Electronic Journal of Qualitative Theory of Differential Equations
2024, No. **39**, 1–18; https://doi.org/10.14232/ejqtde.2024.1.39 www.math.u-szeged.hu/ejqtde/

Note on oscillation of neutral differential equations with multiple delays

Petr Hasil[1], **Michal Pospíšil**[✉ 2,3], **Lucia Pospíšilová Škripková**[4] and **Michal Veselý**[1]

[1]Department of Mathematics and Statistics, Faculty of Science, Masaryk University, Kotlářská 2, 611 37 Brno, Czech Republic
[2]Department of Mathematical Analysis and Numerical Mathematics, Faculty of Mathematics, Physics and Informatics, Comenius University in Bratislava, Mlynská dolina, 842 48 Bratislava, Slovakia
[3]Mathematical Institute, Slovak Academy of Sciences, Štefánikova 49, 814 73 Bratislava, Slovakia
[4]Department of Mathematics, Institute of Computer Science and Mathematics, Faculty of Electrical Engineering and Information Technology, Slovak University of Technology in Bratislava, Ilkovičova 3, 812 19 Bratislava, Slovakia

Received 14 March 2024, appeared 31 July 2024
Communicated by John R. Graef

Abstract. This note is a reaction on a recently published sufficient condition for oscillation of all solutions of a neutral delay differential equation. It is shown by a counterexample that the result is not correct and the problem is explained in details. Several, more general, classes of neutral differential equations with time-dependent discrete, distributed as well as mixed delays are considered. New sufficient conditions for oscillation of all their solutions are proved. Applications are given for illustration.

Keywords: oscillatory solution, discrete delay, distributed delay, mixed delay.

2020 Mathematics Subject Classification: 34K11, 34K06, 34C10.

1 Introduction

Oscillation theory for first order linear neutral differential equations with delay,

$$[x(t) + P(t)\,x(t-\tau)]' + Q(t)\,x(t-\sigma) = 0, \quad t \geq t_0,$$

has attracted researchers' interest for decades (see, e.g., [1–3] and references therein).

Recently, a sufficient condition was proved in [5] for oscillation of all solutions of the neutral delay differential equation

$$[x(t) - x(\tau(t))]' + Q(t)\,x(\sigma(t)) = 0, \quad t \geq t_0 \tag{1.1}$$

for some $t_0 \in \mathbb{R}$, where $Q \in C([t_0,\infty),\mathbb{R}_+)$, $\mathbb{R}_+ = [0,\infty)$, and $\tau, \sigma \in \mathcal{T}_{t_0}$ with

$$\mathcal{T}_\xi = \left\{ f \in C([\xi,\infty),\mathbb{R}) \;\middle|\; \begin{array}{l} f \text{ is strictly increasing;} \\ f(t) < t \;\forall t \geq \xi;\; \lim_{t\to\infty} f(t) = \infty \end{array} \right\}. \tag{1.2}$$

✉ Corresponding author. Email: michal.pospisil@fmph.uniba.sk

However, as we shall show in this paper, the proof is not correct and the statement does not hold. It is worth to mention that the limit property of functions from $\mathcal{T}_\xi$ was not explicitly assumed in [5], but it was applied in the proof.

In this paper, we prove a similar statement by a cautious use of a mathematical induction. We give a short remark explaining the problem of the original proof from [5]. Next, we generalize the result for the case of a convex combination of multiple discrete delays. We also consider neutral differential equations with distributed and mixed delays, and we prove analogous statements. More precisely, in addition to equation (1.1), the following equations are investigated in this paper:

$$\left[x(t) - \sum_{i=1}^{n} \lambda_i \, x(\tau_i(t))\right]' + \sum_{j=1}^{m} Q_j(t)\, x(\sigma_j(t)) = 0, \quad t \geq t_0, \tag{1.3}$$

$$\left[x(t) - \frac{\int_{\underline{\tau}(t)}^{\overline{\tau}(t)} \lambda(s) x(s)\, ds}{\int_{\underline{\tau}(t)}^{\overline{\tau}(t)} \lambda(s)\, ds}\right]' + Q(t) \int_{\underline{\sigma}(t)}^{\overline{\sigma}(t)} R(s) x(s)\, ds = 0, \quad t \geq t_0, \tag{1.4}$$

and

$$\left[x(t) - \left(\sum_{i=1}^{n_1} \lambda_i(t) x(\tau_i(t)) + \sum_{i=1}^{n_2} \int_{\underline{\tau}_i(t)}^{\overline{\tau}_i(t)} \vartheta_i(s) x(s)\, ds\right)\right]' + \sum_{j=1}^{m_1} Q_j(t) x(\sigma_j(t)) + \sum_{j=1}^{m_2} S_j(t) \int_{\underline{\sigma}_j(t)}^{\overline{\sigma}_j(t)} R_j(s) x(s)\, ds = 0, \quad t \geq t_0 \tag{1.5}$$

with appropriate parameters (see Theorems 3.7, 3.9, and 3.11 below).

The paper is organized as follows. In the next section, we conclude preliminary results and introduce an auxiliary function. Section 3 is devoted to the main results of this paper – sufficient conditions for oscillation of all solutions of various classes of neutral differential equations with delays. In the final section, applications to equations with particular types of delays, and concrete examples are given for illustration.

Throughout the paper, we denote by $\mathbb{N}$ ($\mathbb{N}_0$) the set of all positive (nonnegative) integers.

2 Preliminaries

In this section, we introduce some notation and prove auxiliary results.

Let us fix $\xi \in \mathbb{R}$ and consider $\tau \in \mathcal{T}_\xi$. Analogously to the iterations of function τ: $\tau^k = \tau \circ \tau^{k-1}$, $k \in \mathbb{N}$, $\tau^0 = id$, we denote $\tau^{-k} = \tau^{-1} \circ \tau^{-(k-1)}$, $k \in \mathbb{N}$ the iterations of the inverse function $\tau^{-1}\colon [\tau(\xi), \infty) \to [\xi, \infty)$. Then the following result holds.

Lemma 2.1. *Let $\xi \in \mathbb{R}$ and $\tau \in \mathcal{T}_\xi$. For any $\zeta \in [\tau(\xi), \infty)$, the sequence $\{\tau^{-k}(\zeta)\}_{k=1}^{\infty}$ is strictly increasing to ∞.*

Proof. Let $\zeta \in [\tau(\xi), \infty)$ be arbitrary and fixed. Then, $\tau(\zeta) < \zeta$ implies $\zeta < \tau^{-1}(\zeta)$, which yields $\tau^{-1}(\zeta) < \tau^{-2}(\zeta)$, etc. So, by induction, one can see that $\{\tau^{-k}(\zeta)\}_{k=1}^{\infty}$ is a strictly increasing sequence. Now, suppose by contrary that $\lim_{k\to\infty} \tau^{-k}(\zeta) = C < \infty$. Then,

$$C = \lim_{k\to\infty} \tau^{-k}(\zeta) = \lim_{k\to\infty} \tau^{-1}(\tau^{-(k-1)}(\zeta)) = \tau^{-1}\left(\lim_{k\to\infty} \tau^{-(k-1)}(\zeta)\right) = \tau^{-1}(C)$$

is a contradiction, and the proof is complete. □

For any $\zeta \in [\tau(\xi), \infty)$, we define a function $N_\zeta^\tau \colon [\zeta, \infty) \to \mathbb{N}$ such that for any $t \in [\zeta, \infty)$, $N_\zeta^\tau(t)$ satisfies

$$\tau^{-(N_\zeta^\tau(t)-1)}(\zeta) \le t < \tau^{-N_\zeta^\tau(t)}(\zeta). \tag{2.1}$$

Due to Lemma 2.1, function N_ζ^τ is well defined. Note that

$$N_\zeta^\tau([\tau^{-k}(\zeta), \tau^{-(k+1)}(\zeta))) = k + 1 \tag{2.2}$$

for each $k \in \mathbb{N}_0$. Then, it is easy to see that N_ζ^τ in nondecreasing on $[\zeta, \infty)$ and unbounded from above. Another important property of N_ζ^τ is proved in the next lemma.

Lemma 2.2. *Let $\xi \in \mathbb{R}$, $\tau \in \mathcal{T}_\xi$, $\alpha_1 \in [\xi, \infty)$, $\alpha_2 = \tau^{-k}(\alpha_1)$ for some $k \in \mathbb{N}_0$. Then*

$$N_{\alpha_1}^\tau(t) = N_{\alpha_2}^\tau(t) + k, \quad t \ge \alpha_2.$$

Proof. For any $t \ge \alpha_2$,

$$\begin{aligned}
\tau^{-(N_{\alpha_2}^\tau(t)+k-1)}(\alpha_1) &= \tau^{-(N_{\alpha_2}^\tau(t)+k-1)}(\tau^k(\alpha_2)) \\
&= \tau^{-(N_{\alpha_2}^\tau(t)-1)}(\alpha_2) \le t < \tau^{-N_{\alpha_2}^\tau(t)}(\alpha_2) \\
&= \tau^{-N_{\alpha_2}^\tau(t)}(\tau^{-k}(\alpha_1)) = \tau^{-(N_{\alpha_2}^\tau(t)+k)}(\alpha_1).
\end{aligned}$$

But we know that for any $t \ge \alpha_2$ (even for any $t \ge \alpha_1$), there is a unique $\kappa \in \mathbb{N}$ satisfying $\tau^{-(\kappa-1)}(\alpha_1) \le t < \tau^{-\kappa}(\alpha_1)$, and it is given by $\kappa = N_{\alpha_1}^\tau(t)$. Therefore, $N_{\alpha_1}^\tau(t) = N_{\alpha_2}^\tau(t) + k$. $\square$

We will investigate solutions of equation (1.1) in the sense of the following definitions.

Definition 2.3. Let $t_0 \in \mathbb{R}$ and $Q \in C([t_0, \infty), \mathbb{R}_+)$, $\tau, \sigma \in \mathcal{T}_{t_0}$, $\varphi \in C([\min\{\tau(t_0), \sigma(t_0)\}, t_0], \mathbb{R})$ be given functions. We say that

$$x \in C([\min\{\tau(t_0), \sigma(t_0)\}, \infty), \mathbb{R})$$

is a solution of equation (1.1) along with the initial condition

$$x(t) = \varphi(t), \quad t \in [\min\{\tau(t_0), \sigma(t_0)\}, t_0] \tag{2.3}$$

if $x(t) - x(\tau(t))$ is continuously differentiable for all $t \in [t_0, \infty)$ and x satisfies (1.1), (2.3).

In the rest of the paper, we often omit initial condition (2.3). So, x is a solution of (1.1) if there exists a suitable function φ such that x solves initial value problem (1.1), (2.3).

Definition 2.4. Let $t_0 \in \mathbb{R}$ and $Q \in C([t_0, \infty), \mathbb{R}_+)$, $\tau, \sigma \in \mathcal{T}_{t_0}$ be given functions. Solution x of (1.1) is called eventually positive (eventually negative) if there is $T > t_0$ such that $x(t) > 0$ ($x(t) < 0$) for all $t \ge T$. In this case, x is called nonoscillatory. Otherwise, we say that x oscillates or that it is oscillatory.

In other neutral differential equations studied in the paper, their solutions are understood in an analogous sense.

Finally, in this section, we present an auxiliary lemma.

Lemma 2.5. *Let $A \ge B \ge 0$ and $\alpha > 1$. Then*

$$(A - B)^{\frac{1}{\alpha}} \ge A^{\frac{1}{\alpha}} - B^{\frac{1}{\alpha}}.$$

Proof. If $A = 0$, the statement is obvious. Now, let $A \neq 0$ and consider the function $f(x) = (1-x)^{\frac{1}{\alpha}} - (1 - x^{\frac{1}{\alpha}})$ for $x \in [0,1]$. Then $f(0) = 0 = f(1)$. The derivative,

$$f'(x) = \frac{1}{\alpha}\left(x^{\frac{1-\alpha}{\alpha}} - (1-x)^{\frac{1-\alpha}{\alpha}}\right)$$

vanishes if and only if $x = \frac{1}{2}$. Since

$$f\left(\frac{1}{2}\right) = \frac{2 - 2^{\frac{1}{\alpha}}}{2^{\frac{1}{\alpha}}} > 0,$$

we get that $f(x) \geq 0$ for all $x \in [0,1]$. In particular, $f(\frac{B}{A}) \geq 0$ which proves the statement. □

3 Main results

Here, we recall the result from [5] and provide a counterexample to show that it does not hold. Next, by correcting the wrong proof from [5], we prove a new sufficient condition for oscillation of all solutions of equation (1.1). Then, we give a generalization to multiple discrete delays. In Subsection 3.2, an analogous problem is studied for neutral differential equations with distributed and mixed delays.

3.1 Discrete delays

In [5], the next result was stated (we use the quotation marks to warn readers that the result is not correct):

"Theorem" 3.1. *Let $t_0 > 0$ and $Q \in C([t_0,\infty),\mathbb{R}_+)$, $\tau,\sigma \in \mathcal{T}_{t_0}$ be given functions. If*

$$\int_{t_0}^{\infty} Q(s)\,ds = \infty \tag{3.1}$$

or

$$\int_{t_0}^{\infty} s\,Q(s)\,ds = \infty, \tag{3.2}$$

then every solution of equation (1.1) oscillates.

It will be shown in the proof of Theorem 3.3 below (and it was correctly proved in [5]) that inequality (3.1) is indeed a sufficient condition for oscillation of all solutions of (1.1). In the next example, we illustrate that if (3.1) does not hold, inequality (3.2) does not guarantee the oscillation of all solutions of (1.1).

Example 3.2. Let us consider the following equation

$$\left[x(t) - x\left(\frac{t}{2}\right)\right]' + \frac{1}{t^2}\,x\left(\frac{t}{2 - t\ln(2 - \mathrm{e}^{\frac{1}{t}})}\right) = 0, \quad t \geq t_0 \tag{3.3}$$

for some $t_0 > \frac{1}{\ln 2}$.

Since $0 < 2 - \mathrm{e}^{\frac{1}{t_0}} \leq 2 - \mathrm{e}^{\frac{1}{t}} < 1$, we have $\ln(2 - \mathrm{e}^{\frac{1}{t}}) < 0$ $\forall t \geq t_0$. So,

$$\sigma(t) = \frac{t}{2 - t\ln(2 - \mathrm{e}^{\frac{1}{t}})} < t, \quad t \geq t_0.$$

Furthermore, from

$$\sigma'(t) = \frac{4 - e^{\frac{1}{t}}}{\left(2 - e^{\frac{1}{t}}\right)\left(2 - t\ln(2 - e^{\frac{1}{t}})\right)^2},$$

one can see that $\sigma'(t) > 0$ for all $t \geq t_0$. It is easy to verify that $\sigma(t) \xrightarrow{t\to\infty} \infty$. Thus, $\sigma \in \mathcal{T}_{t_0}$. Clearly, $\tau \in \mathcal{T}_{t_0}$ for $\tau(t) = \frac{t}{2}$. Moreover,

$$\int_{t_0}^{\infty} \frac{ds}{s^2} = \frac{1}{t_0} < \infty, \qquad \int_{t_0}^{\infty} \frac{ds}{s} = \infty,$$

i.e., condition (3.1) is not satisfied, but (3.2) holds. Hence, by "Theorem" 3.1, every solution of equation (3.3) oscillates. However, a positive function $e^{-\frac{1}{t}}$ solves this equation. Indeed, for $x(t) = e^{-\frac{1}{t}}$, the left-hand side of (3.3) reads as

$$\frac{e^{-\frac{1}{t}}}{t^2} - \frac{2\,e^{-\frac{2}{t}}}{t^2} + \frac{1}{t^2}\,e^{-\frac{2 - t\ln(2 - e^{\frac{1}{t}})}{t}} = \frac{1}{t^2}\left[e^{-\frac{1}{t}} - 2\,e^{-\frac{2}{t}} + e^{-\frac{2}{t}}(2 - e^{\frac{1}{t}})\right] = 0.$$

Next, we present our result for equation (1.1).

Theorem 3.3. *Let $t_0 \in \mathbb{R}$ and $Q \in C([t_0, \infty), \mathbb{R}_+)$, $\tau, \sigma \in \mathcal{T}_{t_0}$ be given functions. If condition (3.1) is satisfied or*

$$\int_{\sigma^{-1}(t_0)}^{\infty} (N_{t_0}^{\tau}(\sigma(s)))^{\frac{1}{p}}\, Q(s)\, ds = \infty \tag{3.4$_p$}$$

for some $p > 1$, where $N_{t_0}^{\tau}(t)$ is given by (2.1), then every solution of equation (1.1) oscillates.

Note that in the label of condition $(3.4)_p$, we use the parameter $p > 1$ as the lower index.

Proof. One can easily see that condition (3.1) as well as condition $(3.4)_p$ implies that Q does not vanish for all t sufficiently large, i.e.,

$$\forall t \geq t_0\ \exists T \geq t : Q(T) > 0.$$

Without any loss of generality, we suppose in contrary that x is an eventually positive solution of (1.1). Since $\lim_{t\to\infty} \tau(t) = \lim_{t\to\infty} \sigma(t) = \infty$, there is $t_1 \geq t_0$ such that $x(t)$, $x(\tau(t))$ and $x(\sigma(t))$ are positive for all $t \geq t_1$. For $z(t) = x(t) - x(\tau(t))$, equation (1.1) gives $z'(t) \leq 0$ $\forall t \geq t_1$. Moreover, from the nonvanishing property of Q, we know that for any $t \geq t_1$ there is $T \geq t$ such that $z'(T) < 0$. Hence, $z(t)$ can not vanish for all sufficiently large t, but it is either eventually negative or eventually positive.

If z is eventually negative, then, since it is nonincreasing, there exist $t_2 \geq t_1$ and $\mu > 0$ such that $z(t) \leq -\mu$ for all $t \geq t_2$. Equivalently, we have

$$x(t) \leq x(\tau(t)) - \mu, \quad t \geq t_2.$$

In particular,

$$x(\tau^{-k}(t_2)) \leq x(\tau^{-(k-1)}(t_2)) - \mu \leq \cdots \leq x(t_2) - k\mu$$

for each $k \in \mathbb{N}$. A contradiction with the eventual positivity of x follows, since the right side tends to $-\infty$ as $k \to \infty$.

Hence, z is eventually positive, i.e., there is $t_2 \geq t_1$ such that $z(t) > 0$ $\forall t \geq t_2$. This means that

$$x(t) > x(\tau(t)) > 0, \quad t \geq t_2. \tag{3.5}$$

Therefore,

$$x(t) \geq \min_{s\in[\tau(t_2),t_2]} x(s) =: \omega > 0, \quad t \geq t_2. \tag{3.6}$$

From equation (1.1), we obtain

$$0 = z'(t) + Q(t)x(\sigma(t)) \geq z'(t) + \omega Q(t)$$

for all $t \geq t_3$ for some $t_3 \geq \sigma^{-1}(t_2)$. This gives

$$z(t) \leq z(t_3) - \omega \int_{t_3}^{t} Q(s)\,ds, \quad t \geq t_3.$$

So, if condition (3.1) is satisfied, we get $\lim_{t\to\infty} z(t) = -\infty$ which is a contradiction, and x is oscillatory.

Now, assume that

$$\int_{t_0}^{\infty} Q(s)\,ds < \infty \tag{3.7}$$

and that condition $(3.4)_p$ is satisfied for some $p > 1$. Let us take $t_4 \geq \tau^{-1}(t_2)$ such that $t_4 = \tau^{-\kappa}(t_0)$ for some $\kappa \in \mathbb{N}$.

From

$$x(t) = z(t) + x(\tau(t)), \quad t \geq t_4, \tag{3.8}$$

we get

$$x(t) = z(t) + z(\tau(t)) + \cdots + z(\tau^{(N-1)}(t)) + x(\tau^N(t))$$

for any $t \in [\tau^{-(N-1)}(t_4), \tau^{-N}(t_4))$, $N \in \mathbb{N}$. Since z is nonincreasing and $\tau \in \mathcal{T}_{t_0}$, this identity implies

$$x(t) \geq Nz(t) + x(\tau^N(t)), \quad t \in [\tau^{-(N-1)}(t_4), \tau^{-N}(t_4)),\ N \in \mathbb{N}$$

or, equivalently,

$$x(t) \geq N^{\tau}_{t_4}(t)z(t) + x(\tau^{N^{\tau}_{t_4}(t)}(t)), \quad t \geq t_4$$

(see (2.2)). Note that $\tau^{N^{\tau}_{t_4}(t)}(t) \in [\tau(t_4), t_4) \subset [t_2, \infty)$ for any $t \geq t_4$. Hence, by (3.6),

$$x(t) \geq N^{\tau}_{t_4}(t)z(t) + \omega, \quad t \geq t_4.$$

Next, using the Young inequality,

$$\frac{A^p}{p} + \frac{B^q}{q} \geq AB$$

for $A, B > 0$ and $q = \frac{p}{p-1}$, we derive

$$x(t) \geq \frac{\left(\left(pN^{\tau}_{t_4}(t)z(t)\right)^{\frac{1}{p}}\right)^p}{p} + \frac{\left((q\omega)^{\frac{1}{q}}\right)^q}{q} \geq \left(pN^{\tau}_{t_4}(t)z(t)\right)^{\frac{1}{p}} (q\omega)^{\frac{1}{q}}$$

for all $t \geq t_4$. Let us denote $\omega_1 := p^{\frac{1}{p}}(q\omega)^{\frac{1}{q}} > 0$ and take $t_5 = \sigma^{-1}(t_4)$. Then, (1.1) implies

$$\begin{aligned} z'(t) = -Q(t)x(\sigma(t)) &\leq -\omega_1 Q(t)\left(N^{\tau}_{t_4}(\sigma(t))z(\sigma(t))\right)^{\frac{1}{p}} \\ &\leq -\omega_1 Q(t)\left(N^{\tau}_{t_4}(\sigma(t))z(t)\right)^{\frac{1}{p}}, \quad t \geq t_5 \end{aligned}$$

since z is nonincreasing. Dividing by $z^{\frac{1}{p}}(t)$ and integrating over $[t_5, t]$ yields

$$\int_{t_5}^{t} \frac{z'(s)\,ds}{z^{\frac{1}{p}}(s)} = qz^{\frac{1}{q}}(t) - qz^{\frac{1}{q}}(t_5) \le -\omega_1 \int_{t_5}^{t} \left(N_{t_4}^{\tau}(\sigma(s))\right)^{\frac{1}{p}} Q(s)\,ds$$

for all $t \ge t_5$. Now, it only remains to prove that

$$\int_{t_5}^{\infty} \left(N_{t_4}^{\tau}(\sigma(s))\right)^{\frac{1}{p}} Q(s)\,ds = \infty. \tag{3.9}$$

Consequently, we get a contradiction with the eventual positivity of z, implying that x is oscillatory.

Using Lemmas 2.2, 2.5, we derive

$$\left(N_{t_4}^{\tau}(t)\right)^{\frac{1}{p}} = \left(N_{t_0}^{\tau}(t) - \kappa\right)^{\frac{1}{p}} \ge \left(N_{t_0}^{\tau}(t)\right)^{\frac{1}{p}} - \kappa^{\frac{1}{p}}$$

for all $t \ge t_4$. Therefore, assumptions $(3.4)_p$, (3.7) imply for $t \ge t_5$,

$$\begin{aligned}\int_{t_5}^{t} \left(N_{t_4}^{\tau}(\sigma(s))\right)^{\frac{1}{p}} Q(s)\,ds &\ge \int_{t_5}^{t} \left(N_{t_0}^{\tau}(\sigma(s))\right)^{\frac{1}{p}} Q(s)\,ds - \kappa^{\frac{1}{p}} \int_{t_5}^{t} Q(s)\,ds \\ &= \int_{\sigma^{-1}(t_0)}^{t} \left(N_{t_0}^{\tau}(\sigma(s))\right)^{\frac{1}{p}} Q(s)\,ds - \kappa^{\frac{1}{p}} \int_{t_0}^{t} Q(s)\,ds + C \xrightarrow{t\to\infty} \infty\end{aligned}$$

with an appropriate constant $C \in \mathbb{R}$. □

Remark 3.4. Condition (3.1) was proved in [5], but in the proof of Theorem 3.3, we emphasize were the missing assumption was needed. Namely, to get the existence of t_1.

Remark 3.5. The original proof of "Theorem" 3.1 from [5] contains the following issues:

1. Constant $\tau = \inf_{t \ge t_3}(t - \tau(t))$ was introduced and used as positive. However, the case $\tau(t) \nearrow t$ as $t \to \infty$ was not considered.

2. For fixed t, the value $x(\tau^{N(t)}(t))$ was used, where $N(t) = \left\lfloor \frac{t-t_3}{\tau} \right\rfloor^2$, τ is defined in the previous point of this remark, and $\lfloor \cdot \rfloor$ is the greatest integer function (or the floor function). This can be a problem if $N(t)$ is so large that $\tau^{N(t)}(t) < \tau(t_2)$, because then one can not use the estimation

$$x(\tau^{N(t)-1}(t)) > x(\tau^{N(t)}(t)).$$

   Similarly, we use estimation (3.5), but, in our case, $N_{t_4}^{\tau}(t)$ is bounded for any fixed $t \ge t_4$ (as it does not depend on the infimum).

3. The proof from [5] does not work even if τ is far from zero (e.g., constant delay). The problem is in the power 2 in the definition of $N(t)$ (see the previous point). Because then one can not iterate expansion (3.8) $N(t)$-times, due to $\tau^{N(t)}(t) < \tau(t_4)$.

Remark 3.6. Since $N_{t_0}^{\tau}(t) \in \mathbb{N}$ and $Q(t) \ge 0$ for all $t \ge t_0$, inequality $k^{\frac{1}{p_1}} \le k^{\frac{1}{p_2}}$ for each $k \in \mathbb{N}$ and all $1 \le p_2 \le p_1$ gives that, $(3.4)_{p_1}$ implies $(3.4)_{p_2}$ for any $1 < p_2 \le p_1$. Similarly, (3.1) implies $(3.4)_p$ for all $p > 1$.

Now, we generalize Theorem 3.3 to the case of multiple delays.

Theorem 3.7. *Let $t_0 \in \mathbb{R}$, $n, m \in \mathbb{N}$, $\lambda_i > 0$ for $i = 1, 2, \dots, n$ be such that $\sum_{i=1}^n \lambda_i = 1$, and $Q_j \in C([t_0, \infty), \mathbb{R}_+)$, $\tau_i, \sigma_j \in \mathcal{T}_{t_0}$ for $i = 1, 2, \dots, n$, $j = 1, 2, \dots, m$ be given functions. If there exists $j_0 \in \{1, 2, \dots, m\}$ such that*

$$\int_{t_0}^{\infty} Q_{j_0}(s)\, ds = \infty \tag{3.10}$$

or

$$\int_{\sigma_{j_0}^{-1}(t_0)}^{\infty} (N_{t_0}^{\underline{\tau}}(\sigma_{j_0}(s)))^{\frac{1}{p}} Q_{j_0}(s)\, ds = \infty \tag*{(3.11)$_p$}$$

for some $p > 1$, where $\underline{\tau} = \min_{i=1,2,\dots,n} \tau_i$ and $N_{t_0}^{\underline{\tau}}(t)$ is given by (2.1), then every solution of equation (1.3) oscillates.

Proof. In this proof, we skip some details that are the same as in the proof of Theorem 3.3.

As in the proof of Theorem 3.3, each one of conditions (3.10), (3.11)$_p$ implies that

$$\forall t \geq t_0\ \exists T \geq t : Q_{j_0}(T) > 0.$$

Suppose that x is an eventually positive solution of (1.3). Then, there is $t_1 \geq t_0$ such that $x(t)$, $x(\tau_i(t))$, $x(\sigma_j(t))$, $i = 1, 2, \dots, n$, $j = 1, 2, \dots, m$ are positive for all $t \geq t_1$. From equation (1.3), we get $z'(t) \leq 0$ $\forall t \geq t_1$ for $z(t) = x(t) - \sum_{i=1}^n \lambda_i x(\tau_i(t))$. Again, z can be only eventually negative or eventually positive.

If z is eventually negative, then there exist $t_2 \geq t_1$ and $\mu > 0$ such that $z(t) \leq -\mu$ for all $t \geq t_2$, i.e.,

$$\begin{aligned} x(t) &\leq -\mu + \sum_{i=1}^n \lambda_i x(\tau_i(t)) \leq -\mu + \max_{i=1,2,\dots,n} x(\tau_i(t)) \\ &\leq -\mu + \max_{s \in I(t)} x(s) \end{aligned} \tag{3.12}$$

for all $t \geq t_2$, where $I(t) = [\underline{\tau}(t), \overline{\tau}(t)]$, $\overline{\tau} = \max_{i=1,2,\dots,n} \tau_i$. Note that $\underline{\tau}, \overline{\tau} \in \mathcal{T}_{t_0}$. Denote

$$\begin{aligned} I_\ell &:= \left[\underline{\tau}^{-(\ell-1)}(t_2), \underline{\tau}^{-\ell}(t_2)\right), \quad \ell \in \mathbb{N}_0, \\ I_\ell^k &:= I_\ell \cap \left[\overline{\tau}^{-(k-1)}(\underline{\tau}^{-(\ell-1)}(t_2)), \overline{\tau}^{-k}(\underline{\tau}^{-(\ell-1)}(t_2))\right), \quad \ell \in \mathbb{N},\ k = 1, 2, \dots, K(\ell), \end{aligned} \tag{3.13}$$

where $K(\ell)$ is the largest $k \in \mathbb{N}$ for which $I_\ell^k \neq \emptyset$. Notice that by (2.2), $t \in I_\ell$ for $\ell \in \mathbb{N}_0$ if and only if $N_{t_2}^{\underline{\tau}}(t) = \ell$. Now, if $t \in I_\ell^1$ for $\ell \in \mathbb{N}$, then $\underline{\tau}(t) \in \underline{\tau}(I_\ell) = I_{\ell-1}$ and

$$\begin{aligned} \overline{\tau}(t) &\in \overline{\tau}\left(\left[\underline{\tau}^{-(\ell-1)}(t_2), \overline{\tau}^{-1}(\underline{\tau}^{-(\ell-1)}(t_2))\right)\right) \\ &= \left[\overline{\tau}(\underline{\tau}^{-(\ell-1)}(t_2)), \underline{\tau}^{-(\ell-1)}(t_2)\right) \subset I_{\ell-1}. \end{aligned}$$

Similarly, if $t \in I_\ell^k$ for $\ell \in \mathbb{N}$, $k \in \{2, 3, \dots, K(\ell)\}$, then $\underline{\tau}(t) \in \underline{\tau}(I_\ell) = I_{\ell-1}$ and

$$\begin{aligned} \overline{\tau}(t) &\in \overline{\tau}\left(\left[\overline{\tau}^{-(k-1)}(\underline{\tau}^{-(\ell-1)}(t_2)), \overline{\tau}^{-k}(\underline{\tau}^{-(\ell-1)}(t_2))\right)\right) \\ &= \left[\overline{\tau}^{-(k-2)}(\underline{\tau}^{-(\ell-1)}(t_2)), \overline{\tau}^{-(k-1)}(\underline{\tau}^{-(\ell-1)}(t_2))\right) \subset I_\ell^{k-1}. \end{aligned}$$

Using the above inclusions, we are able to work more precisely with $I(t)$ for particular values of t.

Now, we use the mathematical induction with respect to the intervals $I_1^1, I_1^2, \dots, I_1^{K(1)}$, $I_2^1, \dots$ to prove an estimation of $x(t)$ for all $t \geq t_2$. Let us denote $\Omega := \sup_{s \in I_0} x(s) =$

$\max_{s\in\overline{I_0}} x(s)$ and $C_\ell^k := \operatorname{conv}\{I_{\ell-1}, I_\ell^k\}$, the convex hull of the corresponding sets for $\ell \in \mathbb{N}$, $k \in \{1,2,\ldots,K(\ell)\}$. For $I(I_\ell^k) = \bigcup_{t\in I_\ell^k}[\underline{\tau}(t), \overline{\tau}(t)]$, we get $I(I_\ell^1) \subset \overline{I_{\ell-1}}$ for each $\ell \in \mathbb{N}$. So, if $t \in I_1^1$, then by (3.12),

$$x(t) \le -\mu + \max_{s\in\overline{I(I_1^1)}} x(s) = -\mu + \max_{s\in\overline{I_0}} x(s) = -\mu + \Omega = -N_{t_2}^{\tau}(t)\mu + \Omega.$$

Furthermore, $I(I_\ell^k) \subset \overline{\operatorname{conv}\{I_{\ell-1}, I_\ell^{k-1}\}} = \overline{C_\ell^{k-1}}$ for $l \in \mathbb{N}$, $k \in \{2,3,\ldots,K(\ell)\}$. Let us suppose that $\ell \in \mathbb{N}$, $k \in \{1,2,\ldots,K(\ell)\}$ are fixed and

$$x(t) \le -N_{t_2}^{\tau}(t)\mu + \Omega$$

for all $t \in \operatorname{conv}\{I_0, I_\ell^k\}$ (due to the continuity of x, this estimation is valid for all $t \in \overline{\operatorname{conv}\{I_0, I_\ell^k\}}$). Now, if $k < K(\ell)$, for $t \in I_\ell^{k+1}$ we have $I(t) \subset I(I_\ell^{k+1}) \subset \overline{C_\ell^k}$. Hence, by (3.12),

$$\begin{aligned} x(t) &\le -\mu + \max_{s\in\overline{C_\ell^k}} x(s) \le -\mu + \max_{s\in\overline{C_\ell^k}} (-N_{t_2}^{\tau}(s)\mu + \Omega) \\ &= -\mu - (\ell-1)\mu + \Omega = -\ell\mu + \Omega = -N_{t_2}^{\tau}(t)\mu + \Omega. \end{aligned}$$

On the other side, if $k = K(\ell)$, for $t \in I_{\ell+1}^1$ we obtain

$$\begin{aligned} x(t) &\le -\mu + \max_{s\in\overline{I_\ell}} x(s) \le -\mu + \max_{s\in\overline{I_\ell}} (-N_{t_2}^{\tau}(s)\mu + \Omega) \\ &= -\mu - \ell\mu + \Omega = -(\ell+1)\mu + \Omega = -N_{t_2}^{\tau}(t)\mu + \Omega. \end{aligned}$$

So, we have proved that

$$x(t) \le -N_{t_2}^{\tau}(t)\mu + \Omega, \quad t \ge t_2.$$

Using $N_{t_2}^{\tau}(t) \xrightarrow{t\to\infty} \infty$, for $t \to \infty$ we obtain a contradiction with x being eventually positive.

Therefore, z is eventually positive, i.e., there is $t_2 \ge t_1$ such that

$$x(t) > \sum_{i=1}^{n} \lambda_i x(\tau_i(t)) \ge \min_{i=1,2,\ldots,n} x(\tau_i(t)) \ge \min_{s\in I(t)} x(s) \ge \min_{s\in[\underline{\tau}(t),t]} x(s)$$

for all $t \ge t_2$. In this part of the proof, we adapt the notation from the previous part with this new value of t_2. So, we have

$$x(t) \ge \min_{s\in\overline{I_0}} x(s) =: \omega > 0, \quad t \ge t_2. \tag{3.14}$$

Consequently, from equation (1.3), we get

$$0 = z'(t) + \sum_{j=1}^{m} Q_j(t) x(\sigma_j(t)) \ge z'(t) + \omega \sum_{j=1}^{m} Q_j(t)$$

for all $t \ge t_3$ for some $t_3 \ge \underline{\sigma}^{-1}(t_2)$, $\underline{\sigma} = \min_{j=1,2,\ldots,m} \sigma_j \in \mathcal{T}_{t_0}$. Integrating over $[t_3, t]$ gives

$$z(t) \le z(t_3) - \omega \sum_{j=1}^{m} \int_{t_3}^{t} Q_j(s)\, ds \le z(t_3) - \omega \int_{t_3}^{t} Q_{j_0}(s)\, ds, \quad t \ge t_3.$$

Assuming condition (3.10), this estimation results in a contradiction with eventual positivity of z for $t \to \infty$, which implies that x is oscillatory.

Now, assume that

$$\int_{t_0}^{\infty} Q_j(s)\,ds < \infty$$

for each $j = 1, 2, \ldots, m$, and that condition $(3.11)_p$ is satisfied for some $p > 1$. Then

$$x(t) = z(t) + \sum_{i=1}^{n} \lambda_i x(\tau_i(t)) \geq z(t) + \min_{i=1,2,\ldots,n} x(\tau_i(t)) \geq z(t) + \min_{s \in I(t)} x(s)$$

for all $t \geq t_4$, where $t_4 \geq \underline{\tau}^{-1}(t_2)$ is such that $t_4 = \underline{\tau}^{-\kappa}(t_0)$ for some $\kappa \in \mathbb{N}$. Let us fix arbitrary $T \geq t_4$. Then, due to $z'(t) \leq 0$ for all $t \geq t_4$,

$$x(t) \geq z(T) + \min_{s \in I(t)} x(s), \quad t \in [t_4, T].$$

Using induction as for (3.12), one can now show that

$$x(t) \geq N_{t_4}^{\underline{\tau}}(t) z(T) + \omega, \quad t \in [t_4, T].$$

In particular, this estimation is valid for $t = T$. Since $T \geq t_4$ was arbitrary, we have

$$x(t) \geq N_{t_4}^{\underline{\tau}}(t) z(t) + \omega, \quad t \geq t_4.$$

Applying Young inequality with $p > 1$ such that $(3.11)_p$ holds yields

$$x(t) \geq \left(p N_{t_4}^{\underline{\tau}}(t) z(t) \right)^{\frac{1}{p}} (q\omega)^{\frac{1}{q}}$$

for all $t \geq t_4$. Denoting $\omega_1 := p^{\frac{1}{p}} (q\omega)^{\frac{1}{q}} > 0$, we have

$$\begin{aligned} z'(t) &= -\sum_{j=1}^{m} Q_j(t) x(\sigma_j(t)) \leq -\omega_1 \sum_{j=1}^{m} Q_j(t) \left(N_{t_4}^{\underline{\tau}}(\sigma_j(t)) z(\sigma_j(t)) \right)^{\frac{1}{p}} \\ &\leq -\omega_1 \sum_{j=1}^{m} Q_j(t) \left(N_{t_4}^{\underline{\tau}}(\sigma_j(t)) z(t) \right)^{\frac{1}{p}}, \quad t \geq t_5, \end{aligned}$$

where $t_5 = \underline{\sigma}^{-1}(t_4)$. Dividing by $z^{\frac{1}{p}}(t)$ and integrating over $[t_5, t]$ yields

$$q z^{\frac{1}{q}}(t) - q z^{\frac{1}{q}}(t_5) \leq -\omega_1 \int_{t_5}^{t} \left(N_{t_4}^{\underline{\tau}}(\sigma_{j_0}(s)) \right)^{\frac{1}{p}} Q_{j_0}(s)\,ds, \quad t \geq t_5.$$

Now, the proof is finished as the proof of Theorem 3.3. □

Remark 3.8. Note that conditions (3.10) and $(3.11)_p$ are equivalent to

$$\sum_{j=1}^{m} \int_{t_0}^{\infty} Q_j(s)\,ds = \infty$$

and

$$\sum_{j=1}^{m} \int_{\sigma_j^{-1}(t_0)}^{\infty} \left(N_{t_0}^{\underline{\tau}}(\sigma_j(s)) \right)^{\frac{1}{p}} Q_j(s)\,ds = \infty,$$

respectively.

3.2 Distributed delays

Here, we consider neutral differential equations with distributed and mixed delays.

Theorem 3.9. *Let $t_0 \in \mathbb{R}$, $\underline{\tau}, \overline{\tau}, \underline{\sigma}, \overline{\sigma} \in \mathcal{T}_{t_0}$ satisfy $\underline{\tau}(t) < \overline{\tau}(t)$ and $\underline{\sigma}(t) \leq \overline{\sigma}(t)$ for all $t \geq t_0$, $\lambda \in C([\underline{\tau}(t_0), \infty), (0, \infty))$, $Q \in C([t_0, \infty), \mathbb{R}_+)$, $R \in C([\underline{\sigma}(t_0), \infty), \mathbb{R}_+)$. If*

$$\int_{t_0}^{\infty} Q(s) \int_{\underline{\sigma}(s)}^{\overline{\sigma}(s)} R(r)\, dr\, ds = \infty \tag{3.15}$$

or

$$\int_{\underline{\sigma}^{-1}(t_0)}^{\infty} Q(s) \int_{\underline{\sigma}(s)}^{\overline{\sigma}(s)} (N_{t_0}^{\underline{\tau}}(r))^{\frac{1}{p}} R(r)\, dr\, ds = \infty \qquad (3.16)_p$$

for some $p > 1$, where $N_{t_0}^{\underline{\tau}}(t)$ is given by (2.1), then every solution of equation (1.4) oscillates.

Proof. Again, the proof is similar to the proofs of Theorems 3.3, 3.7, so we only provide some key points. For brevity, we also denote

$$\Lambda(t) = \left(\int_{\underline{\tau}(t)}^{\overline{\tau}(t)} \lambda(s)\, ds \right)^{-1}, \quad t \geq t_0.$$

As before, each one of conditions (3.15), $(3.16)_p$ implies that

$$\forall t \geq t_0 \quad \exists T \geq t : \quad Q(T) \int_{\underline{\sigma}(T)}^{\overline{\sigma}(T)} R(s)\, ds > 0.$$

We suppose that x is an eventually positive solution of (1.4). Then there exists $t_1 \geq t_0$ such that $x(t)$, $x(\underline{\tau}(t))$, $x(\underline{\sigma}(t))$ are positive for all $t \geq t_1$. Hence, by equation (1.4), $z'(t) \leq 0\ \forall t \geq t_1$ for $z(t) = x(t) - \Lambda(t) \int_{\underline{\tau}(t)}^{\overline{\tau}(t)} \lambda(s) x(s)\, ds$. We know that z is either eventually negative or eventually positive.

If z is eventually negative, there are $t_2 \geq t_1$ and $\mu > 0$ such that

$$x(t) \leq -\mu + \Lambda(t) \int_{\underline{\tau}(t)}^{\overline{\tau}(t)} \lambda(s) x(s)\, ds \leq -\mu + \max_{s \in I(t)} x(s) \tag{3.17}$$

for all $t \geq t_2$, where $I(t) = [\underline{\tau}(t), \overline{\tau}(t)]$. Analogously to inequality (3.12), one can show by induction that estimation (3.17) implies

$$x(t) \leq -N_{t_2}^{\underline{\tau}}(t)\mu + \Omega, \quad t \geq t_2,$$

where $\Omega = \max_{s \in \overline{I_0}} x(s)$ using the notation from the proof of Theorem 3.7. Using $N_{t_2}^{\underline{\tau}}(t) \xrightarrow{t \to \infty} \infty$, a contradiction is obtained for $t \to \infty$ with x being eventually positive.

Therefore, z is eventually positive. So, there is $t_2 \geq t_1$ such that

$$x(t) > \Lambda(t) \int_{\underline{\tau}(t)}^{\overline{\tau}(t)} \lambda(s) x(s)\, ds \geq \min_{s \in I(t)} x(s)$$

for all $t \geq t_2$. Adapting the notation (3.13) for I_ℓ, I_ℓ^k, estimation (3.14) follows. Next, from equation (1.4), we get

$$0 = z'(t) + Q(t) \int_{\underline{\sigma}(t)}^{\overline{\sigma}(t)} R(s) x(s)\, ds \geq z'(t) + \omega Q(t) \int_{\underline{\sigma}(t)}^{\overline{\sigma}(t)} R(s)\, ds$$

for all $t \geq t_3$ for some $t_3 \geq \underline{\sigma}^{-1}(t_2)$. Integration over $[t_3, t]$ results in

$$z(t) \leq z(t_3) - \omega \int_{t_3}^{t} Q(s) \int_{\underline{\sigma}(s)}^{\overline{\sigma}(s)} R(r)\, dr\, ds, \quad t \geq t_3.$$

Assuming condition (3.15), we get a contradiction with eventual positivity of z, since the right side of the latter inequality tends to $-\infty$ as $t \to \infty$.

Now, assume that

$$\int_{t_0}^{\infty} Q(s) \int_{\underline{\sigma}(s)}^{\overline{\sigma}(s)} R(r)\, dr\, ds < \infty \tag{3.18}$$

and that condition $(3.16)_p$ is satisfied for some $p > 1$. Then

$$x(t) = z(t) + \Lambda(t) \int_{\underline{\tau}(t)}^{\overline{\tau}(t)} \lambda(s) x(s)\, ds \geq z(t) + \min_{s \in I(t)} x(s)$$

for all $t \geq t_4$, where $t_4 \geq \underline{\tau}^{-1}(t_2)$ is such that $t_4 = \underline{\tau}^{-\kappa}(t_0)$ for some $\kappa \in \mathbb{N}$. As in the proof of Theorem 3.7, it can be shown that

$$x(t) \geq N_{t_4}^{\underline{\tau}}(t) z(t) + \omega, \quad t \geq t_4,$$

and Young inequality implies

$$x(t) \geq \left(p N_{t_4}^{\underline{\tau}}(t) z(t) \right)^{\frac{1}{p}} (q\omega)^{\frac{1}{q}}$$

for all $t \geq t_4$. Denoting $\omega_1 := p^{\frac{1}{p}} (q\omega)^{\frac{1}{q}} > 0$, from equation (1.4) we derive

$$\begin{aligned} z'(t) &= -Q(t) \int_{\underline{\sigma}(t)}^{\overline{\sigma}(t)} R(s) x(s)\, ds \leq -\omega_1 Q(t) \int_{\underline{\sigma}(t)}^{\overline{\sigma}(t)} R(s) \left(N_{t_4}^{\underline{\tau}}(s) z(s) \right)^{\frac{1}{p}} ds \\ &\leq -\omega_1 Q(t) \int_{\underline{\sigma}(t)}^{\overline{\sigma}(t)} R(s) \left(N_{t_4}^{\underline{\tau}}(s) \right)^{\frac{1}{p}} ds\, z(t)^{\frac{1}{p}}, \quad t \geq t_5, \end{aligned}$$

where $t_5 = \underline{\sigma}^{-1}(t_4)$. Dividing by $z^{\frac{1}{p}}(t)$ and integrating over $[t_5, t]$ gives

$$q z^{\frac{1}{q}}(t) - q z^{\frac{1}{q}}(t_5) \leq -\omega_1 \int_{t_5}^{t} Q(s) \int_{\underline{\sigma}(s)}^{\overline{\sigma}(s)} \left(N_{t_4}^{\underline{\tau}}(r) \right)^{\frac{1}{p}} R(r)\, dr\, ds, \quad t \geq t_5.$$

Now, it only remains to show that

$$\int_{t_5}^{\infty} Q(s) \int_{\underline{\sigma}(s)}^{\overline{\sigma}(s)} \left(N_{t_4}^{\underline{\tau}}(r) \right)^{\frac{1}{p}} R(r)\, dr\, ds = \infty$$

to obtain a contradiction with eventual positivity of z, implying that x is oscillatory. Using Lemmas 2.2, 2.5 (see the proof of Theorem 3.3), we obtain

$$\begin{aligned} &\int_{t_5}^{t} Q(s) \int_{\underline{\sigma}(s)}^{\overline{\sigma}(s)} \left(N_{t_4}^{\underline{\tau}}(r) \right)^{\frac{1}{p}} R(r)\, dr\, ds \\ &\quad \geq \int_{t_5}^{t} Q(s) \int_{\underline{\sigma}(s)}^{\overline{\sigma}(s)} \left(N_{t_0}^{\underline{\tau}}(r) \right)^{\frac{1}{p}} R(r)\, dr\, ds - \kappa^{\frac{1}{p}} \int_{t_5}^{t} Q(s) \int_{\underline{\sigma}(s)}^{\overline{\sigma}(s)} R(r)\, dr\, ds \\ &\quad = \int_{\underline{\sigma}^{-1}(t_0)}^{t} Q(s) \int_{\underline{\sigma}(s)}^{\overline{\sigma}(s)} \left(N_{t_0}^{\underline{\tau}}(r) \right)^{\frac{1}{p}} R(r)\, dr\, ds - \kappa^{\frac{1}{p}} \int_{t_0}^{t} Q(s) \int_{\underline{\sigma}(s)}^{\overline{\sigma}(s)} R(r)\, dr\, ds + C \end{aligned}$$

for an appropriate constant $C \in \mathbb{R}$. Note that, by conditions $(3.16)_p$ and (3.18), the right side tends to ∞ as $t \to \infty$. This completes the proof. □

Remark 3.10. Condition $\lambda \in C([\underline{\tau}(t_0),\infty),(0,\infty))$ in Theorem 3.9 can be weakened to $\lambda \in C([\underline{\tau}(t_0),\infty),\mathbb{R}_+)$ satisfying $\int_{\underline{\tau}(t)}^{\overline{\tau}(t)} \lambda(s)\,ds > 0$ for all $t \geq t_0$.

Finally, we present a result for neutral differential equations with mixed delays and time-dependent coefficients.

Theorem 3.11. *Let $t_0 \in \mathbb{R}$, $n_{1,2}, m_{1,2} \in \mathbb{N}_0$ be such that $n_1 + n_2 \geq 1$, $m_1 + m_2 \geq 1$. Moreover, let the following assumptions be fulfilled:*

1. *$\lambda_i \in C([t_0,\infty),\mathbb{R}_+)$ and $\tau_i \in \mathcal{T}_{t_0}$ for each $i = 1,2,\dots,n_1$,*
2. *$\vartheta_i \in C([\underline{\tau}_i(t_0),\infty),\mathbb{R}_+)$ and $\underline{\tau}_i, \overline{\tau}_i \in \mathcal{T}_{t_0}$ are such that $\underline{\tau}_i(t) \leq \overline{\tau}_i(t)$ for all $t \geq t_0$ and for each $i = 1,2,\dots,n_2$,*
3. *$Q_j \in C([t_0,\infty),\mathbb{R}_+)$ and $\sigma_j \in \mathcal{T}_{t_0}$ for each $j = 1,2,\dots,m_1$,*
4. *$S_j \in C([t_0,\infty),\mathbb{R}_+)$, $R_j \in C([\underline{\sigma}_j(t_0),\infty),\mathbb{R}_+)$, and $\underline{\sigma}_j, \overline{\sigma}_j \in \mathcal{T}_{t_0}$ are such that $\underline{\sigma}_j(t) \leq \overline{\sigma}_j(t)$ for all $t \geq t_0$ and each $j = 1,2,\dots,m_2$,*
5. *for all $t \geq t_0$,*
$$\sum_{i=1}^{n_1} \lambda_i(t) + \sum_{i=1}^{n_2} \int_{\underline{\tau}_i(t)}^{\overline{\tau}_i(t)} \vartheta_i(s)\,ds = 1.$$

If
$$\sum_{j=1}^{m_1} \int_{t_0}^{\infty} Q_j(s)\,ds + \sum_{j=1}^{m_2} \int_{t_0}^{\infty} S_j(s) \int_{\underline{\sigma}_j(s)}^{\overline{\sigma}_j(s)} R_j(r)\,dr\,ds = \infty \tag{3.19}$$
or
$$\begin{aligned} &\sum_{j=1}^{m_1} \int_{\sigma_j^{-1}(t_0)}^{\infty} (N_{t_0}^{\underline{\tau}}(\sigma_j(s)))^{\frac{1}{p}}\, Q_j(s)\,ds \\ &\quad + \sum_{j=1}^{m_2} \int_{\underline{\sigma}_j^{-1}(t_0)}^{\infty} S_j(s) \int_{\underline{\sigma}_j(s)}^{\overline{\sigma}_j(s)} (N_{t_0}^{\underline{\tau}}(r))^{\frac{1}{p}}\, R_j(r)\,dr\,ds = \infty \end{aligned} \tag{3.20$_p$}$$
for some $p > 1$, where $\underline{\tau} = \min\{\min_{i=1,2,\dots,n_1} \tau_i, \min_{i=1,2,\dots,n_2} \underline{\tau}_i\}$ and $N_{t_0}^{\underline{\tau}}(t)$ is given by (2.1), then every solution of equation (1.5) oscillates.

Proof. Each one of conditions (3.19), $(3.20)_p$ implies that
$$\begin{aligned} &\forall t \geq t_0 \quad \exists T \geq t: \quad Q_j(T) > 0 \quad \text{for some } j \in \{1,2,\dots,m_1\} \\ &\text{or} \quad S_j(T) \int_{\underline{\sigma}_j(T)}^{\overline{\sigma}_j(T)} R_j(s)\,ds > 0 \quad \text{for some } j \in \{1,2,\dots,m_2\}. \end{aligned} \tag{3.21}$$

Let us denote
$$\overline{\tau} := \max\left\{ \max_{i=1,2,\dots,n_1} \tau_i, \max_{i=1,2,\dots,n_2} \overline{\tau}_i \right\}, \qquad \underline{\sigma} := \min\left\{ \min_{j=1,2,\dots,m_1} \sigma_j, \min_{j=1,2,\dots,m_2} \underline{\sigma}_j \right\}.$$

Note that $\underline{\tau}, \overline{\tau}, \underline{\sigma} \in \mathcal{T}_{t_0}$. Let us assume without any loss of generality that x is an eventually positive solution of (1.5). Take $t_1 \geq t_0$ such that $x(t)$, $x(\underline{\tau}(t))$ and $x(\underline{\sigma}(t))$ are positive for all $t \geq t_1$. Then, by (1.5), $z'(t) \leq 0$ $\forall t \geq t_1$ for
$$z(t) = x(t) - \left(\sum_{i=1}^{n_1} \lambda_i(t) x(\tau_i(t)) + \sum_{i=1}^{n_2} \int_{\underline{\tau}_i(t)}^{\overline{\tau}_i(t)} \vartheta_i(s) x(s)\,ds \right). \tag{3.22}$$

Due to (3.21), z is either eventually negative or eventually positive.

If z is eventually negative, there are $t_2 \geq t_1$ and $\mu > 0$ such that

$$\begin{aligned} x(t) &\leq -\mu + \sum_{i=1}^{n_1} \lambda_i(t)x(\tau_i(t)) + \sum_{i=1}^{n_2} \int_{\underline{\tau}_i(t)}^{\overline{\tau}_i(t)} \vartheta_i(s)x(s)\,ds \\ &\leq -\mu + \left(\sum_{i=1}^{n_1} \lambda_i(t) + \sum_{i=1}^{n_2} \int_{\underline{\tau}_i(t)}^{\overline{\tau}_i(t)} \vartheta_i(s)\,ds \right) \max_{s\in I(t)} x(s) = -\mu + \max_{s\in I(t)} x(s) \end{aligned}$$

for all $t \geq t_2$, where $I(t) = [\underline{\tau}(t), \overline{\tau}(t)]$. As for (3.12), one can use mathematical induction to show that

$$x(t) \leq -N_{t_2}^{\underline{\tau}}(t)\mu + \Omega, \quad t \geq t_2,$$

where $\Omega = \max_{s\in\overline{I_0}} x(s)$ using the notation from the proof of Theorem 3.7. Consequently, $N_{t_2}^{\underline{\tau}}(t) \xrightarrow{t\to\infty} \infty$ yields a contradiction for $t \to \infty$ with x being eventually positive.

Hence, z is eventually positive. Take $t_2 \geq t_1$ such that

$$\begin{aligned} x(t) &> \sum_{i=1}^{n_1} \lambda_i(t)x(\tau_i(t)) + \sum_{i=1}^{n_2} \int_{\underline{\tau}_i(t)}^{\overline{\tau}_i(t)} \vartheta_i(s)x(s)\,ds \\ &\geq \left(\sum_{i=1}^{n_1} \lambda_i(t) + \sum_{i=1}^{n_2} \int_{\underline{\tau}_i(t)}^{\overline{\tau}_i(t)} \vartheta_i(s)\,ds \right) \min_{s\in I(t)} x(s) \\ &= \min_{s\in I(t)} x(s) \geq \min_{s\in[\underline{\tau}(t),t)} x(s) \geq \min_{s\in\overline{I_0}} x(s) =: \omega \end{aligned}$$

for all $t \geq t_2$, where we used the notation from the proof of Theorem 3.7, again. As a consequence, equation (1.5) implies

$$0 \geq z'(t) + \omega \left(\sum_{j=1}^{m_1} Q_j(t) + \sum_{j=1}^{m_2} S_j(t) \int_{\underline{\sigma}_j(t)}^{\overline{\sigma}_j(t)} R_j(s)\,ds \right)$$

for all $t \geq t_3$ for some $t_3 \geq \underline{\sigma}^{-1}(t_2)$. Integrating the latter inequality over $[t_3, t]$ gives

$$z(t) \leq z(t_3) - \omega \left(\sum_{j=1}^{m_1} \int_{t_3}^{t} Q_j(s)\,ds + \sum_{j=1}^{m_2} \int_{t_3}^{t} S_j(s) \int_{\underline{\sigma}_j(s)}^{\overline{\sigma}_j(s)} R_j(r)\,dr\,ds \right)$$

for all $t \geq t_3$. This results in a contradiction with the eventual positivity of z for $t \to \infty$ if (3.19) holds. So, x is oscillatory.

Now suppose that

$$\sum_{j=1}^{m_1} \int_{t_0}^{\infty} Q_j(s)\,ds + \sum_{j=1}^{m_2} \int_{t_0}^{\infty} S_j(s) \int_{\underline{\sigma}_j(s)}^{\overline{\sigma}_j(s)} R_j(r)\,dr\,ds < \infty \tag{3.23}$$

and that condition $(3.20)_p$ is fulfilled for some $p > 1$. Then, by (3.22) and assumption (5), we get

$$x(t) \geq z(t) + \min_{s\in I(t)} x(s), \quad t \geq t_4,$$

where $t_4 \geq \underline{\tau}^{-1}(t_2)$ is such that $t_4 = \underline{\tau}^{-\kappa}(t_0)$ for some $\kappa \in \mathbb{N}$. By induction as in the proof of Theorem 3.7, we derive

$$x(t) \geq N_{t_4}^{\underline{\tau}}(t)z(t) + \omega, \quad t \geq t_4.$$

Denoting $\omega_1 := p^{\frac{1}{p}}(q\omega)^{\frac{1}{q}} > 0$, Young's inequality yields

$$x(t) \geq \omega_1 \left(N^{\tau}_{t_4}(t)z(t)\right)^{\frac{1}{p}}, \quad t \geq t_4.$$

Then, equation (1.5) gives

$$\begin{aligned} z'(t) &\leq -\omega_1 \left(\sum_{j=1}^{m_1} Q_j(t) \left(N^{\tau}_{t_4}(\sigma_j(t))z(\sigma_j(t))\right)^{\frac{1}{p}} + \sum_{j=1}^{m_2} S_j(t) \int_{\underline{\sigma}_j(t)}^{\overline{\sigma}_j(t)} R_j(s) \left(N^{\tau}_{t_4}(s)z(s)\right)^{\frac{1}{p}} ds \right) \\ &\leq -\omega_1 z(t)^{\frac{1}{p}} \left(\sum_{j=1}^{m_1} Q_j(t)(N^{\tau}_{t_4}(\sigma_j(t)))^{\frac{1}{p}} + \sum_{j=1}^{m_2} S_j(t) \int_{\underline{\sigma}_j(t)}^{\overline{\sigma}_j(t)} R_j(s) \left(N^{\tau}_{t_4}(s)\right)^{\frac{1}{p}} ds \right) \end{aligned}$$

for all $t \geq t_5 = \underline{\sigma}^{-1}(t_4)$. Dividing by $z^{\frac{1}{p}}(t)$ and integrating over $[t_5, t]$ results in

$$\begin{aligned} &qz^{\frac{1}{q}}(t) - qz^{\frac{1}{q}}(t_5) \\ &\quad \leq -\omega_1 \left(\sum_{j=1}^{m_1} \int_{t_5}^{t} (N^{\tau}_{t_4}(\sigma_j(s)))^{\frac{1}{p}} Q_j(s)\, ds + \sum_{j=1}^{m_2} \int_{t_5}^{t} S_j(s) \int_{\underline{\sigma}_j(s)}^{\overline{\sigma}_j(s)} (N^{\tau}_{t_4}(r))^{\frac{1}{p}} R_j(r)\, dr\, ds \right), \quad t \geq t_5. \end{aligned}$$

If the right side tends to $-\infty$ as $t \to \infty$, we get a contradiction with the eventual positivity of z, which implies that x is oscillatory. To see this, we use Lemmas 2.2, 2.5 to estimate

$$\begin{aligned} &\sum_{j=1}^{m_1} \int_{t_5}^{t} (N^{\tau}_{t_4}(\sigma_j(s)))^{\frac{1}{p}} Q_j(s)\, ds + \sum_{j=1}^{m_2} \int_{t_5}^{t} S_j(s) \int_{\underline{\sigma}_j(s)}^{\overline{\sigma}_j(s)} (N^{\tau}_{t_4}(r))^{\frac{1}{p}} R_j(r)\, dr\, ds \\ &\geq \sum_{j=1}^{m_1} \left(\int_{t_5}^{t} (N^{\tau}_{t_0}(\sigma_j(s)))^{\frac{1}{p}} Q_j(s)\, ds - \kappa^{\frac{1}{p}} \int_{t_5}^{t} Q_j(s)\, ds \right) \\ &\quad + \sum_{j=1}^{m_2} \left(\int_{t_5}^{t} S_j(s) \int_{\underline{\sigma}_j(s)}^{\overline{\sigma}_j(s)} (N^{\tau}_{t_0}(r))^{\frac{1}{p}} R_j(r)\, dr\, ds - \kappa^{\frac{1}{p}} \int_{t_5}^{t} S_j(s) \int_{\underline{\sigma}_j(s)}^{\overline{\sigma}_j(s)} R_j(r)\, dr\, ds \right) \\ &= \sum_{j=1}^{m_1} \left(\int_{\sigma_j^{-1}(t_0)}^{t} (N^{\tau}_{t_0}(\sigma_j(s)))^{\frac{1}{p}} Q_j(s)\, ds - \kappa^{\frac{1}{p}} \int_{t_0}^{t} Q_j(s)\, ds \right) \\ &\quad + \sum_{j=1}^{m_2} \left(\int_{\underline{\sigma}_j^{-1}(t_0)}^{t} S_j(s) \int_{\underline{\sigma}_j(s)}^{\overline{\sigma}_j(s)} (N^{\tau}_{t_0}(r))^{\frac{1}{p}} R_j(r)\, dr\, ds - \kappa^{\frac{1}{p}} \int_{t_0}^{t} S_j(s) \int_{\underline{\sigma}_j(s)}^{\overline{\sigma}_j(s)} R_j(r)\, dr\, ds \right) + C \end{aligned}$$

for an appropriate constant $C \in \mathbb{R}$. Condition $(3.20)_p$ and inequality (3.23) imply that the right-hand side tends to ∞ as $t \to \infty$. This completes the proof. □

4 Applications

In this section, we apply the results of Section 3 to concrete neutral differential equations.

4.1 Discrete delays

First, let us consider the neutral differential equation with one constant and one variable delay,

$$[x(t) - x(t-\alpha)]' + Q(t)x(\sigma(t)) = 0, \quad t \geq t_0 \tag{4.1}$$

for some $t_0 \in \mathbb{R}$, $\alpha > 0$, $\sigma \in \mathcal{T}_{t_0}$, and $Q \in C([t_0, \infty), \mathbb{R}_+)$. Then $\tau^k(t) = t - k\alpha$ for $k \in \mathbb{Z}$. Now, inequality (2.1) has the form

$$\zeta + (N_\zeta^\tau(t) - 1)\alpha \leq t < \zeta + N_\zeta^\tau(t)\alpha.$$

Therefrom, we derive

$$\frac{t-\zeta}{\alpha} < N_\zeta^\tau(t) \leq \frac{t-\zeta}{\alpha} + 1$$

that gives

$$N_\zeta^\tau(t) = \left\lfloor \frac{t-\zeta}{\alpha} \right\rfloor + 1.$$

Since we are interested in the convergence of the integral on the left side of $(3.4)_p$ in a neighborhood of ∞, it is enough to assume that $s \geq \tilde{t}_0$, where $\tilde{t}_0 \geq \sigma^{-1}(t_0)$ is such that $\sigma(\tilde{t}_0) > 0$. Then, dividing the inequality

$$\frac{\sigma(s) - t_0}{\alpha} < N_{t_0}^\tau(\sigma(s)) \leq \frac{\sigma(s) - t_0}{\alpha} + 1$$

by $\sigma(s)/\alpha$ and taking the limit $s \to \infty$, we obtain

$$\lim_{s \to \infty} \frac{\alpha}{\sigma(s)} N_{t_0}^\tau(\sigma(s)) = 1.$$

Therefore, condition $(3.4)_p$ holds if and only if

$$\int_{\tilde{t}_0}^{\infty} (\sigma(s))^{\frac{1}{p}} Q(s)\, ds = \infty. \tag{4.2$_p$}$$

Using Theorem 3.3, one can easily prove the following result.

Proposition 4.1. *Let $t_0 \in \mathbb{R}$, $\alpha > 0$, $\sigma \in \mathcal{T}_{t_0}$, $Q \in C([t_0, \infty), \mathbb{R}_+)$, and $\tilde{t}_0 \geq \sigma^{-1}(t_0)$ be such that $\sigma(\tilde{t}_0) > 0$. Every solution of equation (4.1) oscillates if condition (3.1) or $(4.2)_p$ for some $p > 1$ is satisfied.*

In a particular case of equation (4.1) when $\sigma(t) = t - \beta$, $\beta > 0$, this statement can be simplified. From the inequality

$$\frac{s - \beta - t_0}{\alpha} < N_{t_0}^\tau(\sigma(s)) \leq \frac{s - \beta - t_0}{\alpha} + 1$$

for $s \geq \tilde{t}_0$, where $\tilde{t}_0 \geq t_0 + \beta$ is positive, we get

$$\lim_{s \to \infty} \frac{\alpha}{s} N_{t_0}^\tau(\sigma(s)) = 1.$$

Hence, condition $(3.4)_p$ is equivalent to

$$\int_{\tilde{t}_0}^{\infty} s^{\frac{1}{p}} Q(s)\, ds = \infty. \tag{4.3$_p$}$$

Proposition 4.2. *Let $t_0 \in \mathbb{R}$, $\alpha, \beta > 0$, $Q \in C([t_0, \infty), \mathbb{R}_+)$, and $\tilde{t}_0 \geq t_0 + \beta$ be positive. Every solution of the equation*

$$[x(t) - x(t - \alpha)]' + Q(t)x(t - \beta) = 0, \quad t \geq t_0 \tag{4.4}$$

oscillates if condition (3.1) or $(4.3)_p$ for some $p > 1$ is satisfied.

Remark 4.3. Note that condition (3.1) or $(4.3)_p$ for some $p > 1$ implies

$$\int_{t_0}^{\infty} sQ(s)\,ds = \infty$$

which, by [6], means that equation (4.4) does not have a bounded positive solution.

Remark 4.4. For $Q(t) = t^{-\alpha}$, $1 < \alpha$, equation (4.4) reads as

$$[x(t) - x(t-\alpha)]' + t^{-\alpha}x(t-\beta) = 0, \quad t \geq t_0.$$

This is known [6] to have a bounded positive solution if $\alpha > 2$, since

$$\int_{t_0}^{\infty} s^{1-\alpha}\,ds < \infty.$$

To see that for $1 < \alpha < 2$ every solution is oscillatory, one can verify that

$$\int_{t_0}^{\infty} Q(s) \exp\left\{\frac{1}{\tau}\int_{t_0}^{s} rQ(r)\,dr\right\} ds = \infty$$

with $Q(t) = t^{-\alpha}$ from [4], or take $p = \frac{1}{\alpha-1} > 1$ in $(4.3)_p$ to get

$$\int_{\tilde{t}_0}^{\infty} s^{\frac{1}{p}} Q(s)\,ds = \int_{\tilde{t}_0}^{\infty} s^{\frac{1}{p}-\alpha}\,ds = \int_{\tilde{t}_0}^{\infty} s^{-1}\,ds = \infty.$$

The case $\alpha = 2$ still remains to be unanswered, despite of the fact that in [5, Corollary] the equation is stated to be oscillatory. At least for the variable delays, we proved that the equation has a positive solution (see Example 3.2).

4.2 Distributed delays

Example 4.5. Let us consider the following equation

$$\left[x(t) - \frac{2}{\pi}\int_{t-\pi}^{t-\frac{\pi}{2}} x(s)\,ds\right]' + \frac{2}{\pi(\sin 2\sigma - \sin\sigma)}\int_{t-2\sigma}^{t-\sigma} x(s)\,ds = 0, \quad t \geq t_0 \tag{4.5}$$

for some $t_0 \in \mathbb{R}$, where $\sigma = \frac{1}{3}\left(\pi - 2\arctan\frac{2}{\pi+2}\right) \doteq 0.79988 > 0$.

This equation is of the form (1.4) with $\lambda(t) \equiv 1$, $R(t) \equiv 1$, $Q(t) \equiv \frac{2}{\pi(\sin 2\sigma - \sin\sigma)} \doteq 2.25506 > 0$, $\underline{\tau}(t) = t - \pi$, $\overline{\tau}(t) = t - \frac{\pi}{2}$, $\underline{\sigma}(t) = t - 2\sigma$, and $\overline{\sigma}(t) = t - \sigma$. It is easy to see that condition (3.15) is satisfied. Thus, by Theorem 3.9, every solution of equation (4.5) oscillates. One of such solutions is $x(t) = \sin t$. Indeed, for this function, the left-hand side of (4.5) is equal to

$$\begin{aligned}\left[\sin(t) + \frac{2}{\pi}(\cos t + \sin t)\right]' &+ \frac{2(\cos(t-2\sigma) - \cos(t-\sigma))}{\pi(\sin 2\sigma - \sin\sigma)} \\ &= \left(1 + \frac{2}{\pi}\right)\cos(t) - \frac{2}{\pi}\sin t + \frac{2\cos t}{\pi}\frac{\cos 2\sigma - \cos\sigma}{\sin 2\sigma - \sin\sigma} + \frac{2}{\pi}\sin t.\end{aligned} \tag{4.6}$$

Noting that

$$\begin{aligned}\frac{\cos 2\sigma - \cos\sigma}{\sin 2\sigma - \sin\sigma} &= -\tan\frac{3}{2}\sigma = -\tan\left(\frac{\pi}{2} - \arctan\frac{2}{\pi+2}\right) \\ &= -\cot\left(\arctan\frac{2}{\pi+2}\right) = -\frac{\pi+2}{2} = -1 - \frac{\pi}{2}\end{aligned}$$

makes the right side of (4.6) vanish.

Acknowledgements

P. Hasil and M. Veselý were supported by Grant GA20-11846S of the Czech Science Foundation. M. Pospíšil was partially supported by the Slovak Research and Development Agency under the Contract no. APVV-23-0039, and by the Grants VEGA 1/0084/23 and 2/0062/24.

References

[1] R. P. Agarwal, M. Bohner, W. T. Li, *Nonoscillation and oscillation theory for functional differential equations*, Marcel Dekker Inc., New York, 2004. https://doi.org/10.1201/9780203025741; MR2084730; Zbl 1068.34002

[2] D. D. Bainov, D. P. Mishev, *Oscillation theory for neutral differential equations with delay*, Adam Hilger, Bristol, 1991. MR1147908; Zbl 0747.34037

[3] I. Győri, G. Ladas, *Oscillation theory of delay differential equations: with applications*, Oxford University Press, London, 1991. MR1168471; Zbl 0780.34048

[4] W. T. Li, Oscillation of first order neutral nonlinear differential equations with variable coefficients, *Indian J. Pure Appl. Math.* **29**(1998), No. 6, 605–615. MR1636432; Zbl 0922.34065

[5] Y. Shoukaku, Oscillation of first order neutral differential equations with delay, *Differ. Equat. Dyn. Syst.* **31**(2023), 283–288. https://doi.org/10.1007/s12591-020-00531-x; MR4572170; Zbl 1519.34081

[6] B. G. Zhang, J. S. Yu, Existence of positive solutions for neutral differential equations, *Sci. Sin. (Ser. A)* **35**(1992), No. 11, 1306–1313. MR1223952; Zbl 0768.34048