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Family of quadratic differential systems with invariant parabolas: a complete classification in the space $\mathbb{R}^{12}$

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Abstract. Consider the class **QS** of all non-degenerate planar quadratic differential systems and its subclass **QSP** of all its systems possessing an invariant parabola. This is an interesting family because on one side it is defined by an algebraic geometric property and on the other, it is a family where limit cycles occur. Note that each quadratic differential system can be identified with a point of $\mathbb{R}^{12}$ through its coefficients. In this paper, we provide necessary and sufficient conditions for a system in **QS** to have at least one invariant parabola. We give the global "bifurcation" diagram of the family **QS** which indicates where a parabola is present or absent and in case it is present, the diagram indicates how many parabolas there could be, their reciprocal position and what kind of singular points at infinity (simple or multiple) as well as their multiplicities are the points at infinity of the parabolas. The diagram is expressed in terms of affine invariant polynomials and it is done in the 12-dimensional space of parameters.

Keywords: quadratic vector fields, affine invariant polynomials, invariant algebraic curve, invariant parabola.

2020 Mathematics Subject Classification: 58K30, 34A26, 34C05, 34C40.

1 Introduction and statement of main results

We consider here differential systems of the form

$$\frac{dx}{dt} = P(x,y), \qquad \frac{dy}{dt} = Q(x,y), \tag{1.1}$$

where P, $Q \in \mathbb{R}[x,y]$, i.e. P, Q are polynomials in x, y over $\mathbb{R}$ and their associated vector fields

$$X = P(x,y)\frac{\partial}{\partial x} + Q(x,y)\frac{\partial}{\partial y}.$$

We call *degree* of a system (1.1) the integer $m = \max(\deg P, \deg Q)$. In particular we call *quadratic* a differential system (1.1) with $m = 2$. We denote here by **QS** the whole class of real quadratic differential systems.

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Quadratic systems appear in the modelling of many natural phenomena described in different branches of science, in biological and physical applications and applications of these systems became a subject of interest for the mathematicians. Many papers have been published about quadratic systems, see for example [19] for a bibliographical survey.

Here we always assume that the polynomials P and Q are coprime. Otherwise doing a rescaling of the time, systems (1.1) can be reduced to linear or constant systems. Quadratic systems under this assumption are called *non-degenerate quadratic systems*.

Let V be an open and dense subset of $\mathbb{R}^2$, we say that a nonconstant differentiable function $H : V \to \mathbb{R}$ is a first integral of a system (1.1) on V if $H(x(t), y(t))$ is constant for all of the values of t for which $(x(t), y(t))$ is a solution of this system contained in V. Obviously H is a first integral of systems (1.1) if and only if

$$X(H) = P\frac{\partial H}{\partial x} + Q\frac{\partial H}{\partial y} = 0,$$

for all $(x, y) \in V$. When a system (1.1) has a first integral we say that this system is integrable.

The knowledge of the first integrals is of particular interest in planar differential systems because they allow us to draw their phase portraits.

On the other hand given $f \in \mathbb{C}[x, y]$ we say that the curve $f(x, y) = 0$ is an *invariant algebraic curve* of systems (1.1) if there exists $K \in \mathbb{C}[x, y]$ such that

$$P\frac{\partial f}{\partial x} + Q\frac{\partial f}{\partial y} = Kf.$$

The polynomial K is called the *cofactor* of the invariant algebraic curve $f = 0$. When $K = 0$, f is a polynomial first integral.

Let us suppose that $f(x, y) = 0$ is of degree n:

$$f(x, y) = c_{00} + c_{10}x + c_{01}y + \cdots + c_{n0}x^n + c_{n-1,1}x^{n-1}y + \cdots + c_{0n}y^n$$

with $\hat{c} = (c_{00}, \dots, c_{0n}) \in \mathbb{C}^N$ where $N = (n+1)(n+2)/2$. We note that the equation $\lambda f(x, y) = 0$ where $\lambda \in \mathbb{C}^* = \mathbb{C}\backslash\{0\}$ yields the same locus of complex points in the plane as the locus induced by $f(x, y) = 0$. So, a curve of degree n defined by $\hat{c}$ can be identified with a point $[\hat{c}] = [c_{00} : c_{10} : \cdots : c_{0n}]$ in $P_{N-1}(\mathbb{C})$. We say that a sequence of degree n curves $f_i(x, y) = 0$ converges to a curve $f(x, y) = 0$ if and only if the sequence of points $[\hat{c}_i] = [c_{i00} : c_{i10} \cdots : c_{i0n}]$ converges to $[\hat{c}] = [c_{00} : c_{10} : \cdots : c_{0n}]$ in the topology of $P_{N-1}(\mathbb{C})$.

We observe that if we rescale the time $t' = \lambda t$ by a positive constant λ the geometry of the systems (1.1) does not change. So for our purposes we can identify a system (1.1) of degree m with a point in $[a_{10}, a_{10}, \dots, b_{0m}]$ in $\mathbb{S}^{M-1}(\mathbb{R})$ with $M = (m+1)(m+2)$.

We compactify the space of all the polynomial differential systems of degree m on $\mathbb{S}^{M-1}$ with $M = (m+1)(m+2)$ by multiplying the coefficients of each systems with $1/\big(\sum(a_{ij}^2 + b_{ij}^2)\big)^{1/2}$.

Definition 1.1. We say that an invariant curve $\mathcal{L} : f(x, y) = 0$, $f \in \mathbb{C}[x, y]$ for a polynomial system (S) of degree m has *multiplicity* r if there exists a sequence of real polynomial systems (S_k) of degree m converging to (S) in the topology of $\mathbb{S}^{M-1}$, $M = (m+1)(m+2)$, such that each (S_k) has r distinct invariant curves $\mathcal{L}_{1,k} : f_{1,k}(x, y) = 0, \dots, \mathcal{L}_{r,k} : f_{r,k}(x, y) = 0$ over $\mathbb{C}$, $\deg(f) = \deg(f_{i,k}) = n$, converging to $\mathcal{L}$ as $k \to \infty$, in the topology of $P_{N-1}(\mathbb{C})$, with $N = (n+1)(n+2)/2$ and this does not occur for $r + 1$.

The motivation for studying the systems in the quadratic class is not only because of their usefulness in many applications but also for theoretical reasons, as discussed by Schlomiuk and Vulpe in the introduction of [20]. The study of non–degenerate quadratic systems could be done using normal forms and applying the invariant theory.

Systematic work on quadratic differential systems possessing an invariant conic began towards the end of the XX-th century and the beginning of this century. Quadratic systems having an invariant ellipse as a limit cycle were investigated by Y.-X. Qin [18]; the necessary and sufficient conditions on the coefficients of a quadratic system and also on the coefficients of a conic so as to have the conic as an invariant curve of the system were presented by Druzhkova [8]; Cairó and Llibre in [4] have investigated the Darboux integrability of the quadratic systems having invariant algebraic conics; Oliveira, Rezende and Vulpe [14] provided necessary and sufficient conditions for a system in **QS** to have at least one invariant hyperbola in terms of its coefficients and the necessary and sufficient affine invariant conditions for a system in **QS** so as to have the ellipse as an invariant curve of the system were presented by Oliveira, Rezende, Schlomiuk and Vulpe [16]. In [15] the authors classified the family of quadratic systems possessing an invariant hyperbola in terms of configurations of hyperbolas and presence or absence of invariant lines. This is an invariant classification, independent of specific normal forms. A similar classification in the case of an invariant ellipse is done in [13].

In this work we consider non-degenerate quadratic differential systems possessing an invariant parabola. We denote this family by **QSP**. Our goal in this paper is to obtain a characterization of systems in **QSP** in terms of invariant polynomials. Thus our equalities and inequalities in the bifurcation diagram splitting the parameter space into regions and subsets with distinct dynamics, will not be expressed in terms of coefficients of a fixed normal form or several such forms, coefficients which do not have any geometrical meaning and are rigidly tied to these normal forms. They will be expressed in terms of invariant polynomials which are very supple objects that can be easily be computed by a computer for any specific normal form and allowing us also to easily pass from one normal form to any other.

It is known that the coordinates of an infinite singular point p of a quadratic system (S) are defined by a linear factor in the factorization of the invariant polynomial $C_2(x,y) = yp_2(x,y) - xq_2(x,y)$ over $\mathbb{C}$. Here $p_2(x,y)$ and $q_2(x,y)$ are the corresponding quadratic homogeneous parts of (S). The multiplicity m of the singularity p has two components (see the concepts and notations introduced in [11]). If we denote them by (m^∞, m^f) (i.e. $m = m^\infty + m^f$) then m^∞ (respectively, m^f) is the maximum number of infinite (respectively, finite) singularities which can split from p, in small perturbations of the systems. In this case the number m^∞ coincides with the multiplicity of the linear factor of $C_2(x,y)$ which defines p.

Definition 1.2. By the direction of an invariant parabola of a quadratic system (S) we mean the direction of its axis of symmetry which intersects the invariant line $Z = 0$ at an infinite singular point p of (S) with the multiplicity (m^∞, m^f). We say that this direction of the invariant parabola is simple (respectively, double; triple) if $m^\infty = 1$ (respectively 2; 3). We denote this parabola by $\cup$ (respectively $\overset{2}{\cup}$; $\overset{3}{\cup}$). Moreover, if the infinite invariant line $Z = 0$ is filled up with singularities then we denote by $\overset{\infty}{\cup}$ the invariant parabola which is tangent to the line $Z = 0$ at a non-isolated singular point.

In order to distinguish the invariant parabolas that a quadratic system could have we use the following notations:

- $\cup$ for a simple invariant parabola;

- $\Cup$ for two simple invariant parabolas in the same direction (they could intersect);
- $\cup\subset$ for two simple invariant parabolas in different directions;
- $\bigcup^2$ for one double invariant parabola;
- $\overset{2}{\cup}$ for one simple invariant parabola in double direction;
- $\overset{3}{\cup}$ for one simple invariant parabola in triple direction;
- $\overset{\infty}{\cup}$ for one simple invariant parabola when the line at infinity is filled up with singularities;
- $\overset{2}{\cup}\subset$ for two simple invariant parabola: one in a double direction and one in a simple direction;
- $\Cup\subset$ for three simple invariant parabolas: two in one direction and one in another direction;
- $\overset{2}{\cup}\overset{2}{\Cup}$ for three real invariant parabolas in the same double direction;
- $\overset{2}{\cup}\overset{2}{\Cup}{}^c$ for one real and two complex invariant parabolas in the same double direction;
- $\overset{2}{\cup}\overset{2}{\bigcup}{}^2$ for one simple and one double real invariant parabolas in the same double direction;
- $\overset{2}{\bigcup}{}^3$ for a triple real invariant parabola in a double direction;
- $\infty\overset{2}{\cup}$ for a 1-parameter family of invariant parabolas in the same double direction;
- $\infty\overset{3}{\cup}$ for a 1-parameter family of invariant parabolas in the same triple direction.

Our main results are stated in the following theorem.

Main Theorem. (A) *The condition $\chi_1 = \chi_2 = 0$ is necessary for a non-degenerate quadratic system to possess at least one invariant parabola.*

$(\boldsymbol{B})$ *Assume that for a non-degenerate quadratic system (S) the condition $\chi_1 = \chi_2 = 0$ holds. Then this system possesses at least one invariant parabola if and only if the corresponding conditions indicated below are satisfied, respectively. Furthermore in the case of the existence of an invariant parabola this systems could be brought via an affine transformation and time rescaling to one of the canonical forms presented below, correspondingly:*

$\boldsymbol{\alpha})$ *For $\eta > 0$ the system (S) could only possess one of the following sets of invariant parabolas: $\cup$, $\Cup$, $\bigcup^2$, $\cup\subset$, $\Cup\subset$. Moreover (S) has one of the above sets of parabolas if and only if the corresponding conditions provided by the diagram given in Figure 1 are satisfied. Furthermore the system (S) with an invariant parabola could be brought via an affine transformation and time rescaling to the following canonical form*

$$\dot{x} = m + nx - \frac{1}{2}(1+g)y + gx^2 + xy, \quad \dot{y} = 2mx + 2ny + (g-1)xy + 2y^2 \qquad (S_{\boldsymbol{\alpha}})$$

possessing the invariant parabola $\Phi(x, y) = x^2 - y$.

β) *For $\eta < 0$ the system (S) could only possess one of the following sets of invariant parabolas: $\cup$, $\mathbb{U}$, $\boldsymbol{\cup}^2$. Moreover (S) has one of the above sets of parabolas if and only if the corresponding conditions provided by the diagram given in Figure 1 are satisfied. Furthermore the system (S) with an invariant parabola could be brought via an affine transformation and time rescaling to the following canonical form*

$$\dot{x} = m + \frac{2n-1}{2}x - \frac{g}{2}y + gx^2 - xy, \quad \dot{y} = 2mx + 2ny - x^2 + gxy - 2y^2 \qquad (S_{\boldsymbol{\beta}})$$

possessing the invariant parabola $\Phi(x,y) = x^2 - y$.

γ) *For $\eta = 0$ and $\widetilde{M} \neq 0$ the system (S) could only possess one of the following sets of invariant parabolas: $\cup$, $\mathbb{U}$, $\boldsymbol{\cup}^2$, $\overset{2}{\cup}$, $\overset{2}{\cup}\subset$, $\overset{2}{\cup}\overset{2}{\mathbb{U}}$, $\overset{2}{\cup}\overset{2}{\mathbb{U}}{}^c$, $\overset{2}{\cup}\overset{2}{\boldsymbol{\cup}}{}^2$, $\overset{2}{\boldsymbol{\cup}}{}^3$, $\infty\overset{2}{\cup}$.*

Moreover (S) has one of the above sets of parabolas if and only if the corresponding conditions provided by the diagram given in Figure 1 are satisfied. Furthermore the system (S) with invariant parabola could be brought via an affine transformation and time rescaling to one of the following two normal forms:

$$\dot{x} = m + nx - gy/2 + gx^2 + xy, \quad \dot{y} = 2mx + 2ny + gxy + 2y^2,\ g \in \{0,1\} \qquad (S^1_{\boldsymbol{\gamma}})$$

possessing the invariant parabola $\Phi(x,y) = x^2 - y$, or

$$\dot{x} = 2mx + 2ny + (h-1)xy, \quad \dot{y} = n - (h+1)x/2 + my + hy^2 \qquad (S^2_{\boldsymbol{\gamma}})$$

possessing the invariant parabola $\Phi(x,y) = y^2 - x$.

δ) *For $\eta = \widetilde{M} = 0$ and $C_2 \neq 0$ the system (S) could only possess one of the following sets of invariant parabolas: $\overset{3}{\cup}$, $\infty\overset{3}{\cup}$. Moreover (S) has one of the above sets of parabolas if and only if the corresponding conditions provided by the diagram given in Figure 1 (the branch $C_2 \neq 0$) are satisfied. Furthermore the system (S) with an invariant parabola could be brought via an affine transformation and time rescaling to the following canonical form*

$$\dot{x} = m + (2n-1)x/2 - gy/2 + gx^2, \quad \dot{y} = 2mx + 2ny - x^2 + gxy \qquad (S_{\boldsymbol{\delta}})$$

possessing the invariant parabola $\Phi(x,y) = x^2 - y$.

θ) *For $\eta = \widetilde{M} = C_2 = 0$ the system (S) could only possess an invariant parabola $\overset{\infty}{\cup}$. Moreover (S) has this invariant parabola if and only if the corresponding conditions provided by the diagram given in Figure 1 (the branch $C_2 = 0$) are satisfied. Furthermore the system (S) with an invariant parabola could be brought via an affine transformation and time rescaling to the following canonical form*

$$\dot{x} = m + nx - y/2 + x^2, \quad \dot{y} = 2mx + 2ny + xy \qquad (S_{\boldsymbol{\theta}})$$

possessing the invariant parabola $\Phi(x,y) = x^2 - y$.

The paper is organized as follows. In Section 2 we construct the invariant polynomials which are responsible for the existence of an invariant parabola and obtain the ten equations relating the coefficients of a quadratic system with those of an invariant parabola. In Section 3 we give the proof of the Main Theorem constructing the conditions for the existence of invariant parabolas as well as the corresponding canonical systems.

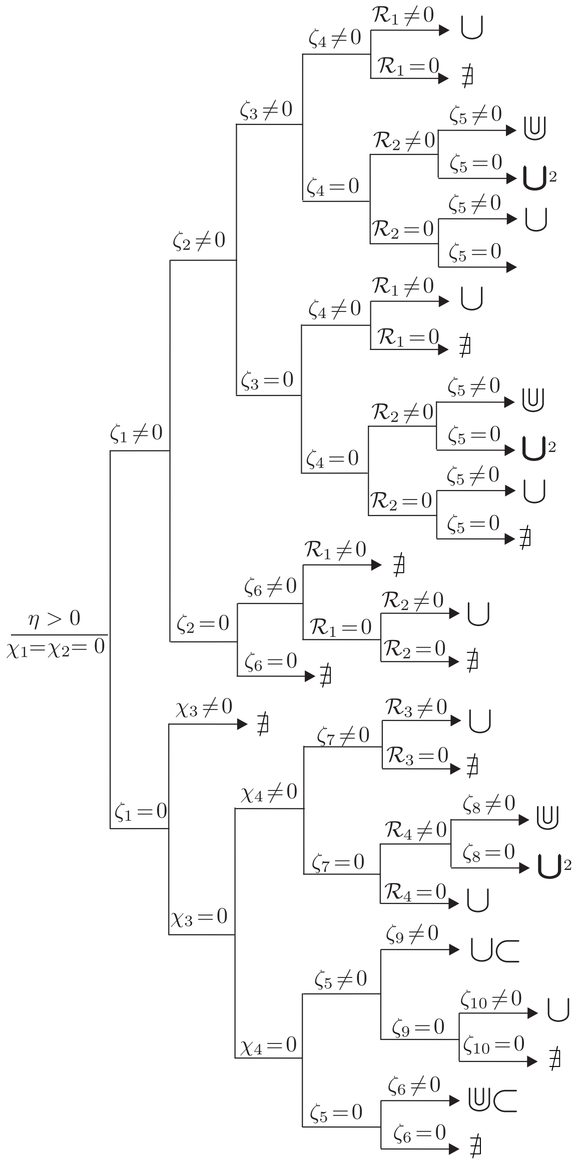

Figure 1.1: Quadratic systems with invariant parabolas: the case $\eta > 0$.

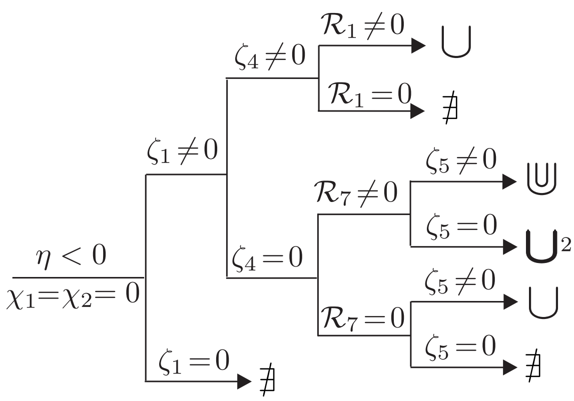

Figure 1.2: Quadratic systems with invariant parabolas: the case $\eta < 0$.

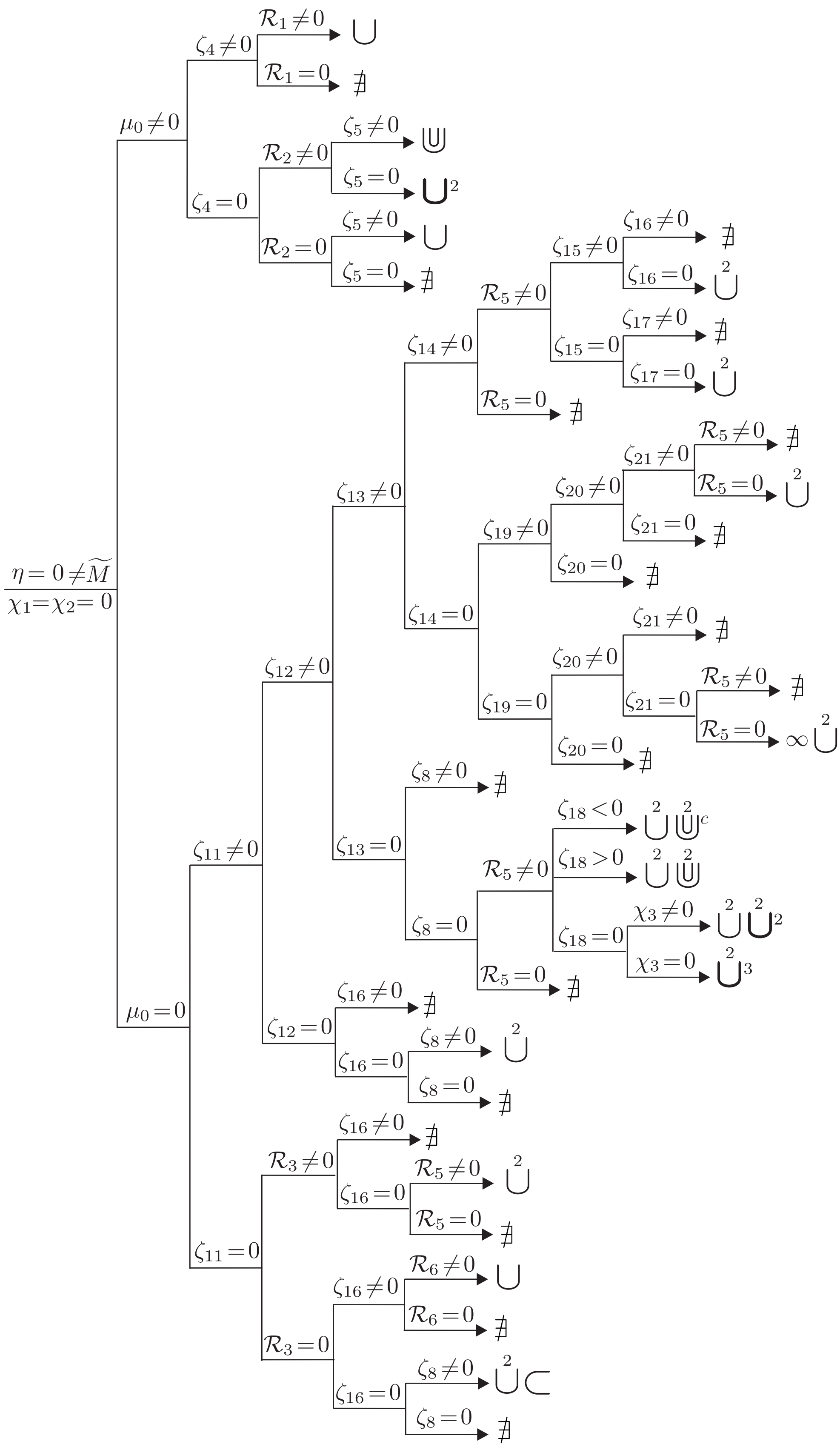

Figure 1.3: Quadratic systems with invariant parabolas: the case $\eta=0\neq\widetilde{M}$.

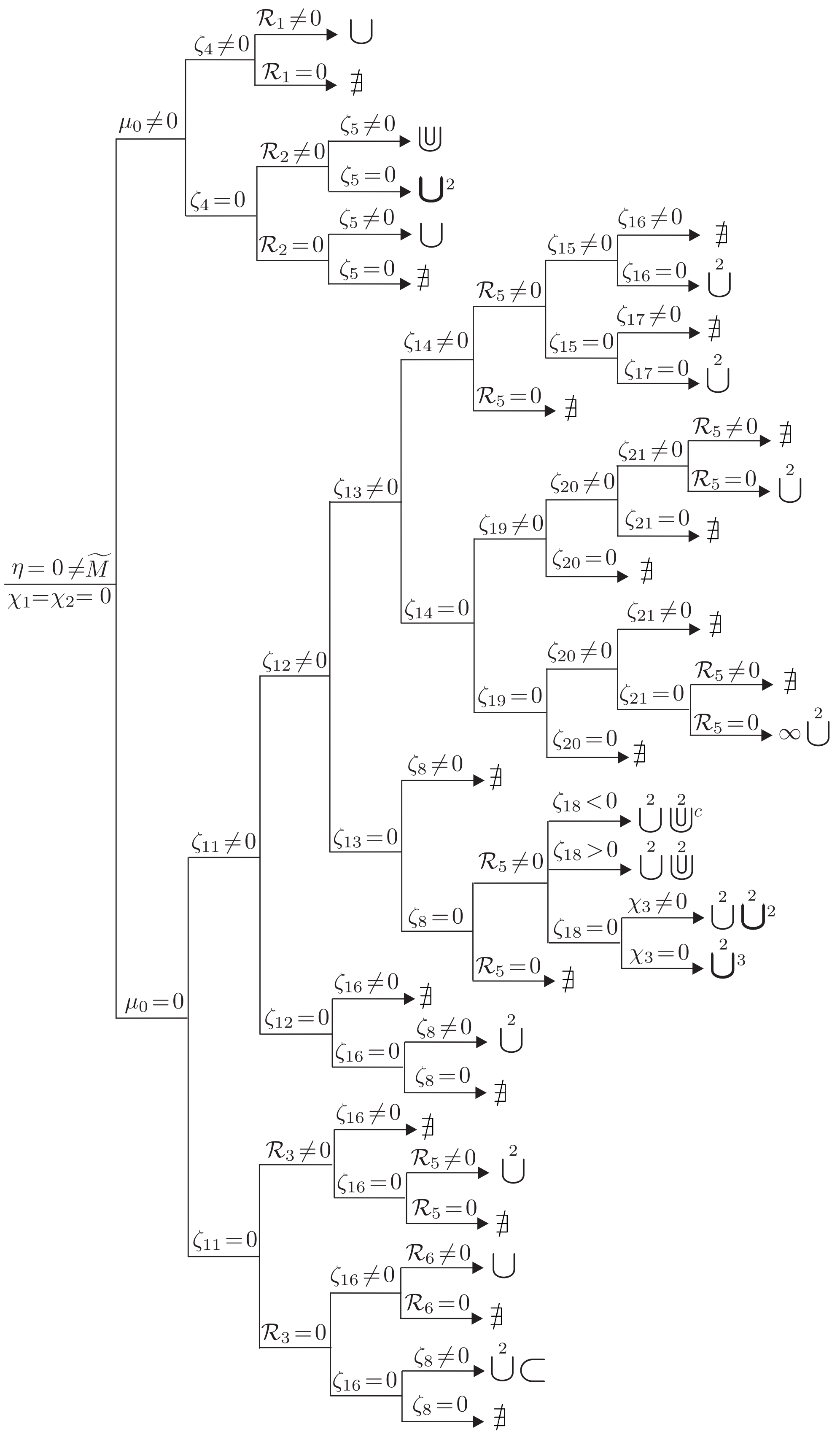

Figure 1.4: Quadratic systems with invariant parabolas: the case $\eta = 0 = \widetilde{M}$.

2 The construction of the invariant polynomials

Consider real quadratic systems of the form

$$\begin{aligned}\frac{dx}{dt} &= p_0 + p_1(x,y) + p_2(x,y) \equiv P(x,y),\\ \frac{dy}{dt} &= q_0 + q_1(x,y) + q_2(x,y) \equiv Q(x,y),\end{aligned} \tag{2.1}$$

with homogeneous polynomials p_i and q_i $(i = 0,1,2)$ of degree i in x, y:

$$\begin{aligned} p_0 &= a_{00}, \quad & p_1(x,y) &= a_{10}x + a_{01}y, \quad & p_2(x,y) &= a_{20}x^2 + 2a_{11}xy + a_{02}y^2,\\ q_0 &= b_{00}, \quad & q_1(x,y) &= b_{10}x + b_{01}y, \quad & q_2(x,y) &= b_{20}x^2 + 2b_{11}xy + b_{02}y^2.\end{aligned}$$

It is known that on the set of quadratic systems acts the group $\mathit{Aff}(2,\mathbb{R})$ of affine transformations on the plane (cf. [21]). For every subgroup $G \subseteq \mathit{Aff}(2,\mathbb{R})$ we have an induced action of G on **QS**. We can identify the set **QS** of systems (2.1) with a subset of $\mathbb{R}^{12}$ via the map $\mathbf{QS} \longrightarrow \mathbb{R}^{12}$ which associates to each system (2.1) the 12-tuple $\tilde{a} = (a_{00}, a_{10}, a_{01}, a_{20}, a_{11}, a_{02}, b_{00}, b_{10}, b_{01}, b_{20}, b_{11}, b_{02})$ of its coefficients. We associate to this group action polynomials in x, y and parameters which behave well with respect to this action, the GL-comitants (GL-invariants), the T-comitants (affine invariants) and the CT-comitants. For their definitions as well as their detailed constructions we refer the reader to the paper [21] (see also [1]).

2.1 Main invariant polynomials associated with invariant parabolas

We single out the following five polynomials, basic ingredients in constructing invariant polynomials for systems (2.1):

$$\begin{aligned} C_i(\tilde{a},x,y) &= y p_i(x,y) - x q_i(x,y),\ (i = 0,1,2),\\ D_i(\tilde{a},x,y) &= \frac{\partial p_i}{\partial x} + \frac{\partial q_i}{\partial y},\ (i = 1,2).\end{aligned} \tag{2.2}$$

As it was shown in [23] these polynomials of degree one in the coefficients of systems (2.1) are GL-comitants of these systems. Let $f, g \in \mathbb{R}[\tilde{a},x,y]$ and

$$(f,g)^{(k)} = \sum_{h=0}^{k} (-1)^h \binom{k}{h} \frac{\partial^k f}{\partial x^{k-h} \partial y^h} \frac{\partial^k g}{\partial x^h \partial y^{k-h}}.$$

The polynomial $(f,g)^{(k)} \in \mathbb{R}[\tilde{a},x,y]$ is called *the transvectant of index k of (f,g)* (cf. [9, 17]).

Proposition 2.1 (see [24]). *Any GL-comitant of systems (2.1) can be constructed from the elements (2.2) by using the operations: $+$, $-$, $\times$, and by applying the differential operation $(*,*)^{(k)}$.*

Remark 2.2. We point out that the elements (2.2) generate the whole set of GL-comitants and hence also the set of affine comitants as well as the set of T-comitants.

We construct the following GL-comitants of the second degree with respect to the coefficients of the initial systems

$$\begin{array}{lll} T_1=(C_0,C_1)^{(1)}, & T_2=(C_0,C_2)^{(1)}, & T_3=(C_0,D_2)^{(1)}, \\ T_4=(C_1,C_1)^{(2)}, & T_5=(C_1,C_2)^{(1)}, & T_6=(C_1,C_2)^{(2)}, \\ T_7=(C_1,D_2)^{(1)}, & T_8=(C_2,C_2)^{(2)}, & T_9=(C_2,D_2)^{(1)}. \end{array} \tag{2.3}$$

Using these GL-comitants as well as the polynomials (2.2) we construct the additional invariant polynomials. In order to be able to calculate the values of the needed invariant polynomials directly for every canonical system we shall define here a family of T-comitants expressed through C_i $(i=0,1,2)$ and D_j $(j=1,2)$:

$$\begin{aligned}
\hat{A} =\ & \left(C_1, T_8-2T_9+D_2^2\right)^{(2)}/144,\\
\widehat{D} =\ & \left[2C_0(T_8-8T_9-2D_2^2)+C_1(6T_7-T_6-(C_1,T_5)^{(1)}+6D_1(C_1D_2-T_5)-9D_1^2C_2\right]/36,\\
\widehat{E} =\ & \left[D_1(2T_9-T_8)-3\,(C_1,T_9)^{(1)}-D_2(3T_7+D_1D_2)\right]/72,\\
\widehat{F} =\ & [6D_1^2(D_2^2-4T_9)+4D_1D_2(T_6+6T_7)+48C_0\,(D_2,T_9)^{(1)}-9D_2^2T_4+288D_1\widehat{E}\\
& -24\left(C_2,\widehat{D}\right)^{(2)}+120\left(D_2,\widehat{D}\right)^{(1)}-36C_1\,(D_2,T_7)^{(1)}+8D_1\,(D_2,T_5)^{(1)}\Big]/144,\\
\widehat{B} =\ & \Big\{16D_1\,(D_2,T_8)^{(1)}\,(3C_1D_1-2C_0D_2+4T_2)+32C_0\,(D_2,T_9)^{(1)}\,(3D_1D_2-5T_6+9T_7)\\
& +2\,(D_2,T_9)^{(1)}\left(27C_1T_4-18C_1D_1^2-32D_1T_2+32\,(C_0,T_5)^{(1)}\right)\\
& +6(D_2,T_7)^{(1)}[8C_0(T_8-12T_9)-12C_1(D_1D_2+T_7)+D_1(26C_2D_1+32T_5)+C_2(9T_4+96T_3)]\\
& +6\,(D_2,T_6)^{(1)}\,[32C_0T_9-C_1(12T_7+52D_1D_2)-32C_2D_1^2]+48D_2\,(D_2,T_1)^{(1)}\left(2D_2^2-T_8\right)\\
& -32D_1T_8\,(D_2,T_2)^{(1)}+9D_2^2T_4\,(T_6-2T_7)-16D_1\,(C_2,T_8)^{(1)}\left(D_1^2+4T_3\right)\\
& +12D_1\,(C_1,T_8)^{(2)}\,(C_1D_2-2C_2D_1)+6D_1D_2T_4\left(T_8-7D_2^2-42T_9\right)\\
& +12D_1\,(C_1,T_8)^{(1)}\,(T_7+2D_1D_2)+96D_2^2\left[D_1\,(C_1,T_6)^{(1)}+D_2\,(C_0,T_6)^{(1)}\right]\\
& -16D_1D_2T_3\left(2D_2^2+3T_8\right)-4D_1^3D_2\left(D_2^2+3T_8+6T_9\right)+6D_1^2D_2^2\,(7T_6+2T_7)\\
& -252D_1D_2T_4T_9\Big\}/(2^8 3^3),\\
\widehat{K} =\ & (T_8+4T_9+4D_2^2)/72, \quad \widehat{H}=(8T_9-T_8+2D_2^2)/72.
\end{aligned}$$

These polynomials in addition to (2.2) and (2.3) will serve as bricks in constructing affine invariant polynomials for systems (2.1).

The following 42 affine invariants $A_1,\dots,A_{42}$ form the minimal polynomial basis of affine invariants up to degree 12. This fact was proved in [2] by constructing $A_1,\dots,A_{42}$ using the above bricks.

$$\begin{array}{llll}
A_1 &= \hat{A}, & A_{22} &= \dfrac{1}{1152}[\![C_2,\widehat{D})^{(1)},D_2)^{(1)},D_2)^{(1)},D_2)^{(1)}D_2)^{(1)},\\
A_2 &= (C_2,\widehat{D})^{(3)}/12, & A_{23} &= [\![\widehat{F},\widehat{H})^{(1)},\widehat{K})^{(2)}/8,\\
A_3 &= [\![C_2,D_2)^{(1)},D_2)^{(1)},D_2)^{(1)}/48, & A_{24} &= [\![C_2,\widehat{D})^{(2)},\widehat{K})^{(1)},\widehat{H})^{(2)}/32,\\
A_4 &= (\widehat{H},\widehat{H})^{(2)}, & A_{25} &= [\![\widehat{D},\widehat{D})^{(2)},\widehat{E})^{(2)}/16,\\
A_5 &= (\widehat{H},\widehat{K})^{(2)}/2, & A_{26} &= (\widehat{B},\widehat{D})^{(3)}/36,
\end{array}$$

$$
\begin{aligned}
A_6 &= (\widehat{E},\widehat{H})^{(2)}/2, \\
A_7 &= \llbracket [C_2,\widehat{E})^{(2)},D_2\big)^{(1)}/8, \\
A_8 &= \llbracket \widehat{D},\widehat{H})^{(2)},D_2\big)^{(1)}/8, \\
A_9 &= \llbracket \widehat{D},D_2)^{(1)},D_2\big)^{(1)},D_2\big)^{(1)}/48, \\
A_{10} &= \llbracket \widehat{D},\widehat{K})^{(2)},D_2\big)^{(1)}/8, \\
A_{11} &= (\widehat{F},\widehat{K})^{(2)}/4, \\
A_{12} &= (\widehat{F},\widehat{H})^{(2)}/4, \\
A_{13} &= \llbracket C_2,\widehat{H})^{(1)},\widehat{H}\big)^{(2)},D_2\big)^{(1)}/24, \\
A_{14} &= (\widehat{B},C_2)^{(3)}/36, \\
A_{15} &= (\widehat{E},\widehat{F})^{(2)}/4, \\
A_{16} &= \llbracket \widehat{E},D_2)^{(1)},C_2\big)^{(1)},\widehat{K}\big)^{(2)}/16, \\
A_{17} &= \llbracket \widehat{D},\widehat{D})^{(2)},D_2\big)^{(1)},D_2\big)^{(1)}/64, \\
A_{18} &= \llbracket \widehat{D},\widehat{F})^{(2)},D_2\big)^{(1)}/16, \\
A_{19} &= \llbracket \widehat{D},\widehat{D})^{(2)},\widehat{H}\big)^{(2)}/16, \\
A_{20} &= \llbracket C_2,\widehat{D})^{(2)},\widehat{F}\big)^{(2)}/16, \\
A_{21} &= \llbracket \widehat{D},\widehat{D})^{(2)},\widehat{K}\big)^{(2)}/16, \\
A_{27} &= \llbracket \widehat{B},D_2)^{(1)},\widehat{H}\big)^{(2)}/24, \\
A_{28} &= \llbracket C_2,\widehat{K})^{(2)},\widehat{D}\big)^{(1)},\widehat{E}\big)^{(2)}/16, \\
A_{29} &= \llbracket \widehat{D},\widehat{F})^{(1)},\widehat{D}\big)^{(3)}/96, \\
A_{30} &= \llbracket C_2,\widehat{D})^{(2)},\widehat{D}\big)^{(1)},\widehat{D}\big)^{(3)}/288, \\
A_{31} &= \llbracket \widehat{D},\widehat{D})^{(2)},\widehat{K}\big)^{(1)},\widehat{H}\big)^{(2)}/64, \\
A_{32} &= \llbracket \widehat{D},\widehat{D})^{(2)},D_2\big)^{(1)},\widehat{H}\big)^{(1)},D_2\big)^{(1)}/64, \\
A_{33} &= \llbracket \widehat{D},D_2)^{(1)},\widehat{F}\big)^{(1)},D_2\big)^{(1)},D_2\big)^{(1)}/128, \\
A_{34} &= \llbracket \widehat{D},\widehat{D})^{(2)},D_2\big)^{(1)},\widehat{K}\big)^{(1)},D_2\big)^{(1)}/64, \\
A_{35} &= \llbracket \widehat{D},\widehat{D})^{(2)},\widehat{E}\big)^{(1)},D_2\big)^{(1)},D_2\big)^{(1)}/128, \\
A_{36} &= \llbracket \widehat{D},\widehat{E})^{(2)},\widehat{D}\big)^{(1)},\widehat{H}\big)^{(2)}/16, \\
A_{37} &= \llbracket \widehat{D},\widehat{D})^{(2)},\widehat{D}\big)^{(1)},\widehat{D}\big)^{(3)}/576, \\
A_{38} &= \llbracket C_2,\widehat{D})^{(2)},\widehat{D}\big)^{(2)},\widehat{D}\big)^{(1)},\widehat{H}\big)^{(2)}/64, \\
A_{39} &= \llbracket \widehat{D},\widehat{D})^{(2)},\widehat{F}\big)^{(1)},\widehat{H}\big)^{(2)}/64, \\
A_{40} &= \llbracket \widehat{D},\widehat{D})^{(2)},\widehat{F}\big)^{(1)},\widehat{K}\big)^{(2)}/64, \\
A_{41} &= \llbracket C_2,\widehat{D})^{(2)},\widehat{D}\big)^{(2)},\widehat{F}\big)^{(1)},D_2\big)^{(1)}/64, \\
A_{42} &= \llbracket \widehat{D},\widehat{F})^{(2)},\widehat{F}\big)^{(1)},D_2\big)^{(1)}/16.
\end{aligned}
$$

In the above list, the bracket "$\llbracket$" is used in order to avoid placing the otherwise necessary up to five parentheses "(".

Using the elements of the minimal polynomial basis given above we construct the affine invariant polynomials

$$
\begin{aligned}
\chi_1 &= 32A_3 + 45A_4 - 160A_5; \\
\chi_2 &= 32A_8(14A_8 - 48A_9 + 37A_{10} + 24A_{11}) \\
&\quad + 16A_5(76A_{17} + 74A_{18} + 313A_{19} - 80A_{20} - 167A_{21}) \\
&\quad + A_4(160A_2^2 + 368A_{18} - 3363A_{19} + 736A_{20} + 2109A_{21}) + 32(17A_{10}^2 + 27A_{10}A_{11} + 24A_{11}^2 \\
&\quad - 48A_9A_{12} + 51A_{10}A_{12} + 24A_{11}A_{12} + 288A_6A_{14} - 96A_7A_{14}); \\
\chi_3 &= 6520480A_{20}(407A_{18} - 2253A_{21}) + 24A_{18}(1057715458A_{19} + 5944853225A_{21}) \\
&\quad + 28800A_{14}(1872476A_{25} - 122259A_{26}) + 144A_{12}(3620283092A_{29} - 1554910481A_{30}) \\
&\quad + 1440A_{15}(107225339A_{25} - 19561440A_{26}) - 72A_{11}(8198511476A_{29} - 2965514443A_{30}) \\
&\quad + 652048(4544A_{18}^2 + 125A_{20}^2 - 8955A_2A_{42}) - 9(264364688A_{19}^2 + 39417454842A_{19}A_{21} \\
&\quad - 54474141921A_{21}^2) + 3448898760A_{19}A_{20}; \\
\chi_4 &= 62713A_{10}^2 + 45787A_{10}A_{11} - 157928A_{11}^2 + 81202A_{10}A_{12} + 353474A_{11}A_{12} - 145848A_{12}^2 \\
&\quad + 64320A_7A_{15} + 28600A_5A_{17}; \\
\zeta_1 &= 13A_4 - 24A_5; \\
\zeta_2 &= -A_4; \\
\zeta_3 &= 16A_5 - 17A_4; \\
\zeta_4 &= 9A_1A_4 - 7A_1A_5 - 2A_{16};
\end{aligned}
$$

$$\zeta_5 = 166A_8 + 384A_9 - 1120A_{10} - 512A_{11} - 62A_{12};$$
$$\zeta_6 = -A_6;$$
$$\zeta_7 = 40(71436A_7A_{20} - 640883A_7A_{21} + 1008622A_1A_{32}) + 12A_{12}(3585035A_{14} + 14919259A_{15}) - 5(8092193A_{10} + 15970731A_{11})A_{14} - (129780821A_{10} + 269944167A_{11})A_{15};$$
$$\zeta_8 = A_2;$$
$$\zeta_9 = 1040(2256A_7A_{15} + 143A_3A_{21}) - 264(162941A_{10} + 315202A_{11})A_{12} + 3A_{11}(25887132A_{10} + 24385177A_{11}) + 20603609A_{10}^2 + 24896016A_{12}^2;$$
$$\zeta_{10} = 250A_1^2 + 34A_{11} - 41A_{12};$$
$$\zeta_{11} = D_2^2 + 28\widehat{H} - 32\widehat{K};$$
$$\zeta_{12} = D_2^2 - 4\widehat{H} - 16\widehat{K};$$
$$\zeta_{13} = D_2^2 - 18\widehat{K};$$
$$\zeta_{14} = D_2^2 - 16\widehat{K};$$
$$\zeta_{15} = \widehat{H};$$
$$\zeta_{16} = A_2(24A_{18} - 42A_{17} - 1024A_{19} - 2A_{20} - 213A_{21}) + 5(420A_1A_{25} - 199A_{38} - 225A_{39} + 60A_{40} + 8A_{41});$$
$$\zeta_{17} = 3456(C_0, T_7)^{(1)}\left[(D_2, T_7)^{(1)}\right]^2 + 81D_1^3(C_1, T_8)^{(2)}(C_1, T_9)^{(2)} - 36D_1(D_2, T_7)^{(1)}\times \times\left[8[\![T_8, C_2)^{(1)}, C_1)^{(2)}, C_0)^{(1)} + [\![C_1, T_5)^{(2)}, 36T_6 - 7D_1D_2)^{(1)}\right] - 4[\![C_1, T_5)^{(2)}, D_2)^{(1)}[\![C_1, T_5)^{(2)}, T_6 + 309D_1D_2)^{(1)} + 70T_4(D_2, T_7)^{(1)}[\![C_1, T_5)^{(2)}, D_2)^{(1)};$$
$$\zeta_{18} = A_{37};$$
$$\zeta_{19} = (C_2, \widetilde{D})^{(1)};$$
$$\zeta_{20} = (C_2, \widetilde{D})^{(2)};$$
$$\zeta_{21} = (C_2, \widetilde{E})^{(1)};$$
$$\zeta_{22} = A_2(3A_2^2 - 4A18) + 72A_1(A25 + 2A26);$$
$$\zeta_{23} = T_4;$$
$$\zeta_{24} = 6C_2D_1^2 + 9C_2T_4 - 4D_1T_5;$$
$$\mathcal{R}_1 = 531A_2A_4 - 1472A_2A_5 - 8352A_1A_6 + 320A_{22} - 3216A_{23} + 1488A_{24};$$
$$\mathcal{R}_2 = 15A_{10} - 10A_8 - 6A_9;$$
$$\mathcal{R}_3 = 4800(6650951968A_{14}A_{15} - 2382132830A_{14}^2 - 9860550485A_{15}^2) + 1600(4765089473A_{11} - 7838161089A_{12})A_{20} + 640(15664652914A_{11} - 50944340271A_{12})A_{18} - 6(20392663986679A_{10} + 34357804389813A_{11} - 739275727012A_{12})A_{21} + 3(46944212550227A_{10} + 83455057317969A_{11} - 22899810934956A_{12})A_{19};$$
$$\mathcal{R}_4 = 251A_1^2 + 25A_{12};$$
$$\mathcal{R}_5 = [\![C_2, C_2)^{(2)}, C_1)^{(2)};$$
$$\mathcal{R}_6 = 851A_2A_{17} - 235A_{41} + 170A_{42};$$
$$\mathcal{R}_7 = 62250A_1^2 + 8956A_9 - 46223A_{10} - 50129A_{11} + 14766A_{12}.$$

2.2 Preliminary results involving the use of polynomial invariants

Considering the GL-comitant $C_2(\tilde{a},x,y) = yp_2(\tilde{a},x,y) - xq_2(\tilde{a},x,y)$ as a cubic binary form of x and y we calculate

$$\eta(\tilde{a}) = \text{Discrim}[C_2,\xi], \quad \widetilde{M}(\tilde{a},x,y) = \text{Hessian}[C_2],$$

where $\xi = y/x$ or $\xi = x/y$. Following [23] (see also [21]) we have the next lemma.

Lemma 2.3. *The number of distinct roots (real and imaginary) of the polynomial $C_2(\tilde{a},x,y)$ is determined by the following conditions:*

(i) 3 real if $\eta > 0$;

(ii) 1 real and 2 imaginary if $\eta < 0$;

(iii) 2 real (1 double) if $\eta = 0$ and $\widetilde{M} \neq 0$;

(iv) 1 real (triple) if $\eta = \widetilde{M} = 0$ and $C_2 \neq 0$;

(v) ∞ if $\eta = \widetilde{M} = C_2 = 0$.

Moreover, for each one of these cases the quadratic systems (2.1) *can be brought via a linear transformation to one of the following canonical systems* $(\mathbf{S}_I)$–$(\mathbf{S}_{IV})$:

$$\begin{cases} \dot{x} = a + cx + dy + gx^2 + (h-1)xy, \\ \dot{y} = b + ex + fy + (g-1)xy + hy^2; \end{cases} \qquad (\mathbf{S}_I)$$

$$\begin{cases} \dot{x} = a + cx + dy + gx^2 + (h+1)xy, \\ \dot{y} = b + ex + fy - x^2 + gxy + hy^2; \end{cases} \qquad (\mathbf{S}_{II})$$

$$\begin{cases} \dot{x} = a + cx + dy + gx^2 + hxy, \\ \dot{y} = b + ex + fy + (g-1)xy + hy^2; \end{cases} \qquad (\mathbf{S}_{III})$$

$$\begin{cases} \dot{x} = a + cx + dy + gx^2 + hxy, \\ \dot{y} = b + ex + fy - x^2 + gxy + hy^2, \end{cases} \qquad (\mathbf{S}_{IV})$$

$$\begin{cases} \dot{x} = a + cx + dy + x^2, \\ \dot{y} = b + ex + fy + xy. \end{cases} \qquad (\mathbf{S}_{V})$$

Some important necessary conditions for a quadratic system to possess invariant parabolas are provided by the next lemma.

Lemma 2.4. *If a quadratic system* (2.1) *possesses an invariant parabola then the conditions $\chi_1 = \chi_2 = 0$ hold.*

Proof. Assume that a quadratic system (2.1) possesses an invariant parabola. It is known that via an affine transformation this parabola could be brought to the canonical form $y = x^2$. Then as it was proved in [5] this quadratic system can be written in the form

$$\dot{x} = c(y - x^2) + (a + bx + gy) + ex, \quad \dot{y} = d(y - x^2) + 2x(a + bx + gy) + 2ey^2,$$

where a,b,c,d,g,h,e are real parameters. A straightforward calculation gives $\chi_1 = \chi_2 = 0$ for the above systems and this completes the proof of the lemma. $\square$

Assume that a conic

$$\Phi(x,y) \equiv p + qx + ry + sx^2 + 2vxy + uy^2 = 0 \tag{2.4}$$

is an affine algebraic invariant curve for quadratic systems (2.1), which we rewrite in the form:

$$\begin{aligned} \frac{dx}{dt} &= a + cx + dy + gx^2 + 2hxy + ky^2 \equiv P(x,y), \\ \frac{dy}{dt} &= b + ex + fy + lx^2 + 2mxy + ny^2 \equiv Q(x,y). \end{aligned} \tag{2.5}$$

Remark 2.5. Following [10] we construct the determinant

$$\Delta = \begin{vmatrix} s & v & q/2 \\ v & u & r/2 \\ q/2 & r/2 & p \end{vmatrix},$$

associated to the conic (2.4). By [10] this conic is irreducible (i.e. the polynomial Φ defining the conic is irreducible over $\mathbb{C}$) if and only if $\Delta \neq 0$.

According to definition of an invariant curve (see page 2) considering the cofactor $K = Ux + Vy + W \in \mathbb{R}[x,y]$ the following identity holds:

$$\frac{\partial\Phi}{\partial x}P(x,y) + \frac{\partial\Phi}{\partial y}Q(x,y) = \Phi(x,y)(Ux + Vy + W).$$

This identity yields a system of 10 equations for determining the 9 unknown parameters p, q, r, s, u, v, U, V, W:

$$\begin{aligned} Eq_1 &\equiv s(2a_{20} - U) + 2b_{20}v = 0, \\ Eq_2 &\equiv 2v(a_{20} + 2b_{11} - U) + s(4a_{11} - V) + 2b_{20}u = 0, \\ Eq_3 &\equiv 2v(2a_{11} + b_{02} - V) + u(4b_{11} - U) + 2a_{02}s = 0, \\ Eq_4 &\equiv u(2b_{02} - V) + 2a_{02}v = 0, \\ Eq_5 &\equiv q(a_{20} - U) + s(2a_{10} - W) + 2b_{10}v + b_{20}r = 0, \\ Eq_6 &\equiv r(2b_{11} - U) + q(2a_{11} - V) + 2v(a_{10} + b_{01} - W) + 2(a_{01}s + b_{10}u) = 0, \\ Eq_7 &\equiv r(b_{02} - V) + u(2b_{01} - W) + 2a_{01}v + a_{02}q = 0, \\ Eq_8 &\equiv q(a_{10} - W) + 2(a_{00}s + b_{00}v) + b_{10}r - pU = 0, \\ Eq_9 &\equiv r(b_{01} - W) + 2(b_{00}u + a_{00}v) + a_{01}q - pV = 0, \\ Eq_{10} &\equiv a_{00}q + b_{00}r - pW = 0. \end{aligned} \tag{2.6}$$

According to [6] (see also [3]) we have the next lemma.

Lemma 2.6. *Suppose that a polynomial system (1.1) of degree n has the invariant algebraic curve $f(x,y) = 0$ of degree m. Let P_n, Q_n and f_m be the homogeneous components of P, Q and f of degree n and m, respectively. Then the irreducible factors of f_m must be factors of $yP_n - xQn$.*

3 The proof of the Main Theorem

Assuming that a quadratic system (2.5) has an invariant parabola (2.4) by Lemma 2.4 we conclude that for this system the conditions $\chi_1 = \chi_2 = 0$ have to be fulfilled.

In what follows considering Lemma 2.3 we examine each one of the families of quadratic systems provided by this lemma.

3.1 Systems with three real infinite singularities

In this case according to Lemma 2.3 systems (2.5) via a linear transformation could be brought to the following family of systems

$$\begin{aligned} \frac{dx}{dt} &= a + cx + dy + gx^2 + (h-1)xy, \\ \frac{dy}{dt} &= b + ex + fy + (g-1)xy + hy^2, \end{aligned} \tag{3.1}$$

for which we have $C_2(x,y) = xy(x-y)$. Therefore the infinite singularities are located at the intersections of the lines $x = 0$, $y = 0$ and $x - y = 0$ with the line $Z = 0$ at infinity. So by Lemma 2.6 it is clear that if a parabola is invariant for these systems, then its homogeneous quadratic part has one of the following forms: (i) kx^2, (ii) ky^2, (iii) $k(x-y)^2$, where k is a real nonzero constant. Obviously we may assume $k = 1$ (otherwise instead of conic (2.4) we could consider $\Phi(x,y)/k = 0$).

According to Lemma 2.4 for the existence of an invariant parabola for a system (3.1) the condition $\chi_1 = 0$ is necessary. For the above systems we calculate

$$\chi_1 = 2(g-2)(h-2)(1+g+h) \tag{3.2}$$

and therefore the condition $\chi_1 = 0$ is equivalent to $(g-2)(h-2)(1+g+h) = 0$.

On the other hand we have the following lemma.

Lemma 3.1. *Assume that a system (3.1) possesses an invariant parabola. Then its quadratic homogeneous part is of the form x^2 (respectively, y^2; $(x-y)^2$) only if the condition $h - 2 = 0$ (respectively, $g - 2 = 0$; $g + h + 1 = 0$) holds.*

Proof. Assume that a system (3.1) possesses an invariant parabola of the form $\Phi(x,y) = p + qx + ry + x^2$ with $r \neq 0$ (otherwise we get a reducible conic). Then considering equations (2.6) we obtain

$$s = 1,\ \ v = u = 0,\ \ Eq_2 = -2 + 2h - V = 0\ \Rightarrow\ V = 2(h-1).$$

Therefore we have $Eq_7 = (2-h)r = 0$ and since $r \neq 0$ this implies $h - 2 = 0$. So the statement of the lemma is true in this case.

If the system possesses an invariant parabola of the form $\Phi(x,y) = p + qx + ry + y^2$ with $q \neq 0$ then considering equations (2.6) we obtain

$$s = v = 0,\ \ u = 1,\ \ Eq_3 = -2 + 2g - U = 0\ \Rightarrow\ U = 2(g-1).$$

In this case we obtain $Eq_5 = (2-g)q = 0$ and due to $q \neq 0$ we get $g - 2 = 0$.

Assume now a system (3.1) possesses an invariant parabola of the form $\Phi(x,y) = p + qx + ry + (x-y)^2$ with $q + r \neq 0$. Then we have

$$s = 1,\ v = -1,\ u = 1,\ \ Eq_1 = 2g - U,\ Eq_4 = 2h - V$$

and therefore the equations $Eq_1 = 0$ and $Eq_4 = 0$ yield $U = 2g$ and $V = 2h$, respectively. Then we calculate $Eq_5 + Eq_6 + Eq_7 = -(1+g+h)(q+r) = 0$ and due to the condition $q + r \neq 0$ we get $1 + g + h = 0$ and this completes the proof of Lemma 3.1. □

Considering (3.2) it is clear that the condition $\chi_1 = 0$ implies either $h = 2$ or $g = 2$ or $h = -(1+g)$. On the other hand for systems (3.1) we have

$$\begin{aligned} \zeta_1 &= 2(g-2)(3+g) && \text{in the case } h = 2;\\ \zeta_1 &= 2(h-2)(3+h) && \text{in the case } g = 2;\\ \zeta_1 &= 2(g-2)(3+g) && \text{in the case } h = -(1+g) \end{aligned}$$

and therefore we conclude that if $\chi_1 = 0$ then the condition $\zeta_1 = 0$ imposes the vanishing of one more factor of the polynomial χ_1.

Remark 3.2. If $(h-2)(g-2)(1+g+h) = 0$ then without losing generality we may assume $h = 2$. Furthermore if two of these factors vanish simultaneously (i.e. $\zeta_1 = 0$) then we may assume $h = 2$ and $g = 2$.

Indeed assume $h - 2 \neq 0$ and suppose first that $g = 2$. We observe that the change

$$(x, y, a, b, c, d, e, f, g, h) \mapsto (y, x, b, a, e, d, f, c, h, g)$$

conserves systems (3.1) and hence the condition $g = 2$ is transformed into $h = 2$.

Admit now that the condition $1 + g + h = 0$ is fulfilled. Then applying to these systems the transformation

$$x_1 = x - y, \quad y_1 = -y$$

and arrive at the systems

$$\dot{x}_1 = a_1 + c_1 x_1 + g_1 x_1^2 + (h_1 - 1)x_1 y_1, \quad \dot{y}_1 = b_1 + f_1 y_1 + (g_1 - 1)x_1 y_1 + h_1 y_1^2$$

where (we are interested only in homogeneous quadratic parts)

$$g_1 = g, \quad h_1 = 1 - g - h, \;\Rightarrow\; g = g_1, \quad h = 1 - g_1 - h_1.$$

Therefore we obtain $1 + h_1 + g_1 = 1 + (1 - g - h) + g = 2 - h$ and hence via the above transformation the condition $1 + g + h = 0$ is reduced to the condition $h - 2 = 0$.

Assume now that two of the factors $(h-2)(g-2)(1+g+h)$ vanish. Then as it was shown above we may assume $h = 2$. In this case other two factors become $g - 2$ and $g + 3$. Supposing $h = 2$ and $g = -3$ systems (3.1) become

$$\dot{x} = a + cx + dy - 3x^2 + xy, \quad \dot{y} = b + ex + fy - 4xy + 2y^2,$$

which via the transformation $x_1 = x$, $y_1 = x - y$ can be brought to the systems

$$\dot{x}_1 = a_1 + c_1 x_1 + d_1 y_1 + 2x_1^2 + x_1 y_1, \quad \dot{y}_1 = b_1 + e_1 x_1 + f_1 y_1 + x_1 y_1 + 2y_1^2.$$

It remains to observe that these systems have the form (3.1) with $h = 2$ and $g = 2$ and we conclude that the statement of Remark 3.2 is valid.

Considering Lemma 3.1 and Remark 3.2 we conclude that for determining the conditions for the existence and the number of invariant parabolas for systems (3.1) it is sufficient to examine when the invariant parabolas have the forms $\Phi(x, y) = p + qx + ry + x^2$ and $\Phi(x, y) = p + qx + ry + y^2$. Moreover as it is mentioned above systems (3.1) could have invariant parabolas only in one direction (if $\chi_1 = 0$ and $\zeta_1 \neq 0$) and they could have invariant parabolas in two directions (if $\chi_1 = 0$ and $\zeta_1 = 0$). In what follows we examine each one of these two possibilities.

3.1.1 The possibility $\chi_1 = 0$ and $\zeta_1 \neq 0$

Then we may assume $h = 2$ and by Lemma 3.1 systems (3.1) could have invariant parabolas of the form $\Phi(x,y) = p + qx + ry + x^2$. Applying the translation $(x,y) \mapsto (x-d,\, y-c+2dg)$ systems (3.1) can be brought to the systems

$$\dot{x} = a + gx^2 + xy, \quad \dot{y} = b + ex + fy + (g-1)xy + 2y^2. \tag{3.3}$$

Coefficient conditions for systems (3.3) to possess invariant parabolas

Lemma 3.3. *A system (3.3) possesses either one or two invariant parabolas or a double invariant parabola of the form $\Phi(x,y) = p + qx + ry + x^2$ $(r \neq 0)$ if and only if $\Omega_1 = 0$ and the corresponding set of conditions are satisfied, respectively:*

$(\mathcal{A}_1)$ $g(g+1) \neq 0$, $2g+1 \neq 0$, $\mathcal{D}_1 \neq 0$, $\mathcal{G}_1 \neq 0 \Rightarrow \cup$;

$(\mathcal{A}_2)$ $g(g+1) \neq 0$, $2g+1 \neq 0$, $\mathcal{D}_1 = 0$, $a \neq 0$, $\mathcal{F}_1 \neq 0 \Rightarrow \Cup$;

$(\mathcal{A}_3)$ $g(g+1) \neq 0$, $2g+1 \neq 0$, $\mathcal{D}_1 = 0$, $a \neq 0$, $\mathcal{F}_1 = 0 \Rightarrow \cup^2$;

$(\mathcal{A}_4)$ $g(g+1) \neq 0$, $2g+1 \neq 0$, $\mathcal{D}_1 = 0$, $a = 0$, $f \neq 0 \Rightarrow \cup$;

$(\mathcal{A}_5)$ $g = -1/2$, $\mathcal{D}_1 \neq 0$, $a \neq 0 \Rightarrow \cup$;

$(\mathcal{A}_6)$ $g = -1/2$, $\mathcal{D}_1 = 0$, $b \neq 0$, $e^2 - 2b \neq 0 \Rightarrow \Cup$;

$(\mathcal{A}_7)$ $g = -1/2$, $\mathcal{D}_1 = 0$, $b \neq 0$, $e^2 - 2b = 0 \Rightarrow \cup^2$;

$(\mathcal{A}_8)$ $g = -1/2$, $\mathcal{D}_1 = 0$, $b = 0$, $e \neq 0 \Rightarrow \cup$;

$(\mathcal{A}_9)$ $g = 0$, $b = a$, $e \neq 0$, $a \neq 0 \Rightarrow \cup$;

$(\mathcal{A}_{10})$ $g = -1$, $b = 0$, $e + f \neq 0$, $a \neq 0 \Rightarrow \cup$,

where

$$\begin{aligned}
\Omega_1 &= 2b^2(1+2g)^2 - b\big[4a(1+g)(1+2g)(1+3g) - (e-fg)(e+f+fg)\big]\\
&\quad + a(1+g)\big[2a(1+g)(1+3g)^2 - (e-2fg)(e+f+fg)\big];\\
\mathcal{D}_1 &= e + f(g+1);\;\; \mathcal{G}_1 = a - b + 4ag - 2bg + 3ag^2;\;\; \mathcal{F}_1 = 8ag(1+g) - f^2(1+2g).
\end{aligned} \tag{3.4}$$

Proof. Considering the equations (2.6) and the form of invariant parabola $\Phi(x,y) = p + qx + ry + x^2$ with $r \neq 0$ we obtain

$$s = 1,\; v = u = 0,\; U = 2g,\; V = 2,\; W = -gq,$$
$$Eq_1 = Eq_2 = Eq_3 = Eq_4 = Eq_5 = Eq_7 = 0.$$

Calculating the remaining equations we obtain

$$Eq_6 = -q - r - gr, \qquad Eq_8 = 2a - 2gp + gq^2 + er,$$
$$Eq_9 = -2p + fr + gqr, \qquad Eq_{10} = aq + gpq + br.$$

It is clear that the equations $Eq_6 = 0$ implies $q = -(1+g)r$ and then $Eq_9 = 0$ gives us $p = r(f - gr - g^2r)/2$. Therefore calculations yield

$$\begin{aligned}
Eq_8 &= g(1+g)(1+2g)r^2 + (e-fg)r + 2a,\\
Eq_{10} &= r\big[g^2(1+g)^2r^2 - fg(1+g)r - 2(a-b+ag)\big]/2 \equiv r\Psi/2
\end{aligned} \tag{3.5}$$

and since $r \neq 0$ the equation $Eq_{10} = 0$ is equivalent to $\Psi = 0$.

According to [12, Lemmas 11, 12] the equations $Eq_8 = 0$ and $\Psi = 0$ have a common solution of degree 2 with respect to the parameter r if and only if

$$Res_r^{(0)}(Eq_8, \Psi) = Res_r^{(1)}(Eq_8, \Psi) = 0$$

where $Res_r^{(1)}$ is the subresultant of order one and $Res_r^{(0)}$ is the subresultant of order zero which coincide with standard resultant (for detailed definition see [12], formula (19)). We calculate

$$Res_r^{(1)}(Eq_8, \Psi) = -g^2(1+g)^2(e+f+fg) \equiv -g^2(1+g)^2\mathcal{D}_1,$$
$$Res_r^{(0)}(Eq_8, \Psi) = 2g^2(g+1)^2\Omega_1.$$

So we examine three possibilities: $g(g+1) \neq 0$, $g = 0$ and $g = -1$.

1: **The possibility** $g(g+1) \neq 0$**.** Considering (3.4) we observe that the polynomial Ω_1 is quadratic with respect to the parameter b if $2g+1 \neq 0$. So we discuss two cases: $2g+1 \neq 0$ and $2g+1 = 0$.

1.1: **The case** $2g+1 \neq 0$**.** We observe that due to the condition $g(g+1) \neq 0$ the subresultant of order one $Res_r^{(1)}(Eq_8, \Psi)$ vanishes if and only if $\mathcal{D}_1 = 0$. So we consider two subcases: $\mathcal{D}_1 \neq 0$ and $\mathcal{D}_1 = 0$.

1.1.1: *The subcase* $\mathcal{D}_1 \neq 0$. Then the invariant parabola exists if and only if $\Omega_1 = 0$ and therefore we have to examine the solutions of the equation $\Omega_1 = 0$. In this case we calculate

$$\text{Discrim}[\Omega_1, b] = -(e+f+fg)^2\left[8ag(1+g)(1+2g) - (e-fg)^2\right] \equiv -\mathcal{D}_1^2\mathcal{E}$$

and hence the equation $\Omega_1 = 0$ has real solutions with respect to the parameter b if and only if either $\mathcal{D}_1 = 0$ or $\mathcal{E} \leq 0$. However since the condition $\mathcal{D}_1 \neq 0$ holds it remains to examine the condition $\mathcal{E} \leq 0$.

In this case setting $\mathcal{E} = -w^2 \leq 0$ we calculate

$$a = \frac{(e-fg)^2 - w^2}{8g(g+1)(2g+1)} \tag{3.6}$$

and then we obtain:

$$\Omega_1 = \frac{B_+B_-}{32g^2(1+2g)^2},$$

where

$$B_\pm = 8bg(1+2g)^2 + (fg - e + \varepsilon w)\left[e(1+g) - fg(3+5g) + \varepsilon w(1+3g)\right], \quad \varepsilon = \pm 1.$$

Then the condition $\Omega_1 = 0$ gives us

$$b = \frac{(e - fg - \varepsilon w)}{8g(1+2g)^2}\left[e(1+g) - fg(3+5g) + \varepsilon w(1+3g)\right] \tag{3.7}$$

where $\varepsilon = 1$ if $B_+ = 0$ and $\varepsilon = -1$ if $B_- = 0$. In this case we obtain that the polynomials Eq_8 and $\Psi(e, f, g, r)$ have the common factor $\zeta = 2g(1+g)(1+2g)r + e - fg - \varepsilon w$ which is linear with respect to the parameter r. Setting $\zeta = 0$ we get

$$r = -\frac{e - fg - \varepsilon w}{2g(1+g)(1+2g)}$$

and we arrive at the family of systems

$$\begin{aligned}\dot{x} &= \frac{(e-fg)^2-w^2}{8g(g+1)(2g+1)}+gx^2+xy,\\ \dot{y} &= \frac{(e-fg-\varepsilon w)}{8g(1+2g)^2}\big[e(1+g)-fg(3+5g)+\varepsilon w(1+3g)\big]+ex+fy+(g-1)xy+2y^2.\end{aligned} \tag{3.8}$$

This family of systems possess the following invariant parabola

$$\begin{aligned}\Phi(x,y) = &-\frac{(e-fg-\varepsilon w)(e+2f+3fg-\varepsilon w)}{8g(1+g)(1+2g)^2}\\ &+\frac{e-fg-\varepsilon w}{2g(1+2g)}\,x-\frac{e-fg-\varepsilon w}{2g(1+g)(1+2g)}\,y+x^2.\end{aligned} \tag{3.9}$$

We observe that this conic is reducible if and only if $e-fg+\varepsilon w=0$.

Considering (3.6) and (3.7) we get

$$w^2=-8ag(1+g)(1+2g)+(e-fg)^2$$

and then we obtain

$$\begin{aligned}b=\frac{1}{8g(1+2g)^2}\big[(e-fg)(e+eg-3fg-5fg^2)-2\varepsilon wg(e+f+fg)-(1+3g)w^2\big]\ \Rightarrow\\ 4b(1+2g)^2-4a(1+g)(1+2g)(1+3g)+(e+f+fg)(e-fg-\varepsilon w)=0.\end{aligned}$$

Since $\mathcal{D}_1=(e+f+fg)\neq 0$ we solve the last equation with respect to εw and we obtain

$$\varepsilon w=\frac{1}{e+f+fg}\big[4b(1+2g)^2-4a(1+g)(1+2g)(1+3g)+(e-fg)(e+f+fg)\big].$$

Then calculations yield

$$r=-\frac{e-fg-\varepsilon w}{2g(1+g)(1+2g)}=-\frac{2(a-b+4ag-2bg+3ag^2)}{g(1+g)(e+f+fg)}=-\frac{2\,\mathcal{G}_1}{g(1+g)(e+f+fg)}\neq 0.$$

This completes the proof of the statement $(\boldsymbol{A_1})$ of Lemma 3.3.

Since systems (3.8) are in the class defined by the condition $\eta>0$, according to the statement $\boldsymbol{\alpha})$ of Main Theorem we have to prove that these systems could be brought via a transformation to the canonical form $(S_{\boldsymbol{\alpha}})$.

Indeed as $g(1+g)(1+2g)\neq 0$ we apply to the parabola (3.9) the translation

$$x=x_1-\frac{e-fg-\varepsilon w}{4g(1+2g)},\quad y=y_1-\frac{e+3eg+fg(3+5g)-(1+3g)\varepsilon w}{8g(1+2g)} \tag{3.10}$$

and we get a simpler form of this parabola:

$$\widetilde{\Phi}(x_1,y_1)=x_1^2-\frac{e-fg-\varepsilon w}{2g(1+g)(1+2g)}\,y_1.$$

On the other hand applying the same translation to the family of systems (3.8) we arrive at the systems

$$\begin{aligned}\dot{x}_1 =& \frac{(e-fg-\varepsilon w)\big[fg(1-g)(3+5g)+e(1+10g+13g^2)+(g-1)(1+3g)\varepsilon w\big]}{32g^2(1+g)(1+2g)^2}-\\ &\frac{fg(3+g)+e(1+7g)-(1+7g)\varepsilon w}{8g(1+2g)}x_1-\frac{e-fg-\varepsilon w}{4g(1+2g)}y_1+gx_1^2+x_1y_1,\\ \dot{y}_1 =& \frac{fg(1-g)(3+5g)+e(1+10g+13g^2)+(g-1)(1+3g)\varepsilon w}{8g(1+2g)}x_1-\\ &\frac{fg(3+g)+e(1+7g)-(1+7g)\varepsilon w}{4g(1+2g)}y_1+(g-1)x_1y_1+2y_1^2.\end{aligned} \tag{3.11}$$

Observation 3.4. We remark that simultaneously applying the same translation on systems (3.8) and on the corresponding invariant parabola (3.9) we arrive at systems (3.11). We point out that the linear parts of these systems together with the coefficients of the transformed parabola $\widetilde{\Phi}(x_1,y_1)$ suggest us the new notations for the simplification of the canonical forms.

Indeed due to the condition $g(1+g)(1+2g)(e-fg-\varepsilon w)\neq 0$ we could set the following new notations:

$$\begin{aligned}&k=\frac{e-fg-\varepsilon w}{2g(1+g)(1+2g)},\quad n=-\frac{fg(3+g)+e(1+7g)-(1+7g)\varepsilon w}{8g(1+2g)},\\ &m=\frac{fg(1-g)(3+5g)+e(1+10g+13g^2)+(g-1)(1+3g)\varepsilon w}{16g(1+2g)}\quad\Rightarrow\\ &e=gk-g^3k+2m+n-gn,\quad f=-\frac{(1+g)(1+7g)k-4n}{2},\\ &w=-\frac{g(1+g)(1+3g)k-2(2m+n+gn)}{2\varepsilon}\end{aligned}$$

where $k\neq 0$ due to $e-fg+\varepsilon w\neq 0$. Then we arrive at the following family of systems:

$$\begin{aligned}\dot{x}_1 &= km+nx_1-\frac{k}{2}(g+1)y_1+gx_1^2+xy,\\ \dot{y}_1 &= 2mx_1+2ny_1+(g-1)x_1y_1+2y_1^2.\end{aligned}$$

which possess the invariant parabola $\Phi(x_1,y_1)=x_1^2-ky_1,\quad k\neq 0.$

Remark 3.5. If $k\neq 0$ then due to a rescaling we may assume $k=1$ in the above systems as well as in the invariant parabola.

Indeed since $k\neq 0$ via the rescaling $(x_1,y_1,t_1)\mapsto(kx_1,ky_1,t/k)$ and setting $m/k=m$ and $n/k=n$ we may assume $k=1$ in the above systems. At the same time applying this rescaling to the above parabola we get $\Phi(x_1,y_1)=k^2(x_1^2-y_1)$ and we conclude that the parabola $\widetilde{\Phi}(x_1,y_1)=x_1^2-y_1$ also in invariant for the above systems.

Therefore due to this remark we get the canonical systems $(S_{\boldsymbol{\alpha}})$ provided by the statement $\boldsymbol{\alpha})$ of Main Theorem.

1.1.2: *The subcase* $\mathcal{D}_1=0$. Then $e=-f(1+g)$ and therefore we obtain:

$$\Omega_1=2\big[a(1+g)(1+3g)-b(1+2g)\big]^2=2\mathcal{G}_1^2$$

and since $2g+1 \neq 0$ the condition $\Omega_1 = 0$ implies

$$b = \frac{a(1+g)(1+3g)}{1+2g}.$$

Therefore we determine that in this case the polynomials Eq_8 and Eq_{10} have the following common factor

$$\tilde{\phi} = 2a - f(1+2g)r + g(1+g)(1+2g)r^2.$$

We observe that $\tilde{\phi}$ is quadratic in r with the discriminant

$$\text{Discrim}[\tilde{\phi}, r] = -(1+2g)\big[8ag(1+g) - f^2(1+2g)\big]$$

and setting this discriminant equal to be w^2 we obtain

$$a = \frac{f^2(1+2g)^2 - w^2}{8g(1+g)(1+2g)}. \tag{3.12}$$

Then we arrive at the following expressions for the polynomials Eq_8 and Eq_{10}:

$$Eq_8 = \frac{H_+H_-}{4g(1+g)(1+2g)}, \quad Eq_{10} = \frac{rH_+H_-}{8(1+2g)^2},$$

where

$$H_{\pm} = f(1+2g) - 2g(1+g)(1+2g)r \pm w.$$

Therefore the equations $Eq_8 = Eq_{10} = 0$ imply $H_+H_- = 0$. If $H_+ = 0$ we determine

$$r = \frac{f + 2fg + w}{2g(1+g)(1+2g)} \equiv r^+$$

and we obtain the parabola

$$\Phi_1(x,y) = \frac{f^2(1+2g)^2 - w^2}{8g(g+1)(2g+1)^2} - \frac{f+2fg+w}{2g(1+2g)}\,x + \frac{f+2fg+w}{2g(g+1)(2g+1)}\,y + x^2.$$

In the case $H_- = 0$ we obtain

$$r = \frac{f + 2fg - w}{2g(1+g)(1+2g)} \equiv r^-$$

and we get the parabola

$$\Phi_2(x,y) = \frac{f^2(1+2g)^2 - w^2}{8g(g+1)(2g+1)^2} - \frac{f+2fg-w}{2g(1+2g)}\,x + \frac{f+2fg-w}{2g(g+1)(2g+1)}\,y + x^2.$$

Both these parabolas are invariant for the following family of systems:

$$\begin{aligned}
\dot{x} &= \frac{f^2(1+2g)^2 - w^2}{8g(1+g)(1+2g)} + gx^2 + xy,\\
\dot{y} &= \frac{(3g+1)\big[f^2(1+2g)^2 - w^2\big]}{8g(2g+1)^2} - f(g+1)x + fy + (g-1)xy + 2y^2).
\end{aligned} \tag{3.13}$$

We observe that both parabolas $\Phi_i(x,y) = 0$ $(i = 1,2)$ exist (i.e. are not reducible) if and only if $r^+r^- \neq 0$ and this is equivalent to

$$(f + 2fg + w)(f + 2fg - w) = f^2(1+2g)^2 - w^2 \neq 0$$

and considering (3.12) this is equivalent to $a \neq 0$.

On the other hand if only one of the factors vanishes we have $a = 0$ and

$$r^+ + r^- = (f + 2fg + w) + (f + 2fg - w) = 2f(1 + 2g) \neq 0$$

and due to $1 + 2g \neq 0$ we obtain that the above condition is equivalent to $f \neq 0$. Therefore for $a = 0$ and $f \neq 0$ we could have only one parabola.

We determine that in the case $w = 0$ we obtain $\Phi_1(x, y) = \Phi_2(x, y)$, i.e. the parabolas coalesce when w tends to zero and we obtain a double parabola. On the other hand considering (3.12) for $w = 0$ we obtain to

$$a - \frac{f^2(1+2g)}{8g(1+g)} = \frac{8ag(1+g) - f^2(1+2g)}{8g(1+g)} = \frac{\mathcal{F}_1}{8g(1+g)}$$

and we conclude that these invariant parabolas coalesce if and only if $\mathcal{F}_1 = 0$.

Thus we conclude that the statements $(\boldsymbol{A_2})$, $(\boldsymbol{A_3})$ and $(\boldsymbol{A_4})$ of Lemma 3.3 are proved.

Next we observe that the family of systems (3.13) is a subfamily of (3.8) defined by the condition $e = -f(1+g)$ (i.e. $\mathcal{D}_1 = 0$). Moreover considering (3.9) for $e = -f(1+g)$ we obtain:

$$\Phi(x, y) = \frac{f^2(1+2g)^2 - w^2}{8g(g+1)(2g+1)^2} - \frac{f + 2fg + \varepsilon w}{2g(1+2g)}x + \frac{f + 2fg + \varepsilon w}{2g(g+1)(2g+1)}y + x^2$$

and we observe that for $\varepsilon = 1$ (respectively $\varepsilon = -1$) the above parabola coincides with the invariant parabola $\Phi_1(x, y) = 0$ (respectively $\Phi_2(x, y) = 0$) of systems (3.13).

So taking the invariant parabola $\Phi_1(x, y) = 0$ (obtained for $e = -f(1+g)$ and $\varepsilon = 1$) we could apply the same translation (3.10) in this particular case and we arrive at the subfamily of systems (3.11) defined by the conditions $e = -f(1+g)$ and $\varepsilon = 1$ which possess the following invariant parabola

$$\widetilde{\Phi}_1(x_1, y_1) = x_1^2 + \frac{(f + 2fg + w)}{2g(1+g)(1+2g)}y_1.$$

Since in the considered case we have only three free parameters, we set only two new parameters as follows:

$$\begin{aligned} k &= -\frac{f + 2fg + w}{2g(1+g)(1+2g)}, \quad n = \frac{f + 5fg + 6fg^2 + w + 7gw}{8g(1+2g)} \quad \Rightarrow \\ f &= -\frac{k + 8gk + 7g^2k + 4n}{2}, \quad w = \frac{(1+2g)(k + 4gk + 3g^2k + 4n)}{2}. \end{aligned}$$

In this case after an additional rescaling (to force $k = 1$, see Remark 3.5) we arrive at the subfamily of systems $(S_{\boldsymbol{\alpha}})$ defined by the condition

$$m = (1 + 3g)(1 + 4g + 3g^2 + 2n)/4.$$

***1.2:* The case** $2g + 1 = 0$**.** Then $g = -1/2$ and evaluating Ω_1 and $\mathcal{D}_1$ we obtain

$$\Omega_1 = \left[2b(2e+f)^2 + a(a - 4e^2 - 6ef - 2f^2)\right]/8 = 0, \quad \mathcal{D}_1 = (2e+f)/2. \tag{3.14}$$

So we discuss two subcases: $\mathcal{D}_1 \neq 0$ and $\mathcal{D}_1 = 0$.

1.2.1: *The subcase* $\mathcal{D}_1 \neq 0$. Then $2e + f \neq 0$ and then the condition $\Omega_1 = 0$ gives us

$$b = -\frac{a(a - 4e^2 - 6ef - 2f^2)}{2(2e + f)^2}.$$

In this case the polynomials Eq_8 and Eq_{10} have the common factor $4a + (2e + f)r$. Therefore the equations $Eq_8 = Eq_{10} = 0$ imply $r = -4a/(2e + f)$ and we arrive at the family of systems

$$\begin{aligned} \dot{x} &= a - x^2/2 + xy, \\ \dot{y} &= -\frac{a(a - 4e^2 - 6ef - 2f^2)}{2(2e + f)^2} + ex + fy - 3xy/2 + 2y^2, \end{aligned} \tag{3.15}$$

which possess the invariant parabola

$$\Phi(x, y) = \frac{2a(a - 2ef - f^2)}{(2e + f)^2} + \frac{2a}{2e + f}\, x - \frac{4a}{2e + f}\, y + x^2. \tag{3.16}$$

Evidently this conic is irreducible if and only if $a \neq 0$. This completes the proof of the statement $\boldsymbol{(A_5)}$ of Lemma 3.3.

Next we show that systems (3.15) could be brought via a transformation to the canonical form $(S_{\boldsymbol{\alpha}})$. Indeed since $2e + f \neq 0$ we apply to parabola (3.16) the translation

$$x = x_1 - \frac{a}{2e + f}, \quad y = y_1 + \frac{a - 4ef - 2f^2}{4(2e + f)}.$$

which brings this parabola to the form $\widetilde{\Phi}(x_1, y_1) = x_1^2 - \dfrac{4a}{2e + f}\, y_1$.

On the other hand considering Observation 3.4 we apply the same translation to systems (3.15) and we set the following new notations (suggested by the above parabola and the linear parts of the transformed systems):

$$\begin{aligned} k &= \frac{4a}{2e + f}, \quad n = -\frac{(-5a + 4ef + 2f^2)}{4(2e + f)}, \quad m = \frac{-3a + 16e^2 + 20ef + 6f^2}{16(2e + f)} \quad \Rightarrow \\ a &= -\frac{k(k - 32m - 8n)}{32}, \quad e = \frac{-3k + 16m + 12n}{8}, \quad f = \frac{5k - 16n}{8}. \end{aligned}$$

Then after an additional rescaling (to force $k = 1$, see Remark 3.5) we arrive at the subfamily of systems $(S_{\boldsymbol{\alpha}})$ defined by the condition $g = -1/2$.

1.2.2: *The subcase* $\mathcal{D}_1 = 0$. This implies $f = -2e$ and considering (3.14) we have

$$\Omega_1 = a^2/8 = 0 \;\Rightarrow\; a = 0.$$

Therefore we obtain

$$Eq_8 = 0, \quad Eq_{10} = r(32b - 8er + r^2)/32 \equiv r\phi(b, e, r)/32 = 0.$$

Since $r \neq 0$ and $\mathrm{Discrim}[\phi, r] = 64(e^2 - 2b)$ we must have $e^2 - 2b \geq 0$ and we set $e^2 - 2b = w^2 \geq 0$, i.e. $b = (e^2 - w^2)/2$. Then we obtain

$$\phi = (4e - r - 4w)(4e - r + 4w) \equiv \varphi_1\varphi_2 = 0.$$

If $\varphi_1 = 0$ we obtain $r = 4(e - w) \neq 0$ and we obtain the parabola

$$\Phi_1'(x, y) = -2(e^2 - w^2) - 2(e - w)x + 4(e - w)y + x^2 \tag{3.17}$$

which is invariant for the family of systems

$$\begin{aligned} \dot{x} &= -x^2/2 + xy, \\ \dot{y} &= (e^2 - w^2)/2 + ex - 2ey - 3xy/2 + 2y^2. \end{aligned} \tag{3.18}$$

In the case $\varphi_2 = 0$ we obtain $r = 4(e + w) \neq 0$ and we obtain the parabola

$$\Phi_2'(x, y) = -2(e^2 - w^2) - 2(e + w)x + 4(e + w)y + x^2$$

which is invariant for the same family of systems (3.18).

We observe that both invariant parabolas exist only for $(e - w)(e + w) = e^2 - w^2 \neq 0$ and since $b = (e^2 - w^2)/2$ we obtain that the condition $b \neq 0$ must hold.

On the other hand we could have only one invariant parabola in the case when one of the factors vanishes, i.e. $(e - w)(e + w) = 0$. So we calculate

$$(e - w) + (e + w) = 2e$$

and we conclude that in the case $b = 0$ and $e \neq 0$ systems (3.18) possess only one invariant parabola.

We determine that in the case $w = 0$ we obtain $\Phi_1'(x, y) = \Phi_2'(x, y)$, i.e. the parabolas coalesce when w tends to zero and we obtain a double parabola. On the other hand considering the relation $e^2 - 2b = w^2$ for $w = 0$ we obtain $e^2 - 2b = 0$ and hence in the case $b \neq 0$ we have two distinct invariant parabolas if $e^2 - 2b \neq 0$ and one double invariant parabola if $e^2 - 2b = 0$.

Thus we conclude that the statements $(\boldsymbol{A_6})$, $(\boldsymbol{A_7})$ and $(\boldsymbol{A_8})$ of Lemma 3.3 are proved.

Next we show that systems (3.18) could be brought via a transformation to the canonical form $(S_{\boldsymbol{\alpha}})$. Indeed we apply to parabola (3.17) the translation

$$x = x_1 + e - w, \quad y = y_1 + (3e + w)/4$$

which brings this parabola to the form $\widetilde{\Phi}(x_1, y_1) = x_1^2 + 4(e - w)\, y_1$.

On the other hand considering Observation 3.4 we apply the same translation to systems (3.18) and we set the following new notations (suggested by the above parabola and the linear parts of the transformed systems):

$$\begin{gathered} k = -4(e - w), \;\; n = (-e + 5w)/4 \quad \Rightarrow \\ e = (-5k + 16n)/16, \quad w = (-k + 16n)/16. \end{gathered}$$

Then after an additional rescaling (to force $k = 1$) we arrive at the subfamily of systems $(S_{\boldsymbol{\alpha}})$ defined by the conditions $g = -1/2$ and $m = (1 - 8n)/32$.

***2:* The possibility** $g = 0$**.** Then considering (3.5) we obtain

$$Eq_8 = 2a + er = 0, \quad Eq_{10} = (b - a)r = 0.$$

and since $r \neq 0$ we obtain $b = a$. We discuss two cases: $e \neq 0$ and $e = 0$.

***2.1:* The case** $e \neq 0$**.** Then we get $r = -2a/e$ and we arrive at the family of systems

$$\dot{x} = a + xy, \quad \dot{y} = a + ex + fy - xy + 2y^2, \tag{3.19}$$

which possess the invariant parabola

$$\Phi(x,y) = -\frac{af}{e} + \frac{2a}{e}x - \frac{2a}{e}y + x^2 \tag{3.20}$$

and clearly this conic is irreducible if and only if the condition $a \neq 0$ holds.

2.2: **The case** $e = 0$**.** Then the equation $Eq_8 = 0$ gives us $a = 0$ and then the equation $Eq_{10} = br = 0$ implies $b = 0$. However in this case we get the degenerate system:

$$\dot{x} = xy, \quad \dot{y} = y(f - x + 2y).$$

So a system (3.3) with $g = 0$ possesses an invariant parabola if and only if the condition $ea \neq 0$ is satisfied. This completes the proof of the statement $(\boldsymbol{A_9})$ of Lemma 3.3.

Now we look for a transformation to brings systems (3.19) to the canonical form (S_α). For this we apply to parabola (3.20) the translation

$$x = x_1 - a/e, \quad y = y_1 - \frac{a+ef}{2e}$$

which brings this parabola to the form $\widetilde{\Phi}(x_1, y_1) = x_1^2 - \frac{2a}{e}y_1$.

On the other hand considering Observation 3.4 we apply the same translation to systems (3.15) and we set the following new notations (suggested by the above parabola and the linear parts of the transformed systems):

$$\begin{gathered} k = \frac{2a}{e}, \quad m = \frac{a + 2e^2 + ef}{4e}, \quad n = -\frac{a+ef}{2e} \quad \Rightarrow \\ a = k(2m+n)/2, \quad e = 2m+n, \quad f = -(k+4n)/2. \end{gathered}$$

Then after an additional rescaling (to force $k = 1$) we arrive at the subfamily of systems (S_α) defined by the condition $g = 0$.

3: **The possibility** $g = -1$**.** Then considering (3.5) we obtain

$$Eq_8 = 2a + (e+f)r = 0, \quad Eq_{10} = br = 0.$$

and since $r \neq 0$ we obtain $b = 0$. We discuss two cases: $e + f \neq 0$ and $e + f = 0$.

3.1: **The case** $e + f \neq 0$**.** Then we get $r = -2a/(e+f)$ and we arrive at the family of systems

$$\dot{x} = a - x^2 + xy, \quad \dot{y} = ex + fy - 2xy + 2y^2, \tag{3.21}$$

which possess the invariant parabola

$$\Phi(x,y) = -\frac{af}{e+f} - \frac{2a}{e+f}y + x^2. \tag{3.22}$$

if $a(e+f) \neq 0$.

3.2: **The case** $e + f = 0$**.** Then $f = -e$ and the equation $Eq_8 = 0$ gives us $a = 0$. Since $b = 0$ this leads to the degenerate system:

$$\dot{x} = -x(x-y), \quad \dot{y} = (e-2y)(x-y).$$

Thus we have proved that a system (3.3) with $g = -1$ possesses an invariant parabola if and only if the condition $a(e+f) \neq 0$ holds. This completes the proof of the statement $(\boldsymbol{A_{10}})$.

Next we show that systems (3.21) could be brought via a transformation to the canonical form (S_α). Indeed we apply to parabola (3.22) the translation

$$x = x_1, \quad y = y_1 - f/2$$

which brings this parabola to the form $\widetilde{\Phi}(x_1, y_1) = x_1^2 - \dfrac{2a}{e+f}\, y_1$.

On the other hand considering Observation 3.4 we apply the same translation to systems (3.15) and we set the following new notations (suggested by the above parabola and the linear parts of the transformed systems):

$$k = \frac{2a}{e+f}, \quad m = \frac{e+f}{2}, \quad n = -\frac{f}{2} \quad \Rightarrow$$
$$a = km, \quad e = 2(m+n), \quad f = -2n.$$

Then after an additional rescaling (to force $k = 1$) we arrive at the subfamily of systems (S_α) defined by the conditions $g = -1$.

Since all the cases are examined we deduce that Lemma 3.3 is proved. □

Invariant conditions: the case $\eta > 0$ and $\zeta_1 \neq 0$ Next we determine the affine invariant conditions for a system with $\eta > 0$ and $\zeta_1 \neq 0$ to possess an invariant parabola. According to Lemma 2.4 in this case the condition $\chi_1 = 0$ is necessary.

We prove the following theorem.

Theorem 3.6. *Assume that for a non-degenerate arbitrary quadratic system the conditions $\eta > 0$, $\chi_1 = 0$ and $\zeta_1 \neq 0$ are satisfied. Then this system could possess invariant parabolas only in one direction. More exactly it could only possess one of the following sets of invariant parabolas: $\cup$, $\Cup$ and $\cup^2$. Moreover this system has one of the above sets of parabolas if and only if $\chi_2 = 0$ and one of the following sets of conditions are satisfied, correspondingly:*

$(\mathcal{A}_1)$ $\zeta_2 \neq 0, \zeta_3 \neq 0, \zeta_4 \neq 0, \mathcal{R}_1 \neq 0 \Rightarrow \cup$;
$(\mathcal{A}_2)$ $\zeta_2 \neq 0, \zeta_3 \neq 0, \zeta_4 = 0, \mathcal{R}_2 \neq 0, \zeta_5 \neq 0 \Rightarrow \Cup$;
$(\mathcal{A}_3)$ $\zeta_2 \neq 0, \zeta_3 \neq 0, \zeta_4 = 0, \mathcal{R}_2 \neq 0, \zeta_5 = 0 \Rightarrow \cup^2$;
$(\mathcal{A}_4)$ $\zeta_2 \neq 0, \zeta_3 \neq 0, \zeta_4 = 0, \mathcal{R}_2 = 0, \zeta_5 \neq 0 \Rightarrow \cup$;
$(\mathcal{A}_5)$ $\zeta_2 \neq 0, \zeta_3 = 0, \zeta_4 \neq 0, \mathcal{R}_1 \neq 0 \Rightarrow \cup$;
$(\mathcal{A}_6)$ $\zeta_2 \neq 0, \zeta_3 = 0, \zeta_4 = 0, \mathcal{R}_2 \neq 0\ \zeta_5 \neq 0 \Rightarrow \Cup$;
$(\mathcal{A}_7)$ $\zeta_2 \neq 0, \zeta_3 = 0, \zeta_4 = 0, \mathcal{R}_2 \neq 0, \zeta_5 = 0 \Rightarrow \cup^2$;
$(\mathcal{A}_8)$ $\zeta_2 \neq 0, \zeta_3 = 0, \zeta_4 = 0, \mathcal{R}_2 = 0, \zeta_5 \neq 0 \Rightarrow \cup$;
$(\mathcal{A}_9)$ $\zeta_2 = 0, \ \zeta_6 \neq 0, \mathcal{R}_1 = 0, \mathcal{R}_2 \neq 0 \Rightarrow \cup$.

Proof. Assume that for an arbitrary non-degenerate quadratic system the condition $\eta > 0$ holds. Then according to Lemma 2.3 this system could be brought via a linear transformation to the family of systems (3.1). Forcing the condition $\chi_1 = 0$ to be fulfilled for these systems we get $(h-2)(g-2)(1+g+h) = 0$. Considering Remark 3.2 we may assume $h = 2$ and after an additional translation we arrive at the family of systems (3.3), i.e. at the systems

$$\dot{x} = a + gx^2 + xy, \quad \dot{y} = b + ex + fy + (g-1)xy + 2y^2. \tag{3.23}$$

For these systems we calculate

$$\zeta_1 = 2(g-2)(3+g), \quad \chi_2 = 384(g-2)(3+g)\Omega_1$$

and since $\zeta_1 \neq 0$ the condition $\chi_2 = 0$ is equivalent to $\Omega_1 = 0$.

Following the statements $(\boldsymbol{\mathcal{A}_1})$–$(\boldsymbol{\mathcal{A}_4})$ of the theorem for systems (3.23) we calculate:

$$\zeta_2 = 4g(g+1),\ \zeta_3 = 8(2g+1)^2,\ \zeta_4 = -(g-2)(3+g)\mathcal{D}_1/8,$$
$$\mathcal{R}_1 = 30(g-2)(3+g)(a-b+4ag-2bg+3ag^2) = 30(g-2)(3+g)\mathcal{G}_1.$$

We discuss two cases: $\zeta_2 \neq 0$ and $\zeta_2 = 0$.

1: **The case $\zeta_2 \neq 0$.** Then $g(g+1) \neq 0$ and taking into account Lemma 3.3 we have to consider the condition $2g+1 \neq 0$ which is equivalent to $\zeta_3 \neq 0$.

1.1: **The subcase $\zeta_3 \neq 0$.**

Then we have $2g+1 \neq 0$ and due to $\zeta_1 \neq 0$ (i.e. $(g-2)(3+g) \neq 0$) the condition $\zeta_4 \neq 0$ is equivalent to $\mathcal{D}_1 \neq 0$. So we examine two possibilities: $\zeta_4 \neq 0$ and $\zeta_4 = 0$.

1.1.1: *The possibility $\zeta_4 \neq 0$.* Then we have $\mathcal{D}_1 \neq 0$ and we observe that the condition $\mathcal{R}_1 \neq 0$ is equivalent to $\mathcal{G}_1 \neq 0$ since $\zeta_1 \neq 0$. Therefore all the conditions provided by the statement $(\boldsymbol{A_1})$ of Lemma 3.3 are satisfied and by this lemma systems (3.23) possess one invariant parabola.

1.1.2: *The possibility $\zeta_4 = 0$.* In this case due to $\zeta_1 \neq 0$ we obtain $\mathcal{D}_1 = 0$ and considering (3.4) we get:

$$\mathcal{D}_1 = e + f(1+g) = 0 \quad \Rightarrow \quad e = -f(1+g).$$

Then for systems (3.23) we obtain

$$\Omega_1 = 2\big[a(1+g)(1+3g) - b(1+2g)\big]^2$$

and due to the condition $1+2g \neq 0$ the condition $\Omega_1 = 0$ yields

$$b = \frac{a(1+g)(1+3g)}{1+2g}.$$

Therefore for systems (3.23) with the parameters e and b above determined we calculate

$$\zeta_5 = -\frac{19}{1+2g}(g-2)(3+g)\big[8ag(1+g) - f^2(1+2g)\big] = -\frac{19}{1+2g}(g-2)(3+g)\mathcal{F}_1.$$

So due to the condition $\zeta_1 \neq 0$ (i.e. $(g-2)(3+g) \neq 0$ we obtain that the condition $\mathcal{F}_1 = 0$ is equivalent to $\zeta_5 = 0$.

We determine that in the case under examination the condition $a \neq 0$ is equivalent to $\mathcal{R}_2 \neq 0$, which for systems (3.23) has the value

$$\mathcal{R}_2 = -\frac{a(g-2)(3+g)(8+27g+27g^2)}{4(1+2g)}.$$

Indeed first we observe that $\text{Discrim}[8+27g+27g^2, g] = -135 < 0$ and secondly we have $(g-2)(3+g)(1+2g) \neq 0$ due to the condition $\zeta_1\zeta_3 \neq 0$. So considering Lemma 3.3 we conclude that systems (3.23) possess two parabolas if the conditions

$$\chi_1 = \chi_2 = 0,\ \zeta_1 \neq 0,\ \zeta_2 \neq 0,\ \zeta_3 \neq 0,\ \zeta_4 = 0,\ \mathcal{R}_2 \neq 0$$

hold. Moreover by Lemma 3.3 these invariant parabolas are distinct if $\zeta_5 \neq 0$, i.e. $\mathcal{F}_1 \neq 0$ (see statement $(\boldsymbol{A_2})$) and they coalesce (obtaining a double parabola) if $\zeta_5 = 0$, i.e. $\mathcal{F}_1 = 0$ (see statement $(\boldsymbol{A_3})$).

Assume now that the condition $\mathcal{R}_2 = 0$ (i.e. $a = 0$) holds. Then for systems (3.23) with $\mathcal{D}_1 = 0$ and $\Omega_1 = 0$ we calculate

$$\zeta_5 = 19f^2(g-2)(3+g)$$

and since $(g-2)(3+g) \neq 0$ (due to $\zeta_1 \neq 0$) we conclude that the condition $f \neq 0$ is equivalent to $\zeta_5 \neq 0$. So we get the conditions provided by Lemma 3.3 (see statement $(\boldsymbol{A_4})$) and therefore we have one simple invariant parabola.

***1.2:* The subcase $\zeta_3 = 0$.** Then $1 + 2g = 0$, i.e. $g = -1/2$ and for systems (3.23) calculations yield:

$$\zeta_1 = -25/2,\ \zeta_2 = -1,\ \zeta_4 = 25(2e+f)/64 = 25\mathcal{D}_1/32,$$
$$\chi_2 = -300\big[2b(2e+f)^2 + a(a - 4e^2 - 6ef - 2f^2)\big].$$

So considering Lemma 3.3 (see statements $(\boldsymbol{A_5})$–$(\boldsymbol{A_8})$) we discuss two possibilities: $\zeta_4 \neq 0$ and $\zeta_4 = 0$.

1.2.1: *The possibility $\zeta_4 \neq 0$.* Then $2e + f \neq 0$ and the condition $\chi_2 = 0$ gives us

$$b = -\frac{a(a - 4e^2 - 6ef - 2f^2)}{2(2e+f)^2}.$$

So according to Lemma 3.3 (see statements $(\boldsymbol{A_5})$) systems (3.23) with $g = -1/2$ and the above given value of the parameter b possess one invariant parabola if in addition the condition $a \neq 0$ holds. It remains to observe that this condition is governed by the invariant polynomial $\mathcal{R}_1$ because for these systems we have $\mathcal{R}_1 = 375a/8$.

1.2.2: *The possibility $\zeta_4 = 0$.* Then we have $\mathcal{D}_1 = 0$ which implies $f = -2e$ and then we obtain $\chi_2 = -300a^2 = 0$, i.e. $a = 0$. As a result we arrive at the family of systems

$$\dot{x} = -x^2/2 + xy, \quad \dot{y} = b + ex - 2ey - 3xy/2 + 2y^2 \tag{3.24}$$

for which we calculate

$$\zeta_1 = -25/2,\ \zeta_2 = -1,\ \zeta_3 = \zeta_4 = \mathcal{R}_1 = 0,\ \ \mathcal{R}_2 = -125b/16,\ \zeta_5 = 475(2b - e^2).$$

So considering the statements $(\boldsymbol{A_6})$ and $(\boldsymbol{A_7})$ of Lemma 3.3 we deduce that in the case $\mathcal{R}_2 \neq 0$ (i.e. $b \neq 0$) systems (3.24) possess two distinct invariant parabolas if $\zeta_5 \neq 0$ and one double invariant parabola if $\zeta_5 = 0$.

Assuming $\mathcal{R}_2 = 0$ (i.e. $b = 0$) considering the value of the invariant polynomial ζ_5 given above we get $\zeta_5 = -475e^2$ and hence the condition $e \neq 0$ is equivalent to $\zeta_5 \neq 0$.

So we get the conditions provided by the statement $(\boldsymbol{A_8})$ of Lemma 3.3 and therefore systems (3.24) possess one simple invariant parabola.

***2:* The case $\zeta_2 = 0$.** Then we have $g(g+1) = 0$, i.e. either $g = 0$ or $g = -1$. We discuss each one of these possibilities.

2.1: *The possibility $g = 0$.* Then for for systems (3.23) we calculate

$$\chi_2 = -2304(a-b)(2a - 2b - e^2 - ef),\ \mathcal{R}_1 = -180(a-b).$$

According to the statement $(\boldsymbol{A_9})$ of Lemma 3.3 for the existence of invariant parabola the condition $b = a$ is necessary, i.e. we must have $\mathcal{R}_1 = 0$ and this implies $\chi_2 = 0$. Setting $b = a$ we obtain

$$\zeta_6 = e/2,\ \ \mathcal{R}_2 = 12a$$

and considering the statements (A_9) the condition $\zeta_6\mathcal{R}_2 \neq 0$ must be satisfied for the existence of an invariant parabola.

2.2: *The possibility* $g = -1$. In this case for systems (3.23) we obtain

$$\chi_2 = -2304b(2b + e^2 + ef),\ \mathcal{R}_1 = -180b.$$

Considering the statement (A_{10}) of Lemma 3.3 we deduce that for the existence of an invariant parabola the condition $b = 0$ is necessary, i.e. we must have $\mathcal{R}_1 = 0$ and this implies $\chi_2 = 0$. Setting $b = 0$ we calculate

$$\zeta_6 = -(e + f)/2,\ \ \mathcal{R}_2 = -12a$$

and therefore by the statements (A_{10}) the condition $\zeta_6\mathcal{R}_2 \neq 0$ must be satisfied for the existence of an invariant parabola.

We observe that in both cases $g = 0$ and $g = -1$ we have obtained the same invariant conditions $\mathcal{R}_1 = 0$ and $\zeta_6\mathcal{R}_2 \neq 0$. This completes the proof of the statement $(\boldsymbol{\mathcal{A}_9})$ of Theorem 3.6 as well as the proof of Theorem 3.6. □

3.1.2 The possibility $\chi_1 = \zeta_1 = 0$

Next we consider the case when systems (3.1) could possess invariant parabolas in two directions. Then two factors of χ_1 from (3.2) vanish. According to Remark 3.2 we could consider $h = 2 = g$ and due to the translation $(x, y) \mapsto (x - d, y - e)$ (forcing $d = e = 0$) we arrive at the family of systems

$$\dot{x} = a + cx + 2x^2 + xy, \quad \dot{y} = b + fy + xy + 2y^2. \tag{3.25}$$

Coefficient conditions for systems (3.25) to possess invariant parabolas. By Lemma 3.1 systems (3.25) could possess invariant parabolas either of the form $\Phi(x, y) = p + qx + ry + x^2$ $(r \neq 0)$ or of the form $\Phi(x, y) = p + qx + ry + y^2$ $(q \neq 0)$. We prove the following lemma.

Lemma 3.7. *A system* (3.25) *possesses either one or two invariant parabolas or a double invariant parabola of the indicated form if and only if the corresponding set of conditions are satisfied, respectively:*

$(\boldsymbol{B})$ $\Phi(x, y) = p + qx + ry + x^2 \quad \Leftrightarrow \quad \Omega'_1 = 0$ *and either*

- $(\boldsymbol{B_1})$ $\mathcal{D}'_1 \neq 0$, $\mathcal{G}'_1 \neq 0 \Rightarrow$ *one invariant parabola; or*
- $(\boldsymbol{B_2})$ $\mathcal{D}'_1 = 0$, $a \neq 0$, $\mathcal{F}'_1 \neq 0 \Rightarrow$ *two invariant parabolas; or*
- $(\boldsymbol{B_3})$ $\mathcal{D}'_1 = 0$, $a \neq 0$, $\mathcal{F}'_1 = 0 \Rightarrow$ *one double invariant parabola; or*
- $(\boldsymbol{B_4})$ $\mathcal{D}'_1 = 0$, $a = 0$, $c \neq 0 \Rightarrow$ *one invariant parabola.*

$(\boldsymbol{B'})$ $\Phi(x, y) = p + qx + ry + y^2 \quad \Leftrightarrow \quad \Omega'_2 = 0$ *and either*

- $(\boldsymbol{B'_1})$ $\mathcal{D}'_2 \neq 0$, $\mathcal{G}'_2 \neq 0 \Rightarrow$ *one invariant parabola; or*
- $(\boldsymbol{B'_2})$ $\mathcal{D}'_2 = 0$, $b \neq 0$, $\mathcal{F}'_2 \neq 0 \Rightarrow$ *two invariant parabolas; or*
- $(\boldsymbol{B'_3})$ $\mathcal{D}'_2 = 0$, $b \neq 0$, $\mathcal{F}'_2 = 0 \Rightarrow$ *one double invariant parabola; or*
- $(\boldsymbol{B'_4})$ $\mathcal{D}'_2 = 0$, $b = 0$, $f \neq 0 \Rightarrow$ *one invariant parabola.*

where

$$
\begin{aligned}
&\Omega_1' = 50b^2 + b(109c^2 - 420a - 53cf - 6f^2) + 3(6a - c^2 + cf)(49a - 6c^2 - cf + 2f^2);\\
&\mathcal{D}_1' = 13c - 3f;\ \ \mathcal{G}_1' = 21a - 5b - 10c^2 + 5cf;\\
&\mathcal{F}_1' = 15a - 15b - 2c^2 + 2f^2;\\
&\Omega_2' = 50a^2 + a(109f^2 - 420b - 53cf - 6c^2) + 3(6b - f^2 + cf)(49b - 6f^2 - cf + 2c^2);\\
&\mathcal{D}_2' = 13f - 3c;\ \ \mathcal{G}_2' = 21b - 5a - 10f^2 + 5cf;\\
&\mathcal{F}_2' = 15b - 15a - 2f^2 + 2c^2.
\end{aligned}
\tag{3.26}
$$

Proof. Considering the equations (2.6) we examine each one of the statements of the above lemma.

(*B*) $\Phi(x,y) = p + qx + ry + x^2$ with $r \neq 0$. In this case we obtain

$$
\begin{gathered}
s = 1,\ v = u = 0,\ U = 4,\ V = 2,\ W = 2(c - q),\\
Eq_1 = Eq_2 = Eq_3 = Eq_4 = Eq_5 = Eq_7 = 0.
\end{gathered}
$$

Calculating the remaining equations we have

$$
\begin{gathered}
Eq_6 = -q - 3r,\ \ Eq_8 = 2a - 4p - cq + 2q^2,\\
Eq_9 = -2p - 2cr + fr + 2qr,\ \ Eq_{10} = aq - 2cp + 2pq + br.
\end{gathered}
$$

It is clear that the equations $Eq_6 = 0$ implies $q = -3r$ whereas $Eq_9 = 0$ gives us $p = -r(2c - f + 6r)/2$. Therefore calculations yield

$$
\begin{aligned}
&Eq_8 = 2a + (7c - 2f)r + 30r^2,\\
&Eq_{10} = r\big[b - 3a + 2c^2 - cf + 3(4c - f)r + 18r^2\big] \equiv r\Psi'(a,b,c,f,r)
\end{aligned}
$$

and since $r \neq 0$ the equation $Eq_{10} = 0$ is equivalent to $\Psi' = 0$.

According to [12, Lemmas 11,12] the equations $Eq_8 = 0$ and $\Psi' = 0$ have a common solution of degree 2 with respect to the parameter r if and only if

$$
Res_r^{(0)}\,(Eq_8, \Psi') = Res_r^{(1)}\,(Eq_8, \Psi) = 0
$$

where $Res_r^{(1)}$ is the subresultant of order one and $Res_r^{(0)}$ is the subresultant of order zero which coincide with standard resultant (for detailed definition see [12], formula (19)). We calculate

$$
Res_r^{(1)}\,(Eq_8, \Psi) = 18(13c - 3f) \equiv 18\mathcal{D}_1',\quad Res_r^{(0)}\,(Eq_8, \Psi)\ \ = 18\Omega_1'.
$$

So we examine two possibilities: $\mathcal{D}_1' \neq 0$ and $\mathcal{D}_1' = 0$.

***1:* The possibility $\mathcal{D}_1' \neq 0$.** Therefore the equations $Eq_8 = 0$ and $\Psi' = 0$ could have a unique common solution with respect to the parameter r and for this it is necessary and sufficient $\Omega_1' = 0$. So we have to examine the solutions of the equation $\Omega_1' = 0$. In this case we calculate

$$
\text{Discrim}[\Omega_1', b] = -(13c - 3f)^2(240a - 49c^2 + 28cf - 4f^2) \equiv -\mathcal{D}_1'^2\mathcal{E}'
$$

and hence the equation $\Omega_1' = 0$ has real solutions with respect to the parameter b if and only if either $\mathcal{D}_1' = 0$ or $\mathcal{E}' \leq 0$. However since the condition $\mathcal{D}_1' \neq 0$ holds it remains to examine the condition $\mathcal{E}' \leq 0$. In this case setting $\mathcal{E}' = -w^2 \leq 0$ we calculate

$$
a = \frac{(7c - 2f)^2 - w^2}{240}
\tag{3.27}
$$

and then we obtain $\Omega_1' = \big(E_+E_-\big)/3200$, where

$$E_\pm = 400b + 93c^2 - 16cf - 52f^2 + 4\varepsilon(13c - 3f)w + 7w^2, \quad \varepsilon = \pm 1.$$

Then the condition $\Omega_1' = 0$ gives us

$$b = -\frac{1}{400}(3c + 2f + \varepsilon w)(31c - 26f + 7\varepsilon w) \tag{3.28}$$

where $\varepsilon = 1$ if $E_+ = 0$ and $\varepsilon = -1$ if $E_- = 0$. In this case we obtain that the polynomials Eq_8 and $\Psi(c, f, r)$ have the common factor $\zeta = (7c - 2f + 60r - \varepsilon w)$ which is linear with respect to the parameter r. Setting $\zeta = 0$ we get

$$r = \frac{2f - 7c + \varepsilon w}{60}$$

and we arrive at the family of systems

$$\begin{aligned} \dot{x} &= \frac{(7c-2f)^2 - w^2}{240} + cx + 2x^2 + xy, \\ \dot{y} &= -\frac{(3c + 2f + \varepsilon w)(31c - 26f + 7\varepsilon w)}{400} + fy + xy + 2y^2. \end{aligned} \tag{3.29}$$

This family of systems possess the following invariant parabola

$$\Phi(x, y) = \frac{(7c - 2f - \varepsilon w)(13c - 8f + \varepsilon w)}{1200} + \frac{(7c - 2f - \varepsilon w)}{20}x - \frac{(7c - 2f - \varepsilon w)}{60}y + x^2. \tag{3.30}$$

We observe that this conic is reducible if and only if $7c - 2f - \varepsilon w = 0$.

Considering (3.27) and (3.28) we get

$$w^2 = -240a + (7c - 2f)^2$$

and then we obtain

$$\begin{gathered} b = -\frac{1}{400}\big[(31c - 26f)(3c + 2f) + 4(13c - 3f)\varepsilon w + 7w^2\big] \;\Rightarrow \\ 100b - 420a + 109c^2 - 53cf - 6f^2 + (13c - 3f)\varepsilon w = 0. \end{gathered}$$

Since $\mathcal{D}_1' = 13c - 3f \neq 0$ we solve the last equation with respect to εw and we obtain

$$\varepsilon w = \frac{1}{13c - 3f}\big(420a - 100b - 109c^2 + 53cf + 6f^2\big).$$

Then calculations yield

$$r = \frac{(-7c + 2f + \varepsilon w)}{60} = \frac{21a - 5b - 10c^2 + 5cf}{3(13c - 3f)} = \frac{\mathcal{G}_1'}{3(13c - 3f)} \neq 0.$$

This completes the proof of the statement $(\boldsymbol{B_1})$ of Lemma 3.3.

Next we show that systems (3.29) could be brought via a transformation to the canonical form $(S_{\boldsymbol{\alpha}})$. Indeed we apply to parabola (3.30) the translation

$$x = x_1 + \frac{2f - 7c + \varepsilon w}{40}, \quad y = y_1 + \frac{31c - 26f + 7\varepsilon w}{80}.$$

which brings this parabola to the form $\widetilde{\Phi}(x_1, y_1) = x_1^2 + \dfrac{2f - 7c + \varepsilon w}{60}\, y_1$.

On the other hand considering Observation 3.4 we apply the same translation to systems (3.29) and we set the following new notations (suggested by the above parabola and the linear parts of the transformed systems):

$$k = -\frac{2f - 7c + \varepsilon w}{60}, \quad m = -\frac{31c - 26f + 7\varepsilon w}{160}, \quad n = \frac{11c - 2f + 3\varepsilon w}{16} \quad \Rightarrow$$
$$c = 6k - 2m + n, \quad f = \frac{3k - 16m + 4n}{2}, \quad \varepsilon w = -21k + 2m + 3n.$$

Then after an additional rescaling (to force $k = 1$) we arrive at the subfamily of systems $(S_{\boldsymbol{\alpha}})$ defined by the conditions $g = 2$.

2: The possibility $\mathcal{D}'_1 = 0$. Then $f = 13c/3$ and therefore we obtain:

$$\Omega'_1 = 2(63a - 15b + 35c^2)^2/9$$

and hence the condition $\Omega'_1 = 0$ implies

$$b = 7(9a + 5c^2)/15.$$

Therefore we determine that in this case the polynomials Eq_8 and Eq_{10} have the following common factor

$$\phi' = 6a - 5cr + 90r^2.$$

We observe that ϕ' is quadratic in r with the discriminant

$$\text{Discrim}[\phi', r] = -5(432a - 5c^2)$$

and setting this discriminant equal to be w^2 we obtain

$$a = \frac{25c^2 - w^2}{2160}. \tag{3.31}$$

Then we arrive at the following expressions for the polynomials Eq_8 and Eq_{10}:

$$Eq_{10} = \frac{3r}{5} Eq_8 = \frac{r(5c - 180r + w)(5c - 180r - w)}{1800} = \frac{rU_+U_-}{1800}.$$

Therefore the equations $Eq_8 = Eq_{10} = 0$ imply $U_+U_- = 0$. If $U_+ = 0$ we determine

$$r = \frac{5c + w}{180} \equiv r^+$$

and we obtain the parabola

$$\Phi'_1(x, y) = \frac{(65c - w)(5c + w)}{10800} - \frac{5c + w}{60}\, x + \frac{5c + w}{180}\, y + x^2. \tag{3.32}$$

In the case $U_- = 0$ we obtain

$$r = \frac{5c + w}{180} \equiv r^-$$

and we get the parabola

$$\Phi'_2(x, y) = \frac{(65c + w)(5c - w)}{10800} - \frac{5c - w}{60}\, x + \frac{5c - w}{180}\, y + x^2.$$

Both these parabolas are invariant for the following family of systems:

$$\dot{x} = \frac{25c^2 - w^2}{2160} + cx + 2x^2 + xy, \quad \dot{y} = \frac{7(1225c^2 - w^2)}{3600} + \frac{13c}{3}y + xy + 2y^2. \tag{3.33}$$

We observe that both parabolas $\Phi_i'(x,y) = 0$ $(i = 1,2)$ exist (i.e. are not reducible) if and only if $r^+r^- \neq 0$ and this is equivalent to

$$(5c+w)(5c-w) = 25c^2 - w^2$$

and considering (3.31) this is equivalent to $a \neq 0$.

On the other hand if only one of the factors vanishes we have $a = 0$ and

$$r^+ + r^- = (5c+w) + (5c-w) = 10c \neq 0.$$

Therefore for $a = 0$ and $c \neq 0$ we could have only one parabola.

We determine that in the case $w = 0$ we obtain $\Phi_1'(x,y) = \Phi_2'(x,y)$, i.e. the parabolas coalesce when w tends to zero and we obtain a double parabola. On the other hand considering (3.31) for $w = 0$ we obtain

$$a - \frac{25c^2}{2160} = \frac{432a - 5c^2}{432} = -\frac{9}{432}\mathcal{F}_1'$$

and we conclude that these invariant parabolas coalesce if and only if $\mathcal{F}_1' = 0$.

Thus we conclude that the statements $(\boldsymbol{B_2})$, $(\boldsymbol{B_3})$ and $(\boldsymbol{B_4})$ of Lemma 3.7 are proved.

Next we show that systems (3.33) could be brought via a transformation to the canonical form $(S_{\boldsymbol{\alpha}})$. Indeed we could apply to parabola (3.32) the translation

$$x = x_1 + \frac{5c+w}{120}, \quad y = y_1 - \frac{7(35c-w)}{240}$$

which brings this parabola to the form $\widetilde{\Phi}(x_1,y_1) = x_1^2 + \dfrac{5c+w}{180}y_1$.

On the other hand considering Observation 3.4 we apply the same translation to systems (3.29) and we set the following new notations (suggested by the above parabola and the linear parts of the transformed systems):

$$k = -\frac{5c+w}{180}, \quad n = \frac{7c+3w}{48} \quad \Rightarrow \quad c = -3(45k+4n)/2, \quad w = 15(21k+4n)/2.$$

Then after an additional rescaling (to force $k = 1$) we arrive at the subfamily of systems $(S_{\boldsymbol{\alpha}})$ defined by the conditions $g = 2$ and $m = 7(21k+2n)/4$.

$(\boldsymbol{B'})$ $\Phi(x,y) = p + qx + ry + y^2$ with $q \neq 0$ otherwise we get a reducible conic. It is not too difficult to detect that this case can be brought to the case $(\boldsymbol{B})$ if we apply two changes: one in systems (3.25) and another in the formula of conic (2.4). More precisely the change

$$(x,y,a,b,c,f) \mapsto (y,x,b,a,f,c) \tag{3.34}$$

conserves systems (3.25) whereas the change

$$(x,y,p,q,r,s,v,u) \to (y,x,p,r,q,u,v,s)$$

conserves the conic (2.4). We observe that the second change transfers the parabola $\Phi(x,y) = p + qx + ry + x^2$ to the parabola $\Phi(x,y) = p + qx + ry + y^2$ and at the same time the first change transfers the conditions $(\boldsymbol{B_i})$, $i = 1,2,3,4$ from the statement $(\boldsymbol{B})$ of Lemma 3.7 to the conditions $(\boldsymbol{B_i'})$, $i = 1,2,3,4$ from the statement $(\boldsymbol{B'})$ of the same lemma, correspondingly. Since the conditions of the statement $(\boldsymbol{B})$ are proved, we conclude that the conditions of the statement $(\boldsymbol{B'})$ of Lemma 3.7 are also valid. This completes the proof of Lemma 3.7. □

We point out that Theorem 3.6 provides the necessary and sufficient conditions for the existence of invariant parabolas for an arbitrary quadratic systems with the conditions $\eta > 0$, $\chi_1 = 0$ and $\zeta_1 \neq 0$. As it was mentioned earlier (see page 16) the condition $\zeta_1 \neq 0$ does not allow this system to possess invariant parabolas in two directions.

Invariant conditions: the case $\eta > 0$ and $\zeta_1 = 0$ Next we consider the class of quadratic systems for which the conditions $\eta > 0$ and $\zeta_1 = 0$, which could possess invariant parabolas in two directions.

We prove the following theorem.

Theorem 3.8. *Assume that for a non-degenerate arbitrary quadratic system the conditions $\eta > 0$ and $\chi_1 = \zeta_1 = 0$ are satisfied. Then this system could possess invariant parabolas in one or two directions. More exactly it could only possess one of the following sets of invariant parabolas: $\cup$, $\Cup$, $\cup^2$, $\cup\subset$ and $\Cup\subset$. Moreover this system has one of the above sets of invariant parabolas if and only if $\chi_3 = 0$ and one of the following sets of conditions are satisfied, correspondingly:*

$$
\begin{array}{lll}
(\boldsymbol{\mathcal{B}_1}) & \chi_4 \neq 0,\ \zeta_7 \neq 0,\ \mathcal{R}_3 \neq 0 & \Rightarrow \cup; \\
(\boldsymbol{\mathcal{B}_2}) & \chi_4 \neq 0,\ \zeta_7 = 0,\ \mathcal{R}_4 \neq 0,\ \zeta_8 \neq 0 & \Rightarrow \Cup; \\
(\boldsymbol{\mathcal{B}_3}) & \chi_4 \neq 0,\ \zeta_7 = 0,\ \mathcal{R}_4 \neq 0,\ \zeta_8 = 0 & \Rightarrow \cup^2; \\
(\boldsymbol{\mathcal{B}_4}) & \chi_4 \neq 0,\ \zeta_7 = 0,\ \mathcal{R}_4 = 0 & \Rightarrow \cup; \\
(\boldsymbol{\mathcal{B}_5}) & \chi_4 = 0,\ \zeta_5 \neq 0,\ \zeta_9 \neq 0 & \Rightarrow \cup\subset; \\
(\boldsymbol{\mathcal{B}_6}) & \chi_4 = 0,\ \zeta_5 \neq 0,\ \zeta_9 = 0,\ \zeta_{10} \neq 0 & \Rightarrow \cup; \\
(\boldsymbol{\mathcal{B}_7}) & \chi_4 = 0,\ \zeta_5 = 0,\ \zeta_6 \neq 0 & \Rightarrow \Cup\subset.
\end{array}
$$

Proof. As it was shown earlier (see page 29) if for a quadratic system with three real infinite singularities the conditions $\chi_1 = \zeta_1 = 0$ are satisfied, then via an affine transformation and time rescaling this system can be brought to the form (3.25). Thus in what follows we consider the family of quadratic systems

$$\dot{x} = a + cx + 2x^2 + xy, \quad \dot{y} = b + fy + xy + 2y^2. \tag{3.35}$$

Considering (3.26) for these systems we calculate

$$\chi_1 = \chi_2 = \zeta_1 = 0, \ \chi_3 = 2^4 3^4 5^3 83 \cdot 491\, \Omega_1' \Omega_2', \ \chi_4 = 123750(\Omega_1' + \Omega_2')$$

and therefore the condition $\chi_3 = 0$ yields $\Omega_1'\Omega_2' = 0$, i.e. one of the necessary conditions provided either by the statement $(\boldsymbol{B})$ of Lemma 3.7 or by the statement $(\boldsymbol{B'})$ of this lemma is satisfied. We discuss two cases: $\chi_4 \neq 0$ and $\chi_4 = 0$.

1: **The case $\chi_4 \neq 0$.** Then $\Omega_1' + \Omega_2' \neq 0$ and we conclude that only one of the polynomials Ω_1' or Ω_2' vanishes. Considering the change (3.34) we may assume without losing generality that the conditions $\Omega_1' = 0$ and $\Omega_2' \neq 0$ are fulfilled.

On the other hand for systems (3.35) we calculate

$$\zeta_7 = 105750\big(\mathcal{D}_2'\Omega_1' + \mathcal{D}_1'\Omega_2'\big), \quad \mathcal{R}_3 = 5134081342500\big(\mathcal{G}_2'\Omega_1' + \mathcal{G}_1'\Omega_2'\big).$$

Therefore since $\Omega_1' = 0$ and $\Omega_2' \neq 0$ we obtain that the condition $\mathcal{D}_1' = 0$ is equivalent to $\zeta_7 = 0$. Moreover in this case the condition $\mathcal{R}_3 \neq 0$ is equivalent to $\mathcal{G}_1' \neq 0$. So we discuss two subcases: $\zeta_7 \neq 0$ and $\zeta_7 = 0$.

1.1: **The subcase $\zeta_7 \neq 0$.** Then $\mathcal{D}_1' \neq 0$ and by Lemma 3.7 (see statement $(\boldsymbol{B_1})$) we deduce that systems (3.25) possess one invariant parabola if and only if $\mathcal{G}_1' \neq 0$. Due to $\Omega_1' = 0$ and

$\Omega'_2 \neq 0$ this condition is equivalent to $\mathcal{R}_3 \neq 0$ and we conclude that the statement $(\boldsymbol{\mathcal{B}_1})$ of Theorem 3.8 is proved.

***1.2:* The subcase** $\zeta_7 = 0$**.** This implies $\mathcal{D}'_1 = 13c - 3f = 0$, i.e. $f = 13c/3$ and then we get

$$\Omega'_1 = 2(63a - 15b + 35c^2)^2/9 = 0 \;\Rightarrow\; b = 7(9a + 5c^2)/15.$$

So we arrive at the family of systems

$$\dot{x} = a + cx + 2x^2 + xy, \quad \dot{y} = \frac{7}{15}(9a + 5c^2) + \frac{13c}{3}y + xy + 2y^2, \tag{3.36}$$

for which we calculate

$$\zeta_8 = -(432a - 5c^2)/9 = \mathcal{F}'_1, \quad \mathcal{R}_4 = 15600a.$$

We observe that the condition $\mathcal{R}_4 \neq 0$ is equivalent to $a \neq 0$ and therefore by Lemma 3.7 in the case $\mathcal{R}_4 \neq 0$ systems (3.36) possess two distinct parabolas in one direction (see statement $(\boldsymbol{B_2})$) if $\zeta_8 \neq 0$ and they possess one double invariant parabola (see statement $(\boldsymbol{B_3})$) if $\zeta_8 = 0$. This means that the statements $(\boldsymbol{\mathcal{B}_2})$ and $(\boldsymbol{\mathcal{B}_3})$ of Theorem 3.8 are proved.

Assume now that the condition $\mathcal{R}_4 = 0$ holds. Then $a = 0$ and for systems (3.36) we have $\chi_4 = 110000c^4 \neq 0$. Then according to the statement $(\boldsymbol{B_4})$ of Lemma 3.7 we conclude that these systems possess one invariant parabola and therefore the statements $(\boldsymbol{\mathcal{B}_4})$ of Theorem 3.8 is valid.

***2:* The case** $\chi_4 = 0$**.** Then we get $\Omega'_1 = \Omega'_2 = 0$ and since for systems (3.35) we have

$$\begin{gathered}\zeta_5 = 25(13c - 3f)(13f - 3c)/4 = 25\mathcal{D}'_1\mathcal{D}'_2/4,\\ \zeta_9 = -990000(21a - 5b - 10c^2 + 5cf)(5a - 21b - 5cf + 10f^2) = 990000\mathcal{G}'_1\mathcal{G}'_2,\\ \zeta_{10} = 5(8a + 8b - 5c^2 + 5cf - 5f^2)/4 = 5(\mathcal{G}'_1 + \mathcal{G}'_2)/8.\end{gathered} \tag{3.37}$$

We examine two subcases: $\zeta_5 \neq 0$ and $\zeta_5 = 0$.

***2.1:* The subcase** $\zeta_5 \neq 0$**.** Then $\mathcal{D}'_1\mathcal{D}'_2 \neq 0$ and by Lemma 3.7 (see statements $(\boldsymbol{A'_1})$ and $(\boldsymbol{B'_1})$) we have one invariant parabola in the direction $x = 0$ if $\mathcal{G}'_1 \neq 0$ and one in the direction $y = 0$ if $\mathcal{G}'_2 \neq 0$. So considering (3.37) we examine two possibilities: $\zeta_9 \neq 0$ and $\zeta_9 = 0$.

2.1.1: *The possibility* $\zeta_9 \neq 0$. This implies $\mathcal{G}'_1\mathcal{G}'_2 \neq 0$ and by Lemma 3.7 in this case we have one invariant parabola in one direction and another invariant parabola in other direction. So the statement $(\boldsymbol{\mathcal{B}_5})$ of Theorem 3.8 is proved.

2.1.2: *The possibility* $\zeta_9 = 0$. Then we have $\mathcal{G}'_1\mathcal{G}'_2 = 0$, i.e. at least one of the factors vanishes. Considering (3.37) we conclude that both factors vanish if and only if $\zeta_{10} = 0$. In this case $\mathcal{G}'_1 = \mathcal{G}'_2 = 0$ and by Lemma 3.7 (see statements $(\boldsymbol{B_1})$ and $(\boldsymbol{B'_1})$) systems (3.25) could not possess any invariant parabolas.

On the other hand in the case $\zeta_{10} \neq 0$ we have $\mathcal{G}'_1 + \mathcal{G}'_2 \neq 0$ and since $\mathcal{G}'_1\mathcal{G}'_2 = 0$, by Lemma 3.7 we have one invariant parabola (either in direction $y = 0$ if $\mathcal{G}'_1 = 0$ or in direction $x = 0$ if $\mathcal{G}'_2 = 0$. This means that the statement $(\boldsymbol{\mathcal{B}_6})$ of Theorem 3.8 is valid.

***2.2:* The subcase** $\zeta_5 = 0$**.** In this case we get $\mathcal{D}'_1\mathcal{D}'_2 = 0$. On the other hand we obtain $\mathcal{D}'_1 + \mathcal{D}'_2 = 10(c + f)$ and hence both $\mathcal{D}'_1$ and $\mathcal{D}'_2$ vanish if and only if $c + f = 0$ and this condition is governed by the invariant polynomial $\zeta_6 = -(c + f)/2$. So we discuss two possibilities: $\zeta_6 \neq 0$ and $\zeta_6 = 0$.

2.2.1: *The possibility* $\zeta_6 \neq 0$. Then only one of the polynomials $\mathcal{D}'_1$ or $\mathcal{D}'_2$ vanishes and due to the change $(x, y, a, b, c, f) \mapsto (y, x, b, a, f, c)$ without losing generality we may assume that for systems (3.25) the condition $\mathcal{D}'_1 = 0$ holds. Considering (3.26) this condition implies $f = 13c/3$ and then we obtain

$$\Omega'_1 = 2(63a - 15b + 35c^2)^2/9 = 0 \;\Rightarrow\; b = 7(9a + 5c^2)/15.$$

Therefore we calculate

$$\Omega'_2 = 8(144a + 5c^2)(2704a + 5c^2)/225, \;\; \zeta_6 = -8c/3 \neq 0$$

and the condition $\Omega'_2 = 0$ gives us either $a = -5c^2/144 \neq 0$ or $a = -5c^2/2704 \neq 0$ (due to $\zeta_6 \neq 0$). In this case we get either

$$\mathcal{F}'_1 = 20c^2/9 \neq 0, \quad \mathcal{G}'_2 = -120c^2 \neq 0$$

if $a = -5c^2/144$ or

$$\mathcal{F}'_1 = 980c^2/1521 \neq 0, \quad \mathcal{G}'_2 = -13720c^2/117 \neq 0$$

if $a = -5c^2/2704$. So considering the statements $(\boldsymbol{B_2})$ and $(\boldsymbol{B'_1})$ of Lemma 3.7 we conclude that systems (3.25) possess two distinct invariant parabolas in the direction $x = 0$ and one invariant parabola in the direction $y = 0$. This means that the statement $(\boldsymbol{\mathcal{B}_7})$ of Theorem 3.8 is valid.

2.2.2: *The possibility* $\zeta_6 = 0$. This condition implies $\mathcal{D}'_1 = \mathcal{D}'_2 = 0$ and considering (3.26) we obtain $c = f = 0$. Then we obtain

$$\Omega'_1 = 2(21a - 5b)^2, \quad \Omega'_2 = 2(5a - 21b)^2$$

and evidently the conditions $\Omega'_1 = \Omega'_2 = 0$ imply $a = b = 0$. Therefore we arrive at the following homogeneous system

$$\dot{x} = x(2x + y), \quad \dot{y} = y(x + 2y)$$

that could not possess any invariant parabola.

Since all the possibilities are examined we conclude that Theorem 3.8 is proved. □

3.2 Systems with one real and two complex infinite singularities

In this case according to Lemma 2.3 systems (2.5) could be brought via a linear transformation to the following family of systems

$$\begin{aligned} \frac{dx}{dt} &= a + cx + dy + gx^2 + (h+1)xy, \\ \frac{dy}{dt} &= b + ex + fy - x^2 + gxy + hy^2. \end{aligned} \tag{3.38}$$

For these systems we calculate

$$C_2(x, y) = x(x^2 + y^2), \;\; \chi_1 = -2(2 + h)\left[g^2 + (h-3)^2\right] \tag{3.39}$$

and by Lemma 2.6 we conclude that the above systems could have invariant parabolas only of the form $\Phi(x, y) = p + qx + ry + x^2$ with $r \neq 0$ (otherwise we get a reducible conic).

On the other hand according to Lemma 2.4 for a system (3.38) to possess an invariant parabola the condition $\chi_1 = 0$ is necessary. Considering (3.39) this condition implies either $h = -2$ or $g = 0 = h - 3$. We claim that in the second case systems (3.38) could not possess any invariant parabola.

Indeed, assuming $g = 0$ and $h = 3$ and using a translation we may assume $c = d = 0$ and we arrive at the family of systems

$$\dot{x} = a + 4xy, \quad \dot{y} = b + ex + fy - x^2 + 3y^2. \tag{3.40}$$

Considering equations (2.6) and the form of the parabola $\Phi(x, y) = p + qx + ry + x^2$ with $r \neq 0$, for systems (3.40) we have

$$s = 1, \ v = u = 0, \ Eq_2 = 8 - V, Eq_7 = r(3 - V).$$

Evidently the conditions $Eq_2 = 0$ and $Eq_7 = 0$ imply $r = 0$, i.e. the conic $\Phi(x, y) = p + qx + ry + x^2$ with $r = 0$ is reducible and this completes the proof of our claim.

For systems (3.38) we calculate

$$\zeta_1 = -2\big[(h-3)(1+h)(2h-1) + g^2(3+2h)\big]$$

and clearly the conditions $g = 0$ and $h = 3$ imply $\zeta_1 = 0$. On the other hand for $h = -2$ we get $\zeta_1 = 2(25 + g^2) \neq 0$ and therefore the condition $h + 2 = 0$ is equivalent to $\chi_1 = 0$ and $\zeta_1 \neq 0$. So we have the next remark.

Remark 3.9. If a system (3.38) possesses an invariant parabola then the conditions $\chi_1 = 0$ and $\zeta_1 \neq 0$ are necessary.

According to this remark we assume that the conditions $\chi_1 = 0$ and $\zeta_1 \neq 0$ are fulfilled for systems (3.38). Then the condition $h = -2$ holds and due to a translation we may consider $c = d = 0$. So we arrive at the family of systems

$$\dot{x} = a + gx^2 - xy, \quad \dot{y} = b + ex + fy - x^2 + gxy - 2y^2. \tag{3.41}$$

3.2.1 Coefficient conditions for systems (3.41) to possess invariant parabolas

We prove the following lemma.

Lemma 3.10. *A system (3.41) possesses either one or two invariant parabolas or a double invariant parabola of the form $\Phi(x, y) = p + qx + ry + x^2$ if and only if $\widetilde{\Omega} = 0$ and the corresponding set of conditions are satisfied, respectively:*

(E_1) $\widetilde{\mathcal{D}} \neq 0, \widetilde{\mathcal{G}} \neq 0 \Rightarrow$ *one invariant parabola;*

(E_2) $\widetilde{\mathcal{D}} = 0, b \neq 0, \widetilde{\mathcal{F}} \neq 0 \Rightarrow$ *two invariant parabolas;*

(E_3) $\widetilde{\mathcal{D}} = 0, b \neq 0, \widetilde{\mathcal{F}} = 0 \Rightarrow$ *one double invariant parabola;*

(E_4) $\widetilde{\mathcal{D}} = 0, b = 0, f \neq 0 \Rightarrow$ *one invariant parabola,*

where

$$\begin{aligned}
&\widetilde{\Omega} = 2a^2(1+3g^2)^2 + a\big[8bg(1+3g^2) - (e-fg)(f+eg+2fg^2)\big] + b(8bg^2 + f^2g^2 - e^2),\\
&\widetilde{\mathcal{D}} = e - fg, \quad \widetilde{\mathcal{G}} = a + 2bg + 3ag^2,\\
&\widetilde{\mathcal{F}} = 608(b+ag)(25+g^2) + 25(49e^2 + 76f^2) - fg(850e + 299fg).
\end{aligned} \tag{3.42}$$

Proof. Considering equations (2.6) and the form of the parabola $\Phi(x,y) = p+qx+ry+x^2$ with $r \neq 0$ for systems (3.41) we obtain

$$s=1,\ v=u=0,\ U=2g,\ V=-2,\ W=-gq-r,$$
$$Eq_1=Eq_2=Eq_3=Eq_4=Eq_5=Eq_7=0,\ \ Eq_6=q-gr.$$

Therefore the condition $Eq_6=0$ gives us $q=gr$ and calculations yield:

$$Eq_9 = 2p+r(f+r+g^2r)=0\ \Rightarrow\ p=-r(f+r+g^2r)/2$$

and then we obtain

$$Eq_8 = 2a+(e+fg)r+2g(1+g^2)r^2,$$
$$Eq_{10} = \frac{r}{2}\left[2(b+ag)-f(1+g^2)r-(1+g^2)^2r^2\right] \equiv \frac{r}{2}\widetilde{\Psi}(a,b,f,g,r).$$

Since $r\neq 0$ the equation $Eq_{10}=0$ is equivalent to $\widetilde{\Psi}=0$.

According to [12, Lemmas 11,12] the equations $Eq_8=0$ and $\widetilde{\Psi}=0$ have a common solution of degree 2 with respect to the parameter r if and only if

$$Res_r^{(0)}\,(Eq_8,\widetilde{\Psi}) = Res_r^{(1)}\,(Eq_8,\widetilde{\Psi})=0$$

where $Res_r^{(1)}$ is the subresultant of order one and $Res_r^{(0)}$ is the subresultant of order zero which coincide with the standard resultant (for detailed definition see [12], formula (19)). We calculate

$$Res_r^{(1)}\,(Eq_8,\widetilde{\Psi}) = (1+g^2)^2(e-fg)\equiv(1+g^2)^2\widetilde{\mathcal{D}},$$
$$Res_r^{(0)}\,(Eq_8,\widetilde{\Psi}) = 2(1+g^2)^2\widetilde{\Omega}.$$

We observe that the subresultant of order one $Res_r^{(1)}\,(Eq_8,\widetilde{\Psi})$ vanishes if and only if $\widetilde{\mathcal{D}}=0$. So we consider two cases: $\widetilde{\mathcal{D}}\neq 0$ and $\widetilde{\mathcal{D}}=0$.

***1:* The case $\widetilde{\mathcal{D}}\neq 0$.** Then the invariant parabola exists if and only if $\widetilde{\Omega}=0$ and therefore we have to examine the solutions of the equation $\widetilde{\Omega}=0$. We calculate

$$\text{Discrim}[\widetilde{\Omega},a] = (e-fg)^2\left[8b(1+g^2)(1+3g^2)+(f+eg+2fg^2)^2\right]\equiv\widetilde{\mathcal{D}}^2\widetilde{\mathcal{E}}$$

and hence the equation $\widetilde{\Omega}=0$ has real solutions in the parameter a if and only if either $\widetilde{\mathcal{D}}=0$ or $\widetilde{\mathcal{E}}\geq 0$. However since the condition $\widetilde{\mathcal{D}}\neq 0$ holds it remains to examine the condition $\widetilde{\mathcal{E}}\geq 0$.

In this case setting $\widetilde{\mathcal{E}}=w^2\geq 0$ we calculate

$$b=-\frac{(f+eg+2fg^2)^2-w^2}{8(1+g^2)(1+3g^2)} \tag{3.43}$$

and then we obtain

$$\widetilde{\Omega}=\frac{G_+G-}{8(1+g^2)^2(1+3g^2)^2},$$

where

$$G_\pm = 4a(1+g^2)(1+3g^2)^2-(f+eg+2fg^2+\varepsilon w)(e+2eg^2+fg^3-\varepsilon gw),\ \ \varepsilon=\pm1.$$

Then the condition $\widetilde{\Omega}=0$ gives us

$$a=\frac{(f+eg+2fg^2+\varepsilon w)(e+2eg^2+fg^3-\varepsilon wg)}{4(1+g^2)(1+3g^2)^2}, \tag{3.44}$$

where $\varepsilon = 1$ if $G_+ = 0$ and $\varepsilon = -1$ if $G_- = 0$. In this case we obtain that the polynomials Eq_8 and $\widetilde{\Psi}$ have the common factor $\zeta = 2(1+g^2)(1+3g^2)r + f + eg + 2fg^2 + \varepsilon w$ which is linear with respect to the parameter r. Setting $\zeta = 0$ we get

$$r = -\frac{f+eg+2fg^2+\varepsilon w}{2(1+g^2)(1+3g^2)}$$

and we arrive at the family of systems

$$\begin{aligned} \dot{x} &= \frac{(f+eg+2fg^2+\varepsilon w)(e+2eg^2+fg^3-g\varepsilon w)}{4(1+g^2)(1+3g^2)^2} + gx^2 - xy, \\ \dot{y} &= -\frac{(f+eg+2fg^2)^2 - w^2}{8(1+g^2)(1+3g^2)} + ex + fy + gxy - 2y^2. \end{aligned} \tag{3.45}$$

This family of systems possess the following invariant parabola

$$\begin{aligned} \Phi(x,y) = &\frac{(f-eg+4fg^2-\varepsilon w)(f+eg+2fg^2+\varepsilon w)}{8(1+g^2)(1+3g^2)^2} - \frac{g(f+eg+2fg^2+\varepsilon w)}{2(1+g^2)(1+3g^2)^2}\,x \\ &- \frac{f+eg+2fg^2+\varepsilon w}{2(1+g^2)(1+3g^2)}\,y + x^2. \end{aligned} \tag{3.46}$$

We observe that this conic is reducible if and only if $f + eg + 2fg^2 + \varepsilon w = 0$.

Considering (3.43) we get

$$w^2 = 8b(1+g^2)(1+3g^2) + (f+eg+2fg^2)^2$$

and then from (3.44) we obtain

$$\begin{aligned} a = \frac{1}{4(1+g^2)(1+3g^2)^2}\big[(f+eg+2fg^2)(e+2eg^2+fg^3) + (e-fg)(1+g^2)\varepsilon w - gw^2\big] \;\Rightarrow \\ 8bg(1+3g^2) + 4a(1+3g^2)^2 - (e-fg)(f+eg+2fg^2) - (e-fg)\varepsilon w = 0. \end{aligned}$$

Since $\widetilde{\mathcal{D}} = (e-fg) \neq 0$ we solve the last equation with respect to εw and we obtain

$$\varepsilon w = \frac{1}{e-fg}\big[4b(1+2g)^2 - 4a(1+g)(1+2g)(1+3g) + (e-fg)(e+f+fg)\big].$$

Then calculations yield

$$r = -\frac{f+eg+2fg^2+\varepsilon w}{2(1+g^2)(1+3g^2)} = -\frac{2(a+2bg+3ag^2)}{(e-fg)(1+g^2)} = -\frac{2\,\widetilde{\mathcal{G}}}{(e-fg)(1+g^2)} \neq 0.$$

This completes the proof of the statement $(\boldsymbol{E_1})$ of Lemma 3.10.

Next we show that systems (3.45) could be brought via a transformation to the canonical form $(S_{\boldsymbol{\beta}})$. Indeed we could apply to parabola (3.46) the translation

$$x = x_1 + \frac{g(f+eg+2fg^2+\varepsilon w)}{4(1+g^2)(1+3g^2)}, \quad y = y_1 + \frac{f(2+9g^2+6g^4) - (2+3g^2)(eg+\varepsilon w)}{8(1+g^2)(1+3g^2)},$$

which brings this parabola to the form $\widetilde{\Phi}(x_1,y_1) = x_1^2 - \dfrac{f+eg+2fg^2+\varepsilon w}{2(1+g^2)(1+3g^2)}\,y_1$.

On the other hand considering Observation 3.4 we apply the same translation to systems (3.45) and we set the following new notations (suggested by the above parabola and the linear parts of the transformed systems):

$$k = \frac{f+eg+2fg^2+\varepsilon w}{2(1+g^2)(1+3g^2)}, \quad n = \frac{g(4e-fg+7eg^2+2fg^3)+(4+7g^2)\varepsilon w}{8(1+g^2)(1+3g^2)},$$
$$m = \frac{(2+3g^2)(4e-fg+7eg^2+2fg^3)-3g(2+g^2)\varepsilon w}{16(1+g^2)(1+3g^2)} \quad \Rightarrow$$
$$e = \frac{gk-2g^3k+4m+2gn}{2}, \quad f = \frac{4k+7g^2k-4n}{2}, \quad w = \frac{2n-2gm+3g^2n}{\varepsilon}.$$

Then after an additional rescaling (to force $k=1$) we arrive at the family of systems (S_β).

2: The case $\widetilde{\mathcal{D}}=0$. Then $e-fg=0$ and we have $e=fg$. Therefore we obtain:

$$\widetilde{\Omega} = 2(a+2bg+3ag^2)^2$$

and the condition $\widetilde{\Omega}=0$ implies

$$a = -\frac{2bg}{1+3g^2}.$$

Therefore we determine that in this case the polynomials Eq_8 and Eq_{10} have the following common factor

$$\tilde{\phi} = 2b - f(1+3g^2)r - (1+g^2)(1+3g^2)r^2.$$

We observe that $\tilde{\phi}$ is quadratic in r with the discriminant

$$\text{Discrim}[\tilde{\phi}, r] = (1+3g^2)(8b+f^2+8bg^2+3f^2g^2)$$

and clearly the condition $(8b+f^2+8bg^2+3f^2g^2)\geq 0$ must hold. Setting

$$8b+f^2+8bg^2+3f^2g^2 = (1+3g^2)w^2 \geq 0,$$

we obtain

$$b = -\frac{(1+3g^2)(f^2-w^2)}{8(1+g^2)}. \tag{3.47}$$

Then we arrive at the following expressions for the polynomials Eq_8 and Eq_{10}:

$$Eq_8 = \frac{gM_+M_-}{2(1+g^2)}, \quad Eq_{10} = -\frac{rM_+M_-}{8}, \quad M_\pm = f+2r+2g^2r\pm w.$$

Therefore the equations $Eq_8 = Eq_{10} = 0$ imply $M_+M_- = 0$.

If $M_+=0$ we determine

$$r = -\frac{f+w}{2(1+g^2)} \equiv r^+$$

and we obtain the parabola

$$\Phi_1(x,y) = \frac{(f-w)(f+w)}{8\,(g^2+1)} - \frac{g(f+w)}{2\,(g^2+1)}\,x - \frac{f+w}{2\,(g^2+1)}\,y + x^2. \tag{3.48}$$

In the case $M_-=0$ we obtain

$$r = -\frac{f-w}{2(1+g^2)} \equiv r^-$$

and we get the parabola

$$\Phi_2(x,y) = \frac{(f-w)(f+w)}{8\,(g^2+1)} - \frac{g(f-w)}{2\,(g^2+1)}\,x - \frac{f-w}{2\,(g^2+1)}\,y + x^2.$$

Both these parabolas are invariant for the following family of systems:

$$\begin{aligned} \dot{x} &= \frac{g(f^2-w^2)}{4(1+g^2)} + gx^2 - xy, \\ \dot{y} &= -\frac{(1+3g^2)(f^2-w^2)}{8(1+g^2)} + fgx + fy + gxy - 2y^2. \end{aligned} \tag{3.49}$$

We observe that both parabolas $\Phi_i(x,y)=0$ $(i=1,2)$ exist (i.e. are not reducible) if and only if $r^+r^- \neq 0$ and this is equivalent to

$$(f-w)(f+w) = f^2 - w^2$$

and considering (3.47) this is equivalent to $b \neq 0$.

On the other hand if only one of the factors vanishes we have $b = 0$ and

$$r^+ + r^- = (f-w) + (f+w) = 2f \neq 0$$

i.e. $f \neq 0$. Therefore for $b = 0$ and $f \neq 0$ we could have only one parabola.

We determine that in the case $w = 0$ we obtain $\Phi_1(x,y) = \Phi_2(x,y)$, i.e. the parabolas coalesced when w tends to zero and we obtain a double parabola. On the other hand considering (3.47) for $w = 0$ we obtain to

$$b + \frac{f^2(1+3g^2)}{8(1+g^2)} = \frac{8b(1+g^2)+f^2(1+3g^2)}{8(1+g^2)} = \frac{(1+3g^2)}{608(25+g^2)(1+g^2)}\,\widetilde{\mathcal{F}}$$

and we conclude that these invariant parabolas coalesce if and only if $\widetilde{\mathcal{F}} = 0$. So the statements $(\boldsymbol{E_2})$–$(\boldsymbol{E_4})$ of Lemma 3.10 are valid.

As all the cases are examined we conclude that Lemma 3.10 is proved. $\square$

Next we show that systems (3.49) could be brought via a transformation to the canonical form $(S_{\boldsymbol{\beta}})$. Indeed we could apply to parabola (3.48) the translation

$$x = x_1 + \frac{g(f+w)}{4(1+g^2)}, \quad y = y_1 + \frac{f(2+g^2)-(2+3g^2)w}{8(1+g^2)}.$$

which brings this parabola to the form $\widetilde{\Phi}(x_1,y_1) = x_1^2 - \dfrac{f+w}{2(1+g^2)}\,y_1$.

On the other hand considering Observation 3.4 we apply the same translation to systems (3.49) and we set the following new notations (suggested by the above parabola and the linear parts of the transformed systems):

$$k = \frac{f+w}{2(1+g^2)}, \quad n = \frac{3fg^2+4w+7g^2w}{8(1+g^2)} \quad \Rightarrow \quad f = \frac{4k+7g^2k-4n}{2}, \quad w = \frac{4n-3g^2k}{2}.$$

Then after an additional rescaling (to force $k = 1$) we arrive at the subfamily of systems $(S_{\boldsymbol{\beta}})$ defined by the conditions $m = 3g(1+3g^2-2n)/4$.

3.2.2 Invariant conditions: the case $\eta < 0$

Next using Lemma 3.10 we shall construct the equivalent affine invariant conditions for a system with $\eta < 0$ to possess an invariant parabola.

We prove the following theorem.

Theorem 3.11. *Assume that for a non-degenerate arbitrary quadratic system the conditions $\eta < 0$, $\chi_1 = 0$ and $\zeta_1 \neq 0$ are satisfied. Then this system could possess invariant parabolas only in one (real) direction. More exactly it could only possess one of the following sets of invariant parabolas: $\cup$, $\Cup$ and $\bigcup^2$. Moreover this system has one of the above sets of invariant parabolas if and only if $\chi_2 = 0$ and one of the following sets of conditions are satisfied, correspondingly:*

$(\mathcal{E}_1)$ $\zeta_4 \neq 0$, $\mathcal{R}_1 \neq 0$ $\Rightarrow \cup$;
$(\mathcal{E}_2)$ $\zeta_4 = 0$, $\mathcal{R}_7 \neq 0$, $\zeta_5 \neq 0$ $\Rightarrow \Cup$;
$(\mathcal{E}_3)$ $\zeta_4 = 0$, $\mathcal{R}_7 \neq 0$, $\zeta_5 = 0$ $\Rightarrow \bigcup^2$;
$(\mathcal{E}_4)$ $\zeta_4 = 0$, $\mathcal{R}_7 = 0$, $\zeta_5 \neq 0$ $\Rightarrow \cup$.

Proof. According to Remark 3.9 for a system (3.38) to possess an invariant parabola the conditions $\chi_1 = 0$ and $\zeta_1 \neq 0$ are necessary. As it was shown earlier (see page 37) if for a quadratic system with one real and two complex infinite singularities the conditions $\chi_1 = 0$ and $\zeta_1 \neq 0$ are satisfied, then via an affine transformation and time rescaling this system can be brought to the form (3.41). Thus in what follows we consider the family of quadratic systems

$$\dot{x} = a + gx^2 - xy, \quad \dot{y} = b + ex + fy - x^2 + gxy - 2y^2, \tag{3.50}$$

for which considering (3.42) we calculate.

$$\begin{aligned} \chi_1 = 0, \ \ \zeta_1 = 2(25+g^2), \ \ \chi_2 = 384(25+g^2)\,\widetilde{\Omega}, \\ \zeta_4 = -(25+g^2)\,\widetilde{\mathcal{D}}/8, \ \ \mathcal{R}_1 = -30(25+g^2)\,\widetilde{\mathcal{G}}. \end{aligned} \tag{3.51}$$

Evidently the condition $\chi_2 = 0$ is equivalent to $\widetilde{\Omega} = 0$ and we consider two cases: $\zeta_4 \neq 0$ and $\zeta_4 = 0$.

1: **The case $\zeta_4 \neq 0$.** Then we have $\widetilde{\mathcal{D}} \neq 0$ and according to Lemma 3.10 in this case a quadratic system possesses an invariant parabola if and only if the condition $\widetilde{\mathcal{G}} \neq 0$ holds. According to (3.51) this condition is governed by the invariant polynomial $\mathcal{R}_1$. So we conclude that the statement $(\mathcal{E}_1)$ of Theorem 3.11 is valid.

2: **The case $\zeta_4 = 0$.** This implies $\widetilde{\mathcal{D}} = 0$ and considering (3.42) we get $e = fg$. Then for systems (3.50) we calculate

$$\chi_2 = 768(25+g^2)(a+2bg+3ag^2)^2 = 0 \ \Rightarrow \ a = -\frac{2bg}{1+3g^2}$$

and in this case we obtain:

$$\zeta_5 = \widetilde{\mathcal{F}}/4, \ \ \mathcal{R}_7 = -64480b.$$

We examine two possibilities: $\mathcal{R}_7 \neq 0$ and $\mathcal{R}_7 = 0$.

2.1: **The possibility $\mathcal{R}_7 \neq 0$.** In this case we get $b \neq 0$. We observe that the condition $\zeta_5 = 0$ is equivalent to $\widetilde{\mathcal{F}} = 0$ and according to Lemma 3.10 due to $b \neq 0$ we get two invariant parabolas for $\zeta_5 \neq 0$ and one double invariant parabola if $\zeta_5 = 0$.

Thus the statements $(\mathcal{E}_2)$ and $(\mathcal{E}_3)$ of Theorem 3.11 are valid.

2.2: **The possibility** $\mathcal{R}_7 = 0$. This implies $b = 0$ and for systems (3.50) with $e = fg$ we calculate

$$\chi_2 = 768a^2(25 + g^2)(1 + 3g^2)^2, \quad \zeta_5 = 19(f^2 + 8ag)(25 + g^2).$$

Therefore the condition $\chi_2 = 0$ gives us $a = 0$ and then we obtain $\zeta_5 = 19f^2(25 + g^2)$. So the condition $f \neq 0$ is equivalent to $\zeta_5 \neq 0$ and considering the statement $(\boldsymbol{E}_4)$ of Lemma 3.10 we conclude that the statement $(\mathcal{E}_4)$ of Theorem 3.11 is valid and this completes the proof of this theorem. □

3.3 Systems with two real distinct infinite singularities

In this case, according to Lemma 2.3, the conditions $\eta = 0$ and $\widetilde{M} \neq 0$ hold and systems (2.5) could be brought via a linear transformation and the additional change $(x, y, a, b, c, d, e, f, g, h) \mapsto (y, x, b, a, f, e, d, c, h, g)$ to the following family of systems

$$\begin{aligned} \frac{dx}{dt} &= a + cx + dy + gx^2 + (h-1)xy, \\ \frac{dy}{dt} &= b + ex + fy + gxy + hy^2. \end{aligned} \tag{3.52}$$

For these systems we calculate

$$C_2(x, y) = -xy^2, \quad \chi_1 = 2g^2(h-2)$$

and by Lemma 2.6 we conclude that the above systems could have invariant parabolas either of the form $\Phi(x, y) = p + qx + ry + x^2$ with $r \neq 0$ (otherwise we get a reducible conic) or of the form $\Phi(x, y) = p + qx + ry + y^2$ with $q \neq 0$.

According to Lemma 2.4 for the existence of an invariant parabola for a system (3.52) the condition $\chi_1 = 0$ is necessary, i.e. $g(h-2) = 0$. We prove the following lemma.

Lemma 3.12. *Assume that a system (3.52) possesses an invariant parabola. Then its quadratic homogeneous part is of the form x^2 (respectively, y^2) only if the condition $h = 2$ (respectively, $g = 0$) holds.*

Proof. Assume that a system (3.52) possesses an invariant parabola of the form $\Phi(x, y) = p + qx + ry + x^2$ with $r \neq 0$ (otherwise we get a reducible conic). Then considering equations (2.6) we obtain

$$s = 1, \quad v = u = 0, \quad Eq_2 = -2 + 2h - V = 0 \ \Rightarrow \ V = 2(h-1).$$

Therefore we have $Eq_7 = -(h-2)r = 0$ and since $r \neq 0$ this implies $h = 2$. So the statement of the lemma is true in this case.

If the system possesses an invariant parabola of the form $\Phi(x, y) = p + qx + ry + y^2$ with $q \neq 0$ then considering equations (2.6) we obtain

$$s = v = 0, \quad u = 1, \quad Eq_3 = 2g - U = 0 \ \Rightarrow \ U = 2g.$$

In this case we obtain $Eq_5 = -gq = 0$ and due to $q \neq 0$ we get $g = 0$. This completes the proof of the lemma. □

Considering Lemma 3.12 we conclude that for determining the conditions for the existence and the number of invariant parabolas for systems (3.52) it is necessary and sufficient to examine the two possibilities: the existence of invariant parabolas of the form $\Phi(x,y) = p+qx+ry+x^2$ ($r \neq 0$) and of the form $\Phi(x,y) = p+qx+ry+y^2$ ($q \neq 0$). By Lemma 3.12 in the first case the condition $h = 2$ holds whereas in the second we have $g = 0$.

Taking into account that for systems (3.52) we have

$$\chi_1 = 2g^2(h-2), \quad \mu_0 = g^2 h$$

we conclude that the case $h-2 = 0$ is equivalent to $\chi_1 = 0$ and $\mu_0 \neq 0$ whereas the case $g = 0$ is equivalent to $\chi_1 = \mu_0 = 0$. In what follows we examine each one of this two possibilities.

3.3.1 The possibility $\chi_1 = 0$ and $\mu_0 \neq 0$

Then we have $g \neq 0$ and $h = 2$ and by Lemma 3.12 systems (3.52) could have invariant parabolas only of the form $\Phi(x,y) = p+qx+ry+x^2$ with $r \neq 0$. Applying the transformation $(x,y) \mapsto (x/g-d,\ y-c+2dg)$ we impose the conditions $g = 1$ and $c = d = 0$ to be fulfilled and we arrive at the family of systems

$$\dot{x} = a + x^2 + xy, \quad \dot{y} = b + ex + fy + xy + 2y^2. \tag{3.53}$$

Coefficient conditions for systems (3.53) to possess invariant parabolas. We prove the following lemma.

Lemma 3.13. *A system (3.53) possesses either one or two invariant parabolas or a double invariant parabola of the form $\Phi(x,y) = p+qx+ry+x^2$ ($r \neq 0$) if and only if $\Upsilon_1 = 0$ and the corresponding set of conditions are satisfied, respectively:*

$(\boldsymbol{H_1})$ $\mathfrak{D}_1 \neq 0,\ \mathfrak{G}_1 \neq 0 \Rightarrow \cup$;

$(\boldsymbol{H_2})$ $\mathfrak{D}_1 = 0,\ a \neq 0,\ \mathfrak{F}_1 \neq 0 \Rightarrow \Cup$;

$(\boldsymbol{H_3})$ $\mathfrak{D}_1 = 0,\ a \neq 0,\ \mathfrak{F}_1 = 0 \Rightarrow \cup^2$;

$(\boldsymbol{H_4})$ $\mathfrak{D}_1 = 0,\ a = 0,\ e \neq 0 \Rightarrow \cup$.

where

$$\begin{aligned} &\Upsilon_1 = 8b^2 - b(24a - e^2 + f^2) + a(18a - e^2 + ef + 2f^2);\\ &\mathfrak{D}_1 = e + f;\quad \mathfrak{G}_1 = 3a - 2b;\quad \mathfrak{F}_1 = 4a - e^2. \end{aligned} \tag{3.54}$$

Proof. Considering the equations (2.6) and the form of invariant parabola $\Phi(x,y) = p+qx+ry+x^2$ with $r \neq 0$ for systems (3.53) we obtain

$$\begin{gathered} s = 1,\ v = u = 0,\ U = 2,\ V = 2,\ W = -q,\\ Eq_1 = Eq_2 = Eq_3 = Eq_4 = Eq_5 = Eq_7 = 0. \end{gathered}$$

Then we have

$$Eq_6 = -q - r = 0,\ \ Eq_9 = -2p + fr + qr = 0 \ \Rightarrow\ q = -r,\ p = r(f-r)/2$$

and calculations yield

$$Eq_8 = 2a + (e-f)r + 2r^2,\ \ Eq_{10} = -\frac{r}{2}\big[2(a-b) + fr - r^2)\big] \equiv -\frac{r}{2}\Psi_1(a,b,f,r).$$

Since $r \neq 0$ the equation $Eq_{10} = 0$ is equivalent to $\Psi_1 = 0$.

According to [12, Lemmas 11,12] the equations $Eq_8 = 0$ and $\Psi_1 = 0$ have a common solution of degree 2 with respect to the parameter r if and only if

$$Res_r^{(0)}(Eq_8, \Psi_1) = Res_r^{(1)}(Eq_8, \Psi_1) = 0$$

where $Res_r^{(1)}$ is the subresultant of order one and $Res_r^{(0)}$ is the subresultant of order zero which coincide with the standard resultant (for detailed definition see [12], formula (19)). We calculate

$$Res_r^{(1)}(Eq_8, \Psi_1) = (e+f) \equiv \mathfrak{D}_1, \quad Res_r^{(0)}(Eq_8, \Psi_1) = 2\Upsilon_1.$$

So we consider two possibilities: $\mathfrak{D}_1 \neq 0$ and $\mathfrak{D}_1 = 0$.

***1:* The possibility $\mathfrak{D}_1 \neq 0$.** Then the invariant parabola exists if and only if $\Upsilon_1 = 0$ and therefore we have to examine the solutions of the equation $\Upsilon_1 = 0$. In this case we calculate ??

$$\mathrm{Discrim}[\Upsilon_1, b] = -(e+f)^2(16a - e^2 + 2ef - f^2) \equiv -\mathfrak{D}_1^2\mathcal{E}$$

and hence due to $\mathfrak{D}_1 \neq 0$ the equation $\Upsilon_1 = 0$ has real solutions with respect to the parameter b if and only if $\mathcal{E} \leq 0$. Then setting $\mathcal{E} = -w^2 \geq 0$ we calculate

$$a = \frac{(e-f)^2 - w^2}{16} \tag{3.55}$$

and then we obtain:

$$\Upsilon_1 = \frac{N_+N_-}{128},$$

where

$$N_\pm = 32b - e^2 + 6ef - 5f^2 + 2(e+f)\varepsilon w + 3w^2, \quad \varepsilon = \pm 1.$$

Then the condition $\Upsilon_1 = 0$ gives us

$$b = \frac{1}{32}(e - 5f - 3\varepsilon w)(e - f + \varepsilon w) \tag{3.56}$$

where $\varepsilon = 1$ if $N_+ = 0$ and $\varepsilon = -1$ if $N_- = 0$. In this case we obtain that the polynomials Eq_8 and $\Psi_1(e, f, g, r)$ have the common factor $\zeta = e - f + 4r + \varepsilon w$ which is linear with respect to the parameter r. Setting $\zeta = 0$ we get

$$r = -\frac{e - f + \varepsilon w}{4}$$

and we arrive at the family of systems

$$\begin{aligned} \dot{x} &= \frac{(e-f)^2 - w^2}{16} + x^2 + xy, \\ \dot{y} &= \frac{1}{32}(e - 5f - 3\varepsilon w)(e - f + \varepsilon w) + ex + fy + xy + 2y^2. \end{aligned} \tag{3.57}$$

This family of systems possess the following invariant parabola

$$\Phi(x,y) = -\frac{(e - f + \varepsilon w)(e + 3f + \varepsilon w)}{32} + \frac{e - f + \varepsilon w}{4}x - \frac{e - f + \varepsilon w}{4}y + x^2. \tag{3.58}$$

We observe that this conic is reducible if and only if $e - f + \varepsilon w = 0$.

Considering (3.55) we get
$$w^2 = -16a + (e-f)^2$$
and then we obtain
$$b = \frac{1}{32}\big[(e-5f)(e-f) - 2(e+f)\varepsilon w - 3w^2\big] \;\Rightarrow$$
$$16b - 24a + (e+f)(e-f+\varepsilon w) = 0.$$
Since $\mathfrak{D}_1 = e+f \neq 0$ we solve the last equation with respect to εw and we obtain
$$\varepsilon w = \frac{1}{e+f}\big(24a - 16b - e^2 + f^2\big).$$
Then calculations yield
$$r = -\frac{e-f+\varepsilon w}{4} = -\frac{2(3a-2b)}{e+f} = -\frac{2\,\mathfrak{G}_1}{e+f} \neq 0.$$
This completes the proof of the statement $\boldsymbol{(H_1)}$ of Lemma 3.13.

Next we show that systems (3.57) could be brought via a transformation to the canonical form $(S^1_{\boldsymbol{\gamma}})$. Indeed we could apply to parabola (3.58) the translation
$$x = x_1 - \frac{e-f+\varepsilon w}{8}, \quad y = y_1 - \frac{3e+5f+3\varepsilon w}{16},$$
which brings this parabola to the form $\widetilde{\Phi}(x_1, y_1) = x_1^2 - \dfrac{e-f+\varepsilon w}{4}\, y_1$.

On the other hand considering Observation 3.4 we apply the same translation to systems (3.57) and we set the following new notations (suggested by the above parabola and the linear parts of the transformed systems):
$$k = \frac{e-f+\varepsilon w}{4}, \quad n = -\frac{7e+f+7\varepsilon w}{16}, \quad m = \frac{13e-5f-3\varepsilon w}{32} \quad \Rightarrow$$
$$e = -k+2m-n, \quad f = -\frac{7k+4n}{2}, \quad w = \frac{3k-4m-2n}{2\varepsilon}.$$
Then after an additional rescaling (to force $k=1$) we arrive at the subfamily of systems $(S^1_{\boldsymbol{\gamma}})$ defined by the conditions $g=1$.

***2:* The possibility** $\mathfrak{D}_1 = 0$**.** Considering (3.54) we have $f=-e$ and then we calculate
$$\Upsilon_1 = 2(3a-2b)^2 = 0 \;\Rightarrow\; b = 3a/2.$$
Therefore we determine that in this case the polynomials Eq_8 and Eq_{10} have the following common factor
$$\phi_1 = a + er + r^2.$$
We observe that $\tilde{\phi}$ is quadratic in r with the discriminant Discrim$[\phi_1, r] = -4a+e^2$ and setting Discrim$[\phi_1, r] = w^2$ we obtain
$$a = \frac{e^2-w^2}{4}. \tag{3.59}$$
Then we arrive at the following expressions for the polynomials Eq_8 and Eq_{10}:
$$Eq_8 = \frac{S_+S_-}{2}, \quad Eq_{10} = r\frac{S_+S_-}{8}, \quad S_\pm = (e+2r\pm w).$$

Therefore the equations $Eq_8 = Eq_{10} = 0$ imply $S_+S_- = 0$. If $S_+ = 0$ we determine

$$r = -\frac{e+w}{2} \equiv r^+$$

and we obtain the parabola

$$\Phi_1(x,y) = \frac{e^2-w^2}{8} + \frac{e+w}{2}x - \frac{e+w}{2}y + x^2. \tag{3.60}$$

In the case $S_- = 0$ we obtain

$$r = -\frac{e-w}{2} \equiv r^-$$

and we get the parabola

$$\Phi_2(x,y) = \frac{e^2-w^2}{8} + \frac{e-w}{2}x - \frac{e-w}{2}y + x^2.$$

Both these parabolas are invariant for the following family of systems:

$$\dot{x} = \frac{e^2-w^2}{4} + x^2 + xy, \quad \dot{y} = \frac{3(e^2-w^2)}{8} + ex - ey + xy + 2y^2. \tag{3.61}$$

We observe that both parabolas $\Phi_i(x,y) = 0$ $(i = 1,2)$ exist (i.e. are not reducible) if and only if $r^+r^- \neq 0$ and this is equivalent to

$$(e+w)(e-w) = e^2 - w^2 \neq 0.$$

Considering (3.59) this is equivalent to $a \neq 0$.

On the other hand if only one of the factors vanishes we have $a = 0$ and

$$r^+ + r^- = (e+w) + (e-w) = 2e \neq 0$$

and we obtain that the above condition is equivalent to $e \neq 0$. Therefore for $a = 0$ and $e \neq 0$ we could have only one parabola and we have no parabolas for $a = e = 0$.

We determine that in the case $w = 0$ we obtain $\Phi_1(x,y) = \Phi_2(x,y)$, i.e. the parabolas coalesce when w tends to zero and we obtain a double parabola. On the other hand considering (3.59) for $w = 0$ we obtain

$$a - \frac{e^2-w^2}{4} = \frac{4a-e^2}{4} = \frac{\mathfrak{F}_1}{4}$$

and we conclude that these invariant parabolas coalesce if and only if $\mathfrak{F}_1 = 0$.

Thus we conclude that the statements $(\boldsymbol{H_2})$,$(\boldsymbol{H_3})$ and $(\boldsymbol{H_4})$ of Lemma 3.13 are proved. □

Next we show that systems (3.61) could be brought via a transformation to the canonical form (S^1_γ). Indeed we could apply to parabola (3.60) the translation

$$x = x_1 - (e+w)/4, \quad y = y_1 + (e-3w)/8,$$

which brings this parabola to the form $\widetilde{\Phi}(x_1,y_1) = x_1^2 - \dfrac{e+w}{2}y_1$.

On the other hand considering Observation 3.4 we apply the same translation to systems (3.61) and we set the following new notations (suggested by the above parabola and the linear parts of the transformed systems):

$$k = (e+w)/2, \;\; n = -(3e+7w)/8 \;\;\Rightarrow\;\; e = (7k+4n)/2, \;\; w = -(3k+4n)/2.$$

Then after an additional rescaling (to force $k = 1$) we arrive at the subfamily of systems (S^1_γ) defined by the conditions $g = 1$ and $m = 3(3+2n)/4$.

Invariant conditions: the case $\eta = 0$, $\widetilde{M} \neq 0$ and $\mu_0 \neq 0$. Next using Lemma 3.13 we shall construct the equivalent affine invariant conditions for a system with $\eta = 0$, $\widetilde{M} \neq 0$ and $\mu_0 \neq 0$ to possess an invariant parabola.

We prove the following theorem.

Theorem 3.14. *Assume that for a non-degenerate arbitrary quadratic system the conditions $\eta = 0$, $\widetilde{M} \neq 0$, $\chi_1 = 0$ and $\mu_0 \neq 0$ are satisfied. Then this system could possess invariant parabolas only in one (simple) direction. More exactly it could only possess one of the following sets of invariant parabolas: $\cup$, $\Cup$ and $\bigcup^2$. Moreover this system has one of the above sets of invariant parabolas if and only if $\chi_2 = 0$ and one of the following sets of conditions are satisfied, correspondingly:*

$(\mathcal{H}_1)$ $\zeta_4 \neq 0$, $\mathcal{R}_1 \neq 0$ $\Rightarrow \cup$;

$(\mathcal{H}_2)$ $\zeta_4 = 0$, $\mathcal{R}_2 \neq 0$, $\zeta_5 \neq 0$ $\Rightarrow \Cup$;

$(\mathcal{H}_3)$ $\zeta_4 = 0$, $\mathcal{R}_2 \neq 0$, $\zeta_5 = 0$ $\Rightarrow \bigcup^2$;

$(\mathcal{H}_4)$ $\zeta_4 = 0$, $\mathcal{R}_2 = 0$, $\zeta_5 \neq 0$ $\Rightarrow \cup$.

Proof. Assume that quadratic system the conditions $\eta = 0$ and $\widetilde{M} \neq 0$ are fulfilled. Then via a linear transformation this system can be brought to the canonical form (3.52). According to Lemma 2.4 for a system (3.52) to possess an invariant parabola the conditions $\chi_1 = \chi_2 = 0$ are necessary. Moreover it was shown earlier (see page 44) that a system (3.52) with $\chi_1 = 0$ and $\mu_0 \neq 0$ via an affine transformation ant time rescaling can be brought to the form (3.53). Thus in what follows we consider the family of quadratic systems

$$\dot{x} = a + x^2 + xy, \quad \dot{y} = b + ex + fy + xy + 2y^2. \tag{3.62}$$

Considering (3.54) for these systems we calculate.

$$\chi_1 = 0, \quad \chi_2 = 384\mathrm{Y}_1, \quad \zeta_4 = -\mathfrak{D}_1/8, \quad \mathcal{R}_1 = 30\mathfrak{G}_1. \tag{3.63}$$

Evidently the condition $\chi_2 = 0$ is equivalent to $\mathrm{Y}_1 = 0$ and we consider two cases: $\zeta_4 \neq 0$ and $\zeta_4 = 0$.

1: **The case $\zeta_4 \neq 0$.** Then we have $\mathfrak{D}_1 \neq 0$ and according to Lemma 3.13 in this case a quadratic system possesses an invariant parabola if and only if the condition $\mathfrak{G}_1 \neq 0$ holds. According to (3.63) this condition is governed by the invariant polynomial $\mathcal{R}_1$. So we conclude that the statement $(\mathcal{H}_1)$ of Theorem 3.14 is valid.

2: **The case $\zeta_4 = 0$.** This implies $\mathfrak{D}_1 = 0$ and considering (3.54) we get $f = -e$. Then for systems (3.62) we calculate

$$\chi_2 = 768(3a - 2b)^2 = 0 \;\Rightarrow\; b = 3a/2$$

and in this case we obtain:

$$\zeta_5 = -19(4a - e^2) = -19\mathfrak{F}_1, \quad \mathcal{R}_2 = -27a/2.$$

We examine two possibilities: $\mathcal{R}_2 \neq 0$ and $\mathcal{R}_2 = 0$.

2.1: **The subcase $\mathcal{R}_2 \neq 0$.** In this case we get $a \neq 0$. We observe that the condition $\zeta_5 = 0$ is equivalent to $\mathfrak{F}_1 = 0$ and according to Lemma 3.13 due to $b \neq 0$ (because $a \neq 0$) we get two invariant parabolas for $\zeta_5 \neq 0$ and one double invariant parabola if $\zeta_5 = 0$.

Thus the statements $(\mathcal{H}_2)$ and $(\mathcal{H}_3)$ of Theorem 3.14 are valid.

2.2: **The subcase** $\mathcal{R}_2 = 0$**.** This implies $a = 0$ and for systems (3.62) with $f = -e$ and $a = b = 0$ we calculate

$$\zeta_5 = 19e^2.$$

So the condition $e \neq 0$ is equivalent to $\zeta_5 \neq 0$ and considering Lemma 3.13 we conclude that the statement $(\boldsymbol{\mathcal{H}_4})$ of Theorem 3.14 is valid. This completes the proof of Theorem 3.14. □

3.3.2 The possibility $\chi_1 = \mu_0 = 0$

In this case considering the conditions $\chi_1 = 2g^2(h-2) = 0$ and $\mu_0 = g^2 h$ for systems (3.52) we obtain $g = 0$ and by Lemma 3.12 these systems could have invariant parabolas of the form $\Phi(x,y) = p + qx + ry + y^2$ with $q \neq 0$ and if in addition $h = 2$ then they could have invariant parabolas of the form $\Phi(x,y) = p + qx + ry + x^2$ with $r \neq 0$.

So we consider the family of systems

$$\dot{x} = a + cx + dy + (h-1)xy, \quad \dot{y} = b + ex + fy + hy^2. \tag{3.64}$$

Coefficient conditions for systems (3.64) to possess invariant parabolas. We prove the following lemma.

Lemma 3.15. *A system* (3.64) *could only possess one of the following sets of invariant parabolas:* $\cup$, $\overset{2}{\cup}$, $\overset{2}{\cup}\subset$, $\overset{2}{\cup}\overset{2}{\mathbb{W}}$, $\overset{2}{\cup}\overset{2}{\mathbb{W}}^c$, $\overset{2}{\cup}\overset{2}{\boldsymbol{\cup}}^2$, $\overset{2}{\boldsymbol{\cup}}^3$ *and* $\infty\overset{2}{\cup}$*. Moreover this system has one of the above sets of invariant parabolas if and only if the corresponding set of conditions are satisfied, respectively:*

$(\boldsymbol{K_1})$ $h+1 \neq 0,\ 3h+1 \neq 0,\ h \neq 0,\ h-2 \neq 0,\ \Upsilon_2 = 0,\ e \neq 0 \Rightarrow \overset{2}{\cup}$;

$(\boldsymbol{K_2})$ $h = 2,\ \Upsilon_3 \neq 0,\ \Upsilon_2 = 0,\ e \neq 0 \Rightarrow \overset{2}{\cup}$;

$(\boldsymbol{K_3})$ $h = 2,\ \Upsilon_3 = 0,\ \Upsilon_2 \neq 0,\ e(a-cd) \neq 0 \Rightarrow \cup$;

$(\boldsymbol{K_4})$ $h = 2,\ \Upsilon_3 = 0,\ \Upsilon_2 = 0,\ e(4c-f) \neq 0 \Rightarrow \overset{2}{\cup}\subset$;

$(\boldsymbol{K_5})$ $h = 0,\ \Upsilon_2 = 0,\ ef \neq 0 \Rightarrow \overset{2}{\cup}$;

$(\boldsymbol{K_6})$ $h = -1/3,\ c = 2f,\ \mathfrak{D}_2 > 0,\ e \neq 0 \ \Rightarrow \overset{2}{\cup}\overset{2}{\mathbb{W}}$;

$(\boldsymbol{K_7})$ $h = -1/3,\ c = 2f,\ \mathfrak{D}_2 < 0,\ e \neq 0 \ \Rightarrow \overset{2}{\cup}\overset{2}{\mathbb{W}}^c$;

$(\boldsymbol{K_8})$ $h = -1/3,\ c = 2f,\ \mathfrak{D}_2 = 0,\ \mathfrak{F}_2 \neq 0,\ e \neq 0 \ \Rightarrow \overset{2}{\cup}\overset{2}{\boldsymbol{\cup}}^2$;

$(\boldsymbol{K_9})$ $h = -1/3,\ c = 2f,\ \mathfrak{D}_2 = 0,\ \mathfrak{F}_2 = 0,\ e \neq 0 \ \Rightarrow \overset{2}{\boldsymbol{\cup}}^3$;

$(\boldsymbol{K_{10}})$ $h = -1,\ e = 0,\ \mathfrak{G}_2 \neq 0,\ \mathfrak{H}_2 \neq 0,\ c-f \neq 0 \ \Rightarrow \overset{2}{\cup}$;

$(\boldsymbol{K_{11}})$ $h = -1,\ e = 0,\ \mathfrak{G}_2 = 0,\ \mathfrak{H}_2 \neq 0,\ c-f = 0 \ \Rightarrow \infty\overset{2}{\cup}$,

where

$$\begin{aligned}
\Upsilon_2 &= aeh(1+3h)^3 - b(1+h)(1+3h)^2(f+2ch-fh) - (f+ch+fh)\big[2c^2h(1+h)\\
&\qquad - cf(1+h)(5h-1) + de - 2f^2 + 6deh + 9deh^2 + 2f^2h^2\big];\ \ \Upsilon_3 = b + 2c^2 - de - cf;\\
\mathfrak{D}_2 &= 256b^3 + 576b^2(de+f^2) + 432b(de+f^2)^2 - 324a^2e^2 - 972ade^2f\\
&\qquad + 27(de-2f^2)^2(4de+f^2);\ \ \mathfrak{F}_2 = 4b + 3de + 3f^2;\ \ \mathfrak{G}_2 = 2a + cd,\ \ \mathfrak{H}_2 = 4b - c^2 + 2cf.
\end{aligned} \tag{3.65}$$

Proof. Considering the equations (2.6) and the form of invariant parabola $\Phi(x,y) = p + qx + ry + y^2$ with $q \neq 0$ we obtain

$$\begin{gathered} s = v = 0,\ u = 1,\ U = 0,\ V = 2h,\ W = 2f - hr, \\ Eq_1 = Eq_2 = Eq_3 = Eq_4 = Eq_5 = Eq_7 = 0,\ Eq_6 = 2e - (h+1)q. \end{gathered} \tag{3.66}$$

So we have to consider two possibilities: $h + 1 \neq 0$ and $h + 1 = 0$.

***1:* The possibility $h + 1 \neq 0$.** Then equation $Eq_6 = 0$ gives us $q = \dfrac{2e}{1+h}$ and since $q \neq 0$ we get $e \neq 0$. Therefore we calculate:

$$Eq_8 = \frac{e\big[2c - 4f + (1+3h)r\big]}{1+h}$$

and we have to examine two cases: $1 + 3h \neq 0$ and $1 + 3h = 0$.

***1.1:* The case $1 + 3h \neq 0$.** Then due to $e \neq 0$ the equation $Eq_8 = 0$ implies $r = -\dfrac{2(c-2f)}{1+3h}$ and calculations yield:

$$\begin{aligned} Eq_9 = {} & \frac{2}{(1+h)(1+3h)^2}\big[de(1+3h)^2 + b(1+h)(1+3h)^2 + (c-2f)(1+h)(f+2ch-fh) \\ & - h(1+h)(1+3h)^2 p\big], \\ Eq_{10} = {} & \frac{2}{(1+h)(1+3h)}\big[ae(1+3h) - b(c-2f)(1+h) - (1+h)(f+ch+fh)p\big]. \end{aligned} \tag{3.67}$$

We observe that both equations are linear with respect to parameter p and we calculate

$$Res_p(Eq_9, Eq_{10}) = -\frac{4}{(1+h)(1+3h)^3}\,\Upsilon_2 = 0.$$

Considering (3.65) we observe that Υ_2 is linear with respect to the parameter a with the coefficient $eh(1+3h)^3$ where $e(1+3h) \neq 0$. So we consider two subcases: $h \neq 0$ and $h = 0$.

1.1.1: *The subcase $h \neq 0$.* Then the condition $\Upsilon_2 = 0$ gives us

$$\begin{aligned} a = {} & \frac{1}{eh(1+3h)^3}\big[b(1+h)(1+3h)^2(f+2ch-fh) \\ & + (f+ch+fh)(de + cf - 2f^2 + 2c^2h + 6deh - 4cfh \\ & + 2c^2h^2 + 9deh^2 - 5cfh^2 + 2f^2h^2)\big] \equiv a' \end{aligned} \tag{3.68}$$

and then we calculate

$$Eq_9 = \frac{2}{(1+h)(1+3h)^2}\Psi(b,c,d,e,f,h), \quad Eq_{10} = \frac{2(f+ch+fh)}{h(1+h)(1+3h)^3}\Psi(b,c,d,e,f,h),$$

where

$$\Psi = b(1+h)(1+3h)^2 - h(1+h)(1+3h)^2 p + de(1+3h)^2 + (c-2f)(1+h)(f+2ch-fh).$$

Therefore the condition $Eq_9 = Eq_{10} = 0$ implies $\Psi = 0$ and we get

$$p = \frac{2}{h(1+h)(1+3h)^2}\big[b(1+h)(1+3h)^2 + de(1+3h)^2 + (c-2f)(1+h)(f+2ch-fh)\big] \equiv p'. \tag{3.69}$$

Thus we arrive at the family of systems

$$\dot{x} = a' + cx + dy + (h-1)xy, \quad \dot{y} \;= b + ex + fy + hy^2, \; e \neq 0, \tag{3.70}$$

where a' is given in (3.68). These systems possess the following invariant parabola:

$$\Phi = p' + \frac{2e}{h+1}x - \frac{2(c-2f)}{3h+1}y + y^2, \quad e \neq 0, \tag{3.71}$$

where p' is given in (3.69).

We recall that according to Lemma 3.12 in the case $h-2=0$ systems (3.64) could possess invariant parabolas of the form $\Phi(x,y) = p + qx + ry + x^2$. So we discuss two cases: $h-2 \neq 0$ and $h-2=0$.

1.1.1.1: *The possibility* $h-2 \neq 0$. Then by Lemma 3.12 systems (3.64) could not possess invariant parabolas in the second direction.

So we proved that in the case $(h+1)(3h+1)h(h-2)e \neq 0$ and $\Upsilon_2 = 0$ systems (3.64) possess an invariant parabola of the form $\Phi(x,y) = p + qx + ry + y^2$.

Thus the proof of the statement $(\boldsymbol{K_1})$ of Lemma 3.15 is completed.

Next we show that systems (3.70) could be brought via a transformation to the canonical form (S^2_{γ}). Indeed we could apply to parabola (3.71) the translation

$$\begin{aligned} x &= x_1 - \frac{1}{2eh(1+3h)^2}\big[de(1+3h)^2 + b(1+h)(1+3h)^2 + (c-2f)(1+h)(f+ch+fh)\big],\\ y &= y_1 + \frac{c-2f}{1+3h}, \end{aligned}$$

which brings this parabola to the form $\widetilde{\Phi}(x_1,y_1) = y_1^2 + \dfrac{2e}{1+h}x_1$.

On the other hand considering Observation 3.4 we apply the same translation to systems (3.61) and we set the following new notations (suggested by the above parabola and the linear parts of the transformed systems):

$$\begin{gathered} k = -\frac{2e}{1+h}, \quad m = \frac{f+2ch-fh}{1+3h}, \quad n = \frac{h+1}{4eh(1+3h)^2}\big[de(1+3h)^2 - \\ b(h-1)(1+3h)^2 - (c-2f)(h-1)(f+ch+fh)\big] \quad \Rightarrow \\ c = \frac{f(h-1)+(1+3h)m}{2h}, \; d = \frac{(h-1)(f^2-m^2)-4h(bh-b-2hkn)}{2h(1+h)k}, \; e = -\frac{(1+h)k}{2}. \end{gathered}$$

Then after an additional rescaling (to force $k=1$) we arrive at the family of systems (S^2_{γ}).

1.1.1.2: *The possibility* $h-2=0$. In this case we examine the conditions for the existence of the invariant parabolas of the form $\Phi(x,y) = p + qx + ry + x^2$ $(r \neq 0)$.

Considering the equations (2.6) for systems (3.64) we obtain

$$\begin{gathered} s=1, \; v=u=0, \; U=0, \; V=2, \; W=2c, \\ Eq_1 = Eq_2 = Eq_3 = Eq_4 = Eq_5 = Eq_7 = 0, \; Eq_6 = 2d - q. \end{gathered}$$

Therefore the equation $Eq_6 = 0$ gives us $q = 2d$ and then calculations yield:

$$Eq_9 = 2d^2 - 2p - 2cr + fr = 0 \;\Rightarrow\; p = (2d^2 - 2cr + fr)/2.$$

In this case we obtain

$$Eq_8 = 2(a-cd) + er, \quad Eq_{10} = 2d(a-cd) + (b+2c^2-cf)r$$

and we claim that the condition $e \neq 0$ must hold in order to have an invariant parabola. Indeed suppose $e = 0$. Then $Eq_8 = 0$ gives us $a = cd$ and therefore due to $r \neq 0$ from $Eq_{10} = 0$ we get $b = c(f-2c)$ and this leads to the following degenerate systems

$$\dot{x} = (d+x)(c+y), \quad \dot{y} = -(2c-f-2y)(c+y).$$

So $e \neq 0$ and we obtain

$$r = -2(a-cd))/e \neq 0.$$

In this case calculations yield

$$Eq_{10} = -2(a-cd)(b+2c^2-de-cf)/e = -2(a-cd)\Upsilon_3/e \tag{3.72}$$

and due to $a - cd \neq 0$ we obtain that $Eq_{10} = 0$ is equivalent to $\Upsilon_3 = 0$. Therefore we discuss two cases: $\Upsilon_3 \neq 0$ and $\Upsilon_3 = 0$.

a) *The case* $\Upsilon_3 \neq 0$. Then systems (3.64) could not possess invariant parabolas of the form $\Phi(x,y) = p + qx + ry + x^2$ $(r \neq 0)$. However these systems could have invariant parabolas of the form $\Phi(x,y) = p + qx + ry + y^2$ $(q \neq 0)$ and for this it is sufficient to force the conditions $\Upsilon_2 = 0$ and $e \neq 0$. Indeed the condition $h = 2$ implies $(h+1)(3h+1)h \neq 0$ and as it was shown above (see p. ***1.1.1:***) in the case $\Upsilon_2 = 0$ and $e \neq 0$ we arrive at the family of systems (3.70) possessing the invariant parabola (3.71) in this particular case with $h = 2$.

Thus we conclude that the statement $(\boldsymbol{K_2})$ of Lemma 3.15 is proved.

b) *The case* $\Upsilon_3 = 0$. Considering (3.65) the condition $\Upsilon_3 = 0$ gives us $b = -2c^2 + de + cf$. Then from (3.72) we get $Eq_{10} = 0$ and we arrive at the following systems

$$\dot{x} = a + cx + dy + xy, \quad \dot{y} = -2c^2 + de + cf + ex + fy + 2y^2, \;\; e \neq 0, \tag{3.73}$$

possessing the invariant parabola

$$\Phi = \frac{a(2c-f) - d(2c^2-de-cf)}{e} + 2dx - \frac{2(a-cd)}{e}y + x^2, \quad e(a-cd) \neq 0. \tag{3.74}$$

Next we show that systems (3.73) could be brought via a transformation to the canonical form (S^1_γ). Indeed we could apply to parabola (3.74) the translation

$$x = x_1 - d, \quad y = y_1 + \frac{2c-f}{2}.$$

which brings this parabola to the form $\widetilde{\Phi}(x_1,y_1) = x_1^2 - \dfrac{2(a-cd)}{e}y_1$.

On the other hand considering Observation 3.4 we apply the same translation to systems (3.73) and we set the following new notations (suggested by the above parabola and the linear parts of the transformed systems):

$$k = \frac{2(a-cd)}{e}, \;\; m = \frac{e}{2}, \;\; n = \frac{4c-f}{2} \quad \Rightarrow$$
$$a = cd + km, \;\; e = 2m, \;\; f = 2(2c-n).$$

Then after an additional rescaling (to force $k = 1$) we arrive at the subfamily of systems (S_γ^1) defined by the condition $g = 0$.

Next we examine the possibility of the existence besides the parabola (3.74) another parabola of the form $\Phi(x, y) = p + qx + ry + y^2$ $(q \neq 0)$. As it was mentioned earlier the condition $h = 2$ implies $(h+1)(3h+1)h \neq 0$ and according to the statement $(\boldsymbol{K_1})$ of Lemma 3.15 to have such a parabola the condition $\Upsilon_2 = 0$ is necessary. So we examine two possibilities: $\Upsilon_2 \neq 0$ and $\Upsilon_2 = 0$.

b.1) *The possibility* $\Upsilon_2 \neq 0$. In this case by the statement $(\boldsymbol{K_1})$ we could not have parabola of the form $\Phi(x, y) = p + qx + ry + y^2$ $(q \neq 0)$ and hence in the case under consideration we have a single parabola (3.74).

Thus we deduce that the conditions provided by the statement $(\boldsymbol{K_3})$ of Lemma 3.15 are valid.

b.2) *The possibility* $\Upsilon_2 = 0$. As it was shown above (see p. ***b)***) for $h = 2$ and $\Upsilon_3 = 0$ systems (3.64) can be brought to the form (3.73). For these systems we calculate

$$\Upsilon_2 = 2(576c^3 + 343ae - 343cde - 432c^2 f + 108cf^2 - 9f^3).$$

So due to $e \neq 0$ we obtain

$$a = \frac{1}{343e}\left[c(343de - 576c^2) + 9f(48c^2 - 12cf + f^2)\right]$$

and we arrive at the family of systems

$$\begin{aligned}
\dot{x} &= \frac{1}{343e}\left[c(343de - 576c^2) + 9f(48c^2 - 12cf + f^2)\right] + cx + dy + xy,\\
\dot{y} &= -2c^2 + de + cf + ex + fy + 2y^2, \quad e \neq 0,
\end{aligned} \tag{3.75}$$

possessing the following two invariant parabolas:

$$\begin{aligned}
\Phi_1 &= d^2 - \frac{9(2c-f)(4c-f)^3}{343e^2} + 2dx + \frac{18(4c-f)^3}{343e^2}\,y + x^2, \quad e(4c-f) \neq 0;\\
\Phi_2 &= \frac{1}{147}(60cf - 141c^2 + 98de + 3f^2) + \frac{2e}{3}\,x - \frac{2(c-2f)}{7}\,y + y^2, \quad e \neq 0.
\end{aligned} \tag{3.76}$$

So the conditions provided by the statement $(\boldsymbol{K_4})$ of Lemma 3.15 are valid.

Next we show that systems (3.75) could be brought via a transformation to the canonical form (S_γ^2). Indeed we could apply to parabola $\Phi_2 = 0$ from (3.76) the translation

$$x = x_1 + \frac{9(4c-f)^2 - 98de}{98e}, \quad y = y_1 + \frac{c-2f}{7}$$

which brings this parabola to the form $\widetilde{\Phi}(x_1, y_1) = y_1^2 + \dfrac{2e}{3}\,x_1$.

On the other hand considering Observation 3.4 we apply the same translation to systems (3.75) and we set the following new notations (suggested by the above parabola and the linear parts of the transformed systems):

$$k = -\frac{2e}{3}, \quad m = \frac{4c-f}{7} \quad \Rightarrow \quad c = \frac{f+7m}{4}, \quad e = -\frac{3k}{2}.$$

Then after an additional rescaling (to force $k = 1$) we arrive at the the subfamily of systems (S_γ^2) defined by the conditions $h = 2$ and $n = -3m^2/2$.

1.1.2: *The subcase* $h=0$. In this case considering (3.67) we obtain

$$Eq_9 = 2(b+de+cf-2f^2), \quad Eq_{10} = 2(ae-bc+2bf-fp)$$

and clearly the condition $Eq_9=0$ gives us $b=-de-cf+2f^2$. We observe that the equation $Eq_{10}=0$ is linear with respect to the parameter p with the coefficient f. So we discuss two possibilities: $f\neq 0$ and $f=0$.

1.1.2.1: *The possibility* $f\neq 0$. Then the condition $Eq_{10}=0$ implies

$$p = \big[ae+(c-2f)(de+cf-2f^2)\big]/f$$

and we arrive at the family of systems

$$\dot{x} = a+cx+dy-xy, \quad \dot{y} = -de-cf+2f^2+ex+fy, \;\; ef\neq 0, \tag{3.77}$$

possessing the following invariant parabola:

$$\Phi = \frac{1}{f}\big[ae+(c-2f)(de+cf-2f^2)\big]+2ex-2(c-2f)y+y^2, \quad ef\neq 0. \tag{3.78}$$

On the other hand for $h=0$ we have

$$\Upsilon_2 = -f(b+de+cf-2f^2)$$

and since $f\neq 0$ we conclude that the condition $\Upsilon_2=0$ is equivalent to $b+de+cf-2f^2=0$.

1.1.2.2: *The possibility* $f=0$. Then considering (3.67) for $h=f=0$ we obtain

$$Eq_9 = 2(b+de)=0, \quad Eq_{10} = 2(ae-bc)=0$$

and due to $e\neq 0$ (since $q\neq 0$) this implies $b=-de$ and $a=-cd$. However in this case we get the degenerate systems

$$\dot{x} = -(d-x)(c-y), \quad \dot{y} = -e(d-x).$$

Thus we have proved that for the existence of invariant parabola of systems (3.64) with $h=0$ the conditions $\Upsilon_2=0$ and $ef\neq 0$ must hold and we deduce that the conditions provided by the statement $(\boldsymbol{K_5})$ of Lemma 3.15 are valid.

Next we show that systems (3.77) could be brought via a transformation to the canonical form (S^2_γ). Indeed we could apply to parabola (3.78) the translation

$$x = x_1 - \frac{a+cd-2df}{2f}, \quad y = y_1+c-2f,$$

which brings this parabola to the form $\widetilde{\Phi}(x_1,y_1) = y_1^2+2e\,x_1$.

On the other hand considering Observation 3.4 we apply the same translation to systems (3.77) and we set the following new notations (suggested by the above parabola and the linear parts of the transformed systems):

$$k=-2e, \;\; m=f, \;\; n=\frac{a+cd}{4f} \quad \Rightarrow$$
$$a=-cd+4mn, \;\; e=-k/2, \;\; f=m.$$

Then after an additional rescaling (to force $k=1$) we arrive at the the subfamily of systems (S^2_γ) defined by the conditions $h=0$.

1.2: **The case** $1+3h=0$**.** Then $h=-1/3$ and considering (2.6) and (3.66) we calculate $Eq_6=2/3(3e-q)=0$ which implies $q=3e\neq 0$. Therefore calculations yield:

$$Eq_8=3e(c-2f),\quad Eq_9=(6b+9de+2p-3fr-r^2)/3$$

and since $e\neq 0$ the equation $Eq_8=0$ implies $c=2f$ and from $Eq_9=0$ we obtain

$$p=\frac{1}{2}\big[r^2+3fr-3(2b+3de)\big].$$

Then we obtain

$$Eq_{10}=-\frac{1}{6}\big[r^3+9fr^2-3(4b+3de-6f^2)r-18(ae+2bf+3def)\big]\equiv-\frac{1}{6}\Psi(r)$$

and we conclude that if r_0 is a solution of the equation $Eq_{10}=0$ (i.e. $\Psi(r_0)=0$) then systems

$$\dot x=a+2fx+dy-4xy/3,\quad \dot y=b+ex+fy-y^2/3,\ \ e\neq 0 \tag{3.79}$$

possess the invariant parabola

$$\Phi_0(x,y)=(r_0^2+3fr_0-6b-9de)/2+3ex+r_0y+y^2. \tag{3.80}$$

On the other hand we calculate

$$\Psi'_r=3(r^2+6fr-4b-3de+6f^2),\quad \Psi''_r=6(3f+r),$$
$$\mathrm{Discrim}[\Psi,r]=27\mathcal{D}_2,\ \ \mathrm{Res}_r(\Psi'_r,\Psi''_r)=-108\mathcal{F}_2,$$

and we conclude that systems (3.79) has the following invariant parabolas of the form (3.80):

- if $\mathcal{D}_2>0\Rightarrow$ three real distinct invariant parabolas;
- if $\mathcal{D}_2<0\Rightarrow$ one real and two complex invariant parabolas;
- if $\mathcal{D}_2=0,\ \mathcal{F}_2\neq 0\ \Rightarrow$ one simple and one double real invariant parabolas;
- if $\mathcal{D}_2=0,\ \mathcal{F}_2=0\ \Rightarrow$ one triple real invariant parabolas.

So we conclude that the conditions provided by the statements $(\boldsymbol{K_6})$–$(\boldsymbol{K_9})$ of Lemma 3.15 are valid.

Next we show that systems (3.79) could be brought via a real transformation to the canonical form (S^2_γ). Indeed we could apply to parabola (3.80) with $r_0\in\mathbb{R}$ the translation

$$x=x_1+\frac{12b+18de-6fr_0-r_0^2}{12e},\quad y=y_1-r_0/2,$$

which brings this parabola to the form $\widetilde{\Phi}(x_1,y_1)=y_1^2+3e\,x_1$.

On the other hand applying the same translation to systems (3.79) we arrive at the systems

$$\begin{aligned}\dot x_1&=-\frac{1}{8e}\Psi(r_0)+\frac{2}{3}(3f+r_0)x_1-\frac{1}{9e}(12b+9de-6fr_0-r_0^2)y_1-\frac{4}{3}x_1y_1,\\ \dot y_1&=\frac{1}{6}(12b+9de-6fr_0-r_0^2)+ex_1+\frac{1}{3}(3f+r_0)y_1-\frac{1}{3}y_1^2.\end{aligned}$$

We recall that $\Psi(r_0)=0$ and we set the following new notations (suggested by the above parabola and the linear parts of the above systems):

$$k=-3e,\ \ m=\frac{3f+r_0}{3},\ \ n=-\frac{12b+9de-6fr_0-r_0^2}{18e}\quad\Rightarrow$$
$$b=\frac{3dk+6kn+6mr_0-r_0^2}{12},\ e=-\frac{k}{3},\ f=\frac{3m-r_0}{3}.$$

Then after an additional rescaling (to force $k=1$) we arrive at the subfamily of systems (S^2_{γ}) defined by the conditions $h=-1/3$.

***2:* The possibility $h+1=0$.** Then $h=-1$ and considering (3.66) for $h=-1$ we have $Eq_6=2e=0$, i.e. $e=0$. Then taking into account (2.6) calculations yield:

$$Eq_8=q(c-2f-r),\ \ Eq_9=2b+2p+dq-fr-r^2$$

and since $q\neq 0$ the condition $Eq_8=0$ implies $c-2f-r=0$ and this gives us $r=c-2f$. Then from $Eq_9=0$ we obtain $p=(c^2-2b-3cf+2f^2-dq)/2$ and we obtain

$$Eq_{10}=\frac{1}{2}\big[(c-f)(4b-c^2+2cf)+(2a+cd)q\big]. \tag{3.81}$$

We observe that for systems (3.64) we have $\mathcal{G}_2=2a+cd$ and therefore we have to consider two cases: $\mathcal{G}_2\neq 0$ and $\mathcal{G}_2=0$.

***2.1:* The case $\mathcal{G}_2\neq 0$.** Then $2a+cd\neq 0$ and the equation $Eq_{10}=0$ implies

$$q=-\frac{(c-f)(4b-c^2+2cf)}{2a+cd}$$

and we obtain the parabola

$$\begin{gathered}\Phi=\frac{-2ab+ac^2-3acf+2af^2+bcd-2bdf}{2a+cd}-\frac{(c-f)(4b-c^2+2cf)}{2a+cd}x+(c-2f)y+y^2,\\ (2a+cd)(c-f)\Big(4b-c^2+2cf\Big)\neq 0\Leftrightarrow\ \mathfrak{G}_2\mathfrak{H}_2(c-f)\neq 0,\end{gathered} \tag{3.82}$$

which is invariant for the family of systems

$$\dot{x}=a+cx+dy-2xy,\quad \dot{y}=b+fy-y^2. \tag{3.83}$$

So we conclude that the conditions provided by the statement $(\boldsymbol{K_{10}})$ of Lemma 3.15 are valid.

Next we show that systems (3.83) could be brought via a transformation to the canonical form (S^2_{γ}). Indeed we could apply to parabola (3.82) the translation

$$x=x_1-\frac{2a-cd+2df}{4(c-f)},\quad y=y_1-\frac{c-2f}{2},$$

which brings this parabola to the form $\widetilde{\Phi}(x_1,y_1)=y_1^2-\dfrac{(c-f)(4b-c^2+2cf)}{2a+cd}x_1$.

On the other hand considering Observation 3.4 we apply the same translation to systems (3.83) and we set the following new notations (suggested by the above parabola and the linear parts of the transformed systems):

$$k=\frac{(c-f)(4b-c^2+2cf)}{2a+cd},\ \ m=c-f,\ \ n=\frac{2a+cd}{4(c-f)}\quad\Rightarrow$$
$$a=(4mn-cd)/2,\ \ b=(2cm-c^2+4kn)/4,\ f=c-m.$$

Then after an additional rescaling (to force $k=1$) we arrive at the the subfamily of systems (S^2_{γ}) defined by the conditions $h=-1$.

***2.2:* The case $\mathcal{G}_2 = 0$.** In this case $2a + cd = 0$, i.e. $a = -cd/2$ and considering (3.81) the equation $Eq_{10} = 0$ yields

$$(c - f)(4b - c^2 + 2cf) = 0 \quad \Rightarrow \quad (c - f)\mathfrak{H}_2 = 0.$$

If $\mathfrak{H}_2 = 0$ then we get $b = c(c - 2f)/4$ and this leads to degenerate systems:

$$\dot{x} = -(d - 2x)(c - 2y)/2, \quad \dot{y} = (c - 2y)(c - 2f + 2y)/4.$$

So the condition $\mathfrak{H}_2 \neq 0$ is necessary and then we have $c - f = 0$. So we get $f = c$ which leads to the family of systems

$$\dot{x} = -cd/2 + cx + dy - 2xy, \quad \dot{y} = b + cy - y^2 \tag{3.84}$$

possessing the following family of invariant parabolas:

$$\Phi = -(2b + dq)/2 + qx - cy + y^2, \quad q \in \mathbb{R}, \quad q \neq 0. \tag{3.85}$$

Next we show that systems (3.84) could be brought via a transformation to the canonical form (S^2_γ). Indeed we could apply to parabola (3.85) the translation

$$x = x_1 + \frac{4b + c^2 + 2dq}{4q}, \quad y = y_1 + \frac{c}{2},$$

which brings this parabola to the form $\widetilde{\Phi}(x_1, y_1) = y_1^2 + q\,x_1$.

On the other hand considering Observation 3.4 we apply the same translation to systems (3.84) and we set the following new notations (suggested by the above parabola and the linear parts of the transformed systems):

$$k = -q, \quad m = c - f, \quad n = -\frac{4b + c^2}{4q} \quad \Rightarrow$$
$$b = (4kn - c^2)/4, \quad q = -k \neq 0.$$

Then after an additional rescaling (to force $k = 1$) we arrive at the subfamily of systems (S^2_γ) defined by the conditions $h = -1$ and $m = 0$. Moreover we observe that this subfamily of systems possess the following family of invariant parabolas:

$$\Phi = -n(1 + q) + qx + y^2, \quad q \in \mathbb{R}, \quad q \neq 0.$$

Evidently for $q = -1$ we get the parabola $y^2 - x = 0$.

Thus the the condition provided by the statement $(\boldsymbol{K_{11}})$ of Lemma 3.15 are valid and this completes the proof of the Lemma 3.15. □

Invariant conditions: the case $\eta = 0$, $\widetilde{M} \neq 0$ and $\mu_0 = 0$. Next we consider the class of quadratic systems for which the conditions $\eta = 0$, $\widetilde{M} \neq 0$, $\mu_0 = 0$, which by Lemma 3.12 could possess invariant parabolas in two directions.

We prove the following theorem.

Theorem 3.16. *Assume that for a non-degenerate arbitrary quadratic system the conditions $\eta = 0$, $\widetilde{M} \neq 0$, $\chi_1 = \mu_0 = 0$ are satisfied. Then this system could possess invariant parabolas in two directions. More exactly it could only possess one of the following sets of invariant parabolas: $\cup$, $\overset{2}{\cup}$, $\overset{2}{\cup}\subset$, $\overset{2}{\cup}\overset{2}{\Cup}$, $\overset{2}{\cup}\overset{2}{\Cup}{}^c$, $\overset{2}{\cup}\overset{2}{\pmb{\cup}}{}^2$, $\overset{2}{\pmb{\cup}}{}^3$ and $\infty\overset{2}{\cup}$. Moreover this system has one of the above sets of invariant parabolas if and only if one of the following sets of conditions are satisfied, correspondingly:*

$(\mathcal{K}_1)$ $\zeta_{11}\zeta_{12}\zeta_{13}\zeta_{14} \neq 0$, $\mathcal{R}_5 \neq 0$, $\begin{cases} \zeta_{15} \neq 0, \zeta_{16} = 0 \\ or\ \zeta_{15} = \zeta_{17} = 0, \end{cases}$ $\Rightarrow \overset{2}{\cup};$

$(\mathcal{K}_2)$ $\zeta_{11} = 0$, $\mathcal{R}_3 \neq 0$, $\zeta_{16} = 0$, $\mathcal{R}_5 \neq 0$ $\Rightarrow \overset{2}{\cup};$

$(\mathcal{K}_3)$ $\zeta_{11} = 0$, $\mathcal{R}_3 = 0$, $\zeta_{16} \neq 0$, $\mathcal{R}_6 \neq 0$ $\Rightarrow \cup;$

$(\mathcal{K}_4)$ $\zeta_{11} = 0$, $\mathcal{R}_3 = 0$, $\zeta_{16} = 0$, $\zeta_8 \neq 0$ $\Rightarrow \overset{2}{\cup}\subset;$

$(\mathcal{K}_5)$ $\zeta_{12} = 0$, $\zeta_{16} = 0$, $\zeta_8 \neq 0$ $\Rightarrow \overset{2}{\cup};$

$(\mathcal{K}_6)$ $\zeta_{13} = 0$, $\zeta_8 = 0$, $\zeta_{18} > 0$, $\mathcal{R}_5 \neq 0$ $\Rightarrow \overset{2}{\cup}\overset{2}{\Cup};$

$(\mathcal{K}_7)$ $\zeta_{13} = 0$, $\zeta_8 = 0$, $\zeta_{18} < 0$, $\mathcal{R}_5 \neq 0$ $\Rightarrow \overset{2}{\cup}\overset{2}{\Cup}{}^c;$

$(\mathcal{K}_8)$ $\zeta_{13} = 0$, $\zeta_8 = 0$, $\zeta_{18} = 0$, $\mathcal{R}_5 \neq 0$, $\chi_3 \neq 0$ $\Rightarrow \overset{2}{\cup}\overset{2}{\boldsymbol{\cup}}{}^2;$

$(\mathcal{K}_9)$ $\zeta_{13} = 0$, $\zeta_8 = 0$, $\zeta_{18} = 0$, $\mathcal{R}_5 \neq 0, \chi_3 = 0$ $\Rightarrow \overset{2}{\boldsymbol{\cup}}{}^3;$

$(\mathcal{K}_{10})$ $\zeta_{14} = 0$, $\zeta_{19} \neq 0$, $\zeta_{20} \neq 0$, $\mathcal{R}_5 = 0$, $\zeta_{21} \neq 0$ $\Rightarrow \overset{2}{\cup};$

$(\mathcal{K}_{11})$ $\zeta_{14} = 0$, $\zeta_{19} = 0$, $\zeta_{20} \neq 0$, $\mathcal{R}_5 = 0$, $\zeta_{21} = 0$ $\Rightarrow \infty\overset{2}{\cup}.$

Proof. Assume that quadratic system the conditions $\eta = 0$ and $\widetilde{M} \neq 0$ are fulfilled. Then via a linear transformation this system can be brought to the canonical form (3.52). According to Lemma 2.4 for a system (3.52) to possess an invariant parabola the conditions $\chi_1 = \chi_2 = 0$ are necessary. Moreover it was shown earlier (see page 49) that a system (3.52) with $\chi_1 = \mu_0 = 0$ via an affine transformation and time rescaling can be brought to the form (3.64). Thus in what follows we consider the family of quadratic systems

$$\dot{x} = a + cx + dy + (h-1)xy, \quad \dot{y} = b + ex + fy + hy^2. \tag{3.86}$$

***1:* The statement $(\mathcal{K}_1)$.** Considering (3.65) for these systems we calculate

$$\begin{gathered} \chi_1 = \chi_2 = 0, \ \ \zeta_{11} = -4(h-2)y^2, \ \ \zeta_{12} = 4hy^2, \ \ \zeta_{13} = (1+3h)y^2, \ \ \zeta_{14} = (1+h)^2y^2, \\ \zeta_{15} = (h-1)^2y^2/4, \ \ \zeta_{16} = 45e^3(h-1)^2\mathrm{Y}_2/8, \ \ \mathcal{R}_5 = 32e. \end{gathered} \tag{3.87}$$

The condition $\zeta_{11} \neq 0$ implies $h - 2 \neq 0$ and according to Lemma 3.12 systems (3.64) could not possess invariant parabolas of the form $\Phi(x, y) = p + qx + ry + x^2$.

On the other hand evidently the condition $\zeta_{12}\zeta_{13}\zeta_{14} \neq 0$ is equivalent to $(1+h)(1+3h)h \neq 0$ and the condition $\mathcal{R}_5 \neq 0$ is equivalent to $e \neq 0$. So considering the statement $(\boldsymbol{K_1})$ of Lemma 3.15 it remains to determine in invariant form the condition which is equivalent to $\mathrm{Y}_2 = 0$. We consider two cases: $\zeta_{15} \neq 0$ and $\zeta_{15} = 0$.

***1.1:* The case $\zeta_{15} \neq 0$.** Then $h - 1 \neq 0$ and due to $\mathcal{R}_5 \neq 0$ (i.e. $e \neq 0$) we conclude that the condition $\mathrm{Y}_2 = 0$ is equivalent to $\zeta_{16} = 0$ and hence the statement $(\mathcal{K}_1)$ of Theorem 3.16 is valid in this case.

***1.2:* The case $\zeta_{15} = 0$.** Then $h = 1$ and we obtain

$$\mathrm{Y}_2 = -4(16bc + c^3 - 16ae + 4cde + 8def - 4cf^2), \quad \zeta_{17} = 13824e^2\mathrm{Y}_2.$$

Therefore due to $e \neq 0$ we conclude that the condition $\mathrm{Y}_2 = 0$ is equivalent to $\zeta_{17} = 0$ and this completes the proof of the statement $(\mathcal{K}_1)$ of Theorem 3.16.

2: **The statements** $(\mathcal{K}_2)$–$(\mathcal{K}_4)$. For systems (3.86) the condition $\zeta_{11} = 0$ gives us $h = 2$. In this case we calculate

$$\mathcal{R}_3 = -503139971565000\, e^4(b + 2c^2 - de - cf) = -503139971565000\, e^4 \Upsilon_3,$$
$$\mathcal{R}_5 = 32e,\ \ \mathcal{R}_6 = 8925(a - cd)e^4/4,\ \ \zeta_{16} = 45e^3\Upsilon_2/8,\ \ \zeta_8 = e(4c - f).$$

Therefore the condition $\mathcal{R}_5 \neq 0$ is equivalent to $e \neq 0$ and for $e \neq 0$ the condition $\mathcal{R}_3 = 0$ (respectively $\zeta_{16} = 0$) is equivalent with $\Upsilon_3 = 0$ (respectively $\Upsilon_2 = 0$). So in the case $\mathcal{R}_3 \neq 0$ we deduce that the conditions of the statements $(\mathcal{K}_2)$ provides the conditions of the statements$(\boldsymbol{K}_2)$ of Lemma 3.15.

Assume now $\mathcal{R}_3 = 0$. We observe that the condition $\mathcal{R}_6 \neq 0$ (respectively $\zeta_8 \neq 0$) implies $e(a - cd) \neq 0$ (respectively $e(4c - f) \neq 0$), i.e. in both cases we have $e \neq 0$. Then the condition $\mathcal{R}_3 = 0$ is equivalent to Υ_3 and the condition $\zeta_{16} = 0$ is equivalent to $\Upsilon_2 = 0$.

Thus considering Lemma 3.15 we conclude that in the case $\zeta_{16} \neq 0$ (respectively $\zeta_{16} = 0$) we get the conditions provided by the statement $(\boldsymbol{K}_3)$ (respectively $(\boldsymbol{K}_4)$) of this lemma. So the invariant conditions provided by the statements statements $(\mathcal{K}_2)$–$(\mathcal{K}_4)$ of Theorem 3.16 are valid.

3: **The statement** $(\mathcal{K}_5)$. Considering (3.87) the condition $\zeta_{12} = 0$ implies $h = 0$ and we calculate:

$$\zeta_{16} = 45e^3\Upsilon_2/8,\ \ \zeta_8 = ef.$$

Clearly the condition $\zeta_8 \neq 0$ implies $ef \neq 0$ and then the condition $\zeta_{16} = 0$ is equivalent to $\Upsilon_2 = 0$. So we get the conditions provided by the statement $(\boldsymbol{K}_5)$ of Lemma 3.15 and this implies the validity of the statement $(\mathcal{K}_5)$ of Theorem 3.16.

4: **The statements** $(\mathcal{K}_6)$–$(\mathcal{K}_9)$. Considering (3.87) we observe that the condition $\zeta_{13} = 0$ implies $h = -1/3$ and we calculate:

$$\mathcal{R}_5 = 32e,\ \ \zeta_8 = -2e(c - 2f)/3.$$

So the condition $\mathcal{R}_5 \neq 0$ implies $e \neq 0$ and then the condition $\zeta_8 = 0$ yields $c = 2f$. In this case we have

$$\zeta_{18} = 64/2187\, e^2\, \mathfrak{D}_2,\ \ \chi_3 = -\frac{15792269387776}{729}\, e^4\, \mathfrak{F}_2^2$$

and hence the condition $\zeta_{18} = 0$ is equivalent to $\mathfrak{D}_2 = 0$ and for $\zeta_{18} \neq 0$ we have $\mathrm{sign}(\zeta_{18}) = \mathrm{sign}(\mathfrak{D}_2)$. Moreover the condition $\chi_3 = 0$ is equivalent to $\mathfrak{F}_2 = 0$.

Thus we obtain that in the case $\mathcal{R}_5 \neq 0$, $\zeta_8 = 0$ and $\zeta_{18} > 0$ (respectively $\zeta_{18} < 0$) then we arrive at the conditions provided by the statement $(\mathcal{K}_6)$ (respectively $(\mathcal{K}_7)$) of Lemma 3.15.

In the case $\zeta_{18} = 0$ (i.e. $\mathfrak{D}_2 = 0$) we obtain the conditions provided by the statement $(\mathcal{K}_8)$ if $\chi_3 \neq 0$ and by the statement $(\mathcal{K}_9)$ if $\chi_3 = 0$ (i.e. $\mathfrak{F}_2 = 0$). This proves the validity of the statements $(\mathcal{K}_6)$–$(\mathcal{K}_9)$ of Theorem 3.16.

5: **The statements** $(\mathcal{K}_{10})$, $(\mathcal{K}_{11})$. From (3.87) we obtain that the condition $\zeta_{14} = 0$ implies $h = -1$ and then we have $\mathcal{R}_5 = 32e$. So the condition $\mathcal{R}_5 = 0$ implies $e = 0$ and we calculate

$$\zeta_{19} = 6(2a + cd)y^4 = 6\mathfrak{G}_2,\ \ \zeta_{20} = 8(4b - c^2 + 2cf)y^2 = 8\mathfrak{H}_2 y^2,\ \ \zeta_{21} = 2(c - f)y^3.$$

So we observe that for $\zeta_{19} \neq 0$, $\zeta_{20} \neq 0$ and $\zeta_{21} \neq 0$ we arrive at the conditions provided by the statement $(\boldsymbol{K}_{10})$ of Lemma 3.15. In the case $\zeta_{19} = 0$, $\zeta_{20} \neq 0$ and $\zeta_{21} = 0$ we get the conditions provided by the statement $(\boldsymbol{K}_{11})$ of the same lemma.

Thus we conclude that the statements $(\mathcal{K}_{10})$ and $(\mathcal{K}_{11})$ of Theorem 3.16 are valid and this completes the proof of this theorem. $\square$

3.4 Systems with a unique infinite singular point which is real

In this case according to Lemma 2.3 systems (2.5) via a linear transformation could be brought to the following family of systems

$$\frac{dx}{dt} = a + cx + dy + gx^2 + hxy, \quad \frac{dy}{dt} = b + ex + fy - x^2 + gxy + hy^2. \tag{3.88}$$

For these systems we calculate

$$C_2(x, y) = x^3, \ \ \chi_1 = -2h^3$$

and by Lemma 2.6 we conclude that the above systems could have invariant parabolas only of the form $\Phi(x, y) = p + qx + ry + x^2$ with $r \neq 0$ (otherwise we get a reducible conic).

According to Lemma 2.4 for a system (3.88) to possess an invariant parabola the condition $\chi_1 = 0$ is necessary and this implies $h = 0$. Moreover we may assume $e = 0$ due to the translation $x \to x + e/2$, $y \to y$ and we arrive at the family of systems

$$\frac{dx}{dt} = a + cx + dy + gx^2, \quad \frac{dy}{dt} = b + fy - x^2 + gxy. \tag{3.89}$$

3.4.1 Coefficient conditions for systems (3.89) to possess invariant parabolas.

We prove the following lemma.

Lemma 3.17. *A system (3.89) could only possess one of the following sets of invariant parabolas:* $\overset{3}{\cup}$ *and* $\infty\overset{3}{\cup}$. *Moreover this system has one of the above sets of invariant parabolas if and only if the corresponding set of conditions are satisfied, respectively:*

$(\boldsymbol{L_1})$ $\ g \neq 0,\ \Upsilon_4 = 0,\ d \neq 0 \Rightarrow \overset{3}{\cup}$;

$(\boldsymbol{L_2})$ $\ g = 0,\ d = 0,\ c - f \neq 0,\ f(2c - f) \neq 0 \ \Rightarrow \overset{3}{\cup}$;

$(\boldsymbol{L_3})$ $\ g = 0,\ d = 0,\ f = c \neq 0 \ \Rightarrow \infty\overset{3}{\cup}$,

where

$$\Upsilon_4 = 27bdg^4 - 9ag^3(d - cg + 2fg) - (2d + cg - 2fg)(d - cg - fg)(2d - 2cg + fg). \tag{3.90}$$

Proof. Considering equations (2.6) and the form of the parabola $\Phi(x, y) = p + qr + ry + x^2$ with $r \neq 0$ (otherwise we get a reducible conic), for systems (3.89) we obtain

$$s = 1,\ v = u = 0,\ U = 2g,\ V = 0,\ W = 2c - gq - r,\ Eq_6 = 2d - gr$$

and clearly we have to discuss two possibilities: $g \neq 0$ and $g = 0$.

1: **The possibility** $g \neq 0$**.** Then the equation $Eq_6 = 0$ yields $r = 2d/g \neq 0$ and we calculate:

$$Eq_8 = 2(a - gp) + q(2d - cg)/g + gq^2 = 0 \ \Rightarrow \ p = \frac{a}{g} - \frac{q(cg - 2d)}{2g^2} + \frac{q^2}{2}.$$

Then we obtain

$$Eq_9 = d(4d - 4cg + 2fg + 3g^2q)/g^2$$

and since $dg \neq 0$ the equation $Eq_8 = 0$ gives us

$$q = 2(2cg - 2d - fg)/(3g^2)$$

and finally we calculate the last equation $Eq_{10} = 0$:

$$\begin{aligned} Eq_{10} &= \frac{2}{27g^5}\big[27bdg^4 - 9ag^3(d - cg + 2fg) - (2d + cg - 2fg)(d - cg - fg)(2d - 2cg + fg)\big] \\ &= \frac{2}{27g^5}\Upsilon_4. \end{aligned}$$

Since $dg \neq 0$ the equation $Eq_{10} = 0$ gives us

$$b = \frac{1}{27dg^4}\big[9ag^3(d - cg + 2fg) + (2d + cg - 2fg)(d - cg - fg)(2d - 2cg + fg)\big] \equiv b_0$$

and we arrive at the family of systems

$$\dot{x} = a + cx + dy + gx^2, \quad \dot{y} = b_0 + fy - x^2 + gxy \tag{3.91}$$

possessing the invariant parabola

$$\Phi(x, y) = \frac{9ag^3 - (2d + cg - 2fg)(2d - 2cg + fg)}{9g^4} - \frac{2(2d - 2cg + fg)}{3g^2}x + \frac{2d}{g}y + x^2. \tag{3.92}$$

This completes the proof of the statement $\boldsymbol{(L_1)}$ of Lemma 3.17.

Next we show that systems (3.91) could be brought via a transformation to the canonical form (S_δ). Indeed we could apply to parabola (3.92) the translation

$$x = x_1 - \frac{2cg - 2d - fg}{3g^2}, \quad y = y_1 + \frac{8d^2 - 2d(5c - f)g + g^2(2c^2 + cf - f^2 - 9ag)}{18dg^3},$$

which brings this parabola to the form $\widetilde{\Phi}(x_1, y_1) = x_1^2 + \dfrac{2d}{g}y_1$.

On the other hand considering Observation 3.4 we apply the same translation to systems (3.91) and we set the following new notations (suggested by the above parabola and the linear parts of the transformed systems):

$$\begin{gathered} k = -\frac{2d}{g},\ n = -\frac{-d + cg - 2fg}{3g},\ m = -\frac{16d^2 - 2d(7c - 5f)g - g^2(2c^2 + cf - f^2 - 9ag)}{36dg^2} \Rightarrow \\ a = \frac{4c^2 - 8ck - 5k^2 - 4n^2 + 4k(8gm + 3n)}{16g},\ d = -\frac{gk}{2},\ f = \frac{2c + k + 6n}{4}. \end{gathered}$$

Then after an additional rescaling (to force $k = 1$) we arrive at the family of systems (S_δ).

***2:* The possibility** $g = 0$. Then the equation $Eq_6 = 0$ yields $d = 0$ and we calculate:

$$Eq_9 = r(f - 2c + r)$$

and due to $r \neq 0$ we get $r = 2c - f \neq 0$. Then calculations yield:

$$Eq_8 = 2a + (c - f)q, \quad Eq_{10} = 2bc - bf - fp + aq$$

and we have to examine two cases: $c - f \neq 0$ and $c - f = 0$.

***2.1:* The case** $c - f \neq 0$**.** Then the equation $Eq_8 = 0$ gives us $q = -2a/(c-f)$ and then we obtain

$$Eq_{10} = \frac{2bc^2 - 2a^2 - 3bcf + bf^2}{c-f} - fp.$$

We claim that for the existence of an invariant parabola the condition $f \neq 0$ must hold. Indeed supposing $f = 0$ we obtain $Eq_{10} = 2(bc - a^2)/c$ and then the condition $Eq_{10} = 0$ implies $b = a^2/c$. However in this case we arrive at the degenerate systems

$$\dot{x} = a + cx, \quad \dot{y} = \frac{(a-cx)(a+cx)}{c^2}$$

and this completes the proof of our claim.

So we have $f \neq 0$ and then the condition $Eq_{10} = 0$ gives us

$$p = \frac{-2a^2 + 2bc^2 - 3bcf + bf^2}{f(c-f)}$$

and we arrive at the parabola

$$\Phi(x,y) = \frac{2bc^2 - 2a^2 - 3bcf + bf^2}{f(c-f)} - \frac{2a}{c-f}x + (2c-f)y + x^2, \quad f(c-f)(2c-f) \neq 0, \tag{3.93}$$

which is invariant for the family of systems:

$$\dot{x} = a + cx, \quad \dot{y} = b + fy - x^2. \tag{3.94}$$

This completes the proof of the statement $(\boldsymbol{L_2})$ of Lemma 3.17.

Next we show that systems (3.94) could be brought via a transformation to the canonical form (S_δ). Indeed we could apply to parabola (3.93) the translation

$$x = x_1 + \frac{a}{c-f}, \quad y = y_1 - \frac{bc^2 - a^2 - 2bcf + bf^2}{(c-f)^2 f},$$

which brings this parabola to the form $\widetilde{\Phi}(x_1, y_1) = x_1^2 + (2c-f)\,y_1$.

On the other hand considering Observation 3.4 we apply the same translation to systems (3.95) and we set the following new notations (suggested by the above parabola and the linear parts of the transformed systems):

$$k = f - 2c, \; m = -\frac{a}{c-f}, \; n = \frac{f}{2} \quad \Rightarrow \quad a = -\frac{m(k+2n)}{2}, \; c = \frac{2n-k}{2}, \; f = 2n.$$

Then after an additional rescaling (to force $k = 1$) we arrive at the subfamily of systems (S_δ) defined by the condition $g = 0$.

***2.2:* The case** $c - f = 0$**.** Then we set $f = c$ and the equation $Eq_8 = 0$ gives us $a = 0$ and we obtain

$$Eq_{10} = c(b - p) = 0.$$

In this case $r = 2c - f = c \neq 0$ and hence the condition $Eq_{10} = 0$ implies $p = b$. Therefore we obtain the family of systems

$$\dot{x} = cx, \quad \dot{y} = b + cy - x^2, \tag{3.95}$$

which possess the family of the invariant parabolas depending on one parameter q.

$$\Phi(x,y) = b + qx + cy + x^2, \quad c \neq 0. \tag{3.96}$$

This completes the proof of the Lemma 3.17. $\square$

Next we show that systems (3.95) could be brought via a transformation to the canonical form (S_δ). Indeed we could apply to parabola (3.96) the translation

$$x = x_1 - \frac{q}{2}, \quad y = y_1 - \frac{4b - q^2}{4c},$$

which brings this parabola to the form $\widetilde{\Phi}(x_1, y_1) = x_1^2 + c\,y_1$.

On the other hand considering Observation 3.4 we apply the same translation to systems (3.95) and we set the following new notations (suggested by the above parabola and the linear parts of the transformed systems):

$$k = -c, \ m = \frac{q}{2} \ \Rightarrow \ c = -k, \ q = 2m.$$

Then after an additional rescaling (to force $k = 1$) arrive at the subfamily of systems (S_δ) defined by the conditions $g = 0$ and $n = -1/2$.

3.4.2 Invariant conditions: the case $\eta = \widetilde{M} = 0$ and $C_2 \neq 0$

Next, using Lemma 3.17 we shall construct the equivalent affine invariant conditions for a system with $\eta = \widetilde{M} = 0$ and $C_2 \neq 0$ to possess an invariant parabola.

We prove the following theorem.

Theorem 3.18. *Assume that for a non-degenerate arbitrary quadratic system the conditions $\eta = \widetilde{M} = 0$, $\chi_1 = 0$ and $C_2 \neq 0$ are satisfied. Then this system could only possess one of the following sets of invariant parabolas: $\overset{3}{\cup}$ and $\infty\overset{3}{\cup}$. Moreover this system has one of the above sets of invariant parabolas if and only if one of the following sets of conditions are satisfied, correspondingly:*

$(\mathcal{L}_1)$ $\quad \zeta_{14} \neq 0, \zeta_{22} = 0, \mathcal{R}_2 \neq 0 \quad \Rightarrow \overset{3}{\cup};$

$(\mathcal{L}_2)$ $\quad \zeta_{14} = 0, \zeta_{20} = 0, \zeta_{23} \neq 0, \zeta_{24} \neq 0 \quad \Rightarrow \overset{3}{\cup};$

$(\mathcal{L}_3)$ $\quad \zeta_{14} = 0, \zeta_{20} = 0, \zeta_{23} = 0, \zeta_{24} \neq 0 \quad \Rightarrow \infty\overset{3}{\cup}.$

Proof. Assume that quadratic system the conditions $\widetilde{M} = 0$ and $C_2 \neq 0$ are fulfilled. Then via a linear transformation this system can be brought to the canonical form (3.88). According to Lemma 2.4 for a system (3.88) to possess an invariant parabola the conditions $\chi_1 = \chi_2 = 0$ are necessary. Moreover it was shown earlier (see page 60) that a system (3.88) with $\chi_1 = 0$ via an affine transformation ant time rescaling can be brought to the form (3.89). Thus in what follows we consider the family of quadratic systems

$$\frac{dx}{dt} = a + cx + dy + gx^2, \quad \frac{dy}{dt} = b + fy - x^2 + gxy. \tag{3.97}$$

1: **The statement $(\mathcal{L}_1)$.** Considering (3.90) for these systems we calculate

$$\chi_1 = \chi_2 = 0, \ \ \zeta_{14} = g^2x^2, \ \ \zeta_{22} = 9d^3g^3\mathrm{Y}_4, \ \ \mathcal{R}_2 = 9d^2g^4/4 \tag{3.98}$$

and clearly the condition $\zeta_{14} \neq 0$ is equivalent to $g \neq 0$ and in this case the condition $\mathcal{R}_2 \neq 0$ gives us $d \neq 0$. Therefore we conclude that for $\zeta_{14}\mathcal{R}_2 \neq 0$ the condition $\zeta_{22} = 0$ is equivalent to $\mathrm{Y}_4 = 0$.

2: **The statement $(\mathcal{L}_2)$.** From (3.98) evidently the condition $\zeta_{14} = 0$ implies $g = 0$ and then we calculate

$$\zeta_{23} = -2(c-f)^2, \quad \zeta_{20} = -12d^2x^2.$$

Therefore the condition $\zeta_{20} = 0$ is equivalent to $d = 0$ whereas $\zeta_{23} \neq 0$ implies $c - f \neq 0$. So for $\zeta_{20} = 0$ we get $d = 0$ and then we calculate

$$\zeta_{24} = 24f(2c-f)x^3$$

and hence the condition $\zeta_{24} \neq 0$ is equivalent to $f(2c-f) \neq 0$. Since the condition $\zeta_{23} \neq 0$ is equivalent to $c - f \neq 0$, considering Lemma 3.17 we conclude that the statement $(\mathcal{L}_2)$ of Theorem 3.18 is proved.

3: **The statement $(\mathcal{L}_3)$.** Since the condition $\zeta_{23} = -2(c-f)^2 = 0$ gives us $f = c$ for systems (3.97) with $g = d = 0$ and $f = c$ we calculate $\zeta_{24} = 24c^2x^3$ and clearly the condition $\zeta_{24} \neq 0$ is equivalent to $c \neq 0$. This completes the proof of Theorem 3.18. □

3.5 Systems with infinite line filled up with singularities

According to Lemma 2.3 in the case $C_2 = 0$ systems (2.5) via a linear transformation could be brought to the systems $(\mathbf{S}_V)$ for which in addition we may assume $e = f = 0$ due to a translation. So we consider the following family of quadratic systems

$$\frac{dx}{dt} = a + cx + dy + x^2, \quad \frac{dy}{dt} = b + xy. \tag{3.99}$$

We prove the following lemma.

Lemma 3.19. *A non-degenerate quadratic system (3.99) could only have invariant parabola of the form $\Phi(x,y) = p + qx + ry + x^2$ with $r \neq 0$. Moreover it possesses an invariant parabola of this form if and only if the following conditions hold:*

$$d \neq 0, \ \Upsilon_5 = 9ac - 2c^3 + 27bd = 0.$$

Proof. Suppose that these systems possess an invariant parabola

$$\Phi(x,y) \equiv p + qx + ry + sx^2 + 2vxy + uy^2 = 0$$

with $v^2 - su = 0$ and $u \neq 0$, i.e. its quadratic part is not of the form sx^2. Then clearly we may assume $u = 1$ and then we obtain $s = v^2$, i.e. we get the parabola

$$\Phi(x,y) \equiv p + qx + ry + (vx+y)^2 = 0, \tag{3.100}$$

for which the condition $q \neq rv$ must hold, otherwise we get a reducible conic.

Considering equations (2.6) and the form of the parabola (3.100) with $q \neq rv$, for systems (3.99) we obtain

$$s = v^2, \ u = 1, \ Eq_3 = 2 - U - 2vV, \ Eq_4 = -V$$

and evidently the equations $Eq_3 = 0$ and $Eq_4 = 0$ imply $V = 0$ and $U = 2$. Then calculations yield

$$Eq_5 = -q + 2cv^2 - v^2W = 0, \ Eq_6 = -r + 2cv + 2dv^2 - 2vW = 0, \ Eq_7 = 2dv - W = 0$$

and we get

$$W = 2dv, \quad q = 2v^2(c - dv), \quad r = 2v(c - dv) \;\Rightarrow\; q = rv.$$

So we obtain a reducible conic.

Thus we have proved that if systems (3.99) possess an invariant parabola then it is necessary of the form

$$\Phi(x, y) \equiv p + qx + ry + sx^2 = 0$$

with $s \neq 0$ and $r \neq 0$, otherwise we get a reducible conic. Then we may assume $s = 1$ and again, considering the ten equations (2.6) and the above parabola, for systems (3.99) we obtain

$$s = 1, \; u = v = 0, \;\; Eq_1 = 2 - U = 0, \;\; Eq_2 = -V = 0 \;\Rightarrow\; V = 0, \; U = 2$$

and then calculations yield:

$$Eq_5 = 2c - q - W = 0, \;\; Eq_6 = 2d - r = 0 \;\Rightarrow\; r = 2d \neq 0, \; W = 2c - q.$$

Therefore evaluating the remaining equations we obtain

$$Eq_8 = 2a - 2p - cq + q^2, \;\; Eq_9 = d(3q - 4c), \;\; Eq_{10} = 2bd - 2cp + aq + pq.$$

Since $d \neq 0$ (due to $r \neq 0$) the equation $Eq_9 = 0$ gives us $q = 4c/3$ and then from $Eq_8 = 0$ we get $p = (9a + 2c^2)/9$. In this case we obtain

$$Eq_{10} = \frac{2}{27}(9ac - 2c^3 + 27bd) = \frac{2}{27}\Upsilon_5 = 0.$$

Since $d \neq 0$ the condition $\Upsilon_5 = 0$ implies $b = \dfrac{c}{27d}(2c^2 - 9a)$ and we arrive at the systems

$$\dot{x} = a + cx + dy + x^2, \quad \dot{y} = \frac{c}{27d}(2c^2 - 9a) + xy \tag{3.101}$$

which possess the following invariant parabola:

$$\Phi(x, y) = \frac{1}{9}\left(9a + 2c^2\right) + \frac{4c}{3}x + 2dy + x^2, \quad d \neq 0. \tag{3.102}$$

This complete the proof of the Lemma 3.19. □

Evaluating for systems (3.99) the invariant polynomials ζ_5 and ζ_{22} we obtain

$$\zeta_5 = -891d^2/4, \;\; \zeta_{22} = 9d^3(9ac - 2c^3 + 27bd) = 9d^3\Upsilon_5.$$

So the condition $d \neq 0$ is equivalent to $\zeta_5 \neq 0$ and in this case the condition $\Upsilon_5 = 0$ is equivalent to $\zeta_{22} = 0$. Therefore considering Lemma 3.19 we conclude that the following theorem is valid.

Theorem 3.20. *Assume that for a non-degenerate quadratic system the condition $C_2 = 0$ holds. Then this system possesses an invariant parabola (which is unique) if and only if the conditions $\zeta_5 \neq 0$ and $\zeta_{22} = 0$ hold.*

In order to determine a simpler canonical form of systems (3.101) we apply to these systems as well as to parabola (3.102) the translation

$$x = x_1 - \frac{2c}{3}, \quad y = y_1 - \frac{9a - 2c^2}{18d}.$$

Then we could set the following new notations:

$$k = -2d, \; m = -\frac{9a - 2c^2}{36d}, \; n = -\frac{c}{3} \; \Rightarrow$$
$$a = 2(km + n^2), \; c = -3n, \; d = -\frac{k}{2},$$

where $k \neq 0$ due to $d \neq 0$. Then we arrive at the family of systems

$$\dot{x}_1 = km + nx_1 - \frac{k}{2} y_1 + x_1^2, \quad \dot{y}_1 = 2mx_1 + 2ny_1 + x_1 y_1,$$

which possess the invariant parabola

$$\Phi(x_1, y_1) = x_1^2 - ky_1, \quad k \neq 0.$$

Finally applying the rescaling $(x_1, y_1, t_1) \mapsto (kx, ky, t/k)$ we arrive at the systems

$$\dot{x} = m + nx - y/2 + x^2, \quad \dot{y} = 2mx + 2ny + xy,$$

which possess the invariant parabola $\Phi(x, y) = x^2 - y$.

As all the cases are investigated we conclude that the Main Theorem is proved.

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