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ON FINITELY G-SUPPLEMENTED MODULES

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Abstract. In this work, some properties of finitely g-supplemented modules are investigated. Let M be a finitely g-supplemented $R-$module and N be a finitely generated or small submodule of M. Then M/N is finitely g-supplemented. Let $f : M \longrightarrow N$ be an $R-$module epimomorphism with small kernel. If M is finitely g-supplemented, then N is also finitely g-supplemented. Let M be a finitely g-supplemented module, $Rad_g M \leq U \leq M$ and U be finitely generated. Then $U/Rad_g M$ is a direct summand of $M/Rad_g M$.

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1. INTRODUCTION

Throughout this paper all rings are associative with identity and all modules are unital left modules.

Let R be a ring and M be an $R-$module. We denote a submodule N of M by $N \leq M$. Let M be an $R-$module and $N \leq M$. If $L = M$ for every submodule L of M such that $M = N + L$, then N is called a *small* (or *superfluous*)submodule of M and denoted by $N \ll M$. A submodule N of an R -module M is called an *essential* submodule, denoted by $N \trianglelefteq M$, in case $K \cap N \neq 0$ for every submodule $K \neq 0$, or equvalently, $N \cap L = 0$ for $L \leq M$ implies that $L = 0$. Let M be an $R-$module and K be a submodule of M. K is called a *generalized small* (briefly, *g-small*) submodule of M if for every essential submodule T of M with the property $M = K + T$ implies that $T = M$, we denote this by $K \ll_g M$ (in [6], it is called an *e-small* submodule of M and denoted by $K \ll_e M$). Let M be an $R-$module and $U, V \leq M$. If $M = U + V$ and V is minimal with respect to this property, or equivalently, $M = U + V$ and $U \cap V \ll V$, then V is called a *supplement* of U in M. M is said to be *supplemented* if every submodule of M has a supplement in M. M is said to be *finitely supplemented* (briefly, *f-supplemented*) if every finitely generated submodule of M has a supplement in M. Let M be an $R-$module and $U, V \leq M$. If $M = U + V$ and $M = U + T$ with $T \trianglelefteq V$ implies that $T =$

V, or equivalently, $M = U + V$ and $U \cap V \ll_g V$, then V is called a *g-supplement* of U in M. M is said to be *g-supplemented* if every submodule of M has a g-supplement in M. The intersection of maximal submodules of an R-module M is called the *radical* of M and denoted by $RadM$. If M have no maximal submodules, then we denote $RadM = M$. The intersection of essential maximal submodules of an R-module M is called a *generalized radical* (briefly, *g-radical*) of M and denoted by Rad_gM (in [6], it is denoted by Rad_eM). If M have no essential maximal submodules, then we denote $Rad_gM = M$. An $R-$module M is said to be *noetherian* if every submodule of M is finitely generated. Let M be an $R-$module and $K \leq V \leq M$. We say V *lies above* K in M if $V/K \ll M/K$.

More details about supplemented modules are in [1, 5]. More informations about g-small submodules and g-supplemented modules are in [2, 3].

Lemma 1. *Let M be an R -module and $K,N \leq M$. Consider the following conditions.*

1. *If $K \leq N$ and N is generalized small submodule of M, then K is a generalized small submodule of M.*
2. *If K is contained in N and a generalized small submodule of N, then K is a generalized small submodule in submodules of M which contain N.*
3. *If $K \ll_g L$ and $N \ll_g T$ with L, $T \leq M$, then $K + N \ll_g L + T$.*
4. *$Rad_gM = \sum_{L \ll_g M} L$.*
5. *Let T be an $R-$module and $f : M \to T$ be an R -module homomorphism. If $K \ll_g M$, then $f(K) \ll_g T$. Here $f(Rad_gM) \leq Rad_gT$.*

Proof. See [3, Lemma 1 and Lemma 3]. □

2. FINITELY G-SUPPLEMENTED MODULES

Definition 1. Let M be an $R-$module. If every finitely generated submodule of M has a g-supplement in M, then M is called a finitely g-supplemented (or briefly fg-supplemented) module. (See also [4])

Clearly we can see that every f-supplemented module is fg-supplemented.

Proposition 1. *Every g-supplemented module is fg-supplemented.*

Proof. Clear from definitions. □

Proposition 2. *Let M be a fg-supplemented $R-$module. If M is noetherian, then M is g-supplemented.*

Proof. Let $U \leq M$. Since M is noetherian, U is finitely generated and since M is fg-supplemented, U has a g-supplement in M. Hence M is g-supplemented. □

Lemma 2. *Let M be a fg-supplemented $R-$module and N be a finitely generated submodule of M. Then M/N is fg-supplemented.*

Proof. Let U/N be a finitely generated submodule of M/N. Since U/N finitely generated, there exists a finitely generated submodule K of M such that $U = K + N$. Since K and N are finitely generated, $U = K + N$ is also finitely generated. By hypothesis, U has a g-supplement V in M. Then by [2, Lemma 4], $(V + N)/N$ is a g-supplement of U/N in M/N. Hence M/N is fg-supplemented. □

Corollary 1. *Let M be a fg-supplemented $R-$module and N be a cyclic submodule of M. Then M/N is fg-supplemented.*

Proof. Clear from Lemma 2. □

Corollary 2. *Let $f : M \longrightarrow N$ be an $R-$module epimorphism and Kef be finitely generated. If M is fg-supplemented, then N is also fg-supplemented.*

Proof. Since M is fg-supplemented and Kef is finitely generated, by Lemma 2, M/Kef is fg-supplemented. Then by $M/Kef \cong N$, N is also fg-supplemented. □

Corollary 3. *Let $f : M \longrightarrow N$ be an $R-$module epimorphism with cyclic kernel. If M is fg-supplemented, then N is also fg-supplemented.*

Proof. Clear from Corollary 2. □

Lemma 3. *Let M be an fg-supplemented module, $Rad_gM \leq U \leq M$ and U be finitely generated. Then U/Rad_gM is a direct summand of M/Rad_gM.*

Proof. Since M is fg-supplemented and U is a finitely generated submodule of M, U has a g-supplement V in M. Here $M = U + V$ and $U \cap V \ll_g V$. By Lemma 1, $U \cap V \leq Rad_gM$. Then $\frac{M}{Rad_gM} = \frac{U+V}{Rad_gM} = \frac{U}{Rad_gM} + \frac{V+Rad_gM}{Rad_gM}$ and $\frac{U}{Rad_gM} \cap \frac{V+Rad_gM}{Rad_gM} = \frac{U\cap V+Rad_gM}{Rad_gM} = \frac{Rad_gM}{Rad_gM} = 0$. Hence $\frac{M}{Rad_gM} = \frac{U}{Rad_gM} \oplus \frac{V+Rad_gM}{Rad_gM}$ and U/Rad_gM is a direct summand of M/Rad_gM. □

Corollary 4. *Let M be a fg-supplemented module and Rad_gM be finitely generated. Then every finitely generated submodule of M/Rad_gM is a direct summand of M/Rad_gM.*

Proof. Let U/Rad_gM be a finitely generated submodule of M/Rad_gM. Then there exists a finitely generated submodule K of M such that $U = K + Rad_gM$. Since K and Rad_gM are finitely generated, $U = K + Rad_gM$ is also finitely generated. Then by Lemma 3, U/Rad_gM is a direct summand of M/Rad_gM. □

Lemma 4. *Let M be a fg-supplemented $R-$module and $N \ll M$. Then M/N is fg-supplemented.*

Proof. Let U/N be a finitely generated submodule of M/N. Then there exists a finitely generated submodule K of M such that $U = K + N$. Since M is fg-supplemented, K has a g-supplement V in M. Here $M = K + V$ and $K \cap V \ll_g V$. Since $K \leq U$, $M = K + V = U + V$. Let $M = U + T$ with $T \trianglelefteq V$. Then $M = U + T = K + N + T$ and since $N \ll M$, $K+T=M$. Since V is a g-supplement of K in M and $T \trianglelefteq V$, by definition, $T=V$. Hence V is a g-supplement of U in M. By [2, Lemma 4], $(V+N)/N$ is a g-supplement of U/N in M/N. Hence M/N is fg-supplemented. □

Corollary 5. *Let $f: M \longrightarrow N$ be an $R-$module epimorphism with small kernel. If M is fg-supplemented, then N is also fg-supplemented.*

Proof. Since M is fg-supplemented and $Kef \ll M$, by Lemma 4, M/Kef is fg-supplemented. Then by $M/Kef \cong N$, N is also fg-supplemented. □

Lemma 5. *Let M be a fg-supplemented $R-$module and $Rad_g M \ll M$. Then every finitely generated submodule of $M/Rad_g M$ is a direct summand of $M/Rad_g M$.*

Proof. Let $U/Rad_g M$ be a finitely generated submodule of $M/Rad_g M$. Then there exists a finitely generated submodule K of M such that $U=K+Rad_g M$. Since M is fg-supplemented, K has a g-supplement V in M. Here $M=K+V$ and $K \cap V \ll_g V$. Since $K \leq U$, $M=K+V=U+V$. Let $M=U+T$ with $T \trianglelefteq V$. Then $M=U+T=K+Rad_g M+T$ and since $Rad_g M \ll M$, $K+T=M$. Since V is a g-supplement of K in M and $T \trianglelefteq V$, by definition, $T=V$. Hence V is a g-supplement of U in M. Here $M=U+V$ and $U \cap V \ll_g V$. By Lemma 1, $U \cap V \leq Rad_g M$. Then $\frac{M}{Rad_g M} = \frac{U+V}{Rad_g M} = \frac{U}{Rad_g M} + \frac{V+Rad_g M}{Rad_g M}$ and $\frac{U}{Rad_g M} \cap \frac{V+Rad_g M}{Rad_g M} = \frac{U \cap V+Rad_g M}{Rad_g M} = \frac{Rad_g M}{Rad_g M} = 0$. Hence $\frac{M}{Rad_g M} = \frac{U}{Rad_g M} \oplus \frac{V+Rad_g M}{Rad_g M}$ and $U/Rad_g M$ is a direct summand of $M/Rad_g M$. □

Corollary 6. *Let M be a fg-supplemented $R-$module and $Rad_g M \ll M$. Then every finitely generated submodule of $M/RadM$ is a direct summand of $M/RadM$.*

Proof. Since $Rad_g M \ll M$, $RadM = Rad_g M$. Then by Lemma 5, every finitely generated submodule of $M/RadM$ is a direct summand of $M/RadM$. □

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