Geosciences and Engineering, Vol. 12, No. 1 (2024), pp. 57–74.
https://doi.org/10.33030/geosciences.2024.01.004

EXPLORING FOURIER TRANSFORMATIONS: BENEFITS, LIMITATIONS, AND APPLICATIONS IN ANALYZING TWO-DIMENSIONAL RIGHT RECTANGULAR PRISM'S MAGNETIC FIELD

KHOULOUD JLAIEL
Institute of Exploration Geosciences, University of Miskolc,
Khouloud.jlail@enis.tn
https://orcid.gov/0009-0005-8503-1934

Abstract: Fourier Transformations are crucial in signal processing, offering a unique approach for complex data analysis. This paper explores their advantages and limitations, explaining key concepts like Fourier Transformation, Fourier series, Discrete Fourier Transform, and Continuous Fourier Transform, focusing on practical applications. The strengths, such as signal decomposition into frequency components, are exemplified through a case study on the total magnetic field of two-dimensional right rectangular prisms. However, limitations arise with non-stationary signals due to the assumption of stationarity. Alternative methods like the Wavelet Transformation and Short-Time Fourier Transformation are briefly discussed. Serving as a practical guide, this paper aids researchers in utilizing Fourier Transformations while recognizing scenarios where alternative techniques may be more suitable.

Keywords: *Discrete Fourier Transform, Continuous Fourier Transform, Wavelet Transform, Short-Time Fourier Transform, Right Rectangular Prisms*

1. INTRODUCTION

The roots of Fourier Transformations can be traced back to the pioneering work of Joseph Fourier, a renowned French mathematician and physicist of the 18th century. Fourier's groundbreaking contributions to the field of mathematics included the development of these transformative techniques. He introduced Fourier series as a powerful tool for analyzing periodic functions and decomposing complex signals into a series of simpler trigonometric components. Over the centuries, Fourier's work laid the foundation for significant advances in various scientific and engineering disciplines (Trigg, 2005) (Baron Fourier, 2003) (Oppenheim, 1999a).

As time progressed, Fourier Transformations evolved from their classical origins into an indispensable mathematical framework for addressing a wide range of contemporary challenges. Today, they are an integral part of fields such as signal processing, image analysis, data interpretation, and more. Their adaptability and utility continue to expand, as they provide valuable insights into the frequency-domain representation of data, enabling us to understand the hidden patterns and structures in complex information (Gray and Goodman, 2012; Briggs and Henson, 1995).

In our data-driven world, the growing complexity and volume of information present both opportunities and challenges. The significance of this study lies in its exploration of the modern applications of Fourier Transformations, highlighting their pivotal role in contemporary data analysis and interpretation. Understanding how to harness these mathematical techniques in the face of increasingly intricate and large datasets is crucial for researchers, scientists, and engineers across numerous domains.

This study aims to shed light on the nuanced and evolving landscape of Fourier Transformations, revealing their applications, benefits, and limitations in a data-centric context. By doing so, it offers valuable insights into addressing the challenges associated with the processing and interpretation of complex data, equipping practitioners with the tools and knowledge needed to make informed decisions and drive innovation in their respective fields. The paper covers Fourier Transformations comprehensively.

2. Fundamental Concepts of Fourier Transformations

2.1. Fourier Transforms

In this section we will introduce the Fourier Transform as a universal problem-solving technique. We investigate the Fourier Transform, its inversion formula, and its basic properties. The transform analysis technique is used to reduce the complexity of the problem so to simplify the problem-solving analysis. Fourier Transform is one of such analysis technique.

2.1.1. Basics of Fourier Transform analysis

The Fourier Transform analyzes signals in the time domain, representing them in the frequency domain. It breaks down complex signals into a sum of simpler, periodic signals with various frequencies, phases, and amplitudes

$$S(f) = \int_{-\infty}^{\infty} s(t) e^{-j2\pi f t}\, dt \tag{1}$$

where s(t) is the waveform to be decomposed into a sum of sinusoids, t is the time, f is the frequency, S(f) is the Fourier Transform of s(t), and $j = \sqrt{-1}$.

The Fourier Transform provides a powerful mathematical tool for analyzing signals and understanding their frequency content. Its pictorial representation can help to visualize and understand the transformation process and the resulting frequency spectrum (Bhattacharyya and Navolio, 1976; Kitney-Hayes et al., 2014).

The Fourier Transformation is an important image-processing tool that divides an Image into sine and cosine components. The transformation's output represents the Fourier or frequency domain image, while the input image is the spatial domain counterpart *(Figure 1)*. Each point in the Fourier domain image represents a different frequency contained in the spatial domain image.

The Fourier Transform is utilized in various applications, including image analysis, image filtering, image reconstruction, and image compression (Bracewell and Bracewell, 1986; Oppenheim, 1999b). One way to understand the Fourier Transform is through its pictorial representation. When a signal is transformed using the Fourier Transform, the resulting representation is called the frequency spectrum. The frequency spectrum can be represented graphically as a plot of amplitude versus frequency. The amplitude represents the strength of each frequency component, and the frequency represents the frequency of the signal. Another way to visualize the Fourier Transform is through its representation as a series of sine and cosine waves.

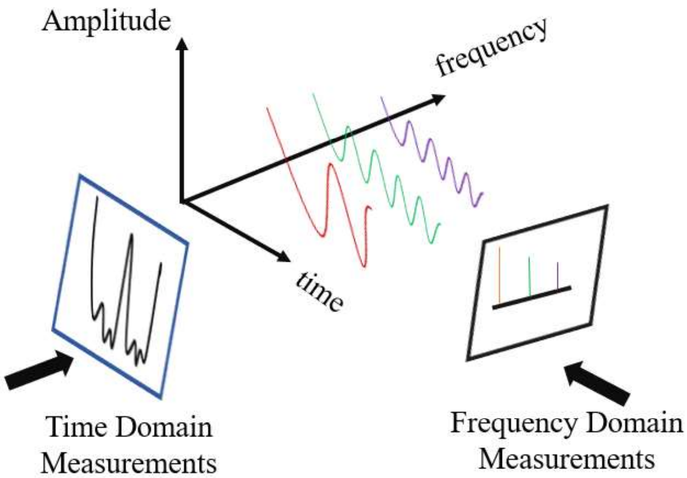

Figure 1

Fourier Transformation of the signal from time domain to frequency domain

2.1.2. Digital computer Fourier analysis

Digital computer Fourier analysis involves applying Fourier analysis algorithms on digital computers. Computational methods are used to calculate the discrete Fourier Transform for analyzing digital signals or data (Cooley and Tukey, 1965). Numerical integration of *Equation 1* implies the relationship

$$S(f_k) = \sum_{i=0}^{N-1} s(t_i)\, e^{-j2\pi f_k t_i}\, (t_{i+1} - t_i) \qquad k=0, 1, \ldots, N\text{-}1 \qquad (2)$$

where $S(f_k)$ denotes the Fourier Transform coefficient at frequency f_k, $s(t_i)$ represents the continuous-time signal sampled at N equally spaced time points t_i , k = 0, 1, ..., N-1 represents the frequency index.

Research and development in this area aimed to reduce the computational complexity and improve the performance for large-scale problems. While the computation of the DFT can be time-consuming for large N, the use of the fast Fourier Transforms algorithms (FFT), invented by Cooley and Tukey (1965), additionally to the advancements in hardware technologies can significantly mitigate this issue.

2.1.3. The inverse Fourier Transform

Mathematically, the Inverse Fourier transform can be defined as follows

$$s(t) = \int_{-\infty}^{+\infty} S(f)e^{j2\pi ft}\, df \quad (3)$$

The Inverse Fourier Transform (IFT) represent the reverses process of the Fourier Transform. It converts a frequency-domain representation of a signal back into the time-domain representation. $s(t)$ and $S(f)$ are called a Fourier transform pair, The IFT is the counterpart of the forward Fourier Transform and is often used in signal processing, image reconstruction, and various other applications. The existence of the Fourier integral is based on certain mathematical conditions. One important condition is the absolute integrability of the function $s(t)$ over its defined interval *(Equation 4)*. The function must be absolutely integrable for the Fourier integral to exist

$$\int_{-\infty}^{+\infty} |s(t)|\, dt < \infty \quad (4)$$

2.2. Properties of the Fourier Transform

The Fourier Transform is a powerful tool in signal processing and analysis, boasting key properties that enhance its effectiveness:

Linearity: The Fourier Transform is a linear operation. If a and b are constants, and $f(t)$ and g(t) are functions and they have the Fourier Transform $F(f)$ and $G(f)$, respectively, then $af(t) + b\, g(t)$a has a Fourier Transform $aF(f) + b\, G(f)$. $(af(t) + b\, g(t))$ and $(aF(f) + b\, G(f))$ are termed a Fourier Transform pair.

Symmetry: If $s(t)$ and $S(f)$ are a Fourier Transform, then $S(t)$ and $s(-f)$ are a Fourier Transform pair.

Time scaling: Time scaling in the Fourier Transform describes the impact of compressing or stretching a function in the time domain on its frequency domain representation (Oppenheim et al., 1997; Folland, 2009). The property states that if a function $f(t)$ is scaled by a factor α in the time domain, its Fourier Transform $F(\omega)$ will be scaled by $1/\alpha$. For a time-scaled version $g(t) = f(\alpha t)$,where α is a positive real number, the Fourier Transform $G(\omega)$ is given by this scaling relationship

$$G(\omega) = (1/|\alpha|)F(\omega/\alpha) \quad (5)$$

Frequency scaling: Frequency scaling in the Fourier Transform involves shifting the frequency components of a function in the time domain and observing the corresponding shift in the frequency domain. If a function $f(t)$ is multiplied by $e^{j\omega_0 t}$, where ω_0 is a real constant, its Fourier Transform $F(\omega)$ will be shifted by ω_0 units. For a frequency-scaled version $g(t) = e^{j\omega_0 t} f(t)$, the Fourier Transform $G(\omega)$ is given by

$$G(\omega) = F(\omega - \omega_0) \tag{6}$$

This property allows manipulation of a signal's frequency characteristics by introducing a constant frequency offset.

Alternate inversion formula: The alternate inversion formula is an alternative expression for the inverse Fourier Transform. It provides a different way to compute the time-domain function from its frequency-domain representation. The alternate inversion formula for the Fourier Transform pair $F(\omega)$ and $f(t)$ is given by

$$f(t) = \left[\int_{-\infty}^{+\infty} F^*(f)\, e^{j2\pi ft}\, df\right]^* \tag{7}$$

where $F^*(f)$ is the conjugate of $F(f)$; if $F(f) = R(f) + j\,I(f)$ then $F^*(f) = R(f) - jI(f)$.

Even functions: An even function $f(x)$ has the property $f(x) = f(-x)$ for all values of x. The Fourier Transform of an even function is real and even.

Odd function: An odd function $f(x)$ has the property $f(x) = -f(-x)$ for all values of x. The Fourier Transform of an odd function is purely imaginary and odd.

Complex time function: The Fourier Transform can be applied to complex time functions to obtain their frequency-domain representation. Understanding complex time functions and their Fourier Transforms is crucial for analyzing and manipulating signals in both the time and frequency domains. It allows for the extraction of frequency information, modulation analysis, filtering, and various other signal-processing operations.

2.3. The Fourier Series: Formulation and Diverse Applications

The Fourier series is a fundamental mathematical tool expanding periodic function, into an infinite sum of trigonometric functions like sines and cosines (Bracewell, 1986; Gray and Goodman, 2012). It elegantly approximates complex waveforms through simpler harmonic components. It extends to generalized Fourier series or transforms, representing functions as linear combinations of orthogonal basis functions, providing a versatile approach to expressing a wide range of periodic and non-periodic functions. These basis functions are typically chosen to be orthogonal where their inner products

$$(f, g) = \int_{\Omega} f(x)\, g(x)\, dx \tag{8}$$

satisfy certain properties (Elias and Stein, 2003). In the Fourier series, traditional basis functions are sine and cosine waves. However, in generalized Fourier series, various basis functions, such as Bessel functions, Legendre polynomials, and Hermite functions, can be used (Trigg, 2005; Silverman, 1972; Abramowitz and Stegun, 1968). A system of Fourier cosine functions represents a collection of cosine functions used in a Fourier series expansion. These functions are derived by taking the real component of equivalent complex exponential functions. The general form of a Fourier cosine series for a function $f(x)$ on [a, a+2π] is as follows

$$f(x) = a_0 + 2\sum_{n=1}^{\infty} a_n \cos nx \qquad (a < x < a + 2\pi), \tag{9}$$

where a_0 and a_n called constant expansion coefficients

$$a_0 = \frac{1}{2\pi}\int_a^{a+2\pi} 1\, f(x)\, dx \tag{10}$$

$$a_n = \frac{1}{2\pi}\int_a^{a+2\pi} f(x) \cos nx \; dx\,, \qquad n = 1, 2, 3, \ldots \tag{11}$$

The Fourier sine system refers to a set of sine functions that can be used as a basis for representing functions in a Fourier series expansion (Bateman, H. and Erdélyi, A., 1954; James et al., 1996). These functions are defined by taking the imaginary part of the corresponding complex exponential functions. The Fourier sine series is defined by the following general form of the Fourier sine series

$$f(x) = 2\sum_{n=1}^{\infty} b_n \sin nx \qquad (a < x < a + 2\pi), \tag{12}$$

$where\ b_n$ constant expansion coefficients are calculated as

$$b_n = \frac{1}{2\pi}\int_a^{a+2\pi} f(x) \sin nx \; dx \qquad ,n = 1, 2, 3, \ldots \tag{13}$$

The Fourier sine system is advantageous for odd functions or those defined on an antisymmetric interval due to its focus on odd terms, aligning with the properties of sine functions. The trigonometric system, which combines Fourier sine and cosine functions, is commonly used for Fourier series expansions, accommodating both antisymmetric and symmetric functions(Bracewell, 1986; Kamen and Heck, 2006). The general form of the Fourier series using this trigonometric system is

$$f(x) = a_0 + 2\sum_{n=1}^{\infty} (a_n \cos nx + \; b_n \sin nx) \tag{14}$$

where

$$a_0 = \frac{1}{2\pi}\int_a^{a+2\pi} f(x)\, dx \tag{15}$$

$$a_n = \frac{1}{2\pi}\int_a^{a+2\pi} f(x) \cos nx \; dx, \qquad n = 1, 2, 3, \ldots \tag{16}$$

$$b_n = \frac{1}{2\pi}\int_a^{a+2\pi} f(x) \sin nx \; dx \qquad ,n = 1, 2, 3, \ldots \tag{17}$$

2.4. The Discrete Fourier Transform: Theory and Applications

The Discrete Fourier Transform (DFT) is crucial for analyzing discrete time domain signals, unveiling their frequency components by converting a sequence of complex numbers representing signal samples. Widely used in signal processing, the DFT helps determine frequency composition, ignores phase information, and facilitates convolution operations (Oppenheim, 1999a; Briggs and Henson, 1995). Mathematically, the DFT is formally defined as

$$DFT(X[k]) = \sum_{n=0}^{N-1}(x[n]e^{-j2\pi nk/N}) \tag{18}$$

where
$X[k]$ represents the k-th frequency component of the signal in the frequency domain,
$x[n]$ denotes the n-th sample of the discrete signal in the time domain,
j is the imaginary unit,
N is the total number of samples.

The DFT is a vital tool in digital signal processing, decomposing discrete signals to reveal their spectral content. Widely used in applications like telecommunications, audio processing, and image analysis, the DFT is integral to tasks such as image compression and enhancement (Bracewell, 1986; Bose and Meyer, 2003; Smith, 2008). Additionally, it enables fast computation through algorithms like the Fast Fourier Transform (FFT), facilitating real-time signal analysis in applications like audio processing and wireless communications (Bagchi and Mitra, 2012; Bracewell, 2004).

2.5. Continuous Fourier Transform: Theory and Applications

The Continuous Fourier Transform (CFT) is a mathematical technique used to analyze continuous, time-domain signals in terms of their frequency components

$$CFT\big(X(f)\big) = \int_{-\infty}^{\infty} x(t)\, e^{-j2\pi ft} dt \tag{19}$$

where
$X(f)$ represents the frequency component of the signal in the frequency domain,
$x(t)$ denotes the continuous signal in the time domain,
j is the imaginary unit,
f is the continuous frequency variable.

The CFT is essential for analyzing continuous time-domain signals, extending the capabilities of the Discrete Fourier Transform (DFT) to non-discretely sampled signals. Widely used in physics and quantum mechanics, the CFT provides insights into the spectral characteristics of quantum particles and various physical phenomena by revealing the frequency components of continuous signals (Lowrie, 2011).

3. Benefits of Fourier Transformations and limitations

3.1. Advantages of Fourier Transformations

A fundamental strength of Fourier Transformations lies in their ability to decompose complex signals into their constituent frequency components. This feature is particularly advantageous when dealing with signals of diverse origins, allowing us to discern the underlying frequencies that make up these signals. This spectral decomposition serves as the foundation for many applications in signal processing and analysis. The Fourier series is a powerful tool for representing periodic signals by expressing them as combinations of sine and cosine functions. It simplifies the analysis of periodic phenomena, providing insights into harmonic content and temporal behavior, especially in fields like music, physics, and engineering (Bracewell, 1986).

In the realm of discrete signals, the DFT plays a pivotal role. Computing the DFT allows access to the frequency domain representation of discrete signals, aiding in understanding underlying frequencies and amplitudes in sampled data. This versatility makes the DFT crucial in applications from audio processing to telecommunications (Smith, 2008).

Real-world applications highlight the benefits of Fourier Transformations in signal processing, image analysis, and audio processing. These examples underscore the advantages of employing Fourier Transformations for effective signal analysis and interpretation.

3.2. Constraints of the Fourier Transformation

While Fourier Transformations offer a multitude of advantages, it is essential to acknowledge their limitations. One significant constraint pertains to the assumption of stationarity. The stationarity assumption implies that the signal's statistical properties remain constant over time (Smith, 1997) . However, many real-world signals exhibit variations in statistical properties, posing a challenge to the application of Fourier Transformations. In this context, it becomes imperative to delve into the specifics of stationarity constraints and their implications.

Another limitation concerns the sensitivity of Fourier Transformations to abrupt signal changes. When signals undergo sudden shifts or contain discontinuities, Fourier Transformations may encounter difficulties in accurately representing the signal's behavior. Addressing the challenges associated with abrupt changes in signals is essential to comprehensively understand the limitations of Fourier Transformations (Bracewell, 1986). The reduced accuracy of Fourier Transformations when applied to real-world data constitutes another critical limitation. Practical data often deviates from the idealized mathematical models assumed in Fourier analysis. As a result, the precision and the reliability of Fourier Transformations may diminish when dealing with complex, noisy, or imperfect data (Oppenheim, 1999c). Exploring the intricacies of this limitation is fundamental to grasping the practical constraints of Fourier Transformations in real-world scenarios.

In addition to theoretical constraints, practical aspects can also impose limitations on the utility of Fourier Transformations (Bracewell, 1986). Understanding the interplay between theoretical and practical constraints is vital in gaining insights into the boundaries of Fourier Transformations when applied in various domains. By comprehensively examining these limitations, we can make informed decisions about when to employ Fourier Transformations and when alternative methods may be more suitable.

4. ADDRESSING LIMITATIONS AND EXPLORING ALTERNATIVES

To address the limitations of Fourier Transformations, the exploration of alternative methods has become essential. These methods provide innovative ways to analyze signals and overcome the constraints associated with Fourier analysis. Alternative techniques often offer unique advantages in specific applications, making them valuable tools for researchers and practitioners working in signal analysis.

4.1. Wavelet Transform: Explanation and Applications

The Wavelet Transform is a mathematical technique used to analyze functions and signals by decomposing them into different scales and frequencies. Unlike the Fourier Transform, which represents a signal in the frequency domain, the Wavelet Transform simultaneously provides information in both the time and frequency domain (Mallat, 1999; Daubechies, 1992). This dual-domain analysis is one of the key advantages of wavelet analysis.

Mathematically, the continuous Wavelet Transform (CWT) of a signal x(t) is computed using the following formula

$$CWT(a,b) = \int x(t)\psi\left[\frac{(t-b)}{a}\right]dt \tag{20}$$

where
CWT (a, b) is the continuous Wavelet Transform at scale a and position b,
x(t) is the input signal in the time domain,
ψ is the complex conjugate of the mother wavelet function, which is a scaled and translated version of the mother wavelet.

The continuous Wavelet Transform examines signal x(t) at different scales (a) and positions (b), providing a time-frequency representation that's ideal for analyzing non-stationary signals. The choice of the mother wavelet function $\psi(t)$ determines the Wavelet Transform's properties, with wavelets like Morlet, Mexican hat, and Haar serving specific applications.

This versatile mathematical tool has diverse applications. In image processing, it aids in compression, denoising, and feature extraction, while data compression techniques efficiently reduce file sizes (Strang and Nguyen, 1996). In pattern recognition, particularly in computer vision and machine learning, it helps identify crucial features (Cohen, 2003). Wavelet analysis benefits biomedical signal processing, environmental monitoring, speech and audio processing, and financial data analysis. Renowned for its adaptability and precise time-frequency information, the Wavelet Transform expands the signal analysis toolkit, offering an alternative to Fourier Transformations, especially for non-stationary signals (Mallat, 1999).

4.2. Short-Time Fourier Transform (STFT): Explanation and Applications

The Short-Time Fourier Transform (STFT) is one such alternative method that merits attention. It is designed to address a critical limitation of the standard Fourier Transform – the assumption of signal stationarity. The STFT overcomes this constraint by analyzing short, overlapping segments of a signal rather than the entire signal at once (Durak and Arikan, 2003). This approach allows for the examination of time-varying frequency components within a signal, making it particularly useful for non-stationary signals like speech, music, or any signal that changes over time

$$\mathrm{STFT}(\mathrm{t},\mathrm{f}) = \int x(\tau)w(\tau t)e^{-j2\pi f\tau}d\tau \tag{21}$$

where

$\mathrm{STFT}(\mathrm{t},\mathrm{f})$ represents the Short-Time Fourier Transform at time t and frequency f,

$x(\tau)$ is the input signal in the time domain,

$w(t)$ is the window function, usually a short, localized function with non-zero values only within a limited time frame,

τ is the time variable used for shifting the window function,

t is a variable represents time,

$e^{-j2\pi f\tau}$ represents the complex exponential term for frequency f.

In the STFT, a window function slides over the signal, capturing a small portion at a time. By applying the Fourier Transform to each windowed segment, we obtain a time-frequency representation of the signal. This representation reveals how the frequency content evolves over time, providing crucial insights into transient events and non-stationary behavior. As a result, the STFT is widely used in applications like audio processing, speech recognition, and the analysis of time-varying phenomena. These are a few examples within the broader family of Fourier Transformations (These are shown in *Table 1*). The choice of which transform to use depends on the specific characteristics of the signal and the goals of the analysis.

Table 1

The system of Fourier Transformation and its characteristics, advantages, and disadvantages of some key transformations within the Fourier family

<table>
<tr><th>System of Fourier Transformations</th><th colspan="2">Characteristics</th><th>Advantages</th><th>Disadvantages</th></tr>
<tr><td rowspan="2">Fourier Transform</td><td>Mathematical Representation</td><td>Continuous Domain</td><td rowspan="2">– Frequency Analysis.
– Widespread Applicability.</td><td rowspan="2">– Computational Complexity
– Limited for Discrete Signals</td></tr>
<tr><td>Converts a function of time (or space) into a function of frequency.
$F(\omega) = \int_{-\infty}^{\infty} f(t) . e^{-j\omega t}\, dt$</td><td>Defined for continuous signals</td></tr>
<tr><td rowspan="2">Discrete Fourier Transform (DFT)</td><td>Discretization</td><td>Finite-Length Signals</td><td rowspan="2">– Digital Implementation
– Exact Representation</td><td rowspan="2">– Computational Complexity</td></tr>
<tr><td>Computes the Fourier Transform for discrete signals</td><td>Suitable for processing finite-length signals</td></tr>
<tr><td>Fast Fourier Transform (FFT)</td><td colspan="2">Efficiency Improvement: An algorithmic approach to compute the DFT with reduced computational complexity.</td><td>– Efficiency
– Widespread Use</td><td>– Requires Power-of-Two Lengths: Some FFT algorithms are most efficient when the signal length is a power of two.</td></tr>
<tr><td rowspan="2">Short-Time Fourier Transform (STFT) (Allen and Rabiner 1977; Cohen 1995)</td><td>Time-Frequency Analysis</td><td>Windowing</td><td rowspan="2">– Time-Frequency Localization
– Adaptability: Suitable for non-stationary signals.</td><td rowspan="2">– Time-Frequency Resolution Tradeoff: The choice of window size affects time and frequency resolution; a smaller window provides better time resolution but poorer frequency resolution, and vice versa.</td></tr>
<tr><td>Represents how the frequency content of a signal changes over time</td><td>Involves dividing the signal into short segments and applying Fourier Transform to each segment.</td></tr>
</table>

5. Exploring the Practical Applications: Total Magnetic Field Analysis of a Two-Dimensional Right Rectangular Prism and its Fourier Transform Representation

The magnetic field of a 2D right rectangular prism, magnetized along the z-direction *(Figure 2)*, is determined using the 2D Fourier Transform. It is represented as the sum of magnetic fields from two infinite lines of dipoles (Kis, 2009). The inverse transform, employing the convolution theorem, yields a comprehensive magnetic field. The resulting expression depends on the prism's dimensions and the observation point's specific location (Griffiths and Inglefield, 2005; Jackson and Fox, 1999).

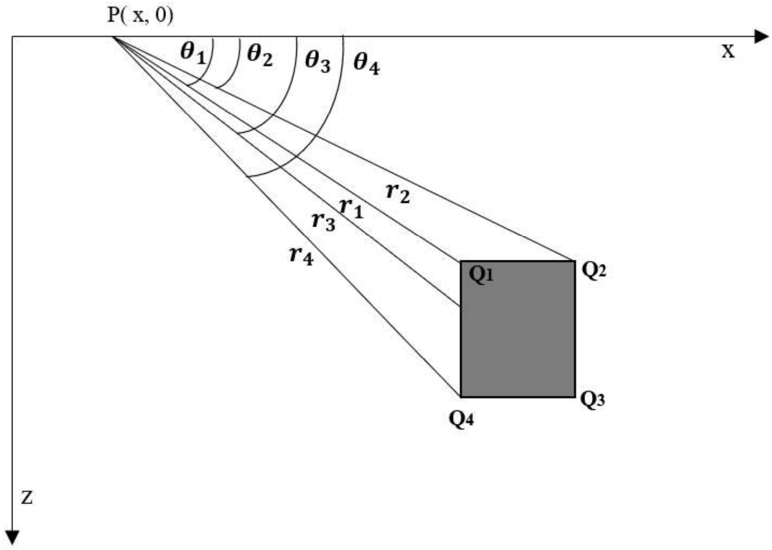

Figure 2
The location of a 2D right rectangular prism in the xz-coordinate system; the distances; r_1 , r_2, r_3, and r_4; and angles θ_1, θ_2, θ_3, and θ_4

The magnetic field produced by a 2D right rectangular prism *(Figure 2)* can be computed using the following analytical formula that involves the prism's dimensions and the coordinates of the observation point

$$\mathrm{T(x,z)} = -\frac{\mu_{0\,\mathrm{J}}}{2\pi}\,(I_5 + I_6 + I_7 + I_8\,) \tag{22}$$

where

$$I_5 = Kk\left(\tan^{-1}\frac{z-z_2}{x-x_2} - \tan^{-1}\frac{z-z_2}{x-x_1} - \tan^{-1}\frac{z-z_1}{x-x_2} + \tan^{-1}\frac{z-z_1}{x-x_1}\right) \tag{23}$$

Let us initiate the angles $\theta_1\,, \theta_2, \theta_3, and\ \theta_4$

$$\theta_1 = \tan^{-1}\frac{z_1-z}{x_1-x}\ ,\ \theta_2 = \tan^{-1}\frac{z_1-z}{x_2-x},\ \theta_3 = \tan^{-1}\frac{z_2-z}{x_2-x},\ \text{and}\ \theta_4 = \tan^{-1}\frac{z_2-z}{x_1-x}$$

$$I_5 = Kk\,[\theta_1 - \theta_2 + \theta_3 - \theta_4]. \qquad \text{and} \qquad I_6 = Nkln\frac{r_1 r_3}{r_2 r_4}$$

With

$$r_1^2 = (x-x_1)^2 + (z-z_1)^2\ ,\ r_2^2 = (x-x_2)^2 + (z-z_1)^2,$$

$$r_3^2 = (x-x_2)^2 + (z-z_2)^2\ ,\ r_4^2 = (x-x_1)^2 + (z-z_2)^2\ ,$$

$$I_7 = Knln\frac{r_1 r_3}{r_2 r_4},\ \text{and} \quad I_8 = Nn(-\theta_1 + \theta_2 - \theta_3 + \theta_4)$$

where $\mathrm{K} = \cos \mathrm{a} \cos(\mathrm{A}-\mathrm{b})$, $\mathrm{N} = \sin \mathrm{a}$, $\mathrm{k} = \cos \mathrm{I} \cos(\mathrm{A}-\mathrm{D})$, $\mathrm{n} = \sin \mathrm{I}$, $\mathrm{l} = \cos \mathrm{I} \cos \mathrm{D}$.

In the direction K, the angle A is the azimuth of the profile measured from the geographic North, a and b are the inclination and declination of the moment, respectively. The total magnetic field of a 2D right rectangular prism can be computed, given specific parameters such as the absolute value of the vector of magnetization (J), dimensions (rd), height (h), inclination of the earth's magnetic field (incl), declination of the earth's magnetic field (decl), and the position at which the magnetic field is to be calculated (x).

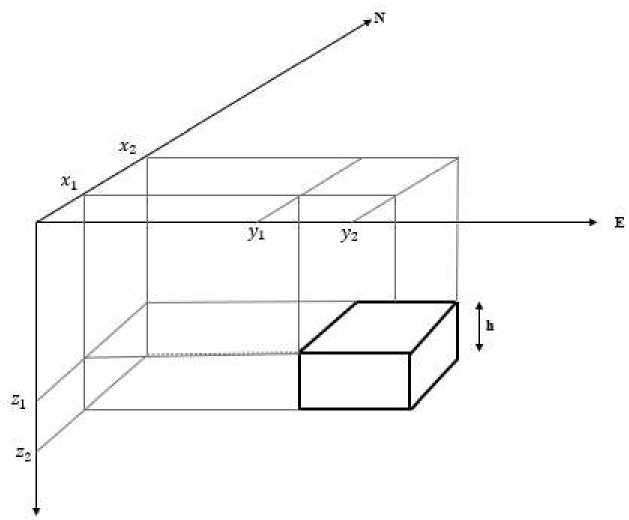

Figure 3

The position of 3D right rectangular prism in the xyz-coordinate system; the coordinates are x_1 , x_2, y_1, y_2 *and* z_1 *and* z_2

The function takes the following inputs as it is shown in the 2D and 3D representation, (*Figure 2* and *3*):
`J` is the magnetization of the prism in A/m,
`rd`is the dimensions of the prism represented as a vector [x_1, x_2, z_1, z_2], where x_1 and x_2 are the x-coordinates of the opposite corners in the x-direction, and z_1 and z_2 are the z-coordinates of the opposite corners in the z-direction,
`h`is the height of the prism in the y-direction,
`incl`is the inclination of the earth's magnetic field, provided as a vector [a, I], where 'a' is the angle of inclination in degrees from the horizontal plane, and 'I' is the angle of inclination in degrees from the northward vertical,
`decl` is the declination of the earth's magnetic field, expressed as a vector [b, D], where 'b' is the angle of declination in degrees from the geographic north to the magnetic north, and 'D' is the angle of declination in degrees from the magnetic north to the northward vertical,
`x` is the position at which the magnetic field is to be calculated, specified in the x-direction.

The function computes the magnetic field (in nT) at a specified x-position using the formula for a 2D rectangular prism, involving inclination and declination angle conversions, direction cosine calculations, and dipole moment vector components. It sums contributions from each prism side to determine the total field. The accompanying plotting script visualizes magnetic field variations for different angle configurations (D = β = 0; I = α = 0°, 30°, 60°, 90°) in *Figures 2.A.a, 2.B.d, 2.C.g*, and *2.D.j*.

Next, we introduce a Fourier Transform for the total field of a 2D right rectangular prism. The computation considers a source with a horizontal extension of 2a (symmetrical to the origin), upper and lower depths (d_1 and d_2), and uniform magnetization. The spatial frequency 'f' (in cycles per spatial unit) is used. The given equation yields result for z = 0 and f < 0

$$T(f) = \mu_0 J\, a\, sinc(2fa)\left((Kk - Nn)\left(e^{-2\pi|f|d_2} - e^{-2\pi|f|d_1}\right) + j\,(Kn + Nk)\left(e^{-2\pi|f|d_2} - e^{-2\pi|f|d_1}\right)\right)$$

For z = 0 and f > 0

$$T(f) = \mu_0 J\, a\, \mathrm{sinc(2fa)}\left((Kk - Nn)\left(e^{-2\pi|f|d_2} - e^{-2\pi|f|d_1}\right) + j\,(Kn + Nk)\left(-e^{-2\pi|f|d_2} + e^{-2\pi|f|d_1}\right)\right)$$

With a thickness t(t = d_2–d_1) the Fourier Transform of the total magnetic field of the 2D right rectangular prism is

$$T(f) = \mu_0 J\, a\, \mathrm{sinc(2fa)}\, e^{-2\pi|f|d_1}(e^{-2\pi|f|t} - 1)((Kk - Nn) - jsign(f)(Kn + Nk)$$

Figures 2 illustrates the real and imaginary parts of the spectrum of the total magnetic field of the 2D right rectangular prism. These functions are plotted with respect to the spatial frequency 'f', and the sampling interval is determined by the upper depth 'd1' (set at 1 km) and the horizontal extension '2a' (equal to 6 km), where 't' is 1 km. *Figures* (*4.A.b* and *4.A.c*), (*4.B.e* and *4.B.f*), (*4.C.h* and *4.C.i*), and (*4.D.k* and *4.D.l*) are generated for D = β = 0; I = α = 0°, 30°, 60°, 90°, respectively.

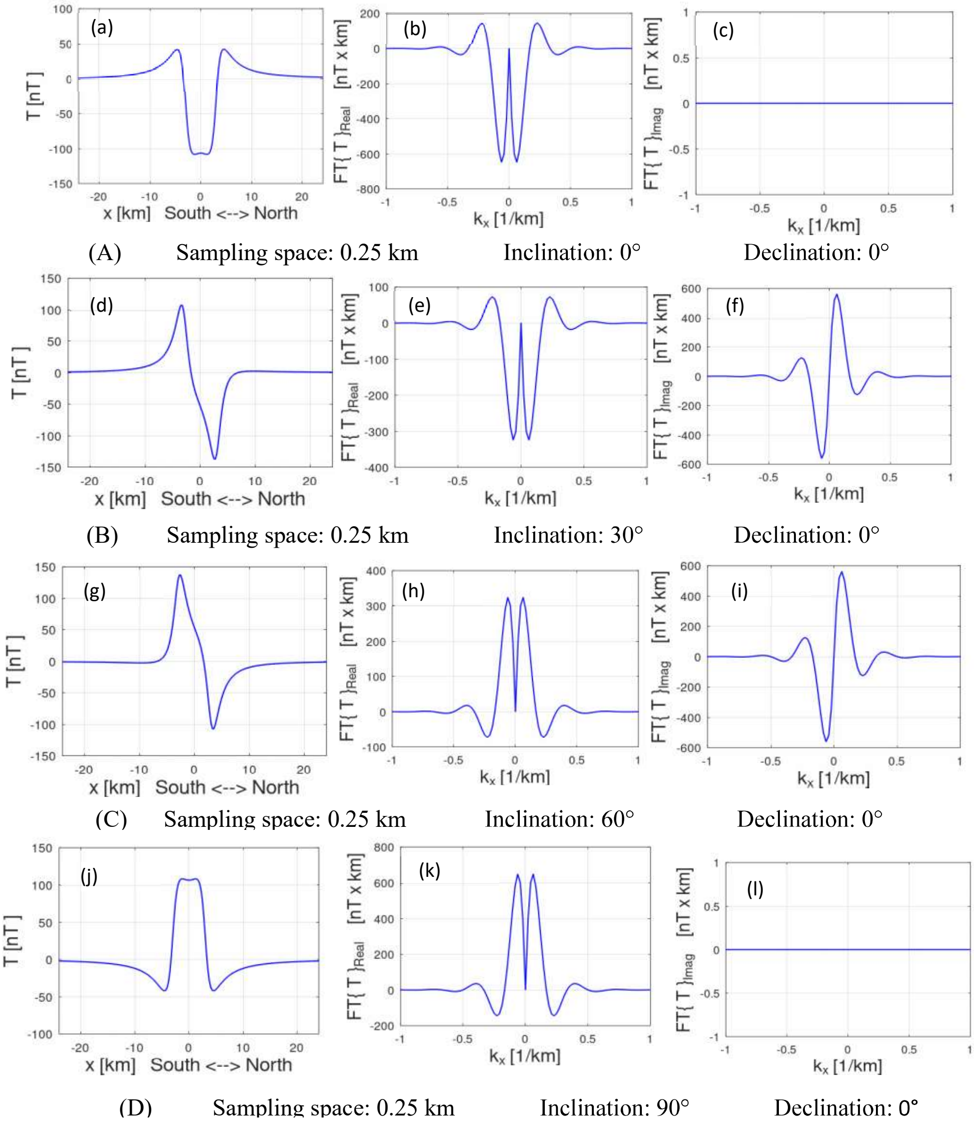

Figure 4

The total magnetic field of a 2D right rectangular prism with its Fourier Transforms representation

6. Conclusions

In conclusion, this paper has provided a comprehensive exploration of the historical significance, theoretical foundations, virtues, and limitations of Fourier Transformations. For professionals in diverse fields, understanding these aspects is essential for informed decision-making in signal analysis. We introduced alternative methods like the Wavelet Transform and Short-Time Fourier Transform, offering a comparative analysis for method selection. As technology advances, ongoing research into innovative methods and interdisciplinary approaches is crucial to address challenges in non-ideal scenarios and real-world data. Embracing a forward-looking perspective will contribute to the continuous advancement of signal analysis.

Acknowledgments

The author extends gratitude to the Stipendium Hungaricum scholarship for enabling their PhD studies in Hungary. Special thanks to prof. dr. Endre Turai and Dr. Péter Tamás Vass for their support. Appreciation is also conveyed to the Mikoviny Sámuel Doctoral School of Earth Sciences and the Department of Geophysics at the University of Miskolc for providing a valuable opportunity and an enriching research environment.

References

Abramowitz, M., Stegun, I. A. (1968). *Handbook of Mathematical Functions with Formulas, Graphs, and Mathematical Tables.* U.S. Government Printing Office.

Allen, J. B., Rabiner, L. R. (1977). A unified approach to short-time Fourier analysis and synthesis. *Proceedings of the IEEE*, 65, pp. 1558–1564.

Bagchi, S., Mitra, S. K. (2012). *The nonuniform discrete Fourier transform and its applications in signal processing.* Springer Science & Business Media.

Baron Fourier, J. B. J. (2003). *The analytical theory of heat.* Courier Corporation.

Bateman, H. and Erdélyi, A. (1954). Tables of Integral Transforms.

Bhattacharyya, B. K., Navolio, M. E. (1976). A Fast Fourier Transform method for rapid computation of gravity and magnetic anomalies due to arbitrary bodies*. *Geophys. Prospect.*, 24, pp. 633–649. https://doi.org/10.1111/j.1365-2478.1976.tb01562.x

Bose, T., Meyer, F., (2003). *Digital signal and image processing.* John Wiley & Sons, Inc.

Bracewell, R. (2004). *Fourier analysis and imaging.* Springer Science & Business Media.

Bracewell, R. N. (1986). *The Fourier Transform and Its Applications.* NY, McGraw-Hill.

Briggs, W. L., Henson, V. E. (1995). The DFT: An Owner's Manual for the Discrete Fourier Transform. *Society for Industrial and Applied Mathematics*. https://doi.org/10.1137/1.9781611971514

Cohen, A. (2003). *Numerical analysis of wavelet methods.* Elsevier.

Cohen, L. (1995). *Time-frequency analysis.* Prentice hall New Jersey.

Cooley, J. W., Tukey, J. W. (1965). An algorithm for the machine calculation of complex Fourier series. *Mathematics of Computation*, 19, pp. 297–301.

Daubechies, I. (1992). Ten Lectures on Wavelets. *Society for Industrial and Applied Mathematics*. https://doi.org/10.1137/1.9781611970104

Durak, L., Arikan, O. (2003). Short-time Fourier transform: two fundamental properties and an optimal implementation. *IEEE Transactions on Signal Processing*, 51, pp. 1231–1242.

Elias, M., Stein, R. S. (2003). *Fourier analysis: An introduction.*

Folland, G. B. (2009). *Fourier analysis and its applications.* American Mathematical Soc.

Gray, R. M., Goodman, J. W. (2012). *Fourier transforms: an introduction for engineers.* Springer Science & Business Media.

Griffiths, D. J., Inglefield, C. (2005). Introduction to Electrodynamics. *American Journal of Physics*, 73, pp. 574–574. https://doi.org/10.1119/1.4766311

Jackson, J. D., Fox, R. F. (1999). Classical Electrodynamics. 3rd ed. *American Journal of Physics*, 67, pp. 841–842. https://doi.org/10.1119/1.19136

James, J. F., Enzweiler, R. N., McKay, S., Christian, W. (1996). A student's guide to Fourier transforms with applications in physics and engineering. *Computers in Physics*, 10, pp. 47–47.

Kamen, E. W., Heck, B. S. (2006). *Fundamentals of signals and systems using the web and matlab.* Prentice-Hall, Inc.

Kis, K. (2009). *Magnetic methods of applied geophysics.* Eötvös University Press.

Kitney-Hayes, K. A., Ferro, A. A., Tiwari, V., Jonas, D. M. (2014). Two-dimensional Fourier transform electronic spectroscopy at a conical intersection. *The Journal of Chemical Physics*, 140, 124312. https://doi.org/10.1063/1.4867996

Lowrie, W. (2011). *A student's guide to geophysical equations.* Cambridge University Press.

Mallat, S. (1999). *A wavelet tour of signal processing.* Elsevier.

Oppenheim, A. V. (1999a). *Discrete-time signal processing.* Pearson Education India.

Oppenheim, A. V. (1999b). *Discrete-time signal processing.* Pearson Education India.

Oppenheim, A. V. (1999c). *Discrete-time signal processing.* Pearson Education India.

Oppenheim, A. V., Willsky, A. S., Nawab, S. H., Ding, J.-J. (1997). *Signals and systems.* Prentice Hall Upper Saddle River, NJ.

Silverman, R. A. (1972). *Special functions and their applications.* Courier Corporation.

Smith, J. O. (2008). *Mathematics of the discrete Fourier transform (DFT): with audio applications.* Julius Smith.

Smith, S. W. (1997). *The scientist and engineer's guide to digital signal processing.*

Strang, G., Nguyen, T. (1996). Wavelets and filter banks. SIAM.

Trigg, G. L. (ed.) (2005). *Mathematical Tools for Physicists.* 1st ed. Wiley. https://doi.org/10.1002/3527607773