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Leighton–Wintner type oscillation criteria for second-order differential equations with $p(t)$-Laplacian

Kōdai Fujimoto✉ [1] and **Masakazu Onitsuka**[2]

[1]Institute of Science and Engineering, Academic Assembly,
Shimane University, Matsue 690-8504, Japan
[2]Department of Applied Mathematics, Okayama University of Science, Okayama 700-0005, Japan

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Abstract. This paper deals with the oscillation problems for nonlinear differential equations of the form $(r(t)|x'|^{p(t)-2}x')' + c(t)f(x) = 0$ involving $p(t)$-Laplacian. The Leighton–Wintner type oscillation criteria are established without any conditions on the limit of $p(t)$. In addition, we discuss the applications to partial differential equations. Some examples are given to illustrate our results.

Keywords: oscillation, second-order differential equation, $p(t)$-Laplacian, quasilinear differential equation, Emden–Fowler differential equation, Riccati technique.

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1 Introduction

In this paper, we consider the second-order nonlinear differential equation

$$\Big(r(t)|x'|^{p(t)-2}x'\Big)' + c(t)f(x) = 0, \quad t \ge t_0 \in \mathbb{R}, \tag{1.1}$$

where $r(t) > 0$, $c(t)$, and $p(t) > 1$ are continuous functions, and $f(u)$ is a continuous function satisfying the condition $uf(u) > 0$ for $u \neq 0$.

A function $x(t)$ is said to be a *solution* of equation (1.1) defined on $[t_0, \tau) \subset \mathbb{R}$, if $x(t)$ and the quasiderivative $r(t)|x'(t)|^{p(t)-2}x'(t)$ are continuously differentiable and $x(t)$ satisfies equation (1.1) on $[t_0, \tau)$. A nontrivial solution $x(t)$ of equation (1.1) is said to be a *singular solution of the first kind*, if there exists a number $T_x > t_0$ such that $x(t) \equiv 0$ for $t \ge T_x$. It is said to be a *singular solution of the second kind* if $\tau < \infty$, which means that $x(t)$ is nonextendable to the right, i.e.,

$$\limsup_{t\to\tau-} \left(|x(t)| + |x'(t)|\right) = \infty$$

✉Corresponding author. Email: kfujimoto@riko.shimane-u.ac.jp

holds. It is said to be a *proper* solution if $x(t)$ is nonsingular. Furthermore, a proper solution $x(t)$ of equation (1.1) can be divided into the following two types. It is called *oscillatory*, if there exists a sequence $\{t_n\}$ of $[t_0, \infty)$ such that $x(t_n) = 0$ and $t_n \to \infty$ as $n \to \infty$. Otherwise, it is called *nonoscillatory*.

A great deal of papers have been devoted to the oscillation problems for the quasilinear differential equation

$$\left(r(t)|x'|^{p-2}x'\right)' + c(t)|x|^{p-2}x = 0 \tag{1.2}$$

involving the classical p-Laplacian. It is easy to see that the constant multiple of a solution of equation (1.2) is also a solution, but the sum of solutions is not always a solution. In this point of view, equation (1.2) is known as a *half-linear* differential equation (see [1, 8]). With this advantage, we can introduce the generalized trigonometric functions and Sturm's separation and comparison theorems as basic tools for p-Laplacian. Moreover, the global existence and uniqueness of solutions of equation (1.2) are guaranteed for initial-value problem, i.e., all nontrivial solutions of equation (1.2) are proper. For example, various results for the oscillation problems for equation (1.2) can be found in [1, 7–9, 14–17, 22] and the references cited therein. Especially, the so-called Leighton–Wintner type oscillation criterion has been obtained as follows.

Theorem A ([1,8])**.** *Suppose that*

$$\int_{t_0}^{\infty} \left(\frac{1}{r(t)}\right)^{1/(p-1)} dt = \infty \quad \textit{and} \quad \int_{t_0}^{\infty} c(t)\, dt = \infty.$$

Then, all nontrivial solutions of equation (1.2) *are oscillatory.*

The differential operator in equation (1.1) is called $p(t)$*-Laplacian*, which is a generalization of p-Laplacian. It is also known as the one-dimensional version of the partial differential operator $p(\mathbf{x})$-Laplacian, which appears in mathematical models of various research fields such as nonlinear elasticity theory, electrorheological fluids, and image processing (see [4, 13, 18]). For example, oscillation problems for quasilinear elliptic partial differential equations with $p(\mathbf{x})$-Laplacian are considered in [23–25]. In particular, sufficient conditions are obtained under which all radial solutions of the equation

$$\operatorname{div}\left(|\nabla u|^{p(\mathbf{x})-2}\nabla u\right) + \frac{1}{|\mathbf{x}|^{\theta(\mathbf{x})}}|u|^{q(\mathbf{x})-2}u = 0 \quad \text{in } \Omega$$

are oscillatory in [25] under certain conditions on the limits of p, θ, and q, where $\Omega = \{\mathbf{x} \in \mathbb{R}^N \mid |x| > r_0\}$ with the Euclidean norm. The proof is based on radialization technique with ordinary differential equation involving $p(t)$-Laplacian. In this way, there has been an increasing interest in the study of asymptotic behavior of solutions for ordinary differential equations involving $p(t)$-Laplacian. For instance, those results can be found in [3,5,6,10–12,19–21]. In [10], a kind of comparison theorem is proved to the oscillation problems for equation (1.1). In addition, the existence of proper solutions and singular solutions of equation (1.1) is treated in [3].

However, we point out that the solution space of the equation

$$\left(r(t)|x'|^{p(t)-2}x'\right)' + c(t)|x|^{p(t)-2}x = 0 \tag{1.3}$$

involving $p(t)$-Laplacian does not have homogeneity unlike equation (1.2). Hence, to the best of our knowledge, generalized trigonometric functions and Sturm's separation and comparison theorems are not obtained for equation (1.3). Hence, not a few results do not rule out the coexistence of oscillatory and nonoscillatory solutions. Moreover, the literature on $p(t)$-Laplacian often assumes certain conditions on the limit of $p(t)$. For example, the log-Hölder decay condition is assumed in [12,25], i.e., there exist $p > 1$, and $M > 0$ such that

$$t^{|p-p(t)|} < M$$

for t sufficiently large. This implies that $p(t) \to p > 1$ as $t \to \infty$.

The purpose of this paper is to establish Leighton–Wintner type oscillation criteria for equation (1.1). This paper is organized as follows. In Section 2, we give two oscillation criteria. In Section 3, we deal with the existence of proper solutions. Finally, we consider an application to partial differential equations in Section 4.

2 Oscillation problem

In this section, we give Leighton–Wintner type oscillation criteria for equation (1.1).

Theorem 2.1. *Assume that $f(u)$ is a smooth function satisfying $f'(u) \geq 0$ for $u \in \mathbb{R}$. Suppose that for any $L > 0$,*

$$\int_{t_0}^{\infty} \left(\frac{L}{r(t)} \right)^{1/(p(t)-1)} dt = \infty \quad \text{and} \quad \int_{t_0}^{\infty} c(t)\, dt = \infty. \tag{2.1}$$

Then, all proper solutions of equation (1.1) are oscillatory.

Proof. Suppose, toward a contradiction, that equation (1.1) has a positive solution. That is to say, there exists $t_1 \geq t_0$ such that $x(t) > 0$ for $t \geq t_1$. Let

$$w(t) = \frac{r(t)|x'(t)|^{p(t)-2}x'(t)}{f(x(t))}.$$

Then, we have

$$w'(t) = -c(t) - \frac{r(t)|x'(t)|^{p(t)} f'(x(t))}{(f(x(t)))^2}.$$

Integrating both sides of this equality from t_1 to $t \geq t_1$, we get

$$w(t) = w(t_1) - \int_{t_1}^{t} c(s)\, ds - \int_{t_1}^{t} \frac{r(s)|x'(s)|^{p(s)} f'(x(s))}{(f(x(s)))^2}\, ds.$$

From (2.1) and $f'(u) \geq 0$ ($u \in \mathbb{R}$), there exists $t_2 \geq t_1$ such that

$$\int_{t_2}^{t} c(s)\, ds \geq 0$$

and $w(t) < 0$ for $t \geq t_2$, which implies $x'(t) < 0$ for $t \geq t_2$.

Integrating both sides of equation (1.1) from t_2 to $t \geq t_2$, we get

$$\begin{aligned}
-r(t)|x'(t)|^{p(t)-1} &= r(t)|x'(t)|^{p(t)-2}x'(t) \\
&= r(t_2)|x'(t_2)|^{p(t_2)-2}x'(t_2) - \int_{t_2}^{t} c(s) f(x(s))\, ds \\
&= r(t_2)|x'(t_2)|^{p(t_2)-2}x'(t_2) - f(x(t)) \int_{t_2}^{t} c(s)\, ds \\
&\quad + \int_{t_2}^{t} f'(x(s))x'(s) \int_{t_2}^{s} c(\tau)\, d\tau\, ds \\
&\leq r(t_2)|x'(t_2)|^{p(t_2)-2}x'(t_2) = -r(t_2)|x'(t_2)|^{p(t_2)-1}.
\end{aligned}$$

Hence, we have

$$-x'(t) \geq \left(\frac{K}{r(t)}\right)^{\frac{1}{p(t)-1}}$$

for $t \geq t_2$, where $K = r(t_2)|x'(t_2)|^{p(t_2)-1} > 0$. Thus, by (2.1) we obtain

$$x(t) \leq x(t_2) - \int_{t_2}^{t} \left(\frac{K}{r(s)}\right)^{\frac{1}{p(s)-1}} ds \to -\infty$$

as $t \to \infty$, which is a contradiction to the positivity of $x(t)$. □

We also prove the following criterion.

Theorem 2.2. *Assume that $c(t) > 0$ for $t \geq t_0$ and there exists a smooth function $g(u)$ such that $ug(u) > 0$ ($u \neq 0$), $g'(u) \geq 0$, and $|f(u)| \geq |g(u)|$ ($u \in \mathbb{R}$). Suppose that (2.1) holds for any $L > 0$. Then, all proper solutions of equation (1.1) are oscillatory.*

Proof. Suppose, toward a contradiction, that equation (1.1) has a positive solution $x(t)$. That is to say, there exists $t_1 \geq t_0$ such that $x(t) > 0$ for $t \geq t_1$. Hence, $x(t)$ satisfies

$$\left(r(t)|x'(t)|^{p(t)-2}x'(t)\right)' + C(t)g(x(t)) = 0, \quad t \geq t_1, \tag{2.2}$$

where $C(t) = c(t)f(x(t))/g(x(t))$. We note that $C(t)$ is continuous because $x(t) > 0$ for $t \geq t_1$ and $ug(u) > 0$ for $u \neq 0$. Since $|f(u)| \geq |g(u)|$ ($u \in \mathbb{R}$), we see that $C(t) \geq c(t)$, and therefore, we get

$$\int_{t_0}^{\infty} C(t)\, dt \geq \int_{t_0}^{\infty} c(t)\, dt = \infty.$$

Proceeding in the same manner as the proof of Theorem 2.1 with (2.2), we see that the assertion holds. □

Remark 2.3. Although the positivity of $c(t)$ is required, we don't need the monotonicity and the smoothness of $f(u)$ in Theorem 2.2.

We consider the special case that $f(u) = |u|^{\lambda-2}u$, where $\lambda > 1$ is a constant. Then, equation (1.1) becomes the equation

$$\left(r(t)|x'|^{p(t)-2}x'\right)' + c(t)|x|^{\lambda-2}x = 0. \tag{2.3}$$

In the rest of this paper, for simplicity, we focus on equation (2.3). By Theorem 2.1, we give the following corollary.

Corollary 2.4. *Suppose that (2.1) holds for any $L > 0$. Then, all proper solutions of equation (2.3) are oscillatory.*

3 Existence of proper solutions

In order to deal with the asymptotic behavior of solutions, we must pay attention to the existence of singular solutions. In fact, for example, when $p(t) \equiv p > 1$ and $r(t) \equiv 1$, equation (2.3) becomes the generalized Emden–Fowler type differential equation

$$\left(|x'|^{p-2}x'\right)' + c(t)|x|^{\lambda-2}x = 0. \tag{3.1}$$

It is known that if $p > \lambda$ (resp., $p < \lambda$) then equation (3.1) has a singular solution of the first (resp., second) kind for certain $c(t)$ (see [2, Theorem 4]).

In this section, we consider the existence of proper solutions of equation (2.3). According to [3], the following theorem is proved.

Theorem B ([3]). *Suppose that $p(t)$ and $(r(t))^{1/(p(t)-1)}$ are continuously differentiable, $p(t)$ is non-decreasing, and $c(t)$ is positive. Then, every nontrivial solutions of equation (2.3) is proper.*

Using Corollary 2.4 and Theorem B, we obtain the following corollary.

Corollary 3.1. *Assume that $p(t)$ and $(r(t))^{1/(p(t)-1)}$ are continuously differentiable, $p(t)$ is nondecreasing, and $c(t)$ is positive. Suppose that (2.1) holds for any $L > 0$. Then, all nontrivial solutions of equation (2.3) are oscillatory.*

We propose an example of Corollary 3.1.

Example 3.2. Let $t_0 = 1$, $r(t) \equiv 1$, $c(t) = 1/t$, and $p(t) = 3 - 1/t$. Then, equation (2.3) becomes

$$\left(|x'|^{3-1/t}x'\right)' + \frac{1}{t}|x|^{\lambda-2}x = 0. \tag{3.2}$$

From Corollary 3.1, all nontrivial solutions of equation (3.2) are oscillatory. Figure 3.1 indicates the solution is proper and oscillatory.

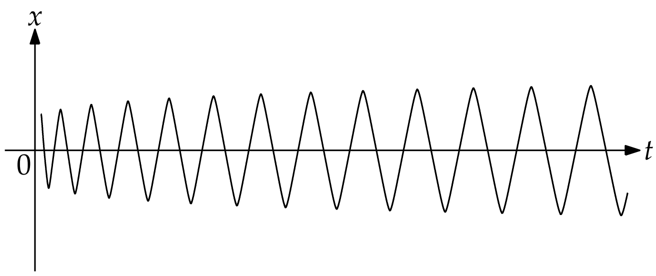

Figure 3.1: A solution $x(t)$ of equation (3.2) with $x(1) = 3$, $x'(1) = 0$, and $\lambda = 5$.

We next consider the case when $p(t)$ does not have monotonicity. For equation (2.3), the following propositions are derived from [3, Theorems 2.1, 2.2].

Proposition 3.3. *Suppose that $p(t) \le \lambda$ for $t \in [t_0, \infty)$. Then, equation (2.3) has no singular solutions of the first kind.*

Proposition 3.4. *Suppose that $p(t) \ge \lambda$ for $t \in [t_0, \infty)$. Then, equation (2.3) has no singular solutions of the second kind.*

In the case when $c(t)$ is negative, then the following result is given from Proposition 3.4.

Theorem 3.5. *Suppose that $p(t) \ge \lambda$ and $c(t) < 0$ for $t \in [t_0, \infty)$. Then, equation (2.3) has proper solutions.*

Proof. Let $x(t)$ be a solution of equation (2.3) satisfying the initial condition $x(t_0) > 0$ and $x'(t_0) > 0$. Since $c(t)$ is negative, we can find $T > t_0$ such that

$$\left(r(t)|x'(t)|^{p(t)-2}x'(t)\right)' = -c(t)|x(t)|^{\lambda-2}x(t) > 0$$

for $t \in [t_0, T)$, which implies that $r(t)|x'(t)|^{p(t)-2}x'(t)$ is positive increasing for $t \in [t_0, T)$. Hence, $x'(t)$ is positive for any $t \in [t_0, \infty)$, and therefore, $x(t)$ is a positive increasing solution. From Proposition 3.4, we see that $x(t)$ is proper. □

However, it is clear that (2.1) does not hold and equation (2.3) has no oscillatory solutions under the assumptions of Theorem 3.5.

In view of Propositions 3.3 and 3.4, we see that all nontrivial solutions of equation (2.3) are proper when $p(t) \equiv \lambda$. Otherwise, we cannot exclude the possibilities of the existence of singular solutions by using these propositions. To illustrate this problem, we introduce an example of Corollary 2.4 and Propositions 3.3, 3.4.

Example 3.6. Let $t_0 = 1$, $r(t) \equiv 1$, $c(t) = 1/t$, and $p(t) = \sin t + 5/2$. Then, equation (2.3) becomes

$$\left(|x'|^{\sin t + 1/2} x'\right)' + \frac{1}{t}|x|^{\lambda - 2}x = 0. \tag{3.3}$$

In the case of $\lambda \geq 7/2$, equation (3.3) has no singular solution of the first kind, as stated in Proposition 3.3. However, we cannot rule out the possibility that equation (3.3) has singular solutions of the second kind. In fact, keen spikes can be observed in Figure 3.2. On the other hand, when $1 < \lambda \leq 3/2$, equation (3.3) has no singular solution of the second kind according to Proposition 3.4. However, we cannot exclude the possibility that equation (3.3) has singular solutions of the first kind. We can identify the points in Figure 3.3 where the derivative of the solution is zero, even though they are not extrema. In the case of $3/2 < \lambda < 7/2$, there are possibilities that equation (3.3) has singular solutions of the first/second kind. In any cases, it can be derived from Corollary 2.4 that all proper solutions of equation (3.3) are oscillatory.

In the case of $p(t) \not\equiv \lambda$, the existence of proper solutions of equation (2.3) is proved by Theorem 4.1 in [3] under the additional assumption $\liminf_{t\to\infty} p(t) > 1$. However, in order to apply this result, the condition

$$\int_{t_0}^{\infty} |c(t)|\, dt < \infty$$

is also required, which is the opposite case of (2.1). It is an open problem if equation (2.3) has proper solutions under $p(t) \not\equiv \lambda$ and (2.1).

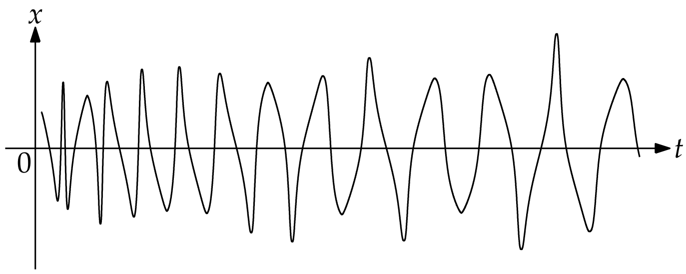

Figure 3.2: A solution $x(t)$ of equation (3.3) with $x(1) = 3$, $x'(1) = 0$, and $\lambda = 4$.

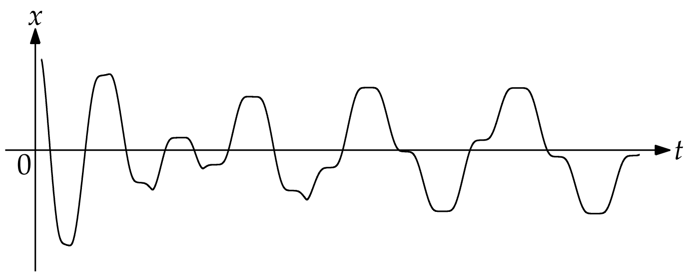

Figure 3.3: A solution $x(t)$ of equation (3.3) with $x(1) = 1$, $x'(1) = 0$, and $\lambda = 3/2$.

4 Applications

In this section, we propose an application to partial differential equations. Let us consider the quasilinear differential equation

$$\operatorname{div}\left(|\nabla u|^{p(\mathbf{x})-2}\nabla u\right) + F(\mathbf{x})|u|^{\lambda-2}u = 0 \quad \text{in } \Omega, \tag{4.1}$$

where $\Omega = \{\mathbf{x} \in \mathbb{R}^N \mid |x| > r_0\}$. If u is a radially symmetric function, i.e., $u(\mathbf{x}) = y(t)$, $t = |\mathbf{x}|$, we can write equation (4.1) as

$$\left(t^{N-1}|y'|^{p(t)-2}y'\right)' + t^{N-1}F(t)|y|^{\lambda-2}y = 0 \quad \text{for } t > r_0. \tag{4.2}$$

We say that a radially symmetric solution $u(\mathbf{x})$ of (4.1) is oscillatory if it keeps neither positive nor negative, that is, the solution $y(t)$ of equation (4.2) corresponding to $u(\mathbf{x})$ is oscillatory. Using Corollary 2.4, we obtain the following theorem.

Theorem 4.1. *Suppose that for any $L > 0$,*

$$\int_{t_0}^{\infty} \left(\frac{L}{t^{N-1}}\right)^{1/(p(t)-1)} dt = \infty \quad \text{and} \quad \int_{t_0}^{\infty} t^{N-1}F(t)\,dt = \infty.$$

Then, all radially symmetric solutions of equation (4.1) are oscillatory.

Example 4.2. Let $N \in \mathbb{N}$, $F(t) = 1/t^N$, and $p(t) = \sin t + N + 3/2$. Then, equation (4.2) becomes

$$\left(t^{N-1}|y'|^{\sin t+N-1/2}y'\right)' + \frac{1}{t}|y|^{\lambda-2}y = 0 \text{ for } t > r_0 \tag{4.3}$$

and it is easy to see that $\int_{r_0}^{\infty} t^{N-1}F(t)\,dt = \infty$. In addition, we have $1/(p(t)-1) \le 2/(2N-1)$. Hence, it is obvious that

$$\int_{r_0}^{\infty} \left(\frac{L}{t^{N-1}}\right)^{1/(p(t)-1)} dt = \infty$$

when $N = 1$. In the case of $N \ge 2$, since $L/t^{N-1} \to 0$ as $t \to \infty$, we can find $r_1 \ge r_0$ such that $L/t^{N-1} < 1$. Hence, we have

$$\begin{aligned}\int_{r_1}^{t} \left(\frac{L}{s^{N-1}}\right)^{1/(p(s)-1)} ds &\ge \int_{r_1}^{t} \left(\frac{L}{s^{N-1}}\right)^{2/(2N-1)} ds = L^{2/(2N-1)} \int_{r_1}^{t} s^{-2(N-1)/(2N-1)}\,ds \\ &= (2N-1)L^{2/(2N-1)}\left(t^{1/(2N-1)} - r_1^{1/(2N-1)}\right) \to \infty\end{aligned}$$

as $t \to \infty$. From Theorem 4.1, all radially symmetric solutions of equation (4.1) are oscillatory.

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