathbb{H}^*(R(\widetilde{E}, Z_K), w_{an})[-2d_{\lambda_m} - 2].
\end{aligned}$$

Proof For the first isomorphism we use the fact that $w_{\lambda_m,0}$ satisfies the matroid rank inequality (cf. Theorem 8.1.1(a)) and it is decreasing on the rectangle $R(0, Z)$ (cf. Proposition 8.4.4). Then the same argument as in the proof of Corollary 6.2.7 (see also Lemma 11.9.2 in [20]) provides the isomorphism

$$\mathbb{H}^*(R(0, mZ_K - mE), w_{\lambda_m}) \cong \mathbb{H}^*(R(Z, mZ_K - mE), w_{\lambda_m}).$$

For the second isomorphism notice, that through Proposition 8.4.4, on the rectangle $R(Z, mZ_K - mE)$ the weight function is

$$\begin{aligned} w_{\lambda_m,0}(l) &= h_{\lambda_m}(l) - h'_{\lambda_m}(l) \\ &= \mathfrak{h}(l-(m-1)Z_K+mE) - \mathfrak{h}'(l-(m-1)Z_K+mE) - d_{\lambda_m} - 1 \\ &= w_{an,0}(l-(m-1)Z_K+mE) - d_{\lambda_m} - 1. \end{aligned} \tag{8.4.8}$$

Note also that if $l \in R(Z, mZ_K - mE) \cap \mathbb{Z}^s$, then $l-(m-1)Z_K+mE \in R(\widetilde{E}, Z_K) \cap \mathbb{Z}^s$, so the affine translation $\tau : l \mapsto l-(m-1)Z_K+mE$ maps $R(Z, mZ_K - mE)$ to $R(\widetilde{E}, Z_K)$ and satisfies the identity:

$$w_{\lambda_m,0}(l) = w_{an,0}(\tau(l)) - d_{\lambda_m} - 1. \tag{8.4.9}$$

Similarly to the proof of Corollary 6.3.3, τ induces a graded cochain isomorphism and thus a graded isomorphism on the cohomology modules

$$\tau^* : \mathbb{H}^*(R(\widetilde{E}, Z_K), w_{an})[-2d_{\lambda_m} - 2] \xrightarrow{\cong} \mathbb{H}^*(R(Z, mZ_K - mE), w_{\lambda_m}).$$

□

Remark 8.4.10 Note that the original Hilbert function $\mathfrak{h}$ on the rectangle $R(\widetilde{E}, E) \cap \mathbb{Z}^s$ is constant 1. This means that $w_{an,0}$ is decreasing in this rectangle (as it satisfies the matroid rank inequality, cf. 5.1.3). Thus, in fact

$$\mathbb{H}^*(R(\widetilde{E}, Z_K), w_{an})[-2d_{\lambda_m} - 2] \cong \mathbb{H}^*(R(E, Z_K), w_{an})[-2d_{\lambda_m} - 2]. \tag{8.4.10}$$

Indeed, use the same proof as for Corollary 6.2.7. Thus, by Corollary 8.4.7 and the above (8.4.10) we obtain the proof of (8.2.2) in this case too.

8.5. Comparison of $\mathbb{H}^*((X, o), w_{\lambda_m})$ and $\mathbb{H}^*_{an,0}(X, o)$ in the $mZ_K - mE \not\geq Z_K$ case

In the $mZ_K - mE \geq Z_K$ case we already saw in 8.1.2 that the $\mathbb{Z}[U]$–modules in the title are isomorphic through a homogeneous graded $\mathbb{Z}[U]$–module isomorphism which shifts the grading by $2d_{\lambda_m}$. In this subsection will prove a similar result for the remaining case. We start with the identity (8.2.2) and we will use the notations of the previous subsections.

8.5.1 An equivalent description of the lattice cohomology [18][20, Theorem 11.1.12].

At this point it is convenient to consider an another description of the lattice cohomology. Let us fix a rectangle R, a cube decomposition (cf. 3.1.2) and a weight function w. For each $n \in \mathbb{Z}$ we define $S_n = S_n(w) \subset R$ as the union of all the cubes $\square_q$ (of any dimension q) with $w(\square_q) \leq n$. Clearly, $S_n = \emptyset$, whenever $n < m_w = \min_{l \in R \cap \mathbb{Z}^s} w_0(l)$. Then, for any $q \geq 0$, we have the following graded $\mathbb{Z}[U]$–module isomorphism of degree zero:

$$\mathbb{H}^q(R, w) \cong \oplus_{n \geq m_w} H^q(S_n, \mathbb{Z}).$$

The $\mathbb{Z}$–grading on the right hand side is the following: the $d = 2n$–homogeneous elements consist of $H^q(S_n, \mathbb{Z})$; while the U–action is given by the restriction map $r_{n+1} : H^q(S_{n+1}, \mathbb{Z}) \to H^q(S_n, \mathbb{Z})$. Moreover, for $q = 0$, a fixed base-point $l_w \in S_{m_w}$ provides an augmentation (splitting) $H^0(S_n, \mathbb{Z}) = \mathbb{Z} \oplus \widetilde{H}^0(S_n, \mathbb{Z})$, which agrees with the augmentation of the graded $\mathbb{Z}[U]$–modules

$$\mathbb{H}^0(R, w) \cong \mathcal{T}^+_{2m_w} \oplus \mathbb{H}^0_{red}(R, w) \text{ and } \mathbb{H}^q(R, w) \cong \mathbb{H}^q_{red}(R, w) \text{ for } q \geq 1.$$

Remark 8.5.2 In the analytic lattice cohomology $w_{an,0}(0) = 0$ and $w_{an,0}(E_v) = 1$ for all E_v-s. Thus the point $\{0\}$ is an isolated point in S_0, which constitutes a distinct connected component of S_0. It 'becomes empty' in S_{-1}, i.e. $\{0\}$ generates a free $\mathbb{Z}$–module $(\{0\}) = \mathbb{Z}\langle\{0\}\rangle$ in $\mathbb{H}^0_{an,0}(X, o)$, and a torsion $\mathbb{Z}[U]$–submodule of $\mathbb{H}^*_{an,0}(X, o)$ with trivial U–action. Hence the quotient $\mathbb{H}^*_{an,0}(X, o)/(\{0\})$ as a double graded $\mathbb{Z}[U]$–module is well-defined.

Theorem 8.5.3 *Suppose we have a non-ADE Gorenstein normal surface singularity with rational homology sphere link. Let $m \in \mathbb{Z}$ be such that $mZ_K - mE \not\geq Z_K$. Then we have the following graded $\mathbb{Z}[U]$–module isomorphism:*

$$\mathbb{H}^*((X, o), w_{\lambda_m}) \cong \mathbb{H}^*_{an,0}(X, o)/(\{0\})[-2d_{\lambda_m} - 2].$$

Proof Since $\mathbb{H}^*_{an,0}(X, o) \cong \mathbb{H}^*(R(0, Z_K), w_{an})$, by (8.2.2) it is enough to prove that

$$\mathbb{H}^*(R(E, Z_K), w_{an}) \cong \mathbb{H}^*(R(0, Z_K), w_{an})/(\{0\}).$$

We proceed in two steps. First we consider the closed cubical subcomplex $\mathcal{X} := \cup_v R(E_v, Z_K)$ of $R := R(0, Z_K)$. This can also be obtained from R if we delete all the relative interiors of the those cubes which contain 0 as a vertex: $\mathcal{X} = R \setminus C$, where $C := \{\sum_v t_v E_v \,|\, 0 \leq t_v < 1 \text{ for all } v\}$. Then we claim that the inclusion $R(E, Z_K) \subset \mathcal{X}$ induces a double graded $\mathbb{Z}[U]$–module isomorphism:

$$\mathbb{H}^*(R(E, Z_K), w_{an}) \cong \mathbb{H}^*(\mathcal{X}, w_{an}). \tag{8.5.4}$$

Indeed, let us fix a vertex v and consider any cycle $l > 0$ with $E_v \not\subset |l|$. Then $\mathfrak{h}(l) = \mathfrak{h}(l+E_v)$ (because if the pullback by ϕ of a function vanishes along some E_w, $w \neq v$, then it vanishes along E_v as well). On the other hand, $h^1(\mathcal{O}_{l+E_v}) \geq h^1(\mathcal{O}_l)$. Hence $w_{an}(l+E_v) \leq w_{an}(l)$. In particular, for any $S_n \subset \mathcal{X}$ the inclusion $S_n \cap R(E_v, Z_K) \subset S_n$ admits a strong deformation retract (by a similar argument as in the proof of Theorem 5.1.4*(a)*, see also Lemma 11.9.2 in [20]). Using this, inductively we proceed as follows. Let us order the vertices $v_1, v_2, \dots, v_s$. Then, all the inclusions of the next pairs induce isomorphisms at the lattice cohomology level: $R(E_{v_1}, Z_K) \subset \mathcal{X}$, $R(E_{v_1} + E_{v_2}, Z_K) \subset R(E_{v_1}, Z_K)$, …, $R(E, Z_K) \subset R(E - E_{v_s}, Z_K)$. Hence (8.5.4) follows.

Finally, we verify the isomorphism

$$\mathbb{H}^*(\mathcal{X}, w_{an}) \cong \mathbb{H}^*(R(0, Z_K), w_{an})/(\{0\}). \tag{8.5.5}$$

First, notice that for any cycle $0 < l \leq E$ the ideal $\{f \in \mathcal{O}_{X,o} : \text{div}(\phi^* f) \geq l\}$ is the maximal ideal $\mathfrak{m}_{X,o}$ of $\mathcal{O}_{X,o}$, hence $\mathfrak{h}(l) \equiv 1$, while $\mathfrak{h}(0) = 0$. Furthermore, $h^1(\mathcal{O}_l) = 0$ for any $0 \leq l \leq E$, hence $w_{an,0}(0) = 0$ and $w_{an,0}(l) = 1$ for any $0 < l \leq E$. Let $\overline{C}$ be the closure of C and $b\overline{C} : \overline{C} \setminus C$. Notice that both C and $b\overline{C}$ are contractible.

Next, consider a level set $S_n = S_n(w_{an,0})$ in $R(0, Z_K)$. Then for $n < 0$ we have $S_n \cap \mathcal{X} = S_n$. For $n = 0$, $S_0 \cap \mathcal{X}$ is obtained from S_0 by eliminating the component $\{0\}$, which consists of a single point. If $n > 0$ then $S_n \cap \mathcal{X}$ contains $b\overline{C}$ and S_n is obtained from $S_n \cap \mathcal{X}$ by gluing $\overline{C}$ to $S_n \cap \mathcal{X}$ along $b\overline{C}$. Hence $S_n \cap \mathcal{X}$ and S_n have the same homotopy type. Thus (8.5.5) follows too. □

9 Examples

9.1 In this section we provide some examples and concrete computations of the plurigenera. For more information and concrete examples see e.g. [20, Sect. 6.8.D], [23] or [29].

Example 9.1.1 [23, Corollary 4.19] or [20, Example 6.8.64]. Under the assumptions of Remark. 7.2.2

$$\gamma_m - p_g = d_{\gamma_m} = -\frac{m(m-1)}{2} \cdot Z_K^2. \tag{9.1.2}$$

This follows basically from the formula from Theorem 7.2.1(e) and the exact sequence

$$0 \to \mathcal{O}_{\widetilde{X}}(-mZ_K) \to \mathcal{O}_{\widetilde{X}}(-Z_K) \to \mathcal{O}_{(m-1)Z_K}(-Z_K) \to 0$$

and the vanishing $h^1(\mathcal{O}_{\widetilde{X}}(-mZ_K)) = 0$ for any $m \geq 1$.

The other terms d_{δ_m} and d_{λ_m} are more arithmetical.

Example 9.1.3 Assume that (X, o) is weighted homogeneous and Gorenstein. Assume that the link is $\mathbb{Q}HS^3$, a star shaped graph with central vertex E_0 with Euler number $-b$ and Seifert invariants $\{(\alpha_i, \omega_i)\}_{i=1}^{\nu}$ (for notations and details see e.g. [20]). Define

$$\mathfrak{r} := \Big(2 - \sum_i \frac{\alpha_i - 1}{\alpha_i}\Big)/\Big(-b + \sum_i \frac{\omega_i}{\alpha_i}\Big).$$

Let $\mathfrak{R}$ be its graded ring $\mathfrak{R} = \oplus_{d\geq 0}\mathfrak{R}_d$. Then $\dim \mathfrak{R}_d = \max\{0, 1 + N(d)\}$, $N(d) = db - \sum_i \lceil d\omega_i/\alpha_i \rceil$. Moreover, see e.g. [20, Example 6.8.62] or [29],

$$\delta_m = \sum_{d \leq m\mathfrak{r}} \dim \mathfrak{R}_d, \;\; \lambda_m = \sum_{d < m\mathfrak{r}} \dim \mathfrak{R}_d,$$

Let us consider the following concrete example. Assume that $b = 3$, $\nu = 5$ and each (α_i, ω_i) is $(2, 1)$. In this case $Z_K = E + E_0$, hence $m(Z_K - E) \not\geq Z_K$. Moreover, $Z_K^2 = -2$. The singularity is minimally elliptic with $p_g = 1$. A computation gives $\gamma_m - p_g = d_{\gamma_m} = m(m-1)$ (cf. (9.1.2)), and

$$\delta_m = \begin{cases} \frac{m^2+3}{4} & \text{if } m \text{ is odd,} \\ \frac{m^2+8}{4} & \text{if } m \text{ is even;} \end{cases} \qquad \lambda_m = \begin{cases} \frac{m^2-2m+9}{4} & \text{if } m \text{ is odd,} \\ \frac{m^2-2m+4}{4} & \text{if } m \text{ is even.} \end{cases}$$

In this case $\mathbb{H}^{\geq 1}_{an,0} = 0$, and $\mathbb{H}^0_{an,0} = \mathcal{T}_0^+ \oplus (\{0\})$, where $(\{0\})$ is a free $\mathbb{Z}$–module of rank 1 and of degree 0, with trivial U–action (as in Remark 8.5.2), cf. [20, Example 11.1.30]. Hence in this case $\mathbb{H}^*_{an,0}(X, o)/(\{0\}) = \mathcal{T}_0^+$ and $\mathbb{H}^*((X, o), w_{\lambda_m}) = \mathcal{T}_0^+[-2\lambda_m]$.

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