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GENERAL DECAY AND BLOW UP OF SOLUTIONS FOR A KIRCHHOFF-TYPE EQUATION WITH VARIABLE-EXPONENTS

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Abstract. A nonlinear Kirchhoff-type equation with logarithmic nonlinearity and variable-exponents is studied. Firstly, the global existence is shown. Next, by using an integral inequality due to Komornik the general decay result is obtained. Finally, the blow-up of the solutions is proved.

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1. INTRODUCTION

In this paper, we are interested in the following problem:

$$\begin{cases} u_{tt} - M\left(\|\nabla u\|_2^2\right)\Delta u(t) + \beta_1 |u_t(t)|^{m(x)-2}u_t(t) = u|u|^{p(x)-2}\ln|u|^k. \\ u(x,0) = u_0(x), \;\; u_t(x,0) = u_1(x), \;\; in \;\; \Omega \\ u(x,t) = 0, \;\; in \;\; \partial\Omega \times (0,\infty), \end{cases} \tag{1.1}$$

where $\Omega \subset \mathbb{R}^N$ is a bounded domain with sufficiently smooth boundary $\partial\Omega$, $\beta_1, k > 0$, $M(s)$ is a positive C^1-function like $M(s) = 1 + s^\gamma$, $\gamma > 0$. $p(.)$ and $m(.)$ are the variable exponents given as measurable functions on $\overline{\Omega}$ such that:

$$\begin{aligned} &2 \le p^- \le p(x) \le p^+ \le p^*, \\ &2 \le m^- \le m(x) \le m^+ \le m^*, \end{aligned} \tag{1.2}$$

where

$$\begin{aligned} p^- &= ess \inf_{x\in\Omega} p(x), \qquad & p^+ &= ess \sup_{x\in\Omega} p(x), \\ m^- &= ess \inf_{x\in\Omega} m(x), \qquad & m^+ &= ess \sup_{x\in\Omega} m(x), \end{aligned} \tag{1.3}$$

and

$$p^*, m^* = \frac{2(n-1)}{n-2}, \text{ if } n \geq 3. \tag{1.4}$$

Of course, the problem we are going to study is of the Kirchhoff-type equation developed by Kirchhoff in the year 1876 [9], which is concerned with the study of describing small vibration amplitude of elastic stings.

The variable-exponents problems are among the most important and fertile research fields in recent years, and many researchers have touched on this matter. These problems appear in many physical phenomena and in many different branches of sciences and their mathematical modeling, such as image processing, electrical fluids, elasticity theory, and linear viscosity. For more in-depth study of this topic, we refer the reader to [1–4, 15]. Also, many results were reached regarding global existence, asymptotic behavior and blow-up of the solutions.

As for logarithmic nonlinearity, it appears in many fields of physical and applied sciences, including quantum mechanics, nuclear physics, and optics. The same goes for inflationary cosmology and symmetry theories. This is what led many mathematicians and researchers to address these applications and work on this type of problems.

In this context, we mention previous works that have a direct relationship with our present work. In case $\gamma = 0$, the authors in [5] determined the existence and uniqueness of a local solution in time by the Faedo-Galerkin method and they proved the blow-up of a solution under suitable conditions (see [6, 8–11]). Recently, in the present of a delay term also in the case $\gamma = 0$ in [14, 16] the authors established a global existence result under suitable conditions on the initial data only without imposing the Sobolev Logarithmic Inequality. After that, by the Komornik's lemma they proved the stability results, and they presented a numerical study that supports their results.

There are other works that study the variable exponent, including the followings. In [12] the authors considered a nonlinear Kirchhoff-type equation with distributed delay and variable-exponents in the presence of the source term ($k = 0$). Under suitable hypotheses they proved the blow-up of solutions, and by using an integral inequality due to Komornik they obtained the general decay result but in the case $b = 0$ (in the absence of the source term). Also, in [12–14], the authors considered a nonlinear $p(x)$-Laplacian equation with time delay and variable exponents, and they proved the blow-up of the solutions. Then, by applying an integral inequality due to Komornik, they obtained the decay result. Starting from all these works and supplementing them, we will try to study our problem (1.1), as we consider the coupling with the logarithmic nonlinearity and the variable exponent, with a thorough study it makes our problem different from what was previously studied.

The remainder of our work is organized as follows: in Section 2, we lay down the hypotheses, concepts and lemmas we need. In Section 3, the global existence is shown and in the Section 4, we obtain the general decay result. Next, in Section 5, we prove the blow-up of the solutions. Finally, we put a general conclusion.

2. Preliminaries

In this section,we introduce some notations and materials that will be used throughout this work. Firstly, let $p\colon \Omega \to [1,\infty)$ be a measurable function. We define the variable-exponent Lebesgue space with a variable exponent $p(.)$ by

$$L^{p(.)}(\Omega)=\left\{u:\Omega\to\mathbb{R};\ \text{ measurable in }\ \Omega:\ \int_{\Omega}|u|^{p(.)}dx<\infty\right\},$$

with a Luxemburg-type norm

$$\|u\|_{p(.)}=\inf\left\{\sigma>0:\ \int_{\Omega}|\frac{u}{\sigma}|^{p(x)}dx\leq 1\right\}.$$

Equipped with this norm, $L^{p(.)}(\Omega)$ is a Banach space (see [4]). Next, we define the variable-exponent Sobolev space $W^{1,p(.)}(\Omega)$ as follows

$$W^{1,p(.)}(\Omega)=\left\{u\in L^{p(.)}(\Omega);\nabla u\ \text{ exists and }\ |\nabla u|\in L^{p(.)}(\Omega)\right\},$$

with

$$\|u\|_{1,p(.)}=\|u\|_{p(.)}+\|\nabla u\|_{p(.)}.$$

$W^{1,p(.)}(\Omega)$ is a Banach space, and the closure of $C_0^{\infty}(\Omega)$ is defined by $W_0^{1,p(.)}(\Omega)$.

For $u\in W_0^{1,p(.)}(\Omega)$, we give the equivalent norm

$$\|u\|_{1,p(.)}=\|\nabla u\|_{p(.)}.$$

$W_0^{-1,p'(.)}(\Omega)$ denotes the dual of $W_0^{1,p(.)}(\Omega)$, where $\frac{1}{p(.)}+\frac{1}{p'(.)}=1$. Also, we suppose that

$$|p(x)-p(y)|\leq-\frac{A}{\log|x-y|}\ \text{ and }\ |m(x)-m(y)|\leq-\frac{B}{\log|x-y|}\ \text{ for all }\ x,y\in\Omega, \tag{2.1}$$

$A,B>0$ and $0<\zeta<1$ with $|x-y|<\zeta$. (Log-Hölder inequality).

Lemma 1 ([2]). *Suppose that $p(.)$ verifies 1.4 and let Ω be a bounded domain of $\mathbb{R}^n$. Then, $\exists c_*=c(\Omega,p^+,p^-)>0$,*

$$\|u\|_{p(.)}\leq c_*\|\nabla u\|_{p(.)},\ \ \forall u\in W_0^{1,p(.)}(\Omega).$$

Lemma 2 ([2]). *If $p:\overline{\Omega}\to[1,\infty)$ is continuous,*

$$2\leq p^-\leq p(x)\leq p^+\leq\frac{2n}{n-2},\ \ n\geq 3, \tag{2.2}$$

is held, then the embedding $H_0^1(\Omega)\to L^{p(.)}(\Omega)$ is continuous.

Lemma 3 (Unit ball property, [3])**.** *Let $p \geq 1$ be a measurable function on Ω. Then,*

$$\|g\|_{p(.)} \leq 1, \text{ if and only if } \rho_{p(.)}(g) \leq 1,$$

where

$$\rho_{p(.)}(g) = \int_{\Omega} |g(x)|^{p(x)} dx.$$

Lemma 4 ([2])**.** *If $p \geq 1$ is a measurable function on Ω, then*

$$\min\left\{\|u\|_{p(.)}^{p^-}, \|u\|_{p(.)}^{p^+}\right\} \leq \rho_{p(.)}(u) \leq \max\left\{\|u\|_{p(.)}^{p^-}, \|u\|_{p(.)}^{p^+}\right\},$$

for all $u \in L^{p(\cdot)}(\Omega)$ and for a.e. $x \in \Omega$.

Theorem 1. *Assume that (1.2) and (2.1) hold. Then, for all $(u_0, u_1) \in H_0^1(\Omega) \times L^2(\Omega)$ there exists a unique solution u of problem (1.1) on $(0,T)$ such that*

$$\begin{aligned} &u \in C((0,T), H_0^1(\Omega)) \cap C^1((0,T), L^2(\Omega)), \\ &u_t \in L^{m(.)}(\Omega \times (0,T)). \end{aligned} \tag{2.3}$$

Firstly, we define the energy functional.

Lemma 5. *The functional of energy E, given by*

$$\begin{aligned} E(t) = &\frac{1}{2}\|u_t\|_2^2 + \frac{1}{2}\|\nabla u(t)\|_2^2 + \frac{1}{2(\gamma+1)}\|\nabla u(t)\|_2^{2(\gamma+1)} \\ &+ k\int_{\Omega} \frac{|u|^{p(x)}}{p^2(x)} dx - \int_{\Omega} \frac{|u|^{p(x)} \ln|u|^k}{p(x)} dx, \end{aligned} \tag{2.4}$$

satisfies

$$E'(t) \leq -\beta_1 \int_{\Omega} |u_t(t)|^{m(x)} dx \leq 0. \tag{2.5}$$

Proof. By the inner product of (1.1) with u_t, we get

$$\begin{aligned} &\frac{d}{dt}\left\{\frac{1}{2}\|u_t\|_2^2 + \frac{1}{2}\|\nabla u(t)\|_2^2 + \frac{1}{2(\gamma+1)}\|\nabla u(t)\|_2^{2(\gamma+1)} + k\int_{\Omega} \frac{|u|^{p(x)}}{p^2(x)} dx - \int_{\Omega} \frac{|u|^{p(x)} \ln|u|^k}{p(x)} dx\right\} \\ &= -\beta_1 \int_{\Omega} |u_t(t)|^{m(x)} dx. \end{aligned} \tag{2.6}$$

Hence, we find (2.4) and (2.5), also E is a decreasing function. This completes the proof. □

3. Global existence

Now, we show that the solution of (1.1) is uniformly bounded and global in time. For this purpose, we set

$$I(t) = \|\nabla u\|_2^2 - \int_\Omega |u|^{p(x)} \ln|u|^k dx, \tag{3.1}$$

$$J(t) = \frac{1}{2}\|\nabla u(t)\|_2^2 + \frac{1}{2(\gamma+1)}\|\nabla u(t)\|_2^{2(\gamma+1)} + k\int_\Omega \frac{|u|^{p(x)}}{p^2(x)}dx - \int_\Omega \frac{|u|^{p(x)}\ln|u|^k}{p(x)}dx. \tag{3.2}$$

Hence,

$$E(t) = J(t) + \frac{1}{2}\|u_t\|_2^2. \tag{3.3}$$

Lemma 6. *Suppose that the initial data $u_0, u_1 \in H^1(\Omega)\times L^2(\Omega)$ satisfying $I(0) > 0$ and*

$$\xi := C_*(p^+,k)\left(\frac{2p^-}{p^- - 2}E(0)\right)^{\frac{p^+ + k - 2}{2}} < 1. \tag{3.4}$$

Then $I(t) > 0$, for any $t \in [0,T]$.

Proof. Of course, for the first case, if $\int_\Omega |u|^{p(x)}\ln|u|^k dx < 0$, the result is clear. Therefore, we can go directly to the second case and impose: $\int_\Omega |u|^{p(x)}\ln|u|^k dx > 0$. Since $I(0) > 0$ we deduce by continuity that there exists $T^* \le T$ such that $I(t) \ge 0$ for all $t \in [0,T^*]$. This implies that $\forall t \in [0,T^*]$,

$$\begin{aligned} J(t) &\ge \frac{1}{2}\|\nabla u(t)\|_2^2 + \frac{1}{2(\gamma+1)}\|\nabla u(t)\|_2^{2(\gamma+1)} + \frac{k}{(p^+)^2}\int_\Omega |u|^{p(x)}dx \\ &\quad - \frac{1}{p^-}\int_\Omega |u|^{p(x)}\ln|u|^k dx \\ &\ge \frac{p^- - 2}{2p^-}\|\nabla u(t)\|_2^2 + \frac{1}{2(\gamma+1)}\|\nabla u(t)\|_2^{2(\gamma+1)} + \frac{k}{(p^+)^2}\int_\Omega |u|^{p(x)}dx + \frac{1}{p^-}I(t) \\ &\ge \frac{p^- - 2}{2p^-}\|\nabla u(t)\|_2^2. \end{aligned}$$

Hence,

$$\|\nabla u(t)\|_2^2 \le \frac{2p^-}{p^- - 2}J(t) \le \frac{2p^-}{p^- - 2}E(t) \le \frac{2p^-}{p^- - 2}E(0). \tag{3.5}$$

On the other hand, using the facts that $\ln|u| < |u|$ and $|u| > 1$, we get

$$\int_\Omega |u|^{p(x)}\ln|u|^k dx < \int_\Omega |u|^{p^+ + k}dx. \tag{3.6}$$

Then, the embedding $H_0^1(\Omega)) \hookrightarrow L^{p^+ + k}(\Omega)$ yields

$$\int_\Omega |u|^{p^+ + k}dx \le C_*(p^+,k)\|\nabla u(t)\|_2^{p^+ + k} = C_*(p^+,k)\|\nabla u(t)\|_2^2\|\nabla u(t)\|_2^{p^+ + k - 2}. \tag{3.7}$$

Using (3.5), we find

$$\int_{\Omega} |u|^{p(x)} \ln |u|^k dx < C_*(p^+,k) \left(\frac{2p^- E(0)}{p^- - 2} \right)^{\frac{p^+ + k - 2}{2}} \|\nabla u(t)\|_2^2 < \xi \|\nabla u(t)\|_2^2, \quad (3.8)$$

where $\xi = C_*(p^+,k) \left(\frac{2p^- E(0)}{p^- - 2} \right)^{\frac{p^+ + k - 2}{2}}$ and $C_*(p^+,k)$ is the embedding constant.

According to (3.1) and (3.5), we get

$$I(t) > (1-\xi)\|\nabla u(t)\|_2^2 > 0, \quad \forall t \in [0,T^*]. \quad (3.9)$$

By repeating this procedure, T^* can be extended to T. This completes the proof. □

Remark 1. Under the conditions of Lemma 6, we have $J(t) \geq 0$ and consequently $E(t) \geq 0, \quad \forall t \in [0,T]$. Hence, by (3.2) and (3.5) we find

$$\begin{aligned} \|u_t(t)\|_2^2 &\leq 2E(0), \\ \|\nabla u(t)\|_{2(\gamma+1)}^2 &\leq 2(\gamma+1)E(0), \\ \int_{\Omega} |u|^{p(x)} dx &\leq \frac{(p^+)^2}{k} E(0). \end{aligned} \quad (3.10)$$

4. General decay

In this section, we state and prove the general decay of system (1.1). The following lemma will be used in the upcoming results.

Lemma 7 (Komornik, [10])**.** *Let $E : \mathbb{R}_+ \to \mathbb{R}_+$ be a non-increasing function and suppose that $\exists \sigma, \omega > 0$ such that*

$$\int_s^{\infty} E^{1+\sigma}(t)dt \leq \frac{1}{\omega} E^{\sigma}(0)E(s) = cE(s), \quad \forall s > 0. \quad (4.1)$$

Then, we have for $\forall t \geq 0$

$$\begin{cases} E(t) \leq cE(0)/(1+t)^{\frac{1}{\sigma}}, & \text{if } \sigma > 0, \\ E(t) \leq cE(0)e^{-\omega t}, & \text{if } \sigma = 0. \end{cases} \quad (4.2)$$

In the next step, we give the third result of our work.

Theorem 2. *Assume that (1.2) and (2.1) hold. Then $\exists c, \lambda > 0$ such that the solution of (1.1) satisfies*

$$\begin{cases} E(t) \leq cE(0)/(1+t)^{\frac{2}{m^+ - 2}}, & \text{if } m^+ > 2, \\ E(t) \leq cE^{-\lambda t}, & \text{if } m(x) = 2. \end{cases} \quad (4.3)$$

Proof. Multiplying (1.1) by $uE^q(t)$, for $q>0$ to be specified later, and integrating the result over $\Omega\times(s,T), s<T$, gives

$$\int_s^T E^q(t)\int_\Omega\left\{uu_{tt}-M\left(\|\nabla u\|_2^2\right)u\Delta u(t)+\beta_1 uu_t|u_t(t)|^{m(x)-2}-|u|^{p(x)}\ln|u|^k\right\}dxdt=0, \tag{4.4}$$

which implies that

$$\int_s^T E^q(t)\int_\Omega\left\{\frac{d}{dt}(uu_t)-|u_t|^2+M\left(\|\nabla u\|_2^2\right)|\nabla u|^2+\beta_1 uu_t|u_t(t)|^{m(x)-2}|u|^{p(x)}\ln|u|^k\right\}dxdt$$
$$=0. \tag{4.5}$$

By (2.4) and the relation

$$\frac{d}{dt}\left(E^q(t)\int_\Omega uu_t dx\right)=qE^{q-1}(t)E'(t)\int_\Omega uu_t dx+E^q(t)\frac{d}{dt}\left(\int_\Omega uu_t dx\right),$$

we get

$$\begin{aligned}
&2\int_s^T E^{q+1}(t)dt\\
&=\underbrace{\int_s^T\frac{d}{dt}\left(E^q(t)\int_\Omega uu_t dx\right)dt}_{I_1}-\underbrace{q\int_s^T\left(E^{q-1}(t)E'(t)\int_\Omega uu_t dx\right)dt}_{I_2}\\
&\quad+\underbrace{2\int_s^T\left(E^q(t)\int_\Omega|\nabla u|^2dx\right)dt}_{I_3}+\underbrace{2k\int_s^T\left(E^q(t)\int_\Omega\frac{|u|^{p(x)}}{p^2(x)}dx\right)dt}_{I_4}\\
&\quad+\underbrace{\frac{\gamma+2}{\gamma+1}\int_s^T\left(E^q(t)\int_\Omega\|\nabla u\|_2^{2\gamma}|\nabla u|^2dx\right)dt}_{I_5}\\
&\quad+\underbrace{\beta_1\int_s^T\left(E^q(t)\int_\Omega uu_t|u_t(t)|^{m(x)-2}dx\right)dt}_{I_6}\\
&\quad-\underbrace{\int_s^T\left(E^q(t)\int_\Omega|u|^{p(x)}\ln|u|^kdx\right)dt}_{I_7}-\underbrace{2\int_s^T\left(E^q(t)\int_\Omega\frac{|u|^{p(x)}\ln|u|^k}{p(x)}dx\right)dt}_{I_8}.
\end{aligned}\tag{4.6}$$

At this point, we estimate $I_i, i=1,\ldots,8$ of the RHS in (4.6), we have

$$\begin{aligned}
I_1&=E^q(T)\int_\Omega uu_t(x,T)dx-E^q(s)\int_\Omega uu_t(x,s)dx\\
&\le\frac{1}{2}E^q(T)\left\{\int_\Omega u^2(x,T)dx+\int_\Omega u_t^2(x,T)dx\right\}+\frac{1}{2}E^q(s)\left\{\int_\Omega u^2(x,s)dx+\int_\Omega u_t^2(x,s)dx\right\}
\end{aligned}$$

$$\leq \frac{1}{2}E^q(T)\left\{c_*\|\nabla u(T)\|_2^2+2E(T)\right\}+E^q(s)\left\{c_*\|\nabla u(s)\|_2^2+2E(s)\right\}$$
$$\leq c_1\left(E^{q+1}(T)+E^{q+1}(s)\right). \tag{4.7}$$

Since E is decreasing, we get

$$I_1 \leq cE^{q+1}(s) \leq E^q(0)E(s) \leq cE(s). \tag{4.8}$$

Similarly, we find

$$I_2 \leq -q\int_s^T E^{q-1}(t)E'(t)\left(c_*E(T)+E(T)\right)dt$$
$$\leq -c\int_s^T E^q(t)E'(t)dt \leq cE^{q+1}(s) \leq cE(s). \tag{4.9}$$

Next, we get

$$I_3 = 2\int_s^T \left(E^q(t)\|\nabla u\|^2\right)dt \leq 2\int_s^T E^q(t)E(t)dt \leq cE^{q+1}(s) \leq cE(s). \tag{4.10}$$

We estimate the next term as follows,

$$I_4 = 2k\int_s^T E^q(t)\int_\Omega \frac{|u|^{p(x)}}{p^2(x)}dxdt \leq c\int_s^T E^q(t)E(t)dt \leq cE^{q+1}(s) \leq cE(s). \tag{4.11}$$

For the fifth term, we have

$$I_5 = 2(\gamma+2)\int_s^T \left(E^q(t)\frac{\|\nabla u\|_2^{2(\gamma+1)}}{2(\gamma+1)}\right)dt \leq 2(\gamma+2)\int_s^T E^q(t)E(t)dt \leq cE^{q+1}(s)$$
$$\leq cE(s), \tag{4.12}$$

and using Young's inequality, we find

$$I_6 = \beta_1\int_s^T \left(E^q(t)\int_\Omega uu_t|u_t(t)|^{m(x)-2}dx\right)dt$$
$$\leq \varepsilon\int_s^T \left(E^q(t)\int_\Omega |u(t)|^{m(x)}dx\right)dt + c\int_s^T \left(E^q(t)\int_\Omega c_\varepsilon(x)|u_t(t)|^{m(x)}dx\right)dt$$
$$\leq \varepsilon\int_s^T E^q(t)\left[\int_{\Omega_+}|u(t)|^{m^+}dx+\int_{\Omega_-}|u(t)|^{m^-}dx\right]dt$$
$$+c\int_s^T \left(E^q(t)\int_\Omega c_\varepsilon(x)|u_t(t)|^{m(x)}dx\right)dt.$$

By using the embeddings $H_0^1(\Omega)\hookrightarrow L^{m^-}(\Omega)$ and $H_0^1(\Omega)\hookrightarrow L^{m^+}(\Omega)$, we get

$$I_6 \leq \varepsilon\int_s^T E^q(t)\left[c\|\nabla u(t)\|_2^{m^+}+c\|\nabla u(t)\|_2^{m^-}\right]dt$$

$$
\begin{aligned}
&+c\int_s^T \left(E^q(t)\int_\Omega c_\varepsilon(x)|u_t(t)|^{m(x)}dx\right)dt \\
&\le \varepsilon\int_s^T E^q(t)\left[cE^{\frac{m^+-2}{2}}(0)E(t)+cE^{\frac{m^--2}{2}}(0)E(t)\right]dt \\
&\quad +c\int_s^T \left(E^q(t)\int_\Omega c_\varepsilon(x)|u_t(t)|^{m(x)}dx\right)dt \\
&\le c\varepsilon\int_s^T E^{q+1}(t)dt + c\int_s^T \left(E^q(t)\int_\Omega c_\varepsilon(x)|u_t(t)|^{m(x)}dx\right)dt. \qquad (4.13)
\end{aligned}
$$

Similarly, we get

$$
\begin{aligned}
I_7 &= \int_s^T \left(E^q(t)\int_\Omega |u|^{p(x)}\ln|u|^k dx\right)dt \\
&\le \xi\int_s^T E^q(t)\|\nabla u(t)\|_2^2 dt \\
&\le c\int_s^T E^{q+1}(t)dt \le cE^{q+1}(s) \le cE(s). \qquad (4.14)
\end{aligned}
$$

Finally, we have

$$
\begin{aligned}
I_8 &= \int_s^T \left(E^q(t)\int_\Omega \frac{|u|^{p(x)}\ln|u|^k}{p(x)}dx\right)dt \\
&\le \frac{1}{p^-}\int_s^T \left(E^q(t)\int_\Omega |u|^{p(x)}\ln|u|^k dx\right)dt \\
&\le cI_7 \le cE(s). \qquad (4.15)
\end{aligned}
$$

By combining (4.6)-(4.15), we find

$$
\int_s^T E^{q+1}(t)dt \le c\varepsilon\int_s^T E^{q+1}(t)dt + c\int_s^T \left(E^q(t)\int_\Omega c_\varepsilon(x)|u_t(t)|^{m(x)}dx\right)dt + cE(s). \qquad (4.16)
$$

Now, we choose ε small enough such that

$$
\int_s^T E^{q+1}(t)dt \le cE(s) + c\int_s^T \left(E^q(t)\int_\Omega c_\varepsilon(x)|u_t(t)|^{m(x)}dx\right)dt. \qquad (4.17)
$$

When ε is fixed, then $c_\varepsilon(x)\le M$ since $m(x)$ is bounded. Then,

$$
\begin{aligned}
\int_s^T E^{q+1}(t)dt &\le cE(s) + cM\int_s^T \left(E^q(t)\int_\Omega |u_t(t)|^{m(x)}dx\right)dt \\
&\le cE(s) - \frac{cM}{\beta_1}\int_s^T E^q(t)E'(t)dt \\
&\le cE(s) + \frac{cM}{\beta_1(q+1)}\left[E^{q+1}(s)-E^{q+1}(T)\right] \le cE(s). \qquad (4.18)
\end{aligned}
$$

Taking $T \to \infty$, we find

$$\int_s^{\infty} E^{q+1}(t)dt \leq cE(s). \tag{4.19}$$

Finally, Komornik's Lemma 7 (with $\sigma = q = \frac{m^+-2}{2}$) implies our result. The proof is complete. □

5. Blow-up

Now, to prove the blow-up, we assume that $E(0) < 0$ and $\int_\Omega |u|^{p(x)} \ln |u|^k dx > 0$. Let

$$\mathbb{H}(t) = -E(t) = \int_\Omega \frac{|u|^{p(x)} \ln |u|^k}{p(x)} dx - k\int_\Omega \frac{|u|^{p(x)}}{p^2(x)} dx$$
$$-\frac{1}{2}\|u_t\|_2^2 - \frac{1}{2}\|\nabla u(t)\|_2^2 - \frac{1}{2(\gamma+1)}\|\nabla u(t)\|_2^{2(\gamma+1)}. \tag{5.1}$$

From (2.5), we have

$$E(t) \leq E(0) < 0. \tag{5.2}$$

Then

$$\mathbb{H}'(t) = -E'(t) \geq \beta_1 \int_\Omega |u_t(t)|^{m(x)} dx \geq 0, \tag{5.3}$$

and

$$0 < \mathbb{H}(0) \leq \mathbb{H}(t) \leq \int_\Omega \frac{|u|^{p(x)} \ln |u|^k}{p(x)} dx \leq \frac{k}{p^-}\int_\Omega |u|^{p(x)} \ln |u|, \tag{5.4}$$

where

$$\rho(u) = \rho_{p(.)}(u) = \int_\Omega |u|^{p(x)} dx.$$

Lemma 8 ([13]). *Assume that (1.2) is held. Then $\exists c = c(\Omega) > 1$, such that*

$$\rho^{s/p^-}(u) \leq c\Big(\|\nabla u\|_2^2 + \rho(u)\Big). \tag{5.5}$$

Then we have

$$\| u\|_{p^-}^s \leq c\Big(\|\nabla u\|_2^2 + \|u\|_{p^-}^{p^-}\Big), \tag{5.6}$$

$$\rho^{s/p^-}(u) \leq c\Big(|\mathbb{H}(t)| + \|u_t\|_2^2 + \rho(u)\Big), \tag{5.7}$$

$$\| u\|_{p^-}^s \leq c\Big(|\mathbb{H}(t)| + \|u_t\|_2^2 + \| u\|_{p^-}^{p^-}\Big), \tag{5.8}$$

$\forall u \in H_0^1(\Omega)$ and $2 \leq s \leq p^-$. Let (u) be a solution of (1.1), then

$$\rho(u) \geq c\|u\|_{p^-}^{p^-} \tag{5.9}$$

and

$$\int_\Omega |u|^{m(x)}dx \le c\Big(\rho^{m^-/p^-}(u)+\rho^{m^+/p^-}(u)\Big). \tag{5.10}$$

Theorem 3. *Suppose that (1.2), (2.1) and $E(0)<0$. Then, the solution of (1.1) blows up in finite time.*

Proof. We set

$$\mathcal{K}(t)=\mathbb{H}^{1-\alpha}(t)+\varepsilon\int_\Omega uu_t dx, \tag{5.11}$$

where $\varepsilon>0$ to be assigned later and

$$0<\alpha<\min\left\{\frac{p^- -2}{2p^-},\frac{p^- -m^+}{p^-(m^+-1)}\right\}. \tag{5.12}$$

By multiplying (1.1) by u and with a derivative of (5.11), we get

$$\begin{aligned}\mathcal{K}'(t)&=(1-\alpha)\mathbb{H}^{-\alpha}(t)\mathbb{H}'(t)+\varepsilon\|u_t\|_2^2-\varepsilon\|\nabla u\|_2^{2(\gamma+1)}\\&\quad-\varepsilon\|\nabla u\|_2^2+\varepsilon k\int_\Omega |u|^{p(x)}\ln|u|dx-\varepsilon\beta_1\int_\Omega uu_t|u_t|^{m(x)-2}dx.\end{aligned} \tag{5.13}$$

Using Young's inequality, we find for $\delta_1>0$

$$\varepsilon\beta_1\int_\Omega uu_t|u_t|^{m(x)-2}dx\le\varepsilon\beta_1\left\{\frac{1}{m^-}\int_\Omega\delta_1^{m(x)}|u|^{m(x)}dx+\frac{m^+-1}{m^-}\int_\Omega\delta_1^{-\frac{m(x)}{m(x)-1}}|u_t|^{m(x)}dx\right\}. \tag{5.14}$$

By substituting (5.14) into (5.13), we obtain

$$\begin{aligned}\mathcal{K}'(t)&\ge(1-\alpha)\mathbb{H}^{-\alpha}(t)\mathbb{H}'(t)+\varepsilon\|u_t\|_2^2-\varepsilon\|\nabla u\|_2^{2(\gamma+1)}\\&\quad-\varepsilon\|\nabla u\|_2^2+\varepsilon k\int_\Omega|u|^{p(x)}\ln|u|dx\\&\quad-\varepsilon\beta_1\left\{\frac{1}{m^-}\int_\Omega\delta_1^{m(x)}|u|^{m(x)}dx+\frac{m^+-1}{m^-}\int_\Omega\delta_1^{-\frac{m(x)}{m(x)-1}}|u_t|^{m(x)}dx\right\}.\end{aligned} \tag{5.15}$$

Therefore, by setting δ_1 so that

$$\delta_1^{-\frac{m(x)}{m(x)-1}}=\beta_1\kappa\mathbb{H}^{-\alpha}(t). \tag{5.16}$$

Replacing in (5.15) and by (5.3), we find

$$\begin{aligned}\mathcal{K}'(t)&\ge\Big[(1-\alpha)-\varepsilon\kappa\widehat{m}\Big]\mathbb{H}^{-\alpha}(t)\mathbb{H}'(t)+\varepsilon\|u_t\|_2^2\\&\quad-\varepsilon\|\nabla u\|_2^2-\varepsilon\|\nabla u\|_2^{2(\gamma+1)}+\varepsilon k\int_\Omega|u|^{p(x)}\ln|u|dx\\&\quad-\varepsilon\frac{\beta_1}{m^-}\int_\Omega(\beta_1\kappa)^{1-m(x)}\mathbb{H}^{\alpha(m(x)-1)}(t)|u|^{m(x)}dx,\end{aligned} \tag{5.17}$$

where $\widehat{m}=\frac{m^+-1}{m^-}$. Using (5.4) and (5.10), we have

$$\begin{aligned}\frac{\beta_1}{m^-}\int_\Omega(\beta_1\kappa)^{1-m(x)}\mathbb{H}^{\alpha(m(x)-1)}(t)|u|^{m(x)}dx &\leq \frac{\beta_1}{m^-}\int_\Omega(\beta_1\kappa)^{1-m^-}\mathbb{H}^{\alpha(m^+-1)}(t)|u|^{m(x)}dx\\ &= \frac{C_1}{\kappa^{m^--1}}\mathbb{H}^{\alpha(m^+-1)}(t)\int_\Omega|u|^{m(x)}dx\\ &\leq \frac{C_2}{\kappa^{m^--1}}\left[\rho^{\frac{m^-}{p^-}+\alpha(m^+-1)}(u)+\rho^{\frac{m^+}{p^-}+\alpha(m^+-1)}(u)\right].\end{aligned}\tag{5.18}$$

By (5.12), we find

$$s=m^-+\alpha p^-(m^+-1)\leq p^-,\ \text{ and }\ s=m^++\alpha p^-(m^+-1)\leq p^-.$$

Further, Lemma 8 gives

$$C_1\mathbb{H}^{\alpha(m^+-1)}(t)\int_\Omega|u|^{m(x)}dx\leq\frac{C_3}{\kappa^{m^--1}}\Big(\|\nabla u\|_2^2+\rho(u)\Big).\tag{5.19}$$

Combining (5.17) and (5.19), we obtain

$$\begin{aligned}\mathcal{K}'(t)\geq&\left[(1-\alpha)-\varepsilon\kappa\widehat{m}\right]\mathbb{H}^{-\alpha}(t)\mathbb{H}'(t)+\varepsilon\|u_t\|_2^2-\varepsilon C_4\|\nabla u\|_2^2\\ &-\varepsilon\|\nabla u\|_2^{2(\gamma+1)}-\varepsilon\frac{C_3}{\kappa^{m^--1}}\rho(u)+\varepsilon k\int_\Omega|u|^{p(x)}\ln|u|dx,\end{aligned}\tag{5.20}$$

where $C_4=\frac{C_3}{\kappa^{m^--1}}+1$.

For $0<a<1$, we have from (5.1)

$$\begin{aligned}\frac{\varepsilon}{p^-}\int_\Omega|u|^{p(x)}\ln|u|^kdx\geq{}&\varepsilon\int_\Omega\frac{|u|^{p(x)}}{p(x)}\ln|u|^kdx\\ ={}&\varepsilon(1-a)\mathbb{H}(t)+\frac{\varepsilon(1-a)}{2}\|u_t\|_2^2+\varepsilon(1-a)k\int_\Omega\frac{|u|^{p(x)}}{p^2(x)}dx\\ &+\frac{\varepsilon(1-a)}{2}\|\nabla u\|_2^2+\frac{\varepsilon(1-a)}{2(\gamma+1)}\|\nabla u\|_2^{2(\gamma+1)}\\ &+a\varepsilon\int_\Omega\frac{|u|^{p(x)}}{p(x)}\ln|u|^kdx,\\ \geq{}&\varepsilon(1-a)\mathbb{H}(t)+\frac{\varepsilon(1-a)}{2}\|u_t\|_2^2+\frac{\varepsilon(1-a)k}{(p^+)^2}\int_\Omega|u|^{p(x)}dx\\ &+\frac{\varepsilon(1-a)}{2}\|\nabla u\|_2^2+\frac{\varepsilon(1-a)}{2(\gamma+1)}\|\nabla u\|_2^{2(\gamma+1)}\\ &+\frac{a\varepsilon}{p^+}\int_\Omega|u|^{p(x)}\ln|u|^kdx.\end{aligned}\tag{5.21}$$

We conclude that

$$\begin{aligned}\varepsilon\int_{\Omega}|u|^{p(x)}\ln|u|^{k}dx \geq\ & \varepsilon(1-a)p^{-}\mathbb{H}(t)+\frac{\varepsilon(1-a)p^{-}}{2}\|u_t\|_2^2+\frac{\varepsilon(1-a)kp^{-}}{(p^{+})^2}\int_{\Omega}|u|^{p(x)}dx\\ &+\frac{\varepsilon(1-a)p^{-}}{2}\|\nabla u\|_2^2+\frac{\varepsilon(1-a)p^{-}}{2(\gamma+1)}\|\nabla u\|_2^{2(\gamma+1)}\\ &+\frac{a\varepsilon p^{-}}{p^{+}}\int_{\Omega}|u|^{p(x)}\ln|u|^{k}dx, \end{aligned} \tag{5.22}$$

substituting in (5.20), gives

$$\begin{aligned}\mathcal{K}'(t)\geq\ & \left[(1-\alpha)-\varepsilon\kappa\widehat{m}\right]\mathbb{H}^{-\alpha}(t)\mathbb{H}'(t)+\varepsilon\left[\frac{p^{-}(1-a)}{2}+1\right]\|u_t\|_2^2\\ &+\varepsilon\left[\frac{p^{-}(1-a)}{2}-1-\frac{C_3}{\kappa^{m^{-}-1}}\right]\|\nabla u\|_2^2+\varepsilon p^{-}(1-a)\mathbb{H}(t)\\ &+\varepsilon\left[\frac{p^{-}(1-a)}{2(\gamma+1)}-1\right]\|\nabla u\|_2^{2(\gamma+1)}-\varepsilon\left(\frac{(1-a)kp^{-}}{(p^{+})^2}-\frac{C_3}{\kappa^{m^{-}-1}}\right)\rho(u)\\ &+\frac{a\varepsilon p^{-}}{p^{+}}\int_{\Omega}|u|^{p(x)}\ln|u|^{k}dx. \end{aligned} \tag{5.23}$$

In this stage, we select $a>0$ so small that

$$\frac{p^{-}(1-a)}{2}-1>0 \quad \text{and} \quad \frac{p^{-}(1-a)}{2(\gamma+1)}-1>0,$$

then we choose κ so large that

$$\left(\frac{p^{-}(1-a)}{2}-1\right)-\frac{C_3}{\kappa^{m^{-}-1}}>0 \quad \text{and} \quad \frac{(1-a)kp^{-}}{(p^{+})^2}-\frac{C_3}{\kappa^{m^{-}-1}}>0,$$

when κ, a are fixed. We can choose ε so small that

$$(1-\alpha)-\varepsilon\kappa\widehat{m}>0.$$

Thus, for some $\mu_1>0$, estimate (5.23) becomes

$$\mathcal{K}'(t)\geq\mu_1\left\{\mathbb{H}(t)+\|u_t\|_2^2+\|\nabla u\|_2^2+\|\nabla u\|_2^{2(\gamma+1)}+\rho(u)+\int_{\Omega}|u|^{p(x)}\ln|u|^{k}dx\right\}, \tag{5.24}$$

and

$$\mathcal{K}(t)\geq\mathcal{K}(0)>0, \quad t>0. \tag{5.25}$$

Therefore, by using Hölder's and Young's inequalities, we have

$$\|u\|_2=\left(\int_{\Omega}u^2dx\right)^{\frac{1}{2}}\leq\left[\left(\int_{\Omega}(|u|^2)^{p^{-}/2}dx\right)^{\frac{2}{p^{-}}}.\left(\int_{\Omega}1dx\right)^{1-\frac{2}{p^{-}}}\right]^{\frac{1}{2}}\leq c\|u\|_{p-}, \tag{5.26}$$

and

$$\left|\int_{\Omega}uu_tdx\right|\leq\|u_t\|_2.\|u\|_2\leq c\|u_t\|_2.\|u\|_{p^{-}}.$$

Therefore

$$\begin{aligned} |\int_\Omega uu_t dx|^{\frac{1}{1-\alpha}} &\geq c\|u_t\|_2^{\frac{1}{1-\alpha}}.\|u\|_{p^-}^{\frac{1}{1-\alpha}} \\ &\leq c[\|u_t\|_2^{\frac{\theta}{1-\alpha}} + \|u\|_{p^-}^{\frac{\mu}{1-\alpha}}], \end{aligned} \tag{5.27}$$

where $\frac{1}{\mu}+\frac{1}{\theta}=1$. We take $\theta = 2(1-\alpha)$, to get

$$\frac{\mu}{1-\alpha} = \frac{2}{1-2\alpha} \leq p^-.$$

Subsequently, for $s = \frac{2}{(1-2\alpha)}$, we obtain

$$|\int_\Omega uu_t dx|^{\frac{1}{1-\alpha}} \leq c[\|u_t\|_2^2 + \|u\|_{p^-}^s].$$

Therefore, Lemma 8 gives

$$|\int_\Omega uu_t dx|^{\frac{1}{1-\alpha}} \geq c\Big\{\mathbb{H}(t) + \|u_t\|_2^2 + \rho(u)\Big\}. \tag{5.28}$$

Subsequently,

$$\begin{aligned} \mathcal{K}^{\frac{1}{1-\alpha}}(t) &= \Big\{\mathbb{H}^{1-\alpha} + \varepsilon \int_\Omega uu_t dx\Big\}^{\frac{1}{1-\alpha}} \\ &\leq c\Big\{\mathbb{H}(t) + |\int_\Omega uu_t dx|^{\frac{1}{1-\alpha}}\Big\} \\ &\leq c\Big\{\mathbb{H}(t) + \|u_t\|_2^2 + \rho(u)\Big\} \\ &\leq c\Big\{\mathbb{H}(t) + \|u_t\|_2^2 + \|\nabla u\|_2^2 + \|\nabla u\|_2^{2(\gamma+1)} + \rho(u) + \int_\Omega |u|^{p(x)} \ln |u|^k dx\Big\}. \end{aligned} \tag{5.29}$$

According (5.24) and (5.29), we have

$$\mathcal{K}'(t) \geq \lambda \mathcal{K}^{\frac{1}{1-\alpha}}(t), \tag{5.30}$$

where $\lambda(\mu_1, c) > 0$, and by integration of (5.30), we find

$$\mathcal{K}^{\frac{\alpha}{1-\alpha}}(t) \geq \frac{1}{\mathcal{K}^{\frac{-\alpha}{1-\alpha}}(0) - \lambda\frac{\alpha}{(1-\alpha)}t}.$$

Hence, the solution blows up in a finite time T^*, such that

$$T^* = \frac{1-\alpha}{\lambda\alpha\mathcal{K}^{\alpha/(1-\alpha)}(0)}.$$

Then the proof is completed. □

6. Conclusion

The combination of the logarithmic term and the variable exponent in a nonlinear Kirchhoff-type equations makes our problem different from what was previously studied, and for this was the purpose of this work, as we showed the global existence. Also, we have reached the result of the general decay by using the integral inequality due to [10]. Next, we reached the result of blowing up of the solutions. This work is a solution to the open problem of our proposal in [5, 7].

What we want to perform in the future from the expected research works is an attempt to employ the same way with the same problem, but by adding another component of damping, which contributes to an actual solution to the required problem.

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