Miskolc Mathematical Notes
Vol. 25 (2024), No. 2, pp. 933–938
HU e-ISSN 1787-2413
DOI: 10.18514/MMN.2024.4273

SOME PROPERTIES OF r-SUPPLEMENTED MODULES

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Received 03 June, 2022

Abstract. In this work, r-supplemented modules are defined and some properties of these modules are investigated. It is proved that every factor module and every homomorphic image of an r-supplemented module are r-supplemented. Let M be an $R-$module and $M = M_1 + M_2 + ... + M_n$. If M_i is r-supplemented for each $i = 1,2,...,n$, then M is also r-supplemented. Let M be an r-supplemented module. Then every finitely $M-$generated $R-$module is r-supplemented.

2010 *Mathematics Subject Classification:* 16D10; 16D70

Keywords: small submodules, radical, r-small submodules, supplemented modules

1. Introduction

Throughout this paper all rings will be associative with identity and all modules will be unital left modules.

Let R be a ring and M be an $R-$module. We will denote a submodule N of M by $N \leq M$. Let M be an $R-$module and $N \leq M$. If $L = M$ for every submodule L of M such that $M = N + L$, then N is called a *small* submodule of M and denoted by $N \ll M$. Let M be an R -module. M is called a *hollow* module if every proper submodule of M is small in M. M is said to be *local* if M has the largest submodule, i.e. a proper submodule which contains all other proper submodules. Let M be an $R-$module and $U,V \leq M$. If $M = U + V$ and V is minimal with respect to this property, or equivalently, $M = U + V$ and $U \cap V \ll V$, then V is called a *supplement* of U in M. M is called a *supplemented* module if every submodule of M has a supplement in M. Let M be an $R-$module and $U \leq M$. If for every $V \leq M$ such that $M = U + V$, U has a supplement V' with $V' \leq V$, we say U has *ample supplements* in M. If every submodule of M has ample supplements in M, then M is called an *amply supplemented* module. The intersection of maximal submodules of an R-module M is called the *radical* of M and denoted by $RadM$. If M have no maximal submodules, then we denote $RadM = M$. Let M be an R-module and $U,V \leq M$. If $M = U + V$

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and $U\cap V\leq RadV$, then V is called a *generalized* (*radical*) *supplement* (briefly, *Rad-supplement*) of U in M. M is said to be *generalized* (*radical*) *supplemented* (briefly, *Rad-supplemented*) if every submodule of M has a Rad-supplement in M. Let M be an $R-$module and $K\leq M$. If $K\ll RadM$, then K is called an *r-small* submodule of M and denoted by $K\ll_r M$. Let M be an $R-$module. It is defined the relation $'\beta^*{}'$ on the set of submodules of an $R-$module M by $X\beta^*Y$ if and only if $Y+K=M$ for every $K\leq M$ such that $X+K=M$ and $X+T=M$ for every $T\leq M$ such that $Y+T=M$. Let M be an $R-$module and $K\leq V\leq M$. We say V *lies above* K in M if $V/K\ll M/K$.

More informations about (amply) supplemented modules are in [2, 3, 8, 9]. More results about Rad-supplemented modules are in [7]. The definition of r-small submodules and some properties of them are in [4, 5]. The definition of β^* relation and some properties of this relation are in [1].

Lemma 1. *Let M be an R -module. The following assertions hold.*

(i) *If $K\ll_r M$, then $K\ll M$.*
(ii) *If $L\ll_r M$ and $K\leq L$, then $K\ll_r M$.*
(iii) *If $K\ll_r L\leq M$, then $K\ll_r M$.*
(iv) *If $K_i\ll_r L_i\leq M$ for $i=1,2,\ldots,n$, then $K_1+K_2+\ldots+K_n\ll_r L_1+L_2+\ldots+L_n$.*
(v) *If $K_i\ll_r M$ for $i=1,2,\ldots,n$, then $K_1+K_2+\ldots+K_n\ll_r M$.*
(vi) *If $K\ll_r M$, then $(K+L)/L\ll_r M/L$ for every $L\leq M$.*
(vii) *If $K\ll M$ and $RadM$ is a supplement submodule in M, then $K\ll_r M$.*
(viii) *Let N be an $R-$module and $f:M\longrightarrow N$ be an $R-$module homomorphism. If $K\ll_r M$, then $f(K)\ll_r f(M)$.*
(ix) *$Rad(RadM)=\sum\limits_{K\ll_r M}K$.*

Proof. See [4, 5]. □

2. r-Supplemented Modules

Definition 1. Let M be an $R-$module and $U,V\leq M$. If $M=U+V$ and $U\cap V\ll_r V$, then V is called an r-supplement of U in M. If every submodule of M has an r-supplement in M, then M is called an r-supplemented module. (See also [6])

Clearly we can see that every r-supplemented module is supplemented. But the converse is not true in general (See Example 1).

Lemma 2. *Let V be a Rad-supplement of U in M. Then V is an r-supplement of U in M if and only if $RadV$ is a supplement of U in $U+RadV$.*

Proof.

($\Longrightarrow$) Let V be an r-supplement of U in M. Then $U\cap V\ll RadV$ and $U\cap RadV=U\cap V\cap RadV\ll RadV$. Hence $RadV$ is a supplement of U in $U+RadV$.

($\Longleftarrow$) Let $RadV$ be a supplement of U in $U+RadV$. Since V is a Rad-supplement of U in M, $M=U+V$ and $U\cap V\leq RadV$. Then $U\cap V=U\cap V\cap RadV=U\cap RadV$. Since $RadV$ is a supplement of U in $U+RadV$, $U\cap RadV\ll RadV$. Hence $U\cap V=U\cap RadV\ll RadV$ and $U\cap V\ll_r V$. Therefore, V is an r-supplement of U in M.

□

Corollary 1. *Let V be an r-supplement of U in M. Then $Rad\,(RadV)=RadV\cap Rad\,(U+RadV)$.*

Proof. Since V is an r-supplement of U in M, by Lemma 2, $RadV$ is a supplement of U in $U+RadV$. Then by [8, 41.1 (5)], $Rad\,(RadV)=RadV\cap Rad\,(U+RadV)$, as desired. □

Lemma 3. *Let V be an r-supplement of U in M and $U\beta^*X$ in $U+RadV$ with $X\leq U+RadV$. Then V is an r-supplement of X in M.*

Proof. Since V is an r-supplement of U in M, by Lemma 2, $RadV$ is a supplement of U in $U+RadV$. Since $U\beta^*X$ in $U+RadV$, by [1, Theorem 2.6 (ii)], $RadV$ is a supplement of X in $U+RadV$. Here $X+RadV=U+RadV$ and $X\cap RadV\ll RadV$. Hence $RadV$ is a supplement of X in $X+RadV$. Let $N=U+RadV$ and $U+K=M$ with $K\leq M$. Since $U+K=M$, $N=N\cap M=N\cap(U+K)=U+N\cap K$ and since $U\beta^*X$ in N, $X+N\cap K=N$. Here $X+K=X+N\cap K+K=N+K=U+N\cap K+K=U+K=M$. Interchanging the roles of U and X, we can see that $U+T=M$ for every $T\leq M$ with $X+T=M$. Hence $U\beta^*X$ in M. Since V is an r-supplement of U in M, V is a supplement of U in M and since $U\beta^*X$ in M, by [1, Theorem 2.6 (ii)], V is a supplement of X in M. Then V is a Rad-supplement of X in M and since $RadV$ is a supplement of X in $X+RadV$, by Lemma 2, V is an r-supplement of X in M. □

Corollary 2. *Let V be an r-supplement of U in M and U lies above X in $U+RadV$. Then V is an r-supplement of X in M.*

Proof. Clear from Lemma 3. □

Lemma 4. *Let V be a supplement of U in M. If $RadV$ is a supplement submodule in V, then V is an r-supplement of U in M.*

Proof. Since V is a supplement of U in M, $M=U+V$ and $U\cap V\ll V$. Since $RadV$ is a supplement submodule in V, by Lemma 1, $U\cap V\ll_r V$. Hence V is an r-supplement of U in M, as desired. □

Corollary 3. *Let V be a supplement of U in M. If $RadV$ is a is direct summand of V, then V is an r-supplement of U in M.*

Proof. Since $RadV$ is a direct summand of V, $RadV$ is a supplement submodule in V. Then by Lemma 4, V is an r-supplement of U in M. □

Corollary 4. *Let V be a supplement of U in M. If $RadV$ is a supplement submodule in M, then V is an r-supplement of U in M.*

Proof. Let $RadV$ be a supplement of X in M. Then $M = X + RadV$ and $X \cap RadV \ll RadV$. Since $M = X + RadV$, by Modular law, $V = V \cap M = V \cap (X + RadV) = V \cap X + RadV$ and since $V \cap X \cap RadV = X \cap RadV \ll RadV$, $RadV$ is a supplement of $V \cap X$ in V. Then by Lemma 4, V is an r-supplement of U in M. □

We can also prove this Corollary 4 as follows:

Proof. Since V is a supplement of U in M, $M = U + V$ and $U \cap V \ll V$. Then $U \cap V \leq RadV$ and $U \cap V \ll M$. Since $RadV$ is a supplement submodule in M, by [8, 41.1 (5)], $U \cap V = U \cap V \cap RadV \ll RadV$. Hence $U \cap V \ll_r V$ and V is an r-supplement of U in M. □

Corollary 5. *Let V be a supplement of U in M. If $RadV$ is a is direct summand of M, then V is an r-supplement of U in M.*

Proof. Clear from Corollary 4. □

Lemma 5. *Let M be a supplemented module. If $RadV$ is a supplement submodule in V for every supplement submodule V in M, then M is r-supplemented.*

Proof. Let $U \leq M$. Since M is supplemented, U has a supplement V in M. By hypothesis, $RadV$ is a supplement submodule in V. Then by Lemma 4, V is an r-supplement of U in M. Hence M is r-supplemented, as desired. □

Corollary 6. *Let M be a supplemented module. If $RadV$ is a direct summand of V for every supplement submodule V in M, then M is r-supplemented.*

Proof. Clear from Lemma 5. □

Proposition 1. *Let M be an r-supplemented module. Then $RadM$ is a supplement submodule in M.*

Proof. Since M is r-supplemented, $RadM$ has an r-supplement V in M. Here $M = RadM + V$ and $V \cap RadM \ll_r V$. Since $V \cap RadM \ll_r V$, $V \cap RadM \ll RadV \leq RadM$. Hence $RadM$ is a supplement of V in M. □

Remark 1. The converse of Proposition 1 is not true in general. Consider the $\mathbb{Z}-$module ${}_{\mathbb{Z}}\mathbb{Q}$. Since $Rad_{\mathbb{Z}}\mathbb{Q} = {}_{\mathbb{Z}}\mathbb{Q}$, $Rad_{\mathbb{Z}}\mathbb{Q}$ is a supplement submodule in ${}_{\mathbb{Z}}\mathbb{Q}$. But ${}_{\mathbb{Z}}\mathbb{Q}$ is not r-supplemented.

Proposition 2. *Let M be an r-supplemented module. Then $M/Rad(RadM)$ is semisimple.*

Proof. Let $\frac{K}{Rad(RadM)}$ be any submodule of $\frac{M}{Rad(RadM)}$. Since M is r-supplemented, K has an r-supplement V in M. Then $M = K + V$ and $K \cap V \ll_r V$. Since $M = K + V$, $\frac{M}{Rad(RadM)} = \frac{K}{Rad(RadM)} + \frac{V + Rad(RadM)}{Rad(RadM)}$. Since $K \cap V \ll_r V$, by Lemma 1, $K \cap V \leq$

$Rad\,(RadM)$. Then $\frac{K}{Rad(RadM)} \cap \frac{V+Rad(RadM)}{Rad(RadM)} = \frac{K\cap V+Rad(RadM)}{Rad(RadM)} = 0$ and $\frac{M}{Rad(RadM)} = \frac{K}{Rad(RadM)} \oplus \frac{V+Rad(RadM)}{Rad(RadM)}$. Hence every submodule of $\frac{M}{Rad(RadM)}$ is a direct summand of $\frac{M}{Rad(RadM)}$ and $\frac{M}{Rad(RadM)}$ is semisimple. □

Lemma 6. *Let M be an $R-$module, $U \leq M$ and $M_1 \leq M$. If X is an r-supplement of $U+M_1$ in M and Y is an r-supplement of $(U+X)\cap M_1$ in M_1, then $X+Y$ is an r-supplement of U in M.*

Proof. Since X is an r-supplement of $U+M_1$ in M, $M = U+M_1+X$ and $X\cap(U+M_1) \ll_r X$. Since Y is an r-supplement of $(U+X)\cap M_1$ in M_1, $M_1 = (U+X)\cap M_1+Y$ and $(U+X)\cap Y = (U+X)\cap M_1\cap Y \ll_r Y$. Then $M = U+M_1+X = U+X+(U+X)\cap M_1+Y = U+X+Y$ and $U\cap(X+Y) \leq (U+X)\cap Y+(U+Y)\cap X \leq (U+M_1)\cap X+(U+X)\cap Y \ll_r X+Y$. Hence $X+Y$ is a r-supplement of U in M. □

Lemma 7. *Let M be an $R-$module, $U \leq M$ and $M_1 \leq M$. If M_1 is r-supplemented and $U+M_1$ has an r-supplement in M, then U has an r-supplement in M.*

Proof. Clear from Lemma 6. □

Corollary 7. *Let M be an $R-$module, $U \leq M$ and $M_i \leq M$ for $i = 1,2,...,n$. If M_i is r-supplemented for every $i = 1,2,...,n$ and $U+M_1+M_2+...+M_n$ has an r-supplement in M, then U has an r-supplement in M.*

Proof. Clear from Lemma 7. □

Lemma 8. *Let $M = M_1+M_2$. If M_1 and M_2 are r-supplemented, then M is also r-supplemented.*

Proof. Let $U \leq M$. Then 0 is an r-supplement of $U+M_1+M_2$ in M. Since M_2 is r-supplemented, by Lemma 7, $U+M_1$ has an r-supplement in M. Since M_1 is r-supplemented, by Lemma 7 again, U has an r-supplement in M. Hence M is r-supplemented. □

Corollary 8. *Let $M = M_1+M_2+...+M_n$. If M_i is r-supplemented for each $i = 1,2,...,n$, then M is also r-supplemented.*

Proof. Clear from Lemma 8. □

Lemma 9. *Let V be an r-supplement of U in M and $K \leq U$. Then $\frac{V+K}{K}$ is an r-supplement of $\frac{U}{K}$ in $\frac{M}{K}$.*

Proof. Since V is an r-supplement of U in M, $M = U+V$ and $U\cap V \ll_r V$. Since $M = U+V$ and $K \leq U$, $\frac{M}{K} = \frac{U+V}{K} = \frac{U}{K}+\frac{V+K}{K}$. Since $U\cap V \ll_r V$, by Lemma 1, $\frac{U}{K}\cap\frac{V+K}{K} = \frac{U\cap V+K}{K} \ll_r \frac{V+K}{K}$. Hence $\frac{V+K}{K}$ is an r-supplement of $\frac{U}{K}$ in $\frac{M}{K}$. □

Lemma 10. *Every factor module of an r-supplemented module is r-supplemented.*

Proof. Let M be an r-supplemented $R-$module and $\frac{M}{K}$ be any factor module of M. Let $\frac{U}{K} \leq \frac{M}{K}$. Since M is r-supplemented, U has an r-supplement V in M. Since $K \leq U$, by Lemma 9, $\frac{V+K}{K}$ is an r-supplement of $\frac{U}{K}$ in $\frac{M}{K}$. Hence $\frac{M}{K}$ is r-supplemented. □

Corollary 9. *Every homomorphic image of an r-supplemented module is r-supplemented.*

Proof. Clear from Lemma 10. □

Corollary 10. *Every direct summand of an r-supplemented module is r-supplemented.*

Proof. Clear from Lemma 10. □

Lemma 11. *Let M be an r-supplemented module. Then every finitely $M-$ generated $R-$module is r-supplemented.*

Proof. Let N be a finitely $M-$generated $R-$module. Then there exist a finite index set Λ and an $R-$module epimorphism $f : M^{(\Lambda)} \longrightarrow N$. Since M is r-supplemented, by Corollary 8, $M^{(\Lambda)}$ is r-supplemented. Then by Corollary 9, N is r-supplemented. □

Proposition 3. *Let R be a ring. Then ${}_RR$ is r-supplemented if and only if every finitely generated $R-$module is r-supplemented.*

Proof. Clear from Lemma 11. □

Lemma 12. *Let M be a local module with $RadM \neq 0$. Then M is not r-supplemented. But M is supplemented.*

Proof. Since M is local, M is the only supplement of $RadM$ in M. Since $M \cap RadM = RadM$ is not small in $RadM$, M is not an r-supplement of $RadM$ in M. Hence M is not r-supplemented. But since M is local, M is supplemented. □

Lemma 13. *Let M be a hollow module with $RadM = M$. Then M is r-supplemented.*

Proof. Clear from definitions. □

Example 1. Consider the $\mathbb{Z}-$module $\mathbb{Z}_8$. Since $\mathbb{Z}_8$ is local and $Rad\mathbb{Z}_8 = \{\overline{0},\overline{2},\overline{4},\overline{6}\} \neq 0$, by Lemma 12, $\mathbb{Z}_8$ is supplemented, but not r-supplemented.

Example 2. Consider the $\mathbb{Z}-$module $\mathbb{Z}_{p^\infty}$. Since $\mathbb{Z}_{p^\infty}$ is hollow and $Rad\mathbb{Z}_{p^\infty} = \mathbb{Z}_{p^\infty}$, by Lemma 13, $\mathbb{Z}_{p^\infty}$ is r-supplemented.

Example 3. Consider the $\mathbb{Z}-$module $M = \mathbb{Z}_{p^\infty} \oplus \mathbb{Z}_5$. Since $\mathbb{Z}_5$ is simple, $\mathbb{Z}_5$ is r-supplemented. Since $\mathbb{Z}_{p^\infty}$ and $\mathbb{Z}_5$ are r-supplemented, by Lemma 8, $M = \mathbb{Z}_{p^\infty} \oplus \mathbb{Z}_5$ is r-supplemented. Here $RadM = Rad\mathbb{Z}_{p^\infty} \oplus Rad\mathbb{Z}_5 = \mathbb{Z}_{p^\infty} \oplus 0 = \mathbb{Z}_{p^\infty} \neq M$.

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