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k-SRIVASTAVA HYPERGEOMETRIC FUNCTIONS AND THEIR INTEGRAL REPRESENTATIONS

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Abstract. In this study, we introduce the k-Srivastava hypergeometric functions by means of the Pochhammer k-symbol. Also, we obtain the relations between k-Srivastava hypergeometric and classical Srivastava hypergeometric functions. Then, we show that with the help of these relations, integral representations of k-Srivastava hypergeometric functions can be easily proved without the need for lengthy proofs.

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1. INTRODUCTION

Various generalizations of special functions have been frequently encountered in recent years [1, 3–5, 7–11, 14]. One of them is the k-generalization of special functions, and some of the studies on this subject are as follows:

In 2007, for $k \in \mathbb{R}^+$, Diaz and Pariguan [4] introduced, respectively, the k-gamma function $\Gamma_k(x)$, k-beta function $B_k(x,y)$ and Pochhammer k-symbol $(\alpha)_{n,k}$ as follows:

$$\Gamma_k(x) = \int_0^\infty t^{x-1} e^{-\frac{t^k}{k}}\, dt, \quad \operatorname{Re}(x) > 0, \tag{1.1}$$

$$B_k(x,y) = \frac{1}{k}\int_0^1 t^{\frac{x}{k}-1}(1-t)^{\frac{y}{k}-1}\, dt, \quad \operatorname{Re}(x) > 0, \operatorname{Re}(y) > 0, \tag{1.2}$$

$$(\alpha)_{n,k} = \begin{cases} \alpha(\alpha+k)(\alpha+2k)\cdots(\alpha+(n-1)k), & n = 1,2,\ldots \\ 1, & n = 0. \end{cases} \tag{1.3}$$

These functions and Pochhammer k-symbol have the following properties [4, 9, 11]:

$$\Gamma_k(x+k) = x\Gamma_k(x),$$

$$B_k(x,y) = \frac{\Gamma_k(x)\Gamma_k(y)}{\Gamma_k(x+y)}, \tag{1.4}$$

$$(\alpha)_{n,k} = \frac{\Gamma_k(\alpha+nk)}{\Gamma_k(\alpha)}, \tag{1.5}$$

$$\frac{(\beta)_{n,k}}{(\gamma)_{n,k}} = \frac{B_k(\beta+kn,\gamma-\beta)}{B_k(\beta,\gamma-\beta)}, \tag{1.6}$$

$$(\alpha)_{m+n,k} = (\alpha)_{m,k}(\alpha+mk)_{n,k}, \tag{1.7}$$

$$\sum_{n=0}^{\infty}(\alpha)_{n,k}\frac{x^n}{n!} = (1-kx)^{-\frac{\alpha}{k}}, \quad |x| < \frac{1}{k}.$$

Clearly, taking $k=1$ in (1.1), (1.2) and (1.3), the classical gamma function $\Gamma(x)$, classical beta function $B(x,y)$ and classical Pochhammer symbol $(\alpha)_n$ are obtained respectively. That is,

$$\Gamma_1(x) = \Gamma(x), \quad B_1(x,y) = B(x,y), \quad (\alpha)_{n,1} = (\alpha)_n.$$

Note that, the relations between the definitions of k-generalizations given by (1.1), (1.2), (1.3) and their corresponding classical definitions (that is, the $k=1$ case), are as follows (see [4]):

$$\Gamma_k(x) = k^{\frac{x}{k}-1}\Gamma\left(\frac{x}{k}\right), \tag{1.8}$$

$$B_k(x,y) = \frac{1}{k}B\left(\frac{x}{k},\frac{y}{k}\right),$$

$$(\alpha)_{n,k} = k^n\left(\frac{\alpha}{k}\right)_n. \tag{1.9}$$

Moreover, Diaz and Pariguan [4] also introduced k-hypergeometric function defined by

$$ {}_2F_{1,k}(\alpha,\beta;\gamma;x) = \sum_{n=0}^{\infty}\frac{(\alpha)_{n,k}(\beta)_{n,k}}{(\gamma)_{n,k}}\frac{x^n}{n!}, \quad |x| < \frac{1}{k} \tag{1.10}$$

$$(k\in\mathbb{R}^+ \text{and } \gamma \neq 0,-k,-2k,\ldots).$$

For $k=1$, k-hypergeometric function reduces to classical hypergeometric function

$$ {}_2F_{1,1}(\alpha,\beta;\gamma;x) = {}_2F_1(\alpha,\beta;\gamma;x)$$

and the relation between them is

$$ {}_2F_{1,k}(\alpha,\beta;\gamma;x) = {}_2F_1\left(\frac{\alpha}{k},\frac{\beta}{k};\frac{\gamma}{k};kx\right). \tag{1.11}$$

In 2012, Mubeen and Habibullah [9] presented an integral representation of k-hypergeometric function as

$$ {}_2F_{1,k}(\alpha,\beta;\gamma;x) = \frac{\Gamma_k(\gamma)}{k\Gamma_k(\beta)\Gamma_k(\gamma-\beta)}\int_0^1 t^{\frac{\beta}{k}-1}(1-t)^{\frac{\gamma-\beta}{k}-1}(1-kxt)^{-\frac{\alpha}{k}}\,dt,$$

$$(\mathrm{Re}(\gamma) > \mathrm{Re}(\beta) > 0).$$

In 2015, Mubeen et al. [10] defined the k-generalization of Appell hypergeometric function F_1 as

$$F_{1,k}(\alpha,\beta,\beta';\gamma;x,y)=\sum_{m,n=0}^{\infty}\frac{(\alpha)_{m+n,k}(\beta)_{m,k}(\beta')_{n,k}}{(\gamma)_{m+n,k}}\frac{x^m}{m!}\frac{y^n}{n!},\quad |x|<\frac{1}{k},|y|<\frac{1}{k} \tag{1.12}$$
$$(k\in\mathbb{R}^{+}\text{and } \gamma\neq 0,-k,-2k,\ldots)$$

and obtained an integral representation of this function as

$$\begin{aligned}F_{1,k}(\alpha,\beta,\beta';\gamma;x,y)=&\frac{\Gamma_k(\gamma)}{k\Gamma_k(\alpha)\Gamma_k(\gamma-\alpha)}\\&\times\int_0^1 t^{\frac{\alpha}{k}-1}(1-t)^{\frac{\gamma-\alpha}{k}-1}(1-kxt)^{-\frac{\beta}{k}}(1-kyt)^{-\frac{\beta'}{k}}\,dt,\\&(\mathrm{Re}(\gamma)>\mathrm{Re}(\alpha)>0).\end{aligned}$$

In 2017, Kıymaz et al. [7] introduced k-generalizations of Appell hypergeometric functions F_2,F_3 and F_4 as

$$F_{2,k}(\alpha,\beta,\beta';\gamma,\gamma';x,y)=\sum_{m,n=0}^{\infty}\frac{(\alpha)_{m+n,k}(\beta)_{m,k}(\beta')_{n,k}}{(\gamma)_{m,k}(\gamma')_{n,k}}\frac{x^m}{m!}\frac{y^n}{n!},\quad |x|+|y|<\frac{1}{k}$$
$$F_{3,k}(\alpha,\alpha',\beta,\beta';\gamma;x,y)=\sum_{m,n=0}^{\infty}\frac{(\alpha)_{m,k}(\alpha')_{n,k}(\beta)_{m,k}(\beta')_{n,k}}{(\gamma)_{m+n,k}}\frac{x^m}{m!}\frac{y^n}{n!},\quad |x|<\frac{1}{k},\ |y|<\frac{1}{k}$$
$$F_{4,k}(\alpha,\beta;\gamma,\gamma';x,y)=\sum_{m,n=0}^{\infty}\frac{(\alpha)_{m+n,k}(\beta)_{m+n,k}}{(\gamma)_{m,k}(\gamma')_{n,k}}\frac{x^m}{m!}\frac{y^n}{n!},\quad \sqrt{|x|}+\sqrt{|y|}<\frac{1}{\sqrt{k}}$$

where $k\in\mathbb{R}^{+}$ and $\gamma,\gamma'\neq 0,-k,-2k,\ldots$. They also presented following relations between k-Appell hypergeometric and classical Appell hypergeometric functions (that is, the $k=1$ case), in the forms

$$F_{1,k}(\alpha,\beta,\beta';\gamma;x,y)=F_1\left(\frac{\alpha}{k},\frac{\beta}{k},\frac{\beta'}{k};\frac{\gamma}{k};kx,ky\right), \tag{1.13}$$

$$F_{2,k}(\alpha,\beta,\beta';\gamma,\gamma';x,y)=F_2\left(\frac{\alpha}{k},\frac{\beta}{k},\frac{\beta'}{k};\frac{\gamma}{k},\frac{\gamma'}{k};kx,ky\right), \tag{1.14}$$

$$F_{3,k}(\alpha,\alpha',\beta,\beta';\gamma;x,y)=F_3\left(\frac{\alpha}{k},\frac{\alpha'}{k},\frac{\beta}{k},\frac{\beta'}{k};\frac{\gamma}{k};kx,ky\right), \tag{1.15}$$

$$F_{4,k}(\alpha,\beta;\gamma,\gamma';x,y)=F_4\left(\frac{\alpha}{k},\frac{\beta}{k};\frac{\gamma}{k},\frac{\gamma'}{k};kx,ky\right). \tag{1.16}$$

In the same article, they gave integral representations and double Mellin transforms to show that many properties of k-Appell hypergeometric functions can be easily deduced using relations (1.13)-(1.16) and the well-known properties of classical Appell hypergeometric functions.

In 2020, Gürel Yılmaz et.al. [5] acquired some transformation formulas, reduction formulas, linear and bilinear generating relations for the k-Appell hypergeometric functions.

The purpose of this study is to firstly introduce the k-Srivastava hypergeometric functions using the Pochhammer k-symbol. Then, it is to obtain the relations between k-Srivastava hypergeometric and classical Srivastava hypergeometric functions corresponding to $k = 1$ case. Finally, it is to point out that many properties of k-Srivastava hypergeometric functions, such as their integral representations, can be easily obtained using these relations and the well-known properties of classical Srivastava hypergeometric functions.

2. k-SRIVASTAVA HYPERGEOMETRIC FUNCTIONS

In this section, k-generalizations of the classical Srivastava hypergeometric functions H_A, H_B and H_C are defined using the Pochhammer k-symbol. The definitions of these k-generalizations called k-Srivastava hypergeometric functions $H_{A,k}$, $H_{B,k}$ and $H_{C,k}$ are as follows:

$$H_{A,k}(\alpha,\beta_1,\beta_2;\gamma_1,\gamma_2;x,y,z) = \sum_{m,n,p=0}^{\infty} \frac{(\alpha)_{m+p,k}(\beta_1)_{m+n,k}(\beta_2)_{n+p,k}}{(\gamma_1)_{m,k}(\gamma_2)_{n+p,k}} \frac{x^m}{m!}\frac{y^n}{n!}\frac{z^p}{p!}, \tag{2.1}$$

$$\big(r<1, s<1, t<(1-r)(1-s)\big);$$

$$H_{B,k}(\alpha,\beta_1,\beta_2;\gamma_1,\gamma_2,\gamma_3;x,y,z) = \sum_{m,n,p=0}^{\infty} \frac{(\alpha)_{m+p,k}(\beta_1)_{m+n,k}(\beta_2)_{n+p,k}}{(\gamma_1)_{m,k}(\gamma_2)_{n,k}(\gamma_3)_{p,k}} \frac{x^m}{m!}\frac{y^n}{n!}\frac{z^p}{p!}, \tag{2.2}$$

$$\big(r+s+t+2\sqrt{rst}<1\big);$$

$$H_{C,k}(\alpha,\beta_1,\beta_2;\gamma;x,y,z) = \sum_{m,n,p=0}^{\infty} \frac{(\alpha)_{m+p,k}(\beta_1)_{m+n,k}(\beta_2)_{n+p,k}}{(\gamma)_{m+n+p,k}} \frac{x^m}{m!}\frac{y^n}{n!}\frac{z^p}{p!}, \tag{2.3}$$

$$\big(r<1, s<1, t<1, r+s+t-2\sqrt{(1-r)(1-s)(1-t)}<2\big)$$

where $k \in \mathbb{R}^+$; $\gamma,\gamma_1,\gamma_2,\gamma_3 \neq 0,-k,-2k,\ldots$ and $r := k|x|, s := k|y|, t := k|z|$.

Throughout this paper, we assume that k is any positive real number.

Remark 1. For $k = 1$, the k-Srivastava hypergeometric functions $H_{A,k}$, $H_{B,k}$ and $H_{C,k}$ in (2.1)-(2.3) are reduced to the classical Srivastava hypergeometric functions H_A, H_B and H_C. That is,

$$H_{A,1}(\alpha,\beta_1,\beta_2;\gamma_1,\gamma_2;x,y,z) = H_A(\alpha,\beta_1,\beta_2;\gamma_1,\gamma_2;x,y,z), \tag{2.4}$$

$$H_{B,1}(\alpha,\beta_1,\beta_2;\gamma_1,\gamma_2,\gamma_3;x,y,z) = H_B(\alpha,\beta_1,\beta_2;\gamma_1,\gamma_2,\gamma_3;x,y,z), \tag{2.5}$$

$$H_{C,1}(\alpha,\beta_1,\beta_2;\gamma;x,y,z)=H_C(\alpha,\beta_1,\beta_2;\gamma;x,y,z). \tag{2.6}$$

Remark 2. Using properties (1.4)-(1.7), the k-Srivastava hypergeometric functions can be also expressed as follows:

$$\begin{aligned}
H_{A,k}(\alpha,\beta_1,\beta_2;\gamma_1,\gamma_2;x,y,z) &= \sum_{m,n,p=0}^{\infty}\frac{(\alpha)_{m+p,k}(\beta_1)_{m+n,k}}{(\gamma_1)_{m,k}}\\
&\qquad\times\frac{B_k(\beta_2+kn+kp,\gamma_2-\beta_2)}{B_k(\beta_2,\gamma_2-\beta_2)}\frac{x^m}{m!}\frac{y^n}{n!}\frac{z^p}{p!},\\
H_{B,k}(\alpha,\beta_1,\beta_2;\gamma_1,\gamma_2,\gamma_3;x,y,z) &= \sum_{m,n,p=0}^{\infty}\frac{(\alpha+\beta_1)_{2m+n+p,k}(\beta_2)_{n+p,k}}{(\gamma_1)_{m,k}(\gamma_2)_{n,k}(\gamma_3)_{p,k}}\\
&\qquad\times\frac{B_k(\alpha+km+kp,\beta_1+km+kn)}{B_k(\alpha,\beta_1)}\frac{x^m}{m!}\frac{y^n}{n!}\frac{z^p}{p!},\\
H_{C,k}(\alpha,\beta_1,\beta_2;\gamma;x,y,z) &= \sum_{m,n,p=0}^{\infty}\frac{(\beta_1)_{m+n,k}(\beta_2)_{n+p,k}}{(\gamma)_{n,k}}\\
&\qquad\times\frac{B_k(\alpha+km+kp,\gamma-\alpha+kn)}{B_k(\alpha,\gamma-\alpha+kn)}\frac{x^m}{m!}\frac{y^n}{n!}\frac{z^p}{p!}.
\end{aligned}$$

Here $B_k(x,y)$ is the k-beta function defined in (1.2).

Remark 3. Considering (1.7) and (1.12), the functions $H_{A,k}$ and $H_{C,k}$ which have the three-fold series representations in (2.1) and (2.3), can also be represented by the single-fold series as follows, respectively.

$$H_{A,k}(\alpha,\beta_1,\beta_2;\gamma_1,\gamma_2;x,y,z)=\sum_{m=0}^{\infty}\frac{(\alpha)_{m,k}(\beta_1)_{m,k}}{(\gamma_1)_{m,k}}F_{1,k}(\beta_2,\beta_1+km,\alpha+km;\gamma_2;y,z)\frac{x^m}{m!},$$

and

$$H_{C,k}(\alpha,\beta_1,\beta_2;\gamma;x,y,z)=\sum_{n=0}^{\infty}\frac{(\beta_1)_{n,k}(\beta_2)_{n,k}}{(\gamma)_{n,k}}F_{1,k}(\alpha,\beta_1+kn,\beta_2+kn;\gamma+kn;x,z)\frac{y^n}{n!}.$$

Here $F_{1,k}$ is the k-Appell hypergeometric function defined in (1.12).

Theorem 1. *The relations between the k-Srivastava hypergeometric functions given in* (2.1)-(2.3) *and the classical Srivastava hypergeometric functions given in* (2.4)-(2.6) *are,*

$$H_{A,k}(\alpha,\beta_1,\beta_2;\gamma_1,\gamma_2;x,y,z)=H_A\left(\frac{\alpha}{k},\frac{\beta_1}{k},\frac{\beta_2}{k};\frac{\gamma_1}{k},\frac{\gamma_2}{k};kx,ky,kz\right), \tag{2.7}$$

$$H_{B,k}(\alpha,\beta_1,\beta_2;\gamma_1,\gamma_2,\gamma_3;x,y,z)=H_B\left(\frac{\alpha}{k},\frac{\beta_1}{k},\frac{\beta_2}{k};\frac{\gamma_1}{k},\frac{\gamma_2}{k},\frac{\gamma_3}{k};kx,ky,kz\right), \tag{2.8}$$

$$H_{C,k}(\alpha,\beta_1,\beta_2;\gamma;x,y,z)=H_C\left(\frac{\alpha}{k},\frac{\beta_1}{k},\frac{\beta_2}{k};\frac{\gamma}{k};kx,ky,kz\right). \tag{2.9}$$

Proof. By using relation (1.9) in the definition of function $H_{A,k}$ in (2.1) and considering the definition of function H_A in (2.4), we obtain

$$\begin{aligned}
H_{A,k}(\alpha,\beta_1,\beta_2;\gamma_1,\gamma_2;x,y,z) &= \sum_{m,n,p=0}^{\infty} \frac{(\alpha)_{m+p,k}(\beta_1)_{m+n,k}(\beta_2)_{n+p,k}}{(\gamma_1)_{m,k}(\gamma_2)_{n+p,k}} \frac{x^m}{m!}\frac{y^n}{n!}\frac{z^p}{p!} \\
&= \sum_{m,n,p=0}^{\infty} \frac{k^{m+p}(\frac{\alpha}{k})_{m+p}k^{m+n}(\frac{\beta_1}{k})_{m+n}k^{n+p}(\frac{\beta_2}{k})_{n+p}}{k^m(\frac{\gamma_1}{k})_m k^{n+p}(\frac{\gamma_2}{k})_{n+p}} \frac{x^m}{m!}\frac{y^n}{n!}\frac{z^p}{p!} \\
&= \sum_{m,n,p=0}^{\infty} \frac{(\frac{\alpha}{k})_{m+p}(\frac{\beta_1}{k})_{m+n}(\frac{\beta_2}{k})_{n+p}}{(\frac{\gamma_1}{k})_m(\frac{\gamma_2}{k})_{n+p}} \frac{(kx)^m}{m!}\frac{(ky)^n}{n!}\frac{(kz)^p}{p!} \\
&= H_A\left(\frac{\alpha}{k},\frac{\beta_1}{k},\frac{\beta_2}{k};\frac{\gamma_1}{k},\frac{\gamma_2}{k};kx,ky,kz\right)
\end{aligned}$$

which completes the proof of (2.7). Similarly, we can prove relation (2.8) by using (1.9), (2.2) and (2.5) as follows:

$$\begin{aligned}
H_{B,k}(\alpha,\beta_1,\beta_2;\gamma_1,\gamma_2,\gamma_3;x,y,z) &= \sum_{m,n,p=0}^{\infty} \frac{(\alpha)_{m+p,k}(\beta_1)_{m+n,k}(\beta_2)_{n+p,k}}{(\gamma_1)_{m,k}(\gamma_2)_{n,k}(\gamma_3)_{p,k}} \frac{x^m}{m!}\frac{y^n}{n!}\frac{z^p}{p!} \\
&= \sum_{m,n,p=0}^{\infty} \frac{k^{m+p}(\frac{\alpha}{k})_{m+p}k^{m+n}(\frac{\beta_1}{k})_{m+n}k^{n+p}(\frac{\beta_2}{k})_{n+p}}{k^m(\frac{\gamma_1}{k})_m k^n(\frac{\gamma_2}{k})_n k^p(\frac{\gamma_3}{k})_p} \frac{x^m}{m!}\frac{y^n}{n!}\frac{z^p}{p!} \\
&= \sum_{m,n,p=0}^{\infty} \frac{(\frac{\alpha}{k})_{m+p}(\frac{\beta_1}{k})_{m+n}(\frac{\beta_2}{k})_{n+p}}{(\frac{\gamma_1}{k})_m(\frac{\gamma_2}{k})_n(\frac{\gamma_3}{k})_p} \frac{(kx)^m}{m!}\frac{(ky)^n}{n!}\frac{(kz)^p}{p!} \\
&= H_B\left(\frac{\alpha}{k},\frac{\beta_1}{k},\frac{\beta_2}{k};\frac{\gamma_1}{k},\frac{\gamma_2}{k},\frac{\gamma_3}{k};kx,ky,kz\right).
\end{aligned}$$

Finally, by using (1.9), (2.3) and (2.6), we have (2.9) as follows:

$$\begin{aligned}
H_{C,k}(\alpha,\beta_1,\beta_2;\gamma;x,y,z) &= \sum_{m,n,p=0}^{\infty} \frac{(\alpha)_{m+p,k}(\beta_1)_{m+n,k}(\beta_2)_{n+p,k}}{(\gamma)_{m+n+p,k}} \frac{x^m}{m!}\frac{y^n}{n!}\frac{z^p}{p!} \\
&= \sum_{m,n,p=0}^{\infty} \frac{k^{m+p}(\frac{\alpha}{k})_{m+p}k^{m+n}(\frac{\beta_1}{k})_{m+n}k^{n+p}(\frac{\beta_2}{k})_{n+p}}{k^{m+n+p}(\frac{\gamma}{k})_{m+n+p}} \frac{x^m}{m!}\frac{y^n}{n!}\frac{z^p}{p!} \\
&= \sum_{m,n,p=0}^{\infty} \frac{(\frac{\alpha}{k})_{m+p}(\frac{\beta_1}{k})_{m+n}(\frac{\beta_2}{k})_{n+p}}{(\frac{\gamma}{k})_{m+n+p}} \frac{(kx)^m}{m!}\frac{(ky)^n}{n!}\frac{(kz)^p}{p!} \\
&= H_C\left(\frac{\alpha}{k},\frac{\beta_1}{k},\frac{\beta_2}{k};\frac{\gamma}{k};kx,ky,kz\right).
\end{aligned}$$

□

Remark 4. The convergence regions of the series (2.1)-(2.3) can easily seen from the relations of (2.7)-(2.9) and the convergence regions of the classic Srivastava hypergeometric series.

3. Integral representations

In this section, we show that the integral representations of k-Srivastava hypergeometric functions can be easily proved using relations (2.7)-(2.9) and the well-known integral representations of classical Srivastava hypergeometric functions given in [2] (see also [6, 12, 13]).

Theorem 2. *The function $H_{A,k}$ has the following integral representations.*

$$H_{A,k}(\alpha,\beta_1,\beta_2;\gamma_1,\gamma_2;x,y,z) = \frac{\Gamma_k(\gamma_1)\Gamma_k(\gamma_2)}{k^2\Gamma_k(\beta_1)\Gamma_k(\beta_2)\Gamma_k(\gamma_1-\beta_1)\Gamma_k(\gamma_2-\beta_2)}$$
$$\times \int_0^1\int_0^1 \xi^{\frac{\beta_1}{k}-1}\eta^{\frac{\beta_2}{k}-1}(1-\xi)^{\frac{\gamma_1-\beta_1}{k}-1}(1-\eta)^{\frac{\gamma_2-\beta_2}{k}-1}$$
$$\times (1-ky\eta)^{\frac{-\beta_1}{k}}(1-kx\xi-kz\eta)^{\frac{-\alpha}{k}}\left(1-\frac{k^2xy\xi\eta}{(1-ky\eta)(1-kx\xi-kz\eta)}\right)^{\frac{-\alpha}{k}} d\xi d\eta, \quad (3.1)$$
$\big(\mathrm{Re}(\gamma_1) > \mathrm{Re}(\beta_1) > 0, \mathrm{Re}(\gamma_2) > \mathrm{Re}(\beta_2) > 0\big)$;

$$H_{A,k}(\alpha,\beta_1,\beta_2;\gamma_1,\gamma_2;x,y,z) = \frac{\Gamma_k(\gamma_2)}{k\Gamma_k(\beta_2)\Gamma_k(\gamma_2-\beta_2)}\int_0^1 \xi^{\frac{\beta_2}{k}-1}(1-\xi)^{\frac{\gamma_2-\beta_2}{k}-1}$$
$$\times (1-ky\xi)^{\frac{-\beta_1}{k}}(1-kz\xi)^{\frac{-\alpha}{k}}\,{}_2F_{1,k}\left(\alpha,\beta_1;\gamma_1;\frac{x}{(1-ky\xi)(1-kz\xi)}\right)d\xi, \quad (3.2)$$
$\big(\mathrm{Re}(\gamma_2) > \mathrm{Re}(\beta_2) > 0\big)$;

$$H_{A,k}(\alpha,\beta_1,\beta_2;\gamma_1,\gamma_2;x,y,z) = \frac{\Gamma_k(\gamma_2)(1+\lambda)^{\frac{\beta_2}{k}}}{k\Gamma_k(\beta_2)\Gamma_k(\gamma_2-\beta_2)}\int_0^1 \xi^{\frac{\beta_2}{k}-1}(1-\xi)^{\frac{\gamma_2-\beta_2}{k}-1}$$
$$\times (1+\lambda\xi)^{\frac{\alpha+\beta_1-\gamma_2}{k}}[1+\lambda\xi-(1+\lambda)ky\xi]^{\frac{-\beta_1}{k}}[1+\lambda\xi-(1+\lambda)kz\xi]^{\frac{-\alpha}{k}}$$
$$\times\, {}_2F_{1,k}\left(\alpha,\beta_1;\gamma_1;\frac{x(1+\lambda\xi)^2}{[1+\lambda\xi-(1+\lambda)ky\xi][1+\lambda\xi-(1+\lambda)kz\xi]}\right)d\xi, \quad (3.3)$$
$\big(\mathrm{Re}(\gamma_2) > \mathrm{Re}(\beta_2) > 0; \lambda > -1\big)$;

$$H_{A,k}(\alpha,\beta_1,\beta_2;\gamma_1,\gamma_2;x,y,z) = \frac{\Gamma_k(\gamma_2)}{k\Gamma_k(\beta_2)\Gamma_k(\gamma_2-\beta_2)}\frac{(b-c)^{\frac{\beta_2}{k}}(a-c)^{\frac{\gamma_2-\beta_2}{k}}}{(b-a)^{\frac{\gamma_2-\alpha-\beta_1}{k}-1}}$$
$$\times \int_a^b (b-\xi)^{\frac{\gamma_2-\beta_2}{k}-1}(\xi-a)^{\frac{\beta_2}{k}-1}(\xi-c)^{\frac{\alpha+\beta_1-\gamma_2}{k}}[(b-a)(\xi-c)-(b-c)(\xi-a)ky]^{\frac{-\beta_1}{k}}$$
$$\times [(b-a)(\xi-c)-(b-c)(\xi-a)kz]^{\frac{-\alpha}{k}}\,{}_2F_{1,k}(\alpha,\beta_1;\gamma_1;x\sigma)d\xi, \quad (3.4)$$
$\big(\mathrm{Re}(\gamma_2) > \mathrm{Re}(\beta_2) > 0; c < a < b\big)$,

$$\sigma := \frac{(b-a)^2(\xi-c)^2}{[(b-a)(\xi-c)-(b-c)(\xi-a)ky][(b-a)(\xi-c)-(b-c)(\xi-a)kz]};$$

$$H_{A,k}(\alpha,\beta_1,\beta_2;\gamma_1,\gamma_2;x,y,z)=\frac{\Gamma_k(\gamma_2)}{k\Gamma_k(\beta_2)\Gamma_k(\gamma_2-\beta_2)}\int_0^\infty \xi^{\frac{\beta_2}{k}-1}(1+\xi)^{\frac{\alpha+\beta_1-\gamma_2}{k}}$$
$$\times(1+\xi-ky\xi)^{\frac{-\beta_1}{k}}(1+\xi-kz\xi)^{\frac{-\alpha}{k}}$$
$$\times {}_2F_{1,k}\left(\alpha,\beta_1;\gamma_1;\frac{x(1+\xi)^2}{(1+\xi-ky\xi)(1+\xi-kz\xi)}\right)d\xi, \tag{3.5}$$
$$\big(\mathrm{Re}(\gamma_2)>\mathrm{Re}(\beta_2)>0\big);$$

$$H_{A,k}(\alpha,\beta_1,\beta_2;\gamma_1,\gamma_2;x,y,z)=\frac{2\Gamma_k(\gamma_2)}{k\Gamma_k(\beta_2)\Gamma_k(\gamma_2-\beta_2)}\int_0^{\frac{\pi}{2}}(\sin^2\xi)^{\frac{\beta_2}{k}-\frac12}(\cos^2\xi)^{\frac{\gamma_2-\beta_2}{k}-\frac12}$$
$$\times(1-ky\sin^2\xi)^{\frac{-\beta_1}{k}}(1-kz\sin^2\xi)^{\frac{-\alpha}{k}}$$
$$\times {}_2F_{1,k}\left(\alpha,\beta_1;\gamma_1;\frac{x}{(1-ky\sin^2\xi)(1-kz\sin^2\xi)}\right)d\xi, \tag{3.6}$$
$$\big(\mathrm{Re}(\gamma_2)>\mathrm{Re}(\beta_2)>0\big).$$

Here ${}_2F_{1,k}$ is the k-hypergeometric function defined in (1.10).

Proof. In [12] (see also [2, 6]), the well-known integral representation of H_A is given by

$$H_A(\alpha,\beta_1,\beta_2;\gamma_1,\gamma_2;x,y,z)=\frac{\Gamma(\gamma_1)\Gamma(\gamma_2)}{\Gamma(\beta_1)\Gamma(\beta_2)\Gamma(\gamma_1-\beta_1)\Gamma(\gamma_2-\beta_2)}$$
$$\times\int_0^1\int_0^1 \xi^{\beta_1-1}\eta^{\beta_2-1}(1-\xi)^{\gamma_1-\beta_1-1}(1-\eta)^{\gamma_2-\beta_2-1}$$
$$\times(1-y\eta)^{-\beta_1}(1-x\xi-z\eta)^{-\alpha}\left(1-\frac{xy\xi\eta}{(1-y\eta)(1-x\xi-z\eta)}\right)^{-\alpha}d\xi d\eta.$$

Using this integral representation in (2.7) and making use of (1.8), we have

$$H_{A,k}(\alpha,\beta_1,\beta_2;\gamma_1,\gamma_2;x,y,z)=H_A\left(\frac{\alpha}{k},\frac{\beta_1}{k},\frac{\beta_2}{k};\frac{\gamma_1}{k},\frac{\gamma_2}{k};kx,ky,kz\right)$$
$$=\frac{\Gamma(\frac{\gamma_1}{k})\Gamma(\frac{\gamma_2}{k})}{\Gamma(\frac{\beta_1}{k})\Gamma(\frac{\beta_2}{k})\Gamma(\frac{\gamma_1-\beta_1}{k})\Gamma(\frac{\gamma_2-\beta_2}{k})}$$
$$\times\int_0^1\int_0^1 \xi^{\frac{\beta_1}{k}-1}\eta^{\frac{\beta_2}{k}-1}(1-\xi)^{\frac{\gamma_1-\beta_1}{k}-1}(1-\eta)^{\frac{\gamma_2-\beta_2}{k}-1}$$
$$\times(1-ky\eta)^{\frac{-\beta_1}{k}}(1-kx\xi-kz\eta)^{\frac{-\alpha}{k}}\left(1-\frac{k^2xy\xi\eta}{(1-ky\eta)(1-kx\xi-kz\eta)}\right)^{\frac{-\alpha}{k}}d\xi d\eta$$
$$=\frac{k^{1-\frac{\gamma_1}{k}}\Gamma_k(\gamma_1)k^{1-\frac{\gamma_2}{k}}\Gamma_k(\gamma_2)}{k^{1-\frac{\beta_1}{k}}\Gamma_k(\beta_1)k^{1-\frac{\beta_2}{k}}\Gamma_k(\beta_2)k^{1-\frac{\gamma_1-\beta_1}{k}}\Gamma_k(\gamma_1-\beta_1)k^{1-\frac{\gamma_2-\beta_2}{k}}\Gamma_k(\gamma_2-\beta_2)}$$

$$\times \int_0^1 \int_0^1 \xi^{\frac{\beta_1}{k}-1} \eta^{\frac{\beta_2}{k}-1} (1-\xi)^{\frac{\gamma_1-\beta_1}{k}-1} (1-\eta)^{\frac{\gamma_2-\beta_2}{k}-1}$$
$$\times (1-ky\eta)^{\frac{-\beta_1}{k}} (1-kx\xi-kz\eta)^{\frac{-\alpha}{k}} \left(1-\frac{k^2 xy\xi\eta}{(1-ky\eta)(1-kx\xi-kz\eta)}\right)^{\frac{-\alpha}{k}} d\xi d\eta$$

which completes the proof of (3.1).

The other integral representations (3.2)-(3.6) of $H_{A,k}$ are proved similarly by considering (1.8) and (1.11) after the well-known integral representations of H_A given in [2] are used in the right-hand side of (2.7). □

Theorem 3. *The function $H_{B,k}$ has the following integral representations.*

$$H_{B,k}(\alpha,\beta_1,\beta_2;\gamma_1,\gamma_2,\gamma_3;x,y,z) = \frac{\Gamma_k(\alpha+\beta_1)}{k\Gamma_k(\alpha)\Gamma_k(\beta_1)} \int_0^1 \xi^{\frac{\alpha}{k}-1}(1-\xi)^{\frac{\beta_1}{k}-1}$$
$$\times X_{4,k}\left(\alpha+\beta_1,\beta_2;\gamma_1,\gamma_2,\gamma_3;x\xi(1-\xi),y(1-\xi),z\xi\right) d\xi, \tag{3.7}$$
$$\left(\min\{\mathrm{Re}(\alpha),\mathrm{Re}(\beta_1)\}>0\right);$$

$$H_{B,k}(\alpha,\beta_1,\beta_2;\gamma_1,\gamma_2,\gamma_3;x,y,z) = \frac{\Gamma_k(\alpha+\beta_1)}{k\Gamma_k(\alpha)\Gamma_k(\beta_1)} \frac{(b-c)^{\frac{\alpha}{k}}(a-c)^{\frac{\beta_1}{k}}}{(b-a)^{\frac{\alpha+\beta_1}{k}-1}}$$
$$\times \int_a^b (b-\xi)^{\frac{\beta_1}{k}-1}(\xi-a)^{\frac{\alpha}{k}-1}(\xi-c)^{\frac{-\alpha-\beta_1}{k}}$$
$$\times X_{4,k}\left(\alpha+\beta_1,\beta_2;\gamma_1,\gamma_2,\gamma_3;x\sigma_1,y\sigma_2,z\sigma_3\right) d\xi, \tag{3.8}$$
$$\left(\min\{\mathrm{Re}(\alpha),\mathrm{Re}(\beta_1)\}>0; c<a<b\right),$$
$$\sigma_1 := \frac{(a-c)(b-c)(\xi-a)(b-\xi)}{(b-a)^2(\xi-c)^2}, \sigma_2 := \frac{(a-c)(b-\xi)}{(b-a)(\xi-c)}, \sigma_3 := \frac{(b-c)(\xi-a)}{(b-a)(\xi-c)};$$

$$H_{B,k}(\alpha,\beta_1,\beta_2;\gamma_1,\gamma_2,\gamma_3;x,y,z) = \frac{2\Gamma_k(\alpha+\beta_1)}{k\Gamma_k(\alpha)\Gamma_k(\beta_1)} \int_0^{\frac{\pi}{2}} (\sin^2\xi)^{\frac{\alpha}{k}-\frac{1}{2}}(\cos^2\xi)^{\frac{\beta_1}{k}-\frac{1}{2}}$$
$$\times X_{4,k}\left(\alpha+\beta_1,\beta_2;\gamma_1,\gamma_2,\gamma_3;x\sigma_1,y\sigma_2,z\sigma_3\right) d\xi, \tag{3.9}$$
$$\left(\min\{\mathrm{Re}(\alpha),\mathrm{Re}(\beta_1)\}>0\right),$$
$$\sigma_1 := \sin^2\xi\cos^2\xi, \sigma_2 := \cos^2\xi, \sigma_3 := \sin^2\xi;$$

$$H_{B,k}(\alpha,\beta_1,\beta_2;\gamma_1,\gamma_2,\gamma_3;x,y,z) = \frac{2\Gamma_k(\alpha+\beta_1)(1+\lambda)^{\frac{\alpha}{k}}}{k\Gamma_k(\alpha)\Gamma_k(\beta_1)}$$
$$\times \int_0^{\frac{\pi}{2}} \frac{(\sin^2\xi)^{\frac{\alpha}{k}-\frac{1}{2}}(\cos^2\xi)^{\frac{\beta_1}{k}-\frac{1}{2}}}{(1+\lambda\sin^2\xi)^{\frac{\alpha+\beta_1}{k}}}$$
$$\times X_{4,k}\left(\alpha+\beta_1,\beta_2;\gamma_1,\gamma_2,\gamma_3;x\sigma_1,y\sigma_2,z\sigma_3\right) d\xi, \tag{3.10}$$
$$\left(\min\{\mathrm{Re}(\alpha),\mathrm{Re}(\beta_1)\}>0; \lambda>-1\right),$$

$$\sigma_1 := \frac{(1+\lambda)\sin^2\xi\cos^2\xi}{(1+\lambda\sin^2\xi)^2}, \sigma_2 := \frac{\cos^2\xi}{1+\lambda\sin^2\xi}, \sigma_3 := \frac{(1+\lambda)\sin^2\xi}{1+\lambda\sin^2\xi};$$

$$\begin{aligned}
H_{B,k}(\alpha,\beta_1,\beta_2;\gamma_1,\gamma_2,\gamma_3;x,y,z) &= \frac{2\Gamma_k(\alpha+\beta_1)\lambda^{\frac{\alpha}{k}}}{k\Gamma_k(\alpha)\Gamma_k(\beta_1)} \\
&\times \int_0^{\frac{\pi}{2}} \frac{(\sin^2\xi)^{\frac{\alpha}{k}-\frac{1}{2}}(\cos^2\xi)^{\frac{\beta_1}{k}-\frac{1}{2}}}{(\cos^2\xi+\lambda\sin^2\xi)^{\frac{\alpha+\beta_1}{k}}} \\
&\times X_{4,k}(\alpha+\beta_1,\beta_2;\gamma_1,\gamma_2,\gamma_3;x\sigma_1,y\sigma_2,z\sigma_3)\,d\xi, \qquad (3.11)\\
&\big(\min\{\mathrm{Re}(\alpha),\mathrm{Re}(\beta_1)\}>0;\lambda>0\big),\\
\sigma_1 &:= \frac{\lambda\sin^2\xi\cos^2\xi}{(\cos^2\xi+\lambda\sin^2\xi)^2}, \sigma_2 := \frac{\cos^2\xi}{\cos^2\xi+\lambda\sin^2\xi}, \sigma_3 := \frac{\lambda\sin^2\xi}{\cos^2\xi+\lambda\sin^2\xi}.
\end{aligned}$$

Here, the k-generalization of the classical Exton hypergeometric function X_4 (see [15]) is defined as

$$X_{4,k}(\alpha,\beta_1;\gamma_1,\gamma_2,\gamma_3;x,y,z) = \sum_{m,n,p=0}^{\infty} \frac{(\alpha)_{2m+n+p,k}(\beta_1)_{n+p,k}}{(\gamma_1)_{m,k}(\gamma_2)_{n,k}(\gamma_3)_{p,k}} \frac{x^m}{m!}\frac{y^n}{n!}\frac{z^p}{p!},$$

which has the following properties:

$$X_{4,1}(\alpha,\beta_1;\gamma_1,\gamma_2,\gamma_3;x,y,z) = X_4(\alpha,\beta_1;\gamma_1,\gamma_2,\gamma_3;x,y,z),$$

and

$$X_{4,k}(\alpha,\beta_1;\gamma_1,\gamma_2,\gamma_3;x,y,z) = X_4\left(\frac{\alpha}{k},\frac{\beta_1}{k};\frac{\gamma_1}{k},\frac{\gamma_2}{k},\frac{\gamma_3}{k};kx,ky,kz\right). \qquad (3.12)$$

Proof. In [2], the well-known integral representation of H_B is given by

$$\begin{aligned}
H_B(\alpha,\beta_1,\beta_2;\gamma_1,\gamma_2,\gamma_3;x,y,z) &= \frac{\Gamma(\alpha+\beta_1)}{\Gamma(\alpha)\Gamma(\beta_1)}\int_0^1 \xi^{\alpha-1}(1-\xi)^{\beta_1-1}\\
&\times X_4(\alpha+\beta_1,\beta_2;\gamma_1,\gamma_2,\gamma_3;x\xi(1-\xi),y(1-\xi),z\xi)\,d\xi.
\end{aligned}$$

Using this integral representation in relation (2.8) and considering (1.8) and (3.12), we obtain

$$\begin{aligned}
H_{B,k}(\alpha,\beta_1,\beta_2;\gamma_1,\gamma_2,\gamma_3;x,y,z) &= H_B\left(\frac{\alpha}{k},\frac{\beta_1}{k},\frac{\beta_2}{k};\frac{\gamma_1}{k},\frac{\gamma_2}{k},\frac{\gamma_3}{k};kx,ky,kz\right)\\
&= \frac{\Gamma(\frac{\alpha+\beta_1}{k})}{\Gamma(\frac{\alpha}{k})\Gamma(\frac{\beta_1}{k})}\int_0^1 \xi^{\frac{\alpha}{k}-1}(1-\xi)^{\frac{\beta_1}{k}-1}\\
&\times X_4\left(\frac{\alpha+\beta_1}{k},\frac{\beta_2}{k};\frac{\gamma_1}{k},\frac{\gamma_2}{k},\frac{\gamma_3}{k};kx\xi(1-\xi),ky(1-\xi),kz\xi\right)d\xi
\end{aligned}$$

$$=\frac{k^{1-\frac{\alpha+\beta_1}{k}}\Gamma_k(\alpha+\beta_1)}{k^{1-\frac{\alpha}{k}}\Gamma_k(\alpha)k^{1-\frac{\beta_1}{k}}\Gamma_k(\beta_1)}\int_0^1\xi^{\frac{\alpha}{k}-1}(1-\xi)^{\frac{\beta_1}{k}-1}$$
$$\times X_{4,k}\left(\alpha+\beta_1,\beta_2;\gamma_1,\gamma_2,\gamma_3;x\xi(1-\xi),y(1-\xi),z\xi\right)d\xi,$$

which completes the proof of (3.7).

The other integral representations (3.8)-(3.11) of $H_{B,k}$ are proved similarly by considering (1.8) and (3.12) after the well-known integral representations of H_B given in [2] are used in the right-hand side of (2.8). □

Theorem 4. *The function $H_{C,k}$ has the following integral representations.*

$$H_{C,k}(\alpha,\beta_1,\beta_2;\gamma;x,y,z)=\frac{\Gamma_k(\gamma)}{k^2\Gamma_k(\alpha)\Gamma_k(\beta_1)\Gamma_k(\gamma-\alpha-\beta_1)}$$
$$\times\int_0^1\int_0^1\xi^{\frac{\alpha}{k}-1}\eta^{\frac{\beta_1}{k}-1}(1-\xi)^{\frac{\gamma-\alpha}{k}-1}(1-\eta)^{\frac{\gamma-\alpha-\beta_1}{k}-1}(1-kx\xi)^{\frac{\beta_2-\beta_1}{k}}$$
$$\times\left(1-kx\xi-ky\eta-kz\xi+ky\xi\eta+k^2xz\xi^2\right)^{\frac{-\beta_2}{k}}d\xi d\eta,\tag{3.13}$$
$$\left(\min\{\mathrm{Re}(\alpha),\mathrm{Re}(\beta_1),\mathrm{Re}(\gamma-\alpha-\beta_1)\}>0\right);$$

$$H_{C,k}(\alpha,\beta_1,\beta_2;\gamma;x,y,z)=\frac{\Gamma_k(\gamma)}{k\Gamma_k(\alpha)\Gamma_k(\gamma-\alpha)}\int_0^1\xi^{\frac{\alpha}{k}-1}(1-\xi)^{\frac{\gamma-\alpha}{k}-1}(1-kx\xi)^{\frac{-\beta_1}{k}}$$
$$\times(1-kz\xi)^{\frac{-\beta_2}{k}}{}_2F_{1,k}\left(\beta_1,\beta_2;\gamma-\alpha;\frac{y(1-\xi)}{(1-kx\xi)(1-kz\xi)}\right)d\xi,\tag{3.14}$$
$$\left(\mathrm{Re}(\gamma)>\mathrm{Re}(\alpha)>0\right);$$

$$H_{C,k}(\alpha,\beta_1,\beta_2;\gamma;x,y,z)=\frac{\Gamma_k(\gamma)(1+\lambda)^{\frac{\alpha}{k}}}{k\Gamma_k(\alpha)\Gamma_k(\gamma-\alpha)}\int_0^1\xi^{\frac{\alpha}{k}-1}(1-\xi)^{\frac{\gamma-\alpha}{k}-1}(1+\lambda\xi)^{\frac{\beta_1+\beta_2-\gamma}{k}}$$
$$\times[1+\lambda\xi-(1+\lambda)kx\xi]^{\frac{-\beta_1}{k}}[1+\lambda\xi-(1+\lambda)kz\xi]^{\frac{-\beta_2}{k}}$$
$$\times{}_2F_{1,k}\left(\beta_1,\beta_2;\gamma-\alpha;\frac{y(1+\lambda\xi)(1-\xi)}{[1+\lambda\xi-(1+\lambda)kx\xi][1+\lambda\xi-(1+\lambda)kz\xi]}\right)d\xi,\tag{3.15}$$
$$\left(\mathrm{Re}(\gamma)>\mathrm{Re}(\alpha)>0;\lambda>-1\right);$$

$$H_{C,k}(\alpha,\beta_1,\beta_2;\gamma;x,y,z)=\frac{\Gamma_k(\gamma)}{k\Gamma_k(\alpha)\Gamma_k(\gamma-\alpha)}\frac{(b-c)^{\frac{\alpha}{k}}(a-c)^{\frac{\gamma-\alpha}{k}}}{(b-a)^{\frac{\gamma-\beta_1-\beta_2}{k}-1}}\int_a^b(b-\xi)^{\frac{\gamma-\alpha}{k}-1}$$
$$\times(\xi-a)^{\frac{\alpha}{k}-1}(\xi-c)^{\frac{\beta_1+\beta_2-\gamma}{k}}[(b-a)(\xi-c)-(b-c)(\xi-a)kx]^{\frac{-\beta_1}{k}}$$
$$\times[(b-a)(\xi-c)-(b-c)(\xi-a)kz]^{\frac{-\beta_2}{k}}{}_2F_{1,k}(\beta_1,\beta_2;\gamma-\alpha;y\sigma)d\xi,\tag{3.16}$$
$$\left(\mathrm{Re}(\gamma)>\mathrm{Re}(\alpha)>0;c<a<b\right),$$

$$\sigma := \frac{(b-a)(a-c)(\xi-c)(b-\xi)}{[(b-a)(\xi-c)-(b-c)(\xi-a)kx][(b-a)(\xi-c)-(b-c)(\xi-a)kz]};$$

$$H_{C,k}(\alpha,\beta_1,\beta_2;\gamma;x,y,z) = \frac{\Gamma_k(\gamma)}{k\Gamma_k(\alpha)\Gamma_k(\gamma-\alpha)} \int_0^\infty \xi^{\frac{\alpha}{k}-1}(1+\xi)^{\frac{\beta_1+\beta_2-\gamma}{k}}$$
$$\times (1+\xi-kx\xi)^{\frac{-\beta_1}{k}}(1+\xi-kz\xi)^{\frac{-\beta_2}{k}} {}_2F_{1,k}(\beta_1,\beta_2;\gamma-\alpha;y\sigma))\,d\xi, \tag{3.17}$$
$$\big(\mathrm{Re}(\gamma) > \mathrm{Re}(\alpha) > 0\big),$$
$$\sigma := \frac{1+\xi}{(1+\xi-kx\xi)(1+\xi-kz\xi)};$$

$$H_{C,k}(\alpha,\beta_1,\beta_2;\gamma;x,y,z) = \frac{2\Gamma_k(\gamma)}{k\Gamma_k(\alpha)\Gamma_k(\gamma-\alpha)} \int_0^{\frac{\pi}{2}} (\sin^2\xi)^{\frac{\alpha}{k}-\frac{1}{2}}(\cos^2\xi)^{\frac{\gamma-\alpha}{k}-\frac{1}{2}}$$
$$\times (1-kx\sin^2\xi)^{\frac{-\beta_1}{k}}(1-kz\sin^2\xi)^{\frac{-\beta_2}{k}} {}_2F_{1,k}(\beta_1,\beta_2;\gamma-\alpha;y\sigma)\,d\xi, \tag{3.18}$$
$$\big(\mathrm{Re}(\gamma) > \mathrm{Re}(\alpha) > 0\big),$$
$$\sigma := \frac{\cos^2\xi}{(1-kx\sin^2\xi)(1-kz\sin^2\xi)}.$$

Here, ${}_2F_{1,k}$ is the k-hypergeometric function defined in (1.10)*.*

Proof. In [13] (see also [2, 6]), the well-known integral representation of H_C is given by

$$H_C(\alpha,\beta_1,\beta_2;\gamma;x,y,z) = \frac{\Gamma(\gamma)}{\Gamma(\alpha)\Gamma(\beta_1)\Gamma(\gamma-\alpha-\beta_1)}$$
$$\times \int_0^1\int_0^1 \xi^{\alpha-1}\eta^{\beta_1-1}(1-\xi)^{\gamma-\alpha-1}(1-\eta)^{\gamma-\alpha-\beta_1-1}(1-x\xi)^{\beta_2-\beta_1}$$
$$\times \left(1-x\xi-y\eta-z\xi+y\xi\eta+xz\xi^2\right)^{-\beta_2} d\xi d\eta.$$

Using this integral representation in (2.9) and applying (1.8), we obtain

$$H_{C,k}(\alpha,\beta_1,\beta_2;\gamma;x,y,z) = H_C\left(\frac{\alpha}{k},\frac{\beta_1}{k},\frac{\beta_2}{k};\frac{\gamma}{k};kx,ky,kz\right)$$
$$= \frac{\Gamma(\frac{\gamma}{k})}{\Gamma(\frac{\alpha}{k})\Gamma(\frac{\beta_1}{k})\Gamma(\frac{\gamma-\alpha-\beta_1}{k})}$$
$$\times \int_0^1\int_0^1 \xi^{\frac{\alpha}{k}-1}\eta^{\frac{\beta_1}{k}-1}(1-\xi)^{\frac{\gamma-\alpha}{k}-1}(1-\eta)^{\frac{\gamma-\alpha-\beta_1}{k}-1}(1-kx\xi)^{\frac{\beta_2-\beta_1}{k}}$$
$$\times \left(1-kx\xi-ky\eta-kz\xi+ky\xi\eta+k^2xz\xi^2\right)^{\frac{-\beta_2}{k}} d\xi d\eta$$
$$= \frac{k^{1-\frac{\gamma}{k}}\Gamma_k(\gamma)}{k^{1-\frac{\alpha}{k}}\Gamma_k(\alpha)k^{1-\frac{\beta_1}{k}}\Gamma_k(\beta_1)k^{1-\frac{\gamma-\alpha-\beta_1}{k}}\Gamma_k(\gamma-\alpha-\beta_1)}$$

$$\times \int_0^1 \int_0^1 \xi^{\frac{\alpha}{k}-1} \eta^{\frac{\beta_1}{k}-1} (1-\xi)^{\frac{\gamma-\alpha}{k}-1} (1-\eta)^{\frac{\gamma-\alpha-\beta_1}{k}-1} (1-kx\xi)^{\frac{\beta_2-\beta_1}{k}}$$
$$\times \left(1-kx\xi-ky\eta-kz\xi+ky\xi\eta+k^2xz\xi^2\right)^{\frac{-\beta_2}{k}} d\xi d\eta$$

which completes the proof of (3.13).

The other integral representations (3.14)-(3.18) of $H_{C,k}$ are similarly proved by considering (1.8) and (1.11) after the well-known integral representations of H_C given in [2] are used in the right-hand side of (2.9). □

Remark 5. For $k = 1$, the integral representations obtained in this study are reduced to the well-known integral representations of the classical Srivastava hypergeometric functions given in [2] (see also [6, 12, 13]).

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References

[1] P. Agarwal, "Some inequalities involving Hadamard-type k-fractional integral operators," *Mathematical Methods in the Applied Sciences*, vol. 40, no. 11, pp. 3882–3891, 2017, doi: 10.1002/mma.4270.

[2] J. Choi, A. Hasanov, H. M. Srivastava, and M. Turaev, "Integral representations for Srivastava's triple hypergeometric functions," *Taiwanese J. Math.*, vol. 15, no. 6, pp. 2751–2762, 2011, doi: 10.11650/twjm/1500406495.

[3] J. Choi and R. K. Parmar, "Generalized Srivastava's triple hypergeometric functions and their associated properties," *J. Nonlinear Sci. Appl*, vol. 10, pp. 817–827, 2017, doi: 10.22436/jnsa.010.02.41.

[4] R. Díaz and E. Pariguan, "On hypergeometric functions and Pochhammer k-symbol," *Divulg. Mat.*, vol. 15, no. 2, pp. 179–192, 2007.

[5] Ö. Gürel Yılmaz, R. Aktaş, and F. Taşdelen, "On some formulas for the k-analogue of Appell functions and generating relations via k-fractional derivative," *Fractal Fract.*, vol. 4, no. 4, p. 48, 2020, doi: 10.3390/fractalfract4040048.

[6] A. Hasanov, H. M. Srivastava, and M. Turaev, "Decomposition formulas for some triple hypergeometric functions," *J. Math. Anal. Appl.*, vol. 324, no. 2, pp. 955–969, 2006, doi: 10.1016/j.jmaa.2006.01.006.

[7] İ. O. Kıymaz, A. Çetinkaya, and P. Agarwal, "A study on the k-generalizations of some known functions and fractional operators," *J. Inequal. Spec. Funct.*, vol. 8, no. 4, pp. 31–41, 2017.

[8] M.-J. Luo, G. V. Milovanovic, and P. Agarwal, "Some results on the extended beta and extended hypergeometric functions," *Applied Mathematics and Computation*, vol. 248, pp. 631–651, 2014, doi: 10.1016/j.amc.2014.09.110.

[9] S. Mubeen and G. M. Habibullah, "An integral representation of some k-hypergeometric functions," *Int. Math. Forum*, vol. 7, no. 4, pp. 203–207, 2012.

[10] S. Mubeen, S. Iqbal, and G. Rahman, "Contiguous function relations and an integral representation for Appell *k*-series $F_{1,k}$," *Inter. J. Math. Research*, vol. 4, no. 2, pp. 53–63, 2015, doi: 10.18488/journal.24/2015.4.2/24.2.53.63.

[11] S. Mubeen and A. Rehman, "A note on *k*-gamma function and Pochhammer *k*-symbol," *J. Inf. Math. Sci.*, vol. 6, no. 2, pp. 93–107, 2014.

[12] H. M. Srivastava, "Hypergeometric functions of three variables," *Gaṇita*, vol. 15, no. 2, pp. 97–108, 1964.

[13] H. M. Srivastava, "Some integrals representing triple hypergeometric functions," *Rend. Circ. Mat. Palermo*, vol. 16, pp. 99–115, 1967, doi: 10.1007/BF02844089.

[14] H. M. Srivastava, P. Agarwal, and S. Jain, "Generating functions for the generalized Gauss hypergeometric functions," *Applied Mathematics and Computation*, vol. 247, pp. 348–352, 2014, doi: 10.1016/j.amc.2014.08.105.

[15] H. M. Srivastava and P. W. Karlsson, *Multiple Gaussian hypergeometric series.* New York, Chichester, Brisbane and Toronto: Halsted Press (Ellis Horwood Limited), John Wiley and Sons, 1985.

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