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# ON SOME FIXED POINT THEOREMS FOR ĆIRIĆ OPERATORS

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*Abstract.* In this paper, we will present existence and localization fixed point theorems and stability results for the fixed point problem involving some very general classes of operators (single-valued and multi-valued), namely for Ćirić type operators. Then, an application to homotopy principles is given. Our results complement and extend the works in the literature.

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## 1. Introduction and Preliminaries

Let $(X,\mathrm{d})$ be a metric space and $\mathrm{f}\colon X \to X$ be an operator. For each $x \in X$, we denote $O(x,\infty) = \{x, \mathrm{f}(x), \ldots, \mathrm{f}^n(x), \ldots\}$.

Let $x_0$ be a given point in $X$ and $r > 0$. The set $B(x_0;r) := \{x \in X \colon \mathrm{d}(x_0,x) < r\}$ is the open ball of center $x_0$ and radius $r$, while $\tilde{B}(x_0;r) := \{x \in X \colon \mathrm{d}(x_0,x) \leq r\}$ is the closed ball of center $x_0$ and radius $r$.

We will now recall some definitions and well-known results, which will be useful throughout the paper.

**Definition 1** ([4, Cirić Definition page 268])**.** Let $(X,\mathrm{d})$ be a metric space and $\mathrm{f}\colon X \to X$ be an operator. Then $X$ is said to be f-orbitally complete if every Cauchy sequence contained in $O(x,\infty)$, for some $x \in X$, converges in $X$.

In the above context, a sequence of Picard iterates starting from $x_0 \in X$ is a sequence $x_n := \mathrm{f}^n(x_0)$, for $n \in \mathbb{N}^*$.

**Definition 2** ([4, Cirić Definition 1])**.** Let $(X,\mathrm{d})$ be a metric space. Then, $\mathrm{f}\colon Y \subseteq X \to X$ is a single-valued Ćirić type operator with constant $q$ if there exists a number $q \in (0,1)$, such that for all $x, y \in Y$ we have

$$\mathrm{d}(\mathrm{f}(x),\mathrm{f}(y)) \leq q \cdot \max\{\mathrm{d}(x,y), \mathrm{d}(x,\mathrm{f}(x)), \mathrm{d}(y,\mathrm{f}(y)), \mathrm{d}(x,\mathrm{f}(y)), \mathrm{d}(y,\mathrm{f}(x))\}.$$

Let $(X,\mathrm{d})$ be a metric space. By $P(X)$ we denote the family of all nonempty subsets of $X$, and the family of all nonempty and closed subsets of $X$ is denoted with $P_{cl}(X)$. Throughout the paper, we consider the following distances (see, e.g., [9, 10]):

(1) The gap functional (generated by d) between a point $a \in X$ and a set $Y \in P(X)$ is
$$\mathrm{D}(a,Y) := \inf\{\mathrm{d}(a,y) \mid y \in Y\}.$$

(2) The Pompeiu-Hausdorff functional (generated by $d$) between two sets $A, B \in P(X)$ is
$$\mathrm{H}(A,B) := \max\left\{\sup_{a\in A}(\inf_{b\in B}\mathrm{d}(a,b)), \sup_{b\in B}(\inf_{a\in A}\mathrm{d}(a,b))\right\}.$$

If $\mathrm{F}\colon X \to P(X)$ is a multi-valued operator, then its fixed point set is denoted by $\mathrm{Fix}(F) := \{x \in X \mid x \in \mathrm{F}(x)\}$, while the graph of F is the set $\mathrm{Graph}(F) := \{(x,y) \in X \times X \mid y \in \mathrm{F}(x)\}$. The set of all strict fixed points of F is denoted by $\mathrm{SFix}(F)$, i.e., there exists $x^* \in X$ such that $\mathrm{F}(x^*) = \{x^*\}$.

In this paper, we will present several existence and localization fixed point theorems and stability results for the fixed point problem involving some very general classes of operators (single-valued and multi-valued), namely for Ćirić type operators. Then, some applications to homotopy principles are given. Our results complement and extend some works in the literature, see e.g. [1–4, 6, 8, 11].

## 2. A STUDY OF THE FIXED POINT EQUATION WITH GENERALIZED ĆIRIĆ OPERATORS

In this section, the single-valued case is taken into consideration. We first recall Ćirić's Theorem which appeared in the well-known paper from 1974, see [4].

**Theorem 1** ([4, Cirić Theorem 1])**.** *Let $(X,\mathrm{d})$ be a metric space and $\mathrm{f}\colon X \to X$ be a Ćirić type operator with constant $q \in (0,1)$. Suppose that $X$ is* f*-orbitally complete. Then:*

*(i)* f *has a unique fixed point $x^*$ in $X$ and $\lim_{n\to\infty} f^n(x) = x^*$, i.e.,* f *is a Picard operator;*

*(ii)* $\mathrm{d}(f^n(x),x^*) \leq \dfrac{q^n}{1-q}\mathrm{d}(x,\mathrm{f}(x))$*, for every $x \in X$ and every $n \in \mathbb{N}^*$.*

Our first main result, which generalizes the above theorem, is an existence, uniqueness and localization for the unique fixed point of a single-valued Ćirić type operator.

**Theorem 2.** *Let $(X,\mathrm{d})$ be a complete metric space, $x_0 \in X$ and $r > 0$. We consider* $\mathrm{f}\colon B(x_0;r) \to X$ *a single-valued Ćirić type operator with constant $q \in \left(0,\frac{1}{2}\right)$. We also suppose that*
$$\mathrm{d}(x_0,\mathrm{f}(x_0)) < \frac{1-2q}{1-q}r.$$

*Then* f *has a unique fixed point* $x^* \in B(x_0;r)$, $f^n(x_0) \in B(x_0;r)$, *for all* $n \in \mathbb{N}$ *and the sequence* $(f^n(x_0))_{n\in\mathbb{N}}$ *of Picard iterates starting from* $x_0$ *converges to* $x^*$ *as* $n \longrightarrow \infty$.

*Proof.* Let $0 < s < r$ such that

$$\mathrm{d}(x_0,\mathrm{f}(x_0)) \leq \frac{1-2q}{1-q}s < \frac{1-2q}{1-q}r.$$

The sequence $(x_n)_{n\in\mathbb{N}}$, with $x_n := \mathrm{f}^n(x_0)$, has the recurrent form $x_{n+1} = \mathrm{f}(x_n)$, for all $n \in \mathbb{N}$. Then,

$$\begin{aligned}
\mathrm{d}(x_1,x_2) &= \mathrm{d}(\mathrm{f}(x_0),\mathrm{f}(x_1))\\
&\leq q\max\{\mathrm{d}(x_0,x_1),\mathrm{d}(x_0,x_1),\mathrm{d}(x_1,x_2),\mathrm{d}(x_0,x_2)\mathrm{d}(x_1,x_1)\}\\
&= q\max\{\mathrm{d}(x_0,x_1),\mathrm{d}(x_1,x_2),\mathrm{d}(x_0,x_2)\}\\
&\leq q\max\{\mathrm{d}(x_0,x_1),\mathrm{d}(x_1,x_2),\mathrm{d}(x_0,x_1)+\mathrm{d}(x_1,x_2)\}\\
&= q\left(\mathrm{d}(x_0,x_1)+\mathrm{d}(x_1,x_2)\right),
\end{aligned}$$

implying

$$\mathrm{d}(x_1,x_2) \leq \frac{q}{1-q}\mathrm{d}(x_0,x_1).$$

We denote $h := \dfrac{q}{1-q}$, thus $\dfrac{1-2q}{1-q} = 1-h$ with $h \in (0,1)$. Using the mathematical induction, we can prove the inequality

$$\mathrm{d}(x_{n-1},x_n) \leq h^{n-1}\mathrm{d}(x_0,x_1)$$

holds for all $n \in \mathbb{N}^*$. We also know that $\mathrm{d}(x_0,x_1) \leq (1-h)s$.

By taking a point $n \in \mathbb{N}^*$ arbitrarily, we obtain

$$\begin{aligned}
\mathrm{d}(x_0,x_n) &\leq \mathrm{d}(x_0,x_1)+\mathrm{d}(x_1,x_2)+\cdots+\mathrm{d}(x_{n-1},x_n)\\
&\leq \mathrm{d}(x_0,x_1)+h\mathrm{d}(x_0,x_1)+\cdots+h^{n-1}\mathrm{d}(x_0,x_1)\\
&= \mathrm{d}(x_0,x_1)(1+h+\cdots+h^{n-1})\\
&= \frac{1-h^n}{1-h}\mathrm{d}(x_0,x_1) \leq \frac{1}{1-h}\mathrm{d}(x_0,x_1) \leq s,
\end{aligned}$$

proving that all elements of the sequence are in the closed ball $\tilde{B}(x_0;s)$.

We will continue by proving that the sequence considered is Cauchy in $X$. Let $m \in \mathbb{N}$ and $n \in \mathbb{N}^*$. We compute

$$\begin{aligned}
\mathrm{d}(x_m,x_{m+n}) &\leq \mathrm{d}(x_m,x_{m+1})+\cdots+\mathrm{d}(x_{m+n-1},x_{m+n})\\
&\leq h^m\mathrm{d}(x_0,x_1)+\cdots+h^{m+n-1}\mathrm{d}(x_0,x_1)\\
&= h^m\mathrm{d}(x_0,x_1)\left(1+h+\cdots+h^{n-1}\right)\\
&= h^m\frac{1-h^n}{1-h}\mathrm{d}(x_0,x_1)
\end{aligned}$$

$$\le \frac{h^m}{1-h} \mathrm{d}(x_0, x_1).$$

This relation leads us to the conclusion that $(x_n)_{n\in\mathbb{N}}$ is a Cauchy sequence. Moreover, due to the completeness of $(X, \mathrm{d})$, we obtain it is also convergent to a point $x^* \in \tilde{B}(x_0; s)$. We will prove that $x^*$ is a fixed point. We compute the following inequality

$$\begin{aligned}
\mathrm{d}(x^*, \mathrm{f}(x^*)) &\le \mathrm{d}(x^*, x_{n+1}) + \mathrm{d}(x_{n+1}, \mathrm{f}(x^*)) \\
&\le \mathrm{d}(x^*, x_{n+1}) + q \max\{\mathrm{d}(x_n, x^*), \mathrm{d}(x_n, x_{n+1}), \mathrm{d}(x^*, \mathrm{f}(x^*)), \\
&\qquad \mathrm{d}(x^*, x_{n+1}), \mathrm{d}(x_n, \mathrm{f}(x^*))\} \\
&\le \mathrm{d}(x^*, x_{n+1}) + q \max\{\mathrm{d}(x_n, x_{n+1}), \mathrm{d}(x^*, x_{n+1}), \mathrm{d}(x_n, x^*) + \mathrm{d}(x^*, \mathrm{f}(x^*))\} \\
&\le \mathrm{d}(x^*, x_{n+1}) + q\left[\mathrm{d}(x_n, x_{n+1}) + \mathrm{d}(x_{n+1}, x^*) + \mathrm{d}(x^*, \mathrm{f}(x^*))\right],
\end{aligned}$$

which implies

$$\mathrm{d}(x^*, \mathrm{f}(x^*)) \le \frac{1+q}{1-q} \mathrm{d}(x_{n+1}, x^*) + \frac{q}{1-q}\left[\mathrm{d}(x_n, x_{n+1})\right], \text{ for all } n \in \mathbb{N}.$$

We only need to let $n \longrightarrow \infty$ in the above inequality, and we will obtain $\mathrm{d}(x^*, \mathrm{f}(x^*)) = 0$, proving that $x^*$ is a fixed point for f.

For the uniqueness of the fixed point, we suppose by contradiction that there exists another fixed point $y^*$, with $x^* \neq y^*$. Then,

$$\begin{aligned}
\mathrm{d}(x^*, y^*) &= \mathrm{d}(\mathrm{f}(x^*), \mathrm{f}(y^*)) \\
&\le q \max\{\mathrm{d}(x^*, y^*), \mathrm{d}(x^*, x^*), \mathrm{d}(y^*, y^*), \mathrm{d}(x^*, y^*), \mathrm{d}(y^*, x^*)\} = q\mathrm{d}(x^*, y^*),
\end{aligned}$$

which is a contradiction due to the fact that $q < \frac{1}{2}$. □

We now denote by $\mathcal{S}(Y, X)$ the family of all operators from $Y$ to $X$, where $X$ is a metric space and $Y$ a closed subset of $X$, and by

$$\mathcal{S}_{\partial Y}(Y, X) := \left\{\mathrm{f} \in \mathcal{S}(Y, X) \text{ such that } \mathrm{f}_{|_{\partial Y}} : \partial Y \to X \text{ is fixed point free}\right\}.$$

We will introduce the concept of a family of single-valued Ćirić type operators with constant $q$.

**Definition 3.** Let $(X, \mathrm{d})$ be a metric space and $(J, \rho)$ be a metric space. We say that $\{\mathrm{f}_\lambda : \lambda \in J\} \subset \mathcal{S}(Y, X)$ is a family of single-valued Ćirić type operators with constant $q \in (0, 1)$ if the following conditions are satisfied: there exist $p \in (0, 1]$ and $M > 0$ such that

(i) for all $x_1, x_2 \in Y$ and $\lambda \in J$, we have

$$\begin{aligned}
\mathrm{d}(\mathrm{f}_\lambda(x_1), \mathrm{f}_\lambda(x_2)) \le q \max\{&\mathrm{d}(x_1, x_2), \mathrm{d}(x_1, \mathrm{f}_\lambda(x_1)), \mathrm{d}(x_2, \mathrm{f}_\lambda(x_2)), \\
&\mathrm{d}(x_1, \mathrm{f}_\lambda(x_2)), \mathrm{d}(x_2, \mathrm{f}_\lambda(x_1))\};
\end{aligned}$$

(ii) for all $x \in Y$ and $\lambda, \mu \in J$, we have

$$\mathrm{d}(\mathrm{f}_\lambda(x), \mathrm{f}_\mu(x)) \le M\left[\rho(\lambda, \mu)\right]^p.$$

The following homotopy result can now be proved.

**Theorem 3.** *Let* $(X,\mathrm{d})$ *be a complete metric space and* $Y$ *be a closed subset such that* $\operatorname{int} Y \neq \varnothing$. *Let* $(J,\rho)$ *be a connected metric space and* $\{\mathrm{f}_\lambda\colon \lambda \in J\}$ *be a family of single-valued Ćirić type operators with constant* $q \in \left(0,\frac{1}{2}\right)$ *from* $\mathcal{S}_{\partial Y}(Y,X)$. *Then the following conclusions occur:*

*(i) If there exists a point* $\lambda_0^* \in J$, *such that the equation* $\mathrm{f}_{\lambda_0^*}(x) = x$ *has a solution, then the equation* $\mathrm{f}_\lambda(x) = x$ *has a unique solution for any* $\lambda \in J$;

*(ii) If* $\mathrm{f}_\lambda(x_\lambda) = x_\lambda$, *for any* $\lambda \in J$, *then the operator*

$$\mathrm{j}\colon J \to \operatorname{int} Y, \mathrm{j}(\lambda) = x_\lambda$$

*is continuous.*

*Proof.* We will begin the proof by considering two fixed points, $x_\lambda$ a fixed point of $\mathrm{f}_\lambda$ and $x_\mu$ a fixed point of $\mathrm{f}_\mu$. Then,

$$\begin{aligned}\mathrm{d}(x_\lambda,x_\mu) &= \mathrm{d}(\mathrm{f}_\lambda(x_\lambda),\mathrm{f}_\mu(x_\mu))\\ &\le \mathrm{d}(\mathrm{f}_\lambda(x_\lambda),\mathrm{f}_\lambda(x_\mu)) + \mathrm{d}(\mathrm{f}_\lambda(x_\mu),\mathrm{f}_\mu(x_\mu)).\end{aligned}$$

Taking $\mathrm{d}(\mathrm{f}_\lambda(x_\lambda),\mathrm{f}_\lambda(x_\mu))$ separately, we compute

$$\begin{aligned}\mathrm{d}(\mathrm{f}_\lambda(x_\lambda),\mathrm{f}_\lambda(x_\mu)) &\le q\max\{\mathrm{d}(x_\lambda,x_\mu),\mathrm{d}(x_\lambda,\mathrm{f}_\lambda(x_\lambda)),\mathrm{d}(x_\mu,\mathrm{f}_\lambda(x_\mu)),\mathrm{d}(x_\lambda,\mathrm{f}_\lambda(x_\mu)),\\ &\qquad \mathrm{d}(x_\mu,\mathrm{f}_\lambda(x_\lambda))\}\\ &= q\max\left\{\mathrm{d}(x_\lambda,x_\mu),\mathrm{d}(x_\mu,\mathrm{f}_\lambda(x_\mu)),\mathrm{d}(x_\lambda,\mathrm{f}_\lambda(x_\mu))\right\}\\ &\le q\max\left\{\mathrm{d}(x_\lambda,x_\mu),\mathrm{d}(x_\lambda,x_\mu)+\mathrm{d}(x_\lambda,\mathrm{f}_\lambda(x_\mu)),\mathrm{d}(x_\lambda,\mathrm{f}_\lambda(x_\mu))\right\}\\ &= q\left[\mathrm{d}(x_\lambda,x_\mu)+\mathrm{d}(x_\lambda,\mathrm{f}_\lambda(x_\mu))\right],\end{aligned}$$

which implies

$$\mathrm{d}(\mathrm{f}_\lambda(x_\lambda),\mathrm{f}_\lambda(x_\mu)) \le \frac{q}{1-q}\mathrm{d}(x_\lambda,x_\mu).$$

Using the latter inequality together with the first one, we obtain

$$\begin{aligned}\mathrm{d}(x_\lambda,x_\mu) &\le \frac{q}{1-q}\mathrm{d}(x_\lambda,x_\mu) + \mathrm{d}(\mathrm{f}_\lambda(x_\mu),\mathrm{f}_\mu(x_\mu))\\ &\le \frac{q}{1-q}\mathrm{d}(x_\lambda,x_\mu) + M\left[\rho(\lambda,\mu)\right]^p\end{aligned}$$

entailing

$$\mathrm{d}(x_\lambda,x_\mu) \le \frac{1-q}{1-2q}M\left[\rho(\lambda,\mu)\right]^p.$$

Let us consider the set

$$Q = \{\lambda \in J \mid \exists x_\lambda \in \operatorname{int} Y \text{ such that } x_\lambda = \mathrm{f}_\lambda(x_\lambda)\}.$$

In addition to $J$ being a connected space, by proving that $Q$ is both closed and open, will lead us to $Q = J$, proving (i). For the closedness of $Q$, let $(\lambda_n)_{n\in\mathbb{N}} \subset Q$ such that

$\lambda_n \to \lambda^*$, and we show that $\lambda^* \in Q$. We consider $x_{\lambda_m} = \mathrm{f}_{\lambda_m}(x_{\lambda_m})$ and $x_{\lambda_n} = \mathrm{f}_{\lambda_n}(x_{\lambda_n})$ and we know that

$$\mathrm{d}(x_{\lambda_m}, x_{\lambda_n}) \leq \frac{1-q}{1-2q} M \left[\rho(\lambda_m, \lambda_n)\right]^p. \tag{2.1}$$

We already know that the sequence $(\lambda_n)_{n\in\mathbb{N}}$ is Cauchy in $J$, which implies that for an arbitrary $\varepsilon > 0$ there exists $n_\varepsilon \in \mathbb{N}$ with $m,n \in \mathbb{N}$, $m,n > n_\varepsilon$ such that

$$\rho(\lambda_m, \lambda_n) < \varepsilon. \tag{2.2}$$

We will denote $\dfrac{\varepsilon^p M(1-q)}{1-2q} =: \varepsilon' > 0$. Using this notation, together with relations (2.1) and (2.2), we get

$$\mathrm{d}(x_{\lambda_m}, x_{\lambda_n}) < \varepsilon',$$

which proves the sequence $(x_{\lambda_n})$ is Cauchy in $X$. Also, since $(X,\mathrm{d})$ is a complete space, we obtain $(x_{\lambda_n})$ is a convergent sequence in $Y$. Let us now denote the limit of this sequence by $x_{\lambda^*}$, and we compute

$$\begin{aligned}
\mathrm{d}(x_{\lambda^*}, \mathrm{f}(x_{\lambda^*})) &\leq \mathrm{d}(x_{\lambda^*}, x_{\lambda_{n+1}}) + \mathrm{d}(x_{\lambda_{n+1}}, \mathrm{f}(x_{\lambda^*})) \\
&\leq \mathrm{d}(x_{\lambda^*}, x_{\lambda_{n+1}}) + q \max\big\{\mathrm{d}(x_{\lambda_n}, x_{\lambda^*}), \mathrm{d}(x_{\lambda_n}, x_{\lambda_{n+1}}), \mathrm{d}(x_{\lambda^*}, \mathrm{f}(x_{\lambda^*})), \\
&\qquad\qquad \mathrm{d}(x_{\lambda_n}, \mathrm{f}(x_{\lambda^*})), \mathrm{d}(x_{\lambda^*}, x_{\lambda_{n+1}})\big\} \\
&\leq \mathrm{d}(x_{\lambda^*}, x_{\lambda_{n+1}}) + q\big[\mathrm{d}(x_{\lambda_n}, x_{\lambda^*}) + \mathrm{d}(x_{\lambda_n}, x_{\lambda_{n+1}}) + \mathrm{d}(x_{\lambda^*}, \mathrm{f}(x_{\lambda^*})) \\
&\qquad\qquad + \mathrm{d}(x_{\lambda_n}, \mathrm{f}(x_{\lambda^*})) + \mathrm{d}(x_{\lambda^*}, x_{\lambda_{n+1}})\big].
\end{aligned}$$

The inequality obtained above implies that for all $n \in \mathbb{N}$,

$$\mathrm{d}(x_{\lambda^*}, \mathrm{f}(x_{\lambda^*})) \leq \frac{1}{1-2q}\left[(1+2q)\mathrm{d}(x_{\lambda_n}, x_{\lambda^*}) + q\left(\mathrm{d}(x_{\lambda_n}, x_{\lambda_{n+1}}) + \mathrm{d}(x_\lambda^*, x_{\lambda_{n+1}})\right)\right],$$

and by letting $n \longrightarrow \infty$, we get that $x_{\lambda^*}$ is a fixed point for f. Since f is fixed point free on its boundary, then $\lambda^*$ belongs to $Q$, proving it is closed.

In order to show that $Q$ is open, we consider $\lambda_0 \in Q$. Then, there exists a point $x_{\lambda_0} \in \operatorname{int} Y$ such that $x_{\lambda_0} = \mathrm{f}_{\lambda_0}(x_{\lambda_0})$. Now, we will prove the existence of an $\varepsilon > 0$ and an open ball $B(\lambda_0;\varepsilon) \subset Q$. Due to $\operatorname{int} Y$ being an open set and $x_{\lambda_0} \in \operatorname{int} Y$, there exists an open ball $B(x_{\lambda_0};r) \subseteq \operatorname{int} Y$. We consider arbitrary $\varepsilon > 0$ such that $\varepsilon^p < \dfrac{1-2q}{M(1-q)} r$ and an arbitrary $\lambda \in B(\lambda_0;\varepsilon)$, and we prove that $\lambda \in Q$. Let us begin by estimating the following distance

$$\begin{aligned}
\mathrm{d}(\mathrm{f}_\lambda(x_{\lambda_0}), x_{\lambda_0}) &= \mathrm{d}(\mathrm{f}_\lambda(x_{\lambda_0}), \mathrm{f}_{\lambda_0}(x_{\lambda_0})) \\
&\leq M(\rho(\lambda,\lambda_0))^p \leq M\varepsilon^p \\
&\leq \frac{1-2q}{1-q} r.
\end{aligned}$$

From the inequality above, we get that the operator

$$f_\lambda \colon B(x_{\lambda_0}; r) \to X$$

is a Ćirić type operator, and using the local fixed point theorem for Ćirić type operators, we obtain that $\mathrm{Fix}(f_\lambda) \neq \varnothing$, implying $\lambda \in Q$.

Based on what we proved so far, the operator j is single-valued. We consider $\lambda,\ \mu \in J$ and we have

$$d(j(\lambda), j(\mu)) \leq \frac{1-q}{1-2q} M\,[\rho(\lambda,\mu)]^p .$$

Letting

$$\rho(\lambda,\mu) < \delta := \left[\frac{\varepsilon(1-2q)}{M(1-q)}\right]^{\frac{1}{p}},$$

we immediately obtain that $d(j(\lambda), j(\mu)) < \varepsilon$, proving that j is a continuous operator. □

## 3. A study of the fixed point equation with multi-valued generalized Ćirić operators

We first consider some notions related to our main results.

**Definition 4.** An operator $F \colon X \to P_{cl}(X)$ is said to be a multi-valued generalized contraction if for every $x, y \in X$ there exist non-negative numbers $p, q, r$, which may depend on both $x$ and $y$, such that $sup\{p + 2q + 2r \mid x, y \in X\} < 1$ and

$$H(F(x), F(y)) \leq p \cdot d(x,y) + q \cdot [D(x, F(x)) + D(y, F(y))] + r \cdot [D(x, F(y)) + D(y, F(x))] .$$

**Definition 5** ([2, A. Amini-Harandi Definition 2.1])**.** Let $(X, d)$ be a metric space. The set-valued map $F \colon Y \subseteq X \to P_{b,cl}(X)$ is said to be a multi-valued Ćirić type operator with constant $k$ (named a $k$-set-valued quasi-contraction in [2]) if

$$H(F(x), F(y)) \leq k \max\{d(x,y), D(x, F(x)), D(y, F(y)), D(x, F(y)), D(y, F(x))\},$$

for any $x, y \in X$, where $0 \leq k < 1$.

We have the following example of a multi-valued Ćirić type operator, which is not a multi-valued generalized contraction.

*Example* 1. Let

$$X_1 = \left\{\frac{m}{n} : m = 0, 1, 2, 4, 6, \ldots;\ n = 1, 3, 7, \ldots, 2k+1, \ldots\right\},$$

$$X_2 = \left\{\frac{m}{n} : m = 1, 2, 4, 6, 8, \ldots;\ n = 2, 5, 8, \ldots, 3k+2, \ldots\right\},$$

where $k \in \mathbb{N}$ and let $X = X_1 \cup X_2$. Let us define $\mathrm{F}\colon X \to X$ by

$$\mathrm{F}(x) = \begin{cases} \left\{ \frac{2}{3}x, \frac{6}{7}x \right\}, & x \in X_1, \\ \frac{1}{5}x, & x \in X_2. \end{cases}$$

The mapping F is a multi-valued Ćirić type operator with $q = \dfrac{6}{7}$. If both $x$ and $y$ are in $X_1$ or in $X_2$, then

$$\mathrm{H}(\mathrm{F}(x), \mathrm{F}(y)) \leq \frac{6}{7}\mathrm{d}(x,y).$$

If we take $x \in X_1$ and $y \in X_2$, then we have that

$$x \geq \frac{7}{30}y \text{ implies } \mathrm{H}(\mathrm{F}(x), \mathrm{F}(y)) = \frac{6}{7}\left(x - \frac{7}{30}y\right) \leq \frac{6}{7}\left(x - \frac{1}{5}y\right) = \frac{6}{7}\mathrm{D}(x, \mathrm{F}(y)),$$

$$x < \frac{7}{30}y \text{ implies } \mathrm{H}(\mathrm{F}(x), \mathrm{F}(y)) = \frac{6}{7}\left(\frac{7}{30}y - x\right) \leq \frac{6}{7}(y - x) = \frac{6}{7}\mathrm{d}(x,y).$$

Therefore, we have that F satisfies the following condition:

$$\mathrm{H}(\mathrm{F}(x), \mathrm{F}(y)) \leq \frac{6}{7}\max\left\{\mathrm{d}(x,y), \mathrm{D}(x, \mathrm{F}(y)), \mathrm{D}(y, \mathrm{F}(x))\right\},$$

and hence, it is a multi-valued Ćirić type operator.

In the following step, we show that F is not a multi-valued generalized contraction on $X$. Let $x = 1$ and $y = \frac{1}{2}$. Then we have that

$$\begin{aligned} p \cdot \mathrm{d}(x,y) + q \cdot [\mathrm{D}(x, \mathrm{F}(x)) + \mathrm{D}(y, \mathrm{F}(y))] + r \cdot [\mathrm{D}(x, \mathrm{F}(y)) + \mathrm{D}(y, \mathrm{F}(x))] = \\ = \frac{1}{2}p + \frac{4}{10}q + \frac{88}{70}r < (p + 2q + 2r)\frac{88}{140} < \\ < \frac{88}{140} < \frac{53}{70} = \mathrm{H}(\mathrm{F}(x), \mathrm{F}(y)), \end{aligned}$$

as $p + 2q + 2r < 1$. Thus, we can see that F is not a multi-valued generalized contraction.

If $(X, \mathrm{d})$ is a metric space and $\mathrm{F}\colon X \to P(X)$ is a multi-valued operator, then a sequence $(x_n)_{n \in \mathbb{N}}$ from $X$ is called a sequence of Picard type starting from $(x,y) \in \mathrm{Graph}(F)$ if $x_0 = x, x_1 = y$ and $x_n \in \mathrm{F}(x_{n-1}), n \in \mathbb{N}^*$.

The following lemma is useful for our following results.

**Lemma 1** (Cauchy's Lemma)**.** *Let $(a_n), (b_n)$ be two sequences of positive numbers such that $\sum_{n \geq 0} a_n < \infty$ and $\lim_{n \to \infty} b_n = 0$. Then $\lim_{n \to \infty}\left(\sum_{k=0}^{n} a_{n-k}b_k\right) = 0$.*

**Theorem 4** ([2, A. Amini-Harandi Theorem 2.2])**.** *Let $(X,\mathrm{d})$ be a complete metric space. Let $\mathrm{F}\colon X \to P_{b,cl}(X)$ be a multi-valued Ćirić type operator with constant $k < \frac{1}{2}$. Then,* F *has a fixed point.*

Here we will give a constructive proof of this theorem, as well as some data dependence and stability results for the fixed point problem $x \in \mathrm{F}(x)$.

**Theorem 5.** *Let $(X,\mathrm{d})$ be a complete metric space. Let $\mathrm{F}\colon X \to P_{cl}(X)$ be a multi-valued Ćirić type operator with constant $k < \frac{1}{2}$. Then:*

*(i)* $\mathrm{Fix}(F) \neq \varnothing$ *and for every $(x,y) \in \mathrm{Graph}(F)$ there exists a sequence $(x_n)_{n\in\mathbb{N}}$ of Picard type starting from $x_0 := x$, $x_1 := y$ which converge to a fixed point $x^*$ of* F*;*

*(ii) the fixed point equation $x \in \mathrm{F}(x)$ has the data dependence property, i.e., for any $x^* \in \mathrm{Fix}(F)$ and any $G\colon X \to P(X)$ such that $\mathrm{Fix}(G) \neq \varnothing$ and the inequality $\mathrm{H}(\mathrm{F}(x),\mathrm{G}(x)) \leq \eta$ holds for all $x \in X$ and some $\eta > 0$, there is $u^* \in \mathrm{Fix}(\mathrm{G})$ such that*

$$\mathrm{d}(x^*,u^*) \leq \frac{(1+k)q}{1-k}\eta,$$

*where* $1 < q < \dfrac{1}{2k}$*;*

*(iii) the fixed point equation is well-posed, i.e., for every sequence $(u_n)_{n\in\mathbb{N}} \subset X$ such that*

$$\mathrm{D}(u_n,\mathrm{F}(u_n)) \longrightarrow 0,$$

*as $n \longrightarrow \infty$, we have that $u_n \longrightarrow x^*$, as $n \longrightarrow \infty$.*

*(iv) if $q < \frac{1}{2}$, then the fixed point equation has the Ostrowski stability property, i.e., for any sequence $(u_n)_{n\in\mathbb{N}} \subset X$ with $\mathrm{D}(u_{n+1},\mathrm{F}(u_n)) \longrightarrow 0$ as $n \longrightarrow \infty$, we have that $u_n \longrightarrow x^*$;*

*Proof.* In order to prove (i), let $x_0 \in X$ and we construct the sequence $(x_n)_{n\in\mathbb{N}}$ of Picard type starting from $x_0 := x$ having the general term $x_n \in \mathrm{F}(x_{n-1}), n \in \mathbb{N}^*$. We prove that this sequence is Cauchy.

Let $x_1 \in \mathrm{F}(x_0)$ and $1 < q < \frac{1}{2k}$. Then, there exists $x_2 \in \mathrm{F}(x_1)$ such that $\mathrm{d}(x_1,x_2) \leq q\mathrm{H}(\mathrm{F}(x_0),\mathrm{F}(x_1))$. Then, we have:

$$\begin{aligned}
\mathrm{d}(x_1,x_2) &\leq qk\cdot\max\{\mathrm{d}(x_0,x_1),\mathrm{D}(x_0,\mathrm{F}(x_0)),\mathrm{D}(x_1,\mathrm{F}(x_1)),\\
&\qquad \mathrm{D}(x_0,\mathrm{F}(x_1)),\mathrm{D}(x_1,\mathrm{F}(x_0))\}\\
&\leq qk\cdot\max\{\mathrm{d}(x_0,x_1),\mathrm{d}(x_1,x_2),\mathrm{D}(x_0,\mathrm{F}(x_1))\}\\
&\leq qk\cdot\max\{\mathrm{d}(x_0,x_1),\mathrm{d}(x_1,x_2),\mathrm{d}(x_0,x_2))\}\\
&\leq qk\cdot\max\{\mathrm{d}(x_0,x_1),\mathrm{d}(x_1,x_2),\mathrm{d}(x_0,x_1)+\mathrm{d}(x_1,x_2)\}\\
&\leq qk(\mathrm{d}(x_0,x_1)+\mathrm{d}(x_1,x_2)).
\end{aligned}$$

Hence,

$$\mathrm{d}(x_1,x_2) \leq \frac{qk}{1-qk}\mathrm{d}(x_0,x_1).$$

We denote $\beta := \dfrac{qk}{1-qk} < 1$. Then $\mathrm{d}(x_1,x_2) \leq \beta \mathrm{d}(x_0,x_1)$. Using mathematical induction we get that:

$$\mathrm{d}(x_n,x_{n+1}) \leq \beta^n \mathrm{d}(x_0,x_1).$$

and

$$\begin{aligned}\mathrm{d}(x_m,x_{m+n}) &\leq \mathrm{d}(x_m,x_{m+1}) + \cdots + \mathrm{d}(x_{m+n+1},x_{m+n}) \\ &\leq \beta^m \mathrm{d}(x_0,x_1) + \cdots + \beta^{m+n+1}\mathrm{d}(x_0,x_1) = \beta^m \frac{1-\beta^n}{1-\beta}\mathrm{d}(x_0,x_1).\end{aligned}$$

It follows that

$$\mathrm{d}(x_m,x_{m+n}) \leq \frac{\beta^m}{1-\beta}\mathrm{d}(x_0,x_1).$$

Due to the fact that the series $\sum \beta^m$ is convergent, we get the sequence $(x_n)_{n\in\mathbb{N}}$ is Cauchy. Since $(X,\mathrm{d})$ is a complete metric space, the sequence $(x_n)_{n\in\mathbb{N}}$ is convergent to an element $x^* \in X$. We show first that $x^* \in \mathrm{Fix}(F)$. Indeed, we have

$$\begin{aligned}0 \leq \mathrm{D}(x^*,\mathrm{F}(x^*)) &\leq \mathrm{d}(x^*,x_{n+1}) + \mathrm{D}(x_{n+1},\mathrm{F}(x^*)) \\ &\leq \mathrm{d}(x^*,x_{n+1}) + \mathrm{H}(\mathrm{F}(x_n),\mathrm{F}(x^*)) \\ &\leq \mathrm{d}(x^*,x_{n+1}) + k\cdot\max\{\mathrm{d}(x_n,x^*),\mathrm{D}(x_n,\mathrm{F}(x_n),\mathrm{D}(x^*,\mathrm{F}(x^*)), \\ &\qquad \mathrm{D}(x_n,\mathrm{F}(x^*)),\mathrm{D}(x^*,\mathrm{F}(x_n))\} \\ &\leq \mathrm{d}(x^*,x_{n+1}) + k\cdot\max\{\mathrm{d}(x_n,x^*),\mathrm{d}(x_n,x_{n+1}),\mathrm{D}(x^*,\mathrm{F}(x^*)), \\ &\qquad \mathrm{D}(x_n,\mathrm{F}(x^*)),\mathrm{d}(x^*,x_{n+1})\}.\end{aligned}$$

In the above inequality if we let $n \longrightarrow \infty$, then we get that

$$0 \leq \mathrm{D}(x^*,\mathrm{F}(x^*)) \leq k\mathrm{D}(x^*,\mathrm{F}(x^*)).$$

Thus $\mathrm{D}(x^*,\mathrm{F}(x^*)) = 0$ and so $x^* \in \mathrm{Fix}(F)$.

For proving (ii), let $x^* \in \mathrm{Fix}(F)$ and $1 < q < \frac{1}{2k}$. Then, there exists $u^* \in G(u^*)$ such that

$$\begin{aligned}\mathrm{d}(x^*,u^*) &\leq q\mathrm{H}(\mathrm{F}(x^*),\mathrm{G}(u^*)) \\ &\leq q\mathrm{H}(\mathrm{F}(x^*),\mathrm{F}(u^*)) + q\eta \\ &\leq qk\cdot\max\{\mathrm{d}(x^*,u^*),\mathrm{D}(u^*,\mathrm{F}(u^*)),\mathrm{D}(x^*,\mathrm{F}(u^*)),\mathrm{D}(u^*,\mathrm{F}(x^*))\} + q\eta \\ &\leq qk\cdot\max\{\mathrm{d}(x^*,u^*),\mathrm{D}(u^*,\mathrm{G}(u^*)) + \eta,\mathrm{D}(x^*,\mathrm{G}(u^*)) + \eta\} + q\eta \\ &\leq qk\cdot\max\{\mathrm{d}(x^*,u^*),\eta,\mathrm{d}(x^*,u^*) + \eta\} + q\eta \\ &\leq qk(\mathrm{d}(x^*,u^*) + \eta) + q\eta,\end{aligned}$$

thus we obtain

$$\mathrm{d}(x^*,u^*) \leq \frac{(1+k)q}{1-kq}\eta.$$

Thus, the fixed point equation with a multi-valued Ćirić type operator has the data dependence property.

Concerning conclusion (iii), in order to prove that the fixed point equation for F is well-posed, we take the sequence $(u_n)_{n\in\mathbb{N}} \subset X$ such that $\mathrm{D}(u_n,\mathrm{F}(u_n)) \longrightarrow 0$, as $n \longrightarrow \infty$. Then, we have $\mathrm{d}(u_n,x^*) \leq \mathrm{D}(u_n,\mathrm{F}(u_n)) + \mathrm{H}(\mathrm{F}(u_n),\mathrm{F}(x^*))$. Furthermore, we can write:

$$\begin{aligned}
\mathrm{d}(u_n,x^*) &\leq \mathrm{D}(u_n,\mathrm{F}(u_n)) + k\cdot\max\{\mathrm{d}(u_n,x^*),\mathrm{D}(u_n,\mathrm{F}(u_n)),\mathrm{D}(x^*,\mathrm{F}(x^*)),\\
&\qquad\qquad \mathrm{D}(x^*,\mathrm{F}(u_n)),\mathrm{D}(u_n,\mathrm{F}(x^*))\}\\
&\leq \mathrm{D}(u_n,\mathrm{F}(u_n)) + k\cdot\max\{\mathrm{d}(u_n,x^*),\mathrm{d}(u_n,x^*)+\mathrm{D}(x^*,\mathrm{F}(u_n)),\\
&\qquad\qquad \mathrm{D}(x^*,\mathrm{F}(u_n)),\mathrm{D}(u_n,\mathrm{F}(x^*))\}\\
&\leq \mathrm{D}(u_n,\mathrm{F}(u_n)) + k(\mathrm{d}(u_n,x^*)+\mathrm{D}(x^*,\mathrm{F}(u_n)))\\
&\leq \mathrm{D}(u_n,\mathrm{F}(u_n)) + k(2\mathrm{d}(u_n,x^*)+\mathrm{D}(u_n,\mathrm{F}(u_n))),
\end{aligned}$$

implying

$$\mathrm{d}(u_n,x^*) \leq \frac{1+k}{1-2k}\mathrm{D}(u_n,\mathrm{F}(u_n)) \longrightarrow 0, n \longrightarrow \infty.$$

Regarding (iv), we will show that the operator $\mathrm{F}\colon X \to P(X)$ has the Ostrowski property. Let us take the sequence $(u_n)_{n\in\mathbb{N}} \subset X$ such that

$$\mathrm{d}(u_{n+1},x^*) \leq \mathrm{D}(u_{n+1},\mathrm{F}(u_n)) + \mathrm{D}(\mathrm{F}(u_n),x^*). \tag{3.1}$$

We take separately $\mathrm{D}(\mathrm{F}(u_n),x^*)$ from the above inequality and we have that

$$\begin{aligned}
\mathrm{D}(\mathrm{F}(u_n),x^*) = \mathrm{H}(\mathrm{F}(u_n),\mathrm{F}(x^*)) &\leq k\cdot\max\{\mathrm{d}(u_n,x^*),\mathrm{D}(u_n,\mathrm{F}(u_n)),\mathrm{D}(x^*,\mathrm{F}(x^*)),\\
&\qquad \mathrm{D}(x^*,\mathrm{F}(u_n)),\mathrm{D}(u_n,\mathrm{F}(x^*))\}\\
&\leq k(\mathrm{d}(u_n,x^*)+\mathrm{D}(x^*,\mathrm{F}(u_n))).
\end{aligned}$$

Thus $\mathrm{D}(\mathrm{F}(u_n),x^*) \leq \frac{k}{1-k}\mathrm{d}(u_n,x^*)$ and denote $\alpha := \frac{k}{1-k} < 1$. We replace this result in the relation (3.1) and it follows that

$$\begin{aligned}
\mathrm{d}(u_{n+1},x^*) &\leq \mathrm{D}(u_{n+1},\mathrm{F}(u_n)) + \alpha\mathrm{d}(u_n,x^*)\\
&\leq \mathrm{D}(u_{n+1},\mathrm{F}(u_n)) + \alpha\mathrm{D}(u_n,\mathrm{F}(u_{n-1})) + \alpha^2\mathrm{d}(u_{n-1},x^*)\\
&\leq \cdots \leq \mathrm{D}(u_{n+1},\mathrm{F}(u_n)) + \alpha\mathrm{D}(u_n,\mathrm{F}(u_{n-1})) + \alpha^2\mathrm{d}(u_{n-1},x^*) + \cdots\\
&\quad + \alpha^n\mathrm{D}(u_1,\mathrm{F}(u_0)) + \alpha^{n+1}\mathrm{d}(u_0,x^*)\\
&= \sum_{k=0}^{n}\alpha^{n-k}\mathrm{D}(u_{k+1},\mathrm{F}(u_k)) + \alpha^{n+1}\mathrm{d}(u_0,x^*)
\end{aligned}$$

Since $\alpha < 1$, using Cauchy's lemma (see 1), we get $\mathrm{d}(u_{n+1}, x^*) \longrightarrow 0$. □

We will now give a theorem that shows that, under an additional condition, the fixed point set and the strict fixed point set of a multi-valued Ćirić type operator coincide.

**Theorem 6.** *Let* $(X, \mathrm{d})$ *be a complete metric space. Let* $\mathrm{F}\colon X \to P_{cl}(X)$ *be a multi-valued Ćirić type operator with constant* $k < 1$. *Suppose that* $\mathrm{SFix(F)} \neq \varnothing$. *Then* $\mathrm{Fix(F)} = \mathrm{SFix(F)} = \{x^*\}$.

*Proof.* We will prove that F has a unique fixed point in $X$. Since $\mathrm{SFix(F)} \neq \varnothing$ we know that there exists $x^* \in X$ such that $\mathrm{F}(x^*) = \{x^*\}$. We suppose that there exists $z \in \mathrm{Fix(F)}$ such that $z \neq x^*$. We have

$$\begin{aligned}\mathrm{d}(x^*, z) &\le \mathrm{H}(\mathrm{F}(x^*), \mathrm{F}(z)) \\ &\le k \max\{\mathrm{d}(z, x^*), \mathrm{D}(z, \mathrm{F}(z)), \mathrm{D}(x^*, \mathrm{F}(x^*)), \mathrm{D}(x^*, \mathrm{F}(z)), \mathrm{D}(z, \mathrm{F}(x^*))\} \\ &\le k\mathrm{d}(z, x^*).\end{aligned}$$

This is a contradiction for $k < 1$. Therefore $\mathrm{SFix}(F) = \mathrm{Fix}(F) = \{x^*\}$. □

Now we will prove a local fixed point theorem.

**Theorem 7.** *Let* $(X, \mathrm{d})$ *be a complete metric space,* $x_0 \in X$ *and* $r > 0$. *We consider the multi-valued operator* $\mathrm{F}\colon \tilde{B}(x_0; r) \to P_{cl}(X)$ *such that there exists* $k \in \left(0, \frac{1}{2}\right)$ *with*

$$\begin{aligned}\mathrm{H}(\mathrm{F}(x), \mathrm{F}(y)) \le k \max\{&\mathrm{d}(x, y), \mathrm{D}(x, \mathrm{F}(x)), \mathrm{D}(y, \mathrm{F}(y)), \mathrm{D}(x, \mathrm{F}(y)), \\ &\mathrm{D}(y, \mathrm{F}(x))\}, \text{ for all } x, y \in \tilde{B}(x_0; r).\end{aligned}$$

*We also suppose that*

$$\mathrm{D}(x_0, \mathrm{F}(x_0)) < \frac{1-2k}{1-k} r.$$

*Then, there exists a sequence* $(x_n)_{n \in \mathbb{N}}$ *of Picard iterates starting from* $x_0$ *which converges to a fixed point of* F.

*Proof.* Since $\mathrm{D}(x_0, \mathrm{F}(x_0)) < \frac{1-2k}{1-k} r$ we get there exists $x_1 \in \mathrm{F}(x_0)$ such that

$$\mathrm{d}(x_0, x_1) < \frac{1-2k}{1-k} r.$$

Moreover,

$$\begin{aligned}\mathrm{H}(\mathrm{F}(x_0), \mathrm{F}(x_1)) &\le k \max\{\mathrm{d}(x_0, x_1), \mathrm{D}(x_0, \mathrm{F}(x_0)), \mathrm{D}(x_1, \mathrm{F}(x_1)), \mathrm{D}(x_0, \mathrm{F}(x_1)), \\ &\qquad \mathrm{D}(x_1, \mathrm{F}(x_0))\} \\ &= k \max\{\mathrm{d}(x_0, x_1), \mathrm{D}(x_0, \mathrm{F}(x_0)), \mathrm{D}(x_1, \mathrm{F}(x_1)), \mathrm{D}(x_0, \mathrm{F}(x_1))\} \\ &\le k \max\{\mathrm{d}(x_0, x_1), \mathrm{D}(x_1, \mathrm{F}(x_1)), \mathrm{d}(x_0, x_1) + \mathrm{D}(x_1, \mathrm{F}(x_1)))\} \\ &\le k \max\{\mathrm{d}(x_0, x_1), \mathrm{H}(\mathrm{F}(x_0), \mathrm{F}(x_1)), \mathrm{d}(x_0, x_1) + \mathrm{H}(\mathrm{F}(x_0), \mathrm{F}(x_1))\}\end{aligned}$$

$$= k \max\left(\mathrm{d}(x_0, x_1) + \mathrm{H}(\mathrm{F}(x_0), \mathrm{F}(x_1))\right),$$

and thus

$$\mathrm{H}(\mathrm{F}(x_0), \mathrm{F}(x_1)) \leq \frac{k}{1-k}\mathrm{d}(x_0, x_1) < \frac{k}{1-k}\frac{1-2k}{1-k}r.$$

We will now denote $h := \dfrac{k}{1-k}$, which immediately implies $\dfrac{1-2k}{1-k} = 1-h$, with $h \in (0,1)$. Hence,

$$\mathrm{H}(\mathrm{F}(x_0), \mathrm{F}(x_1)) < h(1-h)r.$$

Thus, there exists $x_2 \in \mathrm{F}(x_1)$ such that $\mathrm{d}(x_1, x_2) < h(1-h)r$. We assume

$$p(n): \text{ there exists } x_n \in \mathrm{F}(x_{n-1}) \text{ such that } \mathrm{d}(x_{n-1}, x_n) < h^{n-1}(1-h)r,$$

and compute

$$\begin{aligned}
\mathrm{H}(\mathrm{F}(x_{n-1}), \mathrm{F}(x_n)) &\leq k \max\{\mathrm{d}(x_{n-1}, x_n), \mathrm{D}(x_{n-1}, \mathrm{F}(x_{n-1})), \mathrm{D}(x_n, \mathrm{F}(x_n)), \\
&\qquad \mathrm{D}(x_{n-1}, \mathrm{F}(x_n)), \mathrm{D}(x_n, \mathrm{F}(x_{n-1}))\} \\
&\leq k \max\{\mathrm{d}(x_{n-1}, x_n), \mathrm{D}(x_n, \mathrm{F}(x_n)), \mathrm{D}(x_{n-1}, \mathrm{F}(x_n))\} \\
&\leq k \max\{\mathrm{d}(x_{n-1}, x_n), \mathrm{D}(x_n, \mathrm{F}(x_n)), \mathrm{d}(x_{n-1}, x_n) + \mathrm{D}(x_n, \mathrm{F}(x_n))\} \\
&\leq k(\mathrm{d}(x_{n-1}, x_n) + \mathrm{H}(\mathrm{F}(x_{n-1}), \mathrm{F}(x_n))),
\end{aligned}$$

which implies

$$\mathrm{H}(\mathrm{F}(x_{n-1}), \mathrm{F}(x_n)) \leq h\mathrm{d}(x_{n-1}, x_n) < h^n(1-h)r.$$

Using the latter inequality, we get the existence of a point $x_{n+1} \in \mathrm{F}(x_n)$ such that the relation $p(n+1)$ holds, and therefore we proved $p(n)$ by mathematical induction. Again, by means of mathematical induction, one can easily prove the assumption

$$t(n): \mathrm{d}(x_0, x_n) < (1-h^n)r,$$

which shows that all the elements of the sequence $(x_n)_{n\in\mathbb{N}}$ are in the closed ball $\tilde{B}(x_0; r)$. Due to the following inequality

$$\begin{aligned}
\mathrm{d}(x_m, x_{m+n}) &\leq \mathrm{d}(x_m, x_{m+1}) + \cdots + \mathrm{d}(x_{m+n-1}, x_{m+n}) \\
&\leq h^m(1-h)(1+\cdots+h^{n-1})r \leq h^m(1-h)\frac{1-h^n}{1-h}r \leq h^m r,
\end{aligned}$$

the sequence $(x_n)_{n\in\mathbb{N}} \subset B(x_0; s)$ is Cauchy, thus convergent to a point $x^* \in \tilde{B}(x_0; r)$.

We finish the proof with showing $x^* \in \mathrm{Fix}(F)$, for which we compute

$$\begin{aligned}
\mathrm{D}(x^*, \mathrm{F}(x^*)) &\leq \mathrm{d}(x^*, x_{n+1}) + \mathrm{H}(\mathrm{F}(x_n), \mathrm{F}(x^*)) \\
&\leq \mathrm{d}(x^*, x_{n+1}) + k \max\{\mathrm{d}(x_n, x^*), \mathrm{D}(x_n, \mathrm{F}(x_n)), \mathrm{D}(x^*, \mathrm{F}(x^*)), \\
&\qquad \mathrm{D}(x_n, \mathrm{F}(x^*)), \mathrm{D}(x^*, \mathrm{F}(x_n))\} \\
&\leq \mathrm{d}(x^*, x_{n+1}) + k \max\{\mathrm{d}(x_n, x^*) + \mathrm{D}(x_n, \mathrm{F}(x_n)), \\
&\qquad \mathrm{d}(x_n, x^*) + \mathrm{D}(x^*, \mathrm{F}(x^*))\}
\end{aligned}$$

$$\leq \mathrm{d}(x^*, x_{n+1}) + k\mathrm{d}(x_n, x^*) + k\mathrm{d}(x_n, x_{n+1}) + k\mathrm{D}(x^*, \mathrm{F}(x^*)).$$

By considering $n \longrightarrow \infty$, we get the desired conclusion. □

By the above proof, we immediately get the following result.

**Theorem 8.** *Let $(X, \mathrm{d})$ be a complete metric space, $x_0 \in X$ and $r > 0$. We consider the multi-valued operator $\mathrm{F}\colon B(x_0; r) \to P_{cl}(X)$ such that there exists $k \in \left(0, \frac{1}{2}\right)$ with*

$$\mathrm{H}(\mathrm{F}(x), \mathrm{F}(y)) \leq k \max\{\mathrm{d}(x,y), \mathrm{D}(x, \mathrm{F}(x)), \mathrm{D}(y, \mathrm{F}(y)), \mathrm{D}(x, \mathrm{F}(y)), \mathrm{D}(y, \mathrm{F}(x))\}, \textit{ for all } x, y \in B(x_0; r).$$

*We also suppose that*

$$\mathrm{D}(x_0, \mathrm{F}(x_0)) < \frac{1-2k}{1-k} r.$$

*Then, there exists a sequence $(x_n)_{n\in\mathbb{N}}$ of Picard iterates starting from $x_0$ which converges to a fixed point of* F.

*Proof.* Let $s \in (0, r)$ such that

$$\mathrm{D}(x_0, \mathrm{F}(x_0)) < \frac{1-2k}{1-k} s < \frac{1-2k}{1-k} r.$$

For our conclusion, we follow the approach given in the above proof for the operator $\mathrm{F}\colon \tilde{B}(x_0, s) \to P(X)$. □

*Remark* 1. It is an open question to obtain, by the above approach, a local fixed point theorem and related stability results for a multi-valued Ćirić type operators with constant $k \in (0,1)$. For a different approach and a general existence result, see [7].

We now introduce the notion of a family of multi-valued Ćirić type operators with constant $k \in (0,1)$.

**Definition 6.** Let $(X, \mathrm{d})$ be a metric space. Then, the family $(\mathrm{F}_t)_{t\in[0,1]}$ (where $\mathrm{F}_t\colon Y \subseteq X \to P(X)$, for each $t \in [0,1]$) is a family of multi-valued Ćirić type operators with constant $k$ if $k \in (0,1)$ and the following conditions are satisfied:

(i)

$$\mathrm{H}(\mathrm{F}_t(x_1), \mathrm{F}_t(x_2)) \leq k \max\{\mathrm{d}(x_1,x_2), \mathrm{D}(x_1, \mathrm{F}_t(x_1)), \mathrm{D}(x_2, \mathrm{F}_t(x_2)), \mathrm{D}(x_1, \mathrm{F}_t(x_2)), \mathrm{D}(x_2, \mathrm{F}_t(x_1))\}, \text{ for all } x_1, x_2 \in Y, t \in [0,1].$$

(ii) $\mathrm{H}(\mathrm{F}_t(x), \mathrm{F}_s(x)) \leq |\phi(t) - \phi(s)|$, for all $t, s \in [0,1]$ and $x \in Y$,

where $\phi\colon [0,1] \to \mathbb{R}$ is strictly increasing and continuous.

Using the previous definitions, we can state, as an application of the multi-valued local fixed point theorem, a homotopy principle for multi-valued Ćirić type operators. The result generalizes a similar theorem given for multi-valued contraction, given by Frigon and Granas, see [5].

**Theorem 9.** *Let $(X, \mathrm{d})$ be a complete metric space, $U \subset X$ be an open set and* $\mathrm{F}\colon [0,1] \times \overline{U} \to P_{cl}(X)$ *be a multi-valued operator with closed graph. We denote* $\mathrm{F}_t := \mathrm{F}(t,\cdot)$, *for* $t \in [0,1]$. *We suppose:*

*(i)* $(\mathrm{F}_t)_{t\in[0,1]}$ *is a family of multi-valued Ćirić type operators with a constant* $k \in \left(0, \frac{1}{2}\right)$;

*(ii)* $x \notin \mathrm{F}_t(x)$. *for all* $(t,x) \in [0,1] \times \partial U$.

*Then* $\mathrm{F}_0$ *has a fixed point if and only if* $\mathrm{F}_1$ *has a fixed point.*

*Proof.* Let $x^* \in U$ such that $x^* \in \mathrm{F}_0(x^*)$. We define the set

$$Q = \{(t,x) \in [0,1] \times U : x \in \mathrm{Fix}(\mathrm{F}_t)\}.$$

We observe that $Q$ is nonempty, since $(0,x^*) \in Q$. Next, we consider the following partial order relation on $Q$

$$(t,x) \le (s,y) \text{ if and only if } t \le s \text{ and } \mathrm{d}(x,y) \le \frac{2\,(1-k)\,(\phi(s)-\phi(t))}{1-2k},$$

where $\phi$ is the function associated to the family $(\mathrm{F}_t)_{t\in[0,1]}$ of multi-valued Ćirić type operators with constant $k \in (0,1)$. We will use for $Q$ the Kuratowski-Zorn Lemma (saying that if a partially ordered set $Q$ has the property that every chain $P$ in $Q$ has an upper bound in $Q$, then the set $Q$ contains at least one maximal element.)

We consider $P \subset Q$ a totally ordered subset (a chain in $Q$) and define

$$t^* = \sup\{t : (t,x) \in P\}.$$

We also consider a sequence $\{(t_n,x_n)\}$ in $P$ such that

$$(t_n,x_n) \le (t_{n+1},x_{n+1}) \text{ and } t_n \longrightarrow t^*.$$

Then, taking into consideration the partial order relation on $Q$, we obtain that

$$\mathrm{d}(x_m,x_n) \le \frac{2(1-k)\,(\phi(t_m)-\phi(t_n))}{1-2k}, \text{ for all } m > n.$$

As a consequence, the sequence $(x_n)$ is Cauchy, therefore it converges to an element $x^* \in \overline{U}$. Since F has closed graph, and it is fixed point free on the boundary of $U$, we get that $(t^*,x^*) \in Q$. Moreover, we have $(t,x) \le (t^*,x^*)$ for every $(t,x) \in P$, proving that $(t^*,x^*)$ is an upper bound of $P$. Due to the Kuratowski-Zorn lemma, $Q$ admits a maximal element $(t_0,x_0) \in Q$. Thus, $x_0$ is a fixed point of $\mathrm{F}_{t_0}(x_0)$.

We will show now, by contradiction, that $t_0 = 1$. We assume that $t_0 \ne 1$. Hence, there exist $t_1 \in (t_0,1]$ and $r > 0$ such that

$$0 < \frac{(1-k)(\phi(t_1)-\phi(t_0))}{1-2k} < r$$

and $B(x_0;r) \subset U$. We also have the following inequality

$$\mathrm{D}(x_0,\mathrm{F}_{t_1}(x_0)) \le \mathrm{D}(x_0,\mathrm{F}_{t_0}(x_0)) + \mathrm{H}(\mathrm{F}_{t_0}(x_0),\mathrm{F}_{t_1}(x_0)) \le |\phi(t_1)-\phi(t_0)|.$$

This implies

$$D(x_0, F_{t_1}(x_0)) < \frac{1-2k}{1-k} r.$$

Using the local fixed point theorem for multi-valued Ćirić type operators, we obtain that there exists a fixed point $x_1$ of $F_{t_1}$ such that $d(x_0, x_1) \leq r$. Hence, $(t_1, x_1)$ belongs to $Q$ and $(t_0, x_0) < (t_1, x_1)$, which contradicts the maximality of $(t_0, x_0)$.

Conversely, if $F(1, \cdot)$ has a fixed point, then taking $t := 1 - t$ in the previous approach, we get that $F(0, \cdot)$ has a fixed point. The proof is complete. □

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