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Critical points approaches for multiple solutions of a quasilinear periodic boundary value problem

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Abstract. Optimization problems are omnipresent in the mathematical modeling of real world systems and cover a very extensive range of applications becoming apparent in all branches of Economics, Finance, Materials Science, Astronomy, Physics, Structural and Molecular Biology, Engineering, Computer Science, and Medicine. In this paper, we aim to delve deeper into the multiplicity findings concerning a specific class of quasilinear periodic boundary value problems. In fact, as an optimization problem, we look for the critical points of the energy functional related to the problem. Utilizing a corollary derived from Bonanno's local minimum theorem, we investigate the existence of a one solution under certain algebraic conditions on the nonlinear term. Additionally, we explore conditions that lead to the existence of two solutions, incorporating the classical Ambrosetti-Rabinowitz (AR) condition alongside algebraic criteria. Moreover, by employing two critical point theorems one by Averna and Bonanno, and another by Bonanno, we establish the existence of two and three solutions in a particular scenario. To illustrate our findings, we provide an example.

Keywords: multiple solutions, quasilinear periodic, critical point, variational methods.

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1 Introduction

The target of global optimization is to find the best solution of decision models, in presence of the multiple local solutions. Optimization plays an ever-increasing role in mathematics,

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economics, engineering, health sciences, management and life sciences. Many optimization problems have existence in the real world including space planning, networking, logistic management, financial planning, and risk management. The objective of this paper is to ascertain the existence of solutions for the following quasilinear periodic boundary value problem

$$\begin{cases} -p(u')u'' + \zeta(x)u = \lambda f(x, u(x)) + \mu g(x, u(x)), & \text{a.e. } x \in [0,1], \\ u(1) - u(0) = u'(1) - u'(0) = 0 \end{cases} \tag{$P^{f,g}_{\lambda,\mu}$}$$

where $f, g : [0,1] \times \mathbb{R} \to \mathbb{R}$ are L^1-Carathéodory functions, λ is a positive parameter and $\mu \geq 0$. We need the following assumptions:

(Q_1) $p : \mathbb{R} \to (0, \infty)$ is a continuous and nondecreasing on $[0, \infty)$, there exist two positive numbers $M \geq m$ such that

$$m \leq p(x) \leq M, \qquad \forall x \in \mathbb{R} \tag{1.1}$$

(Q_2) $\zeta \in C([0,1])$ and there exist $\zeta_1 \geq \zeta_0 > 0$ such that

$$\zeta_0 \leq \zeta(x) \leq \zeta_1, \qquad \forall x \in [0,1]. \tag{1.2}$$

Exactly, as an optimization problem, we look for the critical points of the energy functional related to the problem which are the solutions of the problem.

In recent years, various fixed-point theorems, critical points, and variational methods have been effectively employed to explore the existence of solutions for quasilinear periodic boundary value problems. References such as [2,9,14,17,20,21,23,26,27] and others have extensively discussed this topic. For instance, Matzakos and Papageorgiou in [21] combined the variational method with techniques involving upper and lower solutions to establish the existence of periodic solutions for quasilinear differential equations. Similarly, Papageorgiou and Papalini in [23] utilized variational arguments, methods from the theory of nonlinear operators of monotone type, and upper and lower solution techniques to demonstrate the existence of at least two nontrivial solutions, one positive and the other negative for the following quasilinear periodic problem

$$\begin{cases} -(|u'(x)|^{p-2}u'(x))' = f(x, u(x)) = 0, & x \in [0, b], \\ u(0) = u(b),\ u'(0) = u'(b),\ 2 \leq p < \infty \end{cases}$$

where $f : T \times \mathbb{R} \to \mathbb{R}$ is a function. In [14], the existence of at least three classical solutions for a Dirichlet quasilinear elliptic system was established through the application of variational methods and critical point theory. Similarly, in [17], the utilization of a recent three critical points theorem by Bonanno and Marano led to the confirmation of at least three solutions for quasilinear second order differential equation on a compact interval $[a, b] \subset \mathbb{R}$

$$\begin{cases} -u'' = (\lambda f(x, u) + g(u))h'(u), & \text{in } (a, b), \\ u(a) - u(b) = 0 \end{cases}$$

where $f : [a, b] \times \mathbb{R} \to \mathbb{R}$ is an L^1-Carathéodory function, $g : \mathbb{R} \to \mathbb{R}$ is a Lipschitz continuous function, was discussed. Shen and Liu, in [26], utilized the symmetric mountain pass theorem and genus properties in critical point theory to explore the existence of infinitely many solutions for second-order quasilinear periodic boundary value problems with impulsive effects.

Meanwhile, Wang et al., in [27], investigated the existence of at least three periodic solutions for the problem ($P^{f,g}_{\lambda,\mu}$) by employing appropriate hypotheses and a three critical points theorem by Ricceri.

Additionally, in [15], variational methods and critical point theorems for smooth functionals defined on reflexive Banach spaces were used to discuss the existence of at least three solutions to an impulsive effects version of the problem ($P^{f,g}_{\lambda,\mu}$). Furthermore, in [19], variational methods were employed to discuss the existence of at least three weak solutions for the problem ($P^{f,g}_{\lambda,\mu}$) in the case $\mu = 0$. In [12], the investigation focused on the existence of infinitely many classical solutions for an impulsive effects version of the problem ($P^{f,g}_{\lambda,\mu}$) in the case $\mu = 0$, utilizing critical point theory. In [13], by using variational methods, the existence of non-zero solutions and the existence of multiple solutions for positive parameter values for the problem ($P^{f,g}_{\lambda,\mu}$) in the case $\mu = 0$, was discussed. Lastly, in [18], the existence of at least one weak solution and infinitely many weak solutions for the problem ($P^{f,g}_{\lambda,\mu}$) in the case $\mu = 0$ was studied based on variational methods.

Our approach employs variational methods, with the primary tools being four local minimum theorems for differentiable functionals. Specifically, we utilize a corollary of Bonanno's local minimum theorem to establish the existence of one solution under certain algebraic conditions on the nonlinear terms, and two solutions for the problem under algebraic conditions alongside the classical Ambrosetti–Rabinowitz (AR) condition on the nonlinear terms (refer to [3]). Furthermore, by leveraging two critical point theorems, one by Averna and Bonanno, and another by Bonanno, we ensure the existence of two and three solutions for the problem ($P^{f,g}_{\lambda,\mu}$) in the case $\mu = 0$.

In comparison to previous findings, we introduce novel assumptions to establish the existence of solutions for the problem ($P^{f,g}_{\lambda,\mu}$), thus extending recent related works.

Here, we present two specific cases of our main results focusing on scenarios with a single impulse.

Theorem 1.1. *Let $\psi : \mathbb{R} \to \mathbb{R}$ be a continuous function. Assume that there exist two positive constants γ and η with the property*

$$\sqrt{\frac{2h(2\eta) + 2h(-2\eta) + 8\zeta_1\eta^2}{\min\{m, \zeta_0\}}} < \gamma$$

and

- *there exist $\nu > 2$ and $R > 0$ such that*

$$0 < \nu \int_0^{\xi} \psi(s)\mathrm{d}s \le \xi\psi(\xi)$$

  for all $|\xi| \ge R$.

Then, for each

$$\lambda \in \left(0, \frac{\min\{m, \zeta_0\}\gamma^2}{8\int_0^{\gamma} \psi(s)\mathrm{d}s}\right)$$

and for every function $g : [0,1] \times \mathbb{R} \to \mathbb{R}$ satisfying the following condition:

- *there exist $\nu > 2$ and $R > 0$ such that*

$$0 < \nu \int_0^{\xi} g(x,s)\mathrm{d}s \le \xi g(x,\xi)$$

  for all $|\xi| \ge R$ and for all $x \in [0,1]$, *there exists $\delta_\lambda > 0$, for each $\mu \in [0, \delta_\lambda[$, the problem*

$$\begin{cases} -p(u')u'' + \zeta(x)u = \lambda e^{-t}\psi(x) + \mu g(x, u(x)), & a.e.\ x \in [0,1], \\ u(1) - u(0) = u'(1) - u'(0) = 0 \end{cases} \tag{1.3}$$

admits at least two solutions u_1 and u_2 in

$$\{u : [0,1] \to \mathbb{R} : u \text{ is absolutely continuous},\ u(1) = u(0),\ u' \in L^2([0,1]\}$$

such that

$$\max_{x\in[0,1]} |u_1(x)| < \gamma.$$

Theorem 1.2. *Assume that $\psi : \mathbb{R} \to \mathbb{R}$ is a nonnegative and continuous function. Moreover, assume that*

$$\lim_{\xi\to 0^+} \frac{\psi(\xi)}{\xi} = \lim_{|\xi|\to\infty} \frac{\psi(\xi)}{|\xi|} = 0$$

and there exists a positive constant $\bar{\eta}$ such that $\int_0^{\bar{\eta}} \psi(s)\mathrm{d}s > 0$. Then, for each $\lambda > \lambda^*$ where

$$\lambda^* = \frac{1}{4\int_{\frac{1}{4}}^{\frac{3}{4}} \theta(x)\mathrm{d}x} \inf_{\bar{\eta}>0} \frac{h(2\bar{\eta}) + h(-2\bar{\eta}) + 4\zeta_1\bar{\eta}^2}{\int_0^{\bar{\eta}} \psi(s)\mathrm{d}s},$$

the problem (1.3) in the case $\mu = 0$, admits at least one nonnegative and one non zero solution in

$$\{u : [0,1] \to \mathbb{R} : u \text{ is absolutely continuous},\ u(1) = u(0),\ u' \in L^2([0,1]\}.$$

The structure of the paper is outlined as follows:

Section 2 presents our fundamental theorems and revisits relevant definitions. Section 3 discusses and proves the existence of one solution for the problem $(P^{f,g}_{\lambda,\mu})$. Section 4 addresses the existence of two solutions for the problem $(P^{f,g}_{\lambda,\mu})$. Section 5 introduces a new multiplicity result aimed at obtaining at least two and three solutions for the problem $(P^{f,g}_{\lambda,\mu})$, specifically in the case $\mu = 0$.

2 Preliminaries

The main tools utilized to prove our results in Sections 3, 4, and 5 are the following theorems.

For the following notations and results, we refer the reader to [22, 24]. Let X be a real Banach space. We say that a continuously Gâteaux differentiable functional $J\colon X \to \mathbb{R}$ satisfies the *Palais–Smale condition* (abbreviated as (PS)-condition) if any sequence $\{u_n\}$ such that $\{J(u_n)\}$ is bounded and $\lim_{n\to\infty} \|J'(u_n)\|_{X^*} = 0$ has a convergent subsequence.

Let $\Phi, \Psi : X \to \mathbb{R}$ be two continuously Gâteaux differentiable functions. Set

$$J = \Phi - \Psi,$$

and fix $r_1, r_2 \in [-\infty, \infty]$ with $r_1 < r_2$. We say that J satisfies the *Palais–Smale condition cut off lower at r_1 and upper at r_2*(in short $^{[r_1]}(\mathrm{PS})^{[r_2]}$-condition) if any sequence $\{u_n\}$ such that $\{J(u_n)\}$ is bounded, $\lim_{n\to\infty} \|J'(u_n)\|_{X^*} = 0$ and $r_1 < \Phi(u_n) < r_2$ for all $n \in \mathbb{N}$, has a convergent subsequence.

Clearly, if $r_1 = -\infty$ and $r_2 = \infty$ it coincides with the classical (PS)-condition. Moreover, if $r_1 = -\infty$ and $r_2 \in \mathbb{R}$ it is denoted by $(\mathrm{PS})^{[r_2]}$, while if $r_1 \in \mathbb{R}$ and $r_2 = \infty$ it is denoted by

${}^{[r_1]}(\mathrm{PS})$. Indeed, if Φ and Ψ be two continuously Gâteaux differentiable functionals defined on a real Banach space X and fix $r \in \mathbb{R}$. The functional $I = \Phi - \Psi$ is said to verify the Palais–Smale condition cut off upper at r (in short $(\mathrm{PS})^{[r]}$) if any sequence $\{u_n\}_{n\in\mathbb{N}}$ in X such that $\{I(u_n)\}$ is bounded, $\lim_{n\to\infty} \|I'(u_n)\|_{X^*} = 0$ and $\Phi(u_n) < r$ for all $n \in \mathbb{N}$, has a convergent subsequence. Furthermore, if J satisfies ${}^{[r_1]}(\mathrm{PS})^{[r_2]}$-condition, then it satisfies ${}^{[\varrho_1]}(\mathrm{PS})^{[\varrho_2]}$-condition for all $\varrho_1, \varrho_2 \in [-\infty, \infty]$ such that $r_1 \le \varrho_1 < \varrho_2 \le r_2$.

In particular, we deduce that if J satisfies the classical (PS)-condition, then it satisfies ${}^{[\varrho_1]}(\mathrm{PS})^{[\varrho_2]}$-condition for all $\varrho_1, \varrho_2 \in [-\infty, \infty]$ with $\varrho_1 < \varrho_2$.

In the proof of our main results, we will apply the following four theorems.

Theorem 2.1 ([7, Theorem 2.3])**.** *Let X be a real Banach space and let $\Phi, \Psi : X \to \mathbb{R}$ be two continuously Gâteaux differentiable functions such that $\inf_{u\in X} \Phi(u) = \Phi(0) = \Psi(0) = 0$. Assume that there exist $r > 0$ and $\bar{u} \in X$, with $0 < \Phi(\bar{u}) < r$, such that:*

(a_1) $\frac{\sup_{\Phi(u)\le r} \Psi(u)}{r} < \frac{\Psi(\bar{u})}{\Phi(\bar{u})}$,

(a_2) *for each $\lambda \in \left(\frac{\Phi(\bar{u})}{\Psi(\bar{u})}, \frac{r}{\sup_{\Phi(u)\le r} \Psi(u)}\right)$, the functional $I_\lambda = \Phi - \lambda\Psi$ satisfies $(\mathrm{PS})^{[r]}$-condition.*

Then, for each

$$\lambda \in \Lambda_r = \left(\frac{\Phi(\bar{u})}{\Psi(\bar{u})}, \frac{r}{\sup_{\Phi(u)\le r} \Psi(u)} \right),$$

there exists $u_{0,\lambda} \in \Phi^{-1}(0, r)$ such that $I_\lambda(u_{0,\lambda}) \equiv \vartheta_{X^*}$ and $I_\lambda(u_{0,\lambda}) \le I_\lambda(u)$ for all $u \in \Phi^{-1}(0, r)$.

Theorem 2.2 ([7, Theorem 3.2])**.** *Let X be a real Banach space, $\Phi, \Psi : X \to \mathbb{R}$ be two continuously Gâteaux differentiable functionals such that Φ is bounded from below and $\Phi(0) = \Psi(0) = 0$. Fix $r > 0$ and assume that, for each*

$$\lambda \in \left(0, \frac{r}{\sup_{u\in\Phi^{-1}(-\infty,r)} \Psi(u)}\right),$$

the functional $I_\lambda = \Phi - \lambda\Psi$ satisfies (PS)-condition and it is unbounded from below. Then, for each

$$\lambda \in \left(0, \frac{r}{\sup_{u\in\Phi^{-1}(-\infty,r)} \Psi(u)}\right),$$

the functional I_λ admits two distinct critical points.

Theorem 2.3 ([4, Theorem A])**.** *Let X be a reflexive real Banach space, $\Phi : X \to \mathbb{R}$ a continuously Gâteaux differentiable and sequentially weakly lower semicontinuous functional whose Gâteaux derivative admits a continuous inverse on X^*, and $\Psi : X \to \mathbb{R}$ a continuously Gâteaux differentiable functional whose Gâteaux derivative is compact. Assume that:*

(b_1) $\lim_{\|u\|\to\infty}(\Phi(u) + \lambda\Psi(u)) = \infty$ *for all $\lambda \in [0, \infty)$;*

(b_2) *there is $r \in \mathbb{R}$ such that*

$$\inf_X \Phi < r,$$

and

$$\varphi_1(r) < \varphi_2(r)$$

where

$$\varphi_1(r) = \inf_{u\in\Phi^{-1}(-\infty,r)} \frac{\Psi(u) - \inf_{\overline{\Phi^{-1}(-\infty,r)}^w} \Psi}{r - \Phi(u)},$$

$$\varphi_2(r) = \inf_{u\in\Phi^{-1}(-\infty,r)} \sup_{v\in\Phi^{-1}[r,\infty)} \frac{\Psi(u) - \Psi(v)}{\Phi(v) - \Phi(u)},$$

and $\overline{\Phi^{-1}(-\infty,r)}^w$ *is the closure of* $\Phi^{-1}(-\infty,r)$ *in the weak topology.*

Then, for each $\lambda \in \left(\frac{1}{\varphi_2(r)}, \frac{1}{\varphi_1(r)}\right)$, *the functional* $\Phi + \lambda\Psi$ *has at least three critical points in* X.

Note that $\varphi_1(r)$ in Theorem 2.3 could be 0. In this and similar cases, here and in the sequel, we agree to read $\frac{1}{0}$ as ∞.

We also use the following two critical points theorem.

Theorem 2.4 ([6, Theorem 1.1])**.** *Let* X *be a reflexive real Banach space, and let* $\Phi, \Psi : X \to \mathbb{R}$ *be two sequentially weakly lower semicontinuous and Gâteaux differentiable functions. Assume that* Φ *is (strongly) continuous and satisfies*

$$\lim_{\|u\|\to\infty} \Phi(u) = \infty.$$

Assume also that there exist two constants r_1 *and* r_2 *such that*

(c_1) $\inf_X \Phi < r_1 < r_2$;

(c_2) $\varphi_1(r_1) < \varphi_2^*(r_1, r_2)$;

(c_3) $\varphi_1(r_2) < \varphi_2^*(r_1, r_2)$, *where* φ_1 *is defined as in Theorem 2.3 and*

$$\varphi_2^*(r_1, r_2) = \inf_{u\in\Phi^{-1}(-\infty,r_1)} \sup_{v\in\Phi^{-1}[r_1,r_2)} \frac{\Psi(u) - \Psi(v)}{\Phi(v) - \Phi(u)}.$$

Then, for each

$$\lambda \in \left(\frac{1}{\varphi_2^*(r_1, r_2)}, \min\left\{\frac{1}{\varphi_1(r_1)}, \frac{1}{\varphi_1(r_2)}\right\}\right),$$

the functional $\Phi + \lambda\Psi$ *admits at least two critical points which lie in* $\Phi^{-1}(-\infty, r_1]$ *and* $\Phi^{-1}[r_1, r_2)$, *respectively.*

We remind the reader that Theorem 2.3 and Theorem 2.4 rely on Ricceri's variational principle [25].

For successful application of Theorems 2.1–2.2, we recommend referring to [8] to ensure the existence of at least one and two solutions for elliptic Dirichlet problems with variable exponent. Additionally, for the utilization of Theorems 2.3–2.4, we suggest consulting [10] to guarantee the existence of at least two and three solutions for a boundary value problem on the half-line. Furthermore, for effective implementations of Theorems 2.1–2.4, we advise referring to [11, 16] to explore the existence of multiple solutions for Kirchhoff-type second-order impulsive differential equations on the half-line and to study an elastic beam equation with local nonlinearities, respectively.

In this section, we will present several fundamental definitions, notations, lemmas, and propositions utilized throughout this paper.

Let us start by defining the finite T-dimensional Banach space

$$X = \{u : [0,1] \to \mathbb{R} : u \text{ is absolutely continuous, } u(1) = u(0),\ u' \in L^2([0,1]\}, \tag{2.1}$$

which is equipped with the norm

$$\|u\| = \left(\int_0^1 \left(|u'(x)|^2 + |u(x)|^2\right) \mathrm{d}x \right)^{\frac{1}{2}}. \tag{2.2}$$

Clearly, X is a Hilbert space and X^* is the dual space of X.

Setting

$$h(y) = \int_0^y \left(\int_0^\tau p(\xi) \mathrm{d}\xi \right) \mathrm{d}\tau$$

for every $y \in \mathbb{R}$, we have

$$h'(y) = \int_0^y p(\xi)\mathrm{d}\xi \quad \text{and} \quad h''(y) = p(y)$$

for every $y \in \mathbb{R}$.

We define functionals Φ, Ψ for every $u \in X$, as follows

$$\Phi(u) = \int_0^1 h(u'(x))\mathrm{d}x + \frac{1}{2}\int_0^1 \zeta(x)|u(x)|^2 \mathrm{d}x \tag{2.3}$$

and

$$\Psi(u) = \int_0^1 \left(\int_0^{u(x)} f(x,\xi)\mathrm{d}\xi \right) \mathrm{d}x + \frac{\mu}{\lambda}\int_0^1 \left(\int_0^{u(x)} g(x,\xi)\mathrm{d}\xi \right) \mathrm{d}x, \tag{2.4}$$

and we put

$$I_\lambda(u) = \Phi(u) - \lambda\Psi(u)$$

for every $u \in X$. Clearly, I_λ is Gâteaux differentiable.

Definition 2.5. We mean by a (weak) solution of the BVP ($P^{f,g}_{\lambda,\mu}$), any function $u \in X$ such that

$$\int_0^1 h'(u'(x))y'(x)\mathrm{d}x + \int_0^1 \zeta(x)u(x)y(x)\mathrm{d}x - \lambda\int_0^1 f(x,u(x))y(x)\mathrm{d}x \\ -\mu\int_0^1 g(x,u(x))y(x)\mathrm{d}x = 0$$

for every $y \in X$.

Lemma 2.6. *If $u \in X$ is a critical point of I_λ in X, iff $u \in X$ is a solution of ($P^{f,g}_{\lambda,\mu}$).*

Proof. If $u \in X$ is a critical point for I_λ, we have

$$\int_0^1 h'(u'(x))y'(x)\mathrm{d}x + \int_0^1 \zeta(x)u(x)y(x)\mathrm{d}x \\ = \lambda\int_0^1 f(x,u(x))y(x)\mathrm{d}x + \mu\int_0^1 g(x,u(x))y(x)\mathrm{d}x$$

for each $y \in X$. This implies that $h' \circ u'$ has a weak derivative which equals $\zeta(x)u(x) - \lambda f(x, u(x)) - \mu g(x, u(x))$ and is thus continuous, so $h' \circ u'$ is $C^1([0,1])$. Since h' is an invertible C^1-function, it follows that u' is also in $C^1([0,1])$, hence x is in $C^2([0,1])$. Set

$$e(x) = -h'(u'(x)) + \int_0^t \zeta(\tau)u(\tau)\mathrm{d}\tau - \lambda \int_0^t f(\tau, u(\tau))\mathrm{d}\tau - \mu \int_0^t g(\tau, u(\tau))\mathrm{d}\tau - C$$

such that $\int_0^1 e(x)\mathrm{d}x = 0$. Let $y(x) = \int_0^t e(\tau)\mathrm{d}\tau$. Then $y(x) \in X$ and $\int_0^1 |e(x)|^2\mathrm{d}x = 0$, that is, $e(x) = 0$ for a.e. $x \in [0,1]$. This shows that

$$\begin{aligned} -(h' \circ u')'(x) + \zeta(x)u(x) &= -h''(u')u''(x) + \zeta(x)u(x) = -p(u'(x))u''(x) + \zeta(x)u(x) \\ &= \lambda f(x, u(x)) + \mu g(x, u(x)) \end{aligned}$$

for all $x \in [0,1]$. Hence we conclude that x is a solution of problem $(P^{f,g}_{\lambda,\mu})$ belongs to $C^2([0,1])$. □

Proposition 2.7 ([27, Proposition 2.3]). *If $p(\cdot)$ satisfies (Q_1), then h' is strongly monotone.*

Proposition 2.8 ([27, Proposition 2.4]). *If $p(\cdot)$ and $\zeta(\cdot)$ satisfy* (1.1) *and* (1.2)*, respectively, then*

(1) *Φ is well-defined in X,*

(2) *Φ is Gâteaux differentiable in X,*

(3) *Φ' is a Lipschitzian operator,*

(4) *Φ is convex in X.*

Put

$$F(x,t) = \int_0^t f(x,s)\mathrm{d}s \quad \text{and} \quad G(x,t) = \int_0^t g(x,s)\mathrm{d}s \quad \text{for all } (x,t) \in [0,1] \times \mathbb{R}.$$

3 Existence of one solution

In this section, we focus on establishing the existence of one solution for the problem $(P^{f,g}_{\lambda,\mu})$. For clarity and convenience, let us define

$$G^\theta = \int_0^1 \sup_{|\xi| \le \theta} G(x,\xi)\mathrm{d}x \quad \text{for all } \theta > 0$$

and

$$G_\eta = \inf_{[0,1]\times[0,\eta]} G(x,\xi) \quad \text{for all } \eta > 0.$$

If g is sign-changing, then clearly $G^\theta \ge 0$ and $G_\eta \le 0$.

For our goal, we fix two positive constants θ and η, put

$$\underline{\delta}_{\lambda,g} = \min\left\{ \frac{\min\{m, \zeta_0\}\theta^2 - 8\lambda \int_0^1 \sup_{|t|\le\theta} F(x,t)\mathrm{d}x}{8G^\theta}, \right. \\ \left. \frac{\frac{1}{4}h(2\eta) + \frac{1}{4}h(-2\eta) + \zeta_1\eta^2 - \lambda \int_{\frac{1}{4}}^{\frac{3}{4}} F(x,\eta)\mathrm{d}x}{G_\eta} \right\}$$

and

$$\overline{\delta}_{\lambda,g} = \min\left\{\underline{\delta}_{\lambda,g}, \frac{1}{\max\left\{0, \frac{8}{\min\{m,\zeta_0\}} \limsup_{|t|\to+\infty} \frac{\sup_{x\in[0,1]} G(x,t)}{x^2}\right\}}\right\} \tag{3.1}$$

where we read $\epsilon/0 = +\infty$, so that, for instance, $\overline{\delta}_{\lambda,g} = +\infty$ when

$$\limsup_{|t|\to+\infty} \frac{\sup_{x\in[0,1]} G(x,t)}{x^2} \le 0,$$

and $G_\eta = G^\theta = 0$.

Theorem 3.1. *Assume that there exist two positive constants γ and η with the property*

$$\sqrt{\frac{2h(2\eta)+2h(-2\eta)+8\zeta_1\eta^2}{\min\{m,\zeta_0\}}} < \gamma$$

such that

(A_1) $f(x,t) \ge 0$ *for every* $(x,t) \in [0,1] \times \left[0,\frac{1}{4}\right] \bigcup \left[\frac{3}{4},1\right]$,

(A_2) $\frac{\int_0^1 \sup_{|t|\le\gamma} F(x,t)\mathrm{d}x}{\gamma^2} < \min\{m,\zeta_0\} \frac{\int_{\frac{1}{4}}^{\frac{3}{4}} F(x,\eta)\mathrm{d}x}{2h(2\eta)+2h(-2\eta)+8\zeta_1\eta^2}$,

(A_3) $\min_{x\in[0,1]} \limsup_{|\xi|\to\infty} \frac{F(x,\xi)}{|\xi|^2} \in (-\infty, 0]$.

Then, for each

$$\lambda \in \Lambda = \left(\frac{h(2\eta)+h(-2\eta)+4\zeta_1\eta^2}{4\int_{\frac{1}{4}}^{\frac{3}{4}} F(x,\eta)\mathrm{d}x}, \frac{\min\{m,\zeta_0\}\gamma^2}{8\int_0^1 \sup_{|t|\le\gamma} F(x,t)\mathrm{d}x}\right)$$

and for every function $g : [0,1] \times \mathbb{R} \to \mathbb{R}$ satisfying the condition

$$\min_{x\in[0,1]} \limsup_{|\xi|\to\infty} \frac{G(x,\xi)}{|\xi|^2} \in (-\infty, 0), \tag{3.2}$$

there exists $\overline{\delta}_{\lambda,g} > 0$ given by (3.1) such that for each $\mu \in [0, \overline{\delta}_{\lambda,g})$, the problem $(P^{f,g}_{\lambda,\mu})$ admits at least one solution u_λ in X such that

$$\max_{x\in[0,1]} |u_\lambda(x)| < \gamma.$$

Proof. Our objective is to apply Theorem 2.1 to the problem $(P^{f,g}_{\lambda,\mu})$. Consider the functionals Φ and Ψ defined in (2.3) and (2.3), respectively. Our task is to demonstrate that these functionals satisfy the necessary conditions outlined in Theorem 2.1. Since $f, g : [0;1] \times \mathbb{R} \to \mathbb{R}$ are L^1-Carathéodory functions, we know that Ψ' is a well-defined and Gâteaux differentiable functional with

$$\Psi'(u)(y) = \int_0^1 f(x,u(x))y(x)\mathrm{d}x + \frac{\mu}{\lambda}\int_0^1 g(x,u(x))y(x)\mathrm{d}x$$

for every $u, y \in X$. Since the embeddings $X \hookrightarrow L^q (q \ge 1)$ and $X \hookrightarrow L^\infty$ are compact (Adams [1]), we have $\Psi' : X \to X^*$ is a continuous and compact operator, and Ψ is sequentially weakly upper semicontinuous. Moreover, Again using the Lebesgue's theorem, from the continuity of h' and the arbitrariness of $\{a_n\}$, we know that Φ is Gâteaux differentiable in X with

$$\Phi'(u)(y) = \int_0^1 h'(u'(x))y'(x)\mathrm{d}x + \int_0^1 \zeta(x)u(x)y(x)\mathrm{d}x \tag{3.3}$$

for every $u, y \in X$. Furthermore, by the definition of Φ, we observe that it is sequentially weakly lower semicontinuous and strongly continuous. Combining this observation with (1.1) and (1.2), we have

$$\int_0^1 h'(u'(x))\mathrm{d}x = \int_0^1 \left(\int_0^{u'(x)} \left(\int_0^\tau p(\xi)\mathrm{d}\xi \right) \mathrm{d}\tau \right) \mathrm{d}x$$

and

$$\begin{aligned} \frac{1}{2}\min\{m, \zeta_0\}\|u\|^2 &\leq \frac{m}{2}\int_0^1 |u'(x)|^2\mathrm{d}x + \frac{\zeta_0}{2}\int_0^1 |u(x)|^2\mathrm{d}x \leq \Phi(u) \\ &\leq \frac{M}{2}\int_0^1 |u'(x)|^2\mathrm{d}x + \frac{\zeta_1}{2}\int_0^1 |u(x)|^2\mathrm{d}x \leq \frac{1}{2}\max\{M, \zeta_1\}\|u\|^2 \end{aligned} \tag{3.4}$$

for every $u \in X$, which implies that Φ is well-defined in X. By using the first inequality in (3.4), it follows

$$\lim_{\|u\| \to +\infty} \Phi(u) = +\infty,$$

namely Φ is coercive. Further, we claim that Φ admits a continuous inverse on X^*. In fact, by (1.1), (1.2), Proposition 2.7 and (3.3), we have

$$\begin{aligned} \langle \Phi'(u) - \Phi'(y), u - y \rangle &= \int_0^1 (h'(u'(x)) - h'(y'(x)), u'(x) - y'(x))\mathrm{d}x \\ &\quad + \int_0^1 \zeta(x)|u(x) - y(x)|^2\mathrm{d}x \\ &\geq \int_0^1 m|u'(x) - y'(x)|^2\mathrm{d}x + \int_0^1 \zeta_0|u(x) - y(x)|^2\mathrm{d}x \\ &\geq \min\{m, \zeta_0\}\|u(x) - y(x)\|^2 \end{aligned}$$

for all $u, y \in X$, which shows that Φ' is uniformly monotone in X. Put $y = 0$, then we have

$$\min\{m, \zeta_0\}\|u\|^2 \leq \langle \Phi'(u), u \rangle \leq \|\Phi'(u)\|_{X^*}.\|u\|$$

$$\Rightarrow \min\{m, \zeta_0\}\|u\| \leq \|\Phi'(u)\|_{X^*}$$

which shows that Φ' is coercive in X. Since Φ' is a Lipschitzian operator, it is hemicontinuous in X. According to Theorem 26.A of [28], Φ admits a continuous inverse on X^*. Additionally, the functional Ψ belongs to $C^1(X, \mathbb{R})$ and has a compact derivative. Given that the embedding $X \hookrightarrow L^q$ (where $q \geq 1$) is compact, there exists a positive constant C such that

$$|u|_{L^q([0,1])} \leq C\|u\|$$

and it follows that

$$|u|_{L^2([0,1])} \leq C_1\|u\|$$

where C_1 is positive constant. Moreover, for $\lambda > 0$, the functional I_λ is coercive. Indeed, since $\mu < \delta_\lambda$ we can fix κ such that

$$\min_{x\in[0,1]} \limsup_{|\xi|\to\infty} \frac{G(x,\xi)}{|\xi|^2} \in (-\infty, \kappa)$$

and $\mu\kappa < \frac{\min\{m,\zeta_0\}}{2C_1^2}$. Therefore, there exists a positive constant ϱ such that

$$G(x,\xi) \le \kappa\xi^2 + \varrho$$

for each $\xi \in \mathbb{R}$ and $x \in [0,1]$. Now, we fix $0 < \varepsilon < \frac{1}{\lambda C_1^2}\left(\frac{\min\{m,\zeta_0\}}{2} - \mu C_1^2\kappa\right)$. From the assumption (A_3) there is a positive constant ρ_ε such that

$$F(x,\xi) \le \varepsilon\xi^2 + \rho_\varepsilon$$

for every $(x,\xi) \in [0,1]\times\mathbb{R}$. It follows that, for each $u \in X$, we have

$$\begin{aligned}
\Phi(u) - \lambda\Psi(u) &\ge \frac{1}{2}\min\{m,\zeta_0\}\|u\|^2 - \lambda\int_0^1 [F(x,u(x)) + \frac{\mu}{\lambda}G(x,u(x))]\mathrm{d}x \\
&\ge \frac{1}{2}\min\{m,\zeta_0\}\|u\|^2 - \lambda\left(\varepsilon\int_0^1 |u(x)|^2\mathrm{d}x + \rho_\varepsilon\right) - \mu\left(\kappa\int_0^1 |u(x)|^2\mathrm{d}x + \varrho\right) \\
&\ge \left(\frac{1}{2}\min\{m,\zeta_0\} - \lambda C_1^2\varepsilon - \mu C_1^2\kappa\right)\|u\|^2 - \lambda\rho_\varepsilon - \mu\varrho
\end{aligned}$$

and thus

$$\lim_{\|u\|\to\infty}(\Phi(u) - \lambda\Psi(u)) = \infty,$$

which means the functional $I_\lambda = \Phi - \lambda\Psi$ is coercive. Thus, by [5, Proposition 2.1] the functional $I_\lambda = \Phi - \lambda\Psi$ verifies $(\mathrm{PS})^{[r]}$-condition for each $r > 0$ and so the condition (a_2) of Theorem 2.1 is verified. Fix $\lambda \in (0,\lambda^*)$, thus

$$\frac{\int_{\frac{1}{4}}^{\frac{3}{4}} F(x,\eta)\mathrm{d}x - \frac{\mu}{\lambda}G_\eta}{\frac{1}{4}h(2\eta) + \frac{1}{4}h(-2\eta) + \zeta_1\eta^2} > \frac{1}{\lambda}.$$

Put $r = \frac{\min\{m,\zeta_0\}}{8}\gamma^2$ and

$$w(x) = \begin{cases} \eta, & x \in \left[0, \frac{1}{4}\right], \\ 2\eta x + \frac{\eta}{2}, & x \in \left[\frac{1}{4}, \frac{1}{2}\right], \\ -2\eta x + \frac{5\eta}{2}, & x \in \left[\frac{1}{2}, \frac{3}{4}\right], \\ \eta, & x \in \left[\frac{3}{4}, 1\right]. \end{cases} \tag{3.5}$$

Clearly, $w \in X$. Then, we have $\Phi(0) = \Psi(0) = 0$,

$$\begin{aligned}
\Phi(w) &= \frac{1}{4}h(2\eta) + \frac{1}{4}h(-2\eta) + \frac{1}{2}\int_0^1 \zeta(x)|w(x)|^2\mathrm{d}x \\
&\le \frac{1}{4}h(2\eta) + \frac{1}{4}h(-2\eta) + \frac{\zeta_1}{2}\int_0^1 |w(x)|^2\mathrm{d}x \\
&= \frac{1}{4}h(2\eta) + \frac{1}{4}h(-2\eta) + \frac{31\zeta_1\eta^2}{2\times 24} \\
&< \frac{1}{4}h(2\eta) + \frac{1}{4}h(-2\eta) + \zeta_1\eta^2
\end{aligned} \tag{3.6}$$

and

$$\begin{aligned}\Phi(w) &\geq \frac{1}{4}h(2\eta)+\frac{1}{4}h(-2\eta)+\frac{\zeta_0}{2}\int_0^1 |w(x)|^2\mathrm{d}x\\ &= \frac{1}{4}h(2\eta)+\frac{1}{4}h(-2\eta)+\frac{31\zeta_0\eta^2}{2\times 24}\\ &> \frac{1}{4}h(2\eta)+\frac{1}{4}h(-2\eta)+\frac{\zeta_0\eta^2}{8}.\end{aligned} \tag{3.7}$$

From $\min_{x\in[\frac{1}{4},\frac{3}{4}]}\{w(x)\}=\eta$, $\max_{x\in[\frac{1}{4},\frac{3}{4}]}\{w(x)\}=\frac{3\eta}{2}$ and the assumption (A_1), we have

$$\begin{aligned}\Psi(w) =& \int_0^{\frac{1}{4}}\int_0^{\eta} f(x,\xi)\mathrm{d}\xi\mathrm{d}x+\int_{\frac{1}{4}}^{\frac{3}{4}}\int_0^{w} f(x,\xi)\mathrm{d}\xi\mathrm{d}x+\int_{\frac{3}{4}}^{1}\int_0^{\eta} f(x,\xi)\mathrm{d}\xi\mathrm{d}x\\ &+\frac{\mu}{\lambda}\int_0^1\left(\int_0^{w(x)} g(x,\xi)\mathrm{d}\xi\right)\mathrm{d}x \geq \int_{\frac{1}{4}}^{\frac{3}{4}}\int_0^{\eta} f(x,\xi)\mathrm{d}\xi\mathrm{d}x+\frac{\mu}{\lambda}G_\eta\\ =& \int_{\frac{1}{4}}^{\frac{3}{4}} F(x,\eta)\mathrm{d}x+\frac{\mu}{\lambda}G_\eta.\end{aligned}$$

Thus, by the assumption

$$\sqrt{\frac{8}{\min\{m,\zeta_0\}}\left(\frac{1}{4}h(2\eta)+\frac{1}{4}h(-2\eta)+\zeta_1\eta^2\right)}<\gamma,$$

we have $0<\Phi(w)<r$. For $u\in X$, taking into account

$$|u(x)|\leq\left|\int_{t_1}^{t} u'(\tau)\mathrm{d}\tau\right|+|u(x_1)|\leq\int_0^1|u'(\tau)|\mathrm{d}\tau+|u(x_1)|$$

and

$$|u(x)|\leq\int_0^1|u'(\tau)|\mathrm{d}\tau+\int_0^1|u(x_1)|\mathrm{d}x_1\leq\left(\int_0^1|u'(\tau)|^2\mathrm{d}\tau\right)^{\frac{1}{2}}+\left(\int_0^1|u(\tau)|^2\mathrm{d}\tau\right)^{\frac{1}{2}},$$

we have

$$\max_{x\in[0,1]}|u(x)|\leq 2\|u\|. \tag{3.8}$$

From the definition of Φ and in view of (3.4) for every $r>0$, one has

$$\begin{aligned}\Phi^{-1}(-\infty,r] =& \{u\in X,\ \Phi(x)\leq r\}\\ &\subseteq\left\{u\in X,\ \max_{x\in[0,1]}|u(x)|\leq\sqrt{\frac{8r}{\min\{m,\zeta_0\}}}\right\}\\ &\subseteq\left\{u\in X,\ \max_{x\in[0,1]}|u(x)|\leq\gamma\right\}.\end{aligned}$$

Hence, we have

$$\sup_{\Phi(u)<r}\Psi(u)=\sup_{u\in\Phi^{-1}(-\infty,r)}\Psi(u)\leq\int_0^1\sup_{|t|\leq\gamma}F(x,t)\mathrm{d}x+\frac{\mu}{\lambda}G^\gamma.$$

Therefore, we have

$$\begin{aligned}\frac{\sup_{u\in\Phi^{-1}(-\infty,r]}\Psi(u)}{r} &= \frac{\sup_{u\in\Phi^{-1}(-\infty,r]}\left(\int_0^1 F(x,u(x))\mathrm{d}x + \frac{\mu}{\lambda}\int_0^1 G(x,u(x))\mathrm{d}x\right)}{r}\\ &\le \frac{\int_0^1 \sup_{|t|\le\gamma} F(x,t)\mathrm{d}x + \frac{\mu}{\lambda}G^{\gamma}}{\frac{\min\{m,\zeta_0\}}{8}\gamma^2}\end{aligned} \tag{3.9}$$

and

$$\begin{aligned}\frac{\Psi(w)}{\Phi(w)} &\ge \frac{\int_{\frac14}^{\frac34} F(x,\eta)\mathrm{d}x + \frac{\mu}{\lambda}\int_0^1 G(x,\eta)\mathrm{d}x}{\frac14 h(2\eta)+\frac14 h(-2\eta)+\zeta_1\eta^2} \\ &\ge \frac{\int_{\frac14}^{\frac34} F(x,\eta)\mathrm{d}x + \frac{\mu}{\lambda}G_{\eta}}{\frac14 h(2\eta)+\frac14 h(-2\eta)+\zeta_1\eta^2}.\end{aligned} \tag{3.10}$$

Since

$$\mu < \frac{\min\{m,\zeta_0\}\gamma^2 - 8\lambda\int_0^1 \sup_{|t|\le\gamma}F(x,t)\mathrm{d}x}{8G^{\gamma}},$$

we have

$$8\frac{\int_0^1 \sup_{|t|\le\gamma} F(x,t)\mathrm{d}x + \frac{\mu}{\lambda}G^{\gamma}}{\min\{m,\zeta_0\}\gamma^2} < \frac{1}{\lambda}.$$

Furthermore,

$$\mu < \frac{\frac14 h(2\eta)+\frac14 h(-2\eta)+\zeta_1\eta^2 - \lambda\int_{\frac14}^{\frac34}F(x,\eta)\mathrm{d}x}{G_{\eta}},$$

this means

$$\frac{\int_{\frac14}^{\frac34} F(x,\eta)\mathrm{d}x + \frac{\mu}{\lambda}G_{\eta}}{\frac14 h(2\eta)+\frac14 h(-2\eta)+\zeta_1\eta^2} > \frac{1}{\lambda}.$$

Then,

$$\frac{8}{\min\{m,\zeta_0\}}\frac{\int_0^1 \sup_{|t|\le\gamma} F(x,t)\mathrm{d}x + \frac{\mu}{\lambda}G^{\gamma}}{\gamma^2} < \frac{1}{\lambda} < \frac{\int_{\frac14}^{\frac34} F(x,\eta)\mathrm{d}x + \frac{\mu}{\lambda}G_{\eta}}{\frac14 h(2\eta)+\frac14 h(-2\eta)+\zeta_1\eta^2}. \tag{3.11}$$

Hence, from (3.9)–(3.11), the condition (a_1) of Applying Theorem 2.1 with $\bar{u}=w$ ensures the existence of a local minimum point u_λ for the functional I_λ such that $0<\Phi(u_\lambda)<r$. Thus, u_λ serves as a nontrivial solution to the problem $(P^{f,g}_{\lambda,\mu})$, satisfying

$$\max_{x\in[0,1]}|u_\lambda(x)| < \gamma.$$

□

Now, we illustrate Theorem 3.1 through the following example.

Example 3.2. We consider the following problem

$$\begin{cases} -p(u')u'' + u = \lambda f(x,u(x)) + \mu g(x,u(x)), & \text{a.e. } x\in[0,1],\\ u(1)-u(0) = u'(1)-u'(0) = 0 \end{cases} \tag{3.12}$$

where $p(t) = 3 - 2\cos(t)$ for every $t \in \mathbb{R}$, $\zeta(x) = 1$ for each $x \in [0,1]$ and

$$f(x,t) = \begin{cases} \frac{4}{10}e^{-x}t^3, & \text{for every } (x,t) \in [0,1] \times (-\infty, 1), \\ \frac{4}{10t}e^{-x}, & \text{for every } (x,t) \in [0,1] \times [1, +\infty). \end{cases}$$

By the expression of f, we have

$$F(x,t) = \begin{cases} \frac{1}{10}e^{-x}t^4, & \text{for every } (x,t) \in [0,1] \times (-\infty, 1), \\ e^{-x}\left(\frac{4}{10}\ln(t) + \frac{1}{10}\right), & \text{for every } (x,t) \in [0,1] \times [1, +\infty). \end{cases}$$

Hence, $\lim_{|\xi|\to\infty} \frac{F(x,\xi)}{|\xi|^2} = 0$, thus (A_3) holds. Choose $\gamma = 10^{-2}$, and $\eta = 1$. By simple calculations, we obtain $m = 1$, $M = 5$ and $\zeta_0 = \zeta_1 = 1$. Since

$$\frac{\int_0^1 \sup_{|t|\le\gamma} F(x,t)\mathrm{d}x}{\gamma^2} = \frac{e-1}{10^9 e} < \frac{e^{0.75} - e^{0.25}}{(160 + 80\cos(2))e} = \frac{\min\{m, \zeta_0\} \int_{\frac{1}{4}}^{\frac{3}{4}} F(x,\eta)\mathrm{d}x}{2h(2\eta) + 2h(-2\eta) + 8\zeta_1\eta^2},$$

thus (A_2) holds true, then all conditions in Theorem 3.1 are satisfied. Therefore, it follows that for each

$$\lambda \in \left(\frac{(80 + 40\cos(2))e}{4(e^{0.75} - e^{0.25})}, \frac{10^9 e}{8(e-1)} \right)$$

and for every function $g : [0,1] \times \mathbb{R} \to \mathbb{R}$ satisfying the condition

$$\min_{x\in[0,1]} \limsup_{|\xi|\to\infty} \frac{G(x,\xi)}{|\xi|^2} \in (-\infty, 0),$$

there exists $\overline{\delta}_{\lambda,g} > 0$ such that for each $\mu \in [0, \overline{\delta}_{\lambda,g})$, the problem (3.12) admits at least one solution u_λ in X such that

$$\max_{x\in[0,1]} |u_\lambda(x)| < 10^{-2}.$$

4 Existence of two solutions

In this section, our objective is to establish the existence of two distinct solutions for the problem $(P^{f,g}_{\lambda,\mu})$. The following result is derived by applying Theorem 2.2, without the need for assumption (A_3).

Theorem 4.1. *Assume that there exist two positive constants γ and η with the property*

$$\sqrt{\frac{2h(2\eta) + 2h(-2\eta) + 8\zeta_1\eta^2}{\min\{m, \zeta_0\}}} < \gamma$$

and

(A_4) *there exist $\nu > 2$ and $R > 0$ such that*

$$0 < \nu F(x,\xi) \le \xi f(x,\xi) \tag{4.1}$$

for all $|\xi| \ge R$ and for all $x \in [0,1]$.

Then, for each

$$\lambda \in \left(0, \frac{\min\{m, \zeta_0\}\gamma^2}{8\int_0^1 \sup_{|t|\le\gamma} F(x,t)\mathrm{d}x}\right),$$

and for every function $g : [0,1] \times \mathbb{R} \to \mathbb{R}$ satisfying the condition (A_4), there exists $\delta_\lambda > 0$, for each $\mu \in [0, \delta_\lambda[$, the problem $(P^{f,g}_{\lambda,\mu})$ admits at least two solutions u_1 and u_2 in X such that

$$\max_{x\in[0,1]} |u_1(x)| < \gamma.$$

Proof. Our aim is to apply Theorem 2.2 to the space X with the norm is defined in (2.2) and to the functionals Φ and Ψ defined in the proof of Theorem 3.1. The functional I_λ satisfies the (PS)-condition. Indeed, assume that $\{u_n\}_{n\in\mathbb{N}} \subset X$ such that $\{I_\lambda(u_n)\}_{n\in\mathbb{N}}$ is bounded and $I'_\lambda(u_n) \to 0$ as $n \to \infty$. Then, there exists a positive constant c_0 such that

$$|I_\lambda(u_n)| \le c_0,\ |I'_\lambda(u_n)| \le c_0 \qquad \forall n \in \mathbb{N}.$$

Therefore, we deduce from the definition of I'_λ and assumption (A_3) that

$$\begin{aligned} c_0 + c_1\|u_n\| \ge \nu I_\lambda(u_n) - I'_\lambda(u_n)(u_n) &\ge \min\{m, \zeta_0\}\left(\frac{\nu}{2} - 1\right)\|u_n\|^2 \\ &\quad - \lambda \int_0^1 (\nu F(x, u_n(x)) - f(x, u_n(x))(u_n(x)))\,\mathrm{d}x \\ &\quad - \mu \int_0^1 (\nu G(x, u_n(x)) - g(x, u_n(x))(u_n(x)))\,\mathrm{d}x \\ &\ge \min\{m, \zeta_0\}\left(\frac{\nu}{2} - 1\right)\|u_n\|^2 \end{aligned}$$

for some $c_1 > 0$. Since $\nu > 2$, this implies that (u_n) is bounded. Consequently, since X is a reflexive Banach space we have, up to a subsequence,

$$u_n \rightharpoonup u \quad \text{in } X.$$

By $I'_\lambda(u_n) \to 0$ and $u_n \rightharpoonup u$ in X, we obtain

$$\left(I'_\lambda(u_n) - I'_\lambda(u)\right)(u_n - u) \to 0. \tag{4.2}$$

From the continuity of f and g, we have

$$\int_0^1 \left(f(x, u_n(x)) - f(x, u(x))\right)(u_n(x) - u(x))\mathrm{d}x \to 0, \quad \text{as } n \to \infty$$

and

$$\int_0^1 \left(g(x, u_n(x)) - g(x, u(x))\right)(u_n(x) - u(x))\mathrm{d}x \to 0, \quad \text{as } n \to \infty.$$

Moreover, an easy computation shows

$$\begin{aligned}
\left(I'_\lambda(u_n) - I'_\lambda(u)\right)(u_n - u) = &\int_0^1 h'(u'_n(x) - u'(x))(u'_n(x) - u'(x))\mathrm{d}x \\
&+ \int_0^1 \zeta(x)(u_n(x) - u(x))(u_n(x) - u(x))\mathrm{d}x \\
&- \lambda \int_0^1 \left(f(x, u_n(x)) - f(x, u(x))\right)(u_n(x) - u(x))\mathrm{d}x \\
&- \mu \int_0^1 \left(g(x, u_n(x) - g(x, u(x))\right)(u_n(x) - u(x))\mathrm{d}x \\
\geq\ &\min\{m, \zeta_0\}\|u_n - u\|^2.
\end{aligned}$$

Thus, the sequence u_n converges strongly to u in X. Therefore, I_λ satisfies the (PS)-condition. Moreover, by integrating the condition (4.1), there exist constants $a_1, a_2, a_3, a_4 > 0$ such that that

$$F(x,t) \geq a_1|t|^\nu - a_2 \quad \text{and} \quad G(x,t) \geq a_3|t|^\nu - a_4$$

for all $x \in [0,1]$ and $t \in \mathbb{R}$. Moreover, for any $u \in X$, one has

$$\frac{1}{2}\min\{m, \zeta_0\}\|u\|^2 \leq \Phi(u) \leq \frac{1}{2}\max\{M, \zeta_1\}\|u\|^2. \tag{4.3}$$

Now, choosing any $u \in X \setminus \{0\}$, for each $\tau > 0$ taking (4.3) into account one has

$$\begin{aligned}
I_\lambda(\tau u) &= (\Phi + \lambda\Psi)(\tau u) \\
&\leq \frac{\max\{M, \zeta_1\}}{2}\|\tau u\|^2 - \lambda \int_0^1 F(x, \tau u(x))\mathrm{d}x - \mu \int_0^1 G(x, \tau u(x))\mathrm{d}x \\
&\leq \frac{\max\{M, \zeta_1\}\tau^2}{2}\|u\|^2 - \lambda\tau^\nu a_1 \int_0^1 |u(x)|^\nu \mathrm{d}x + \mu\tau^\nu a_3 \int_0^1 |u(x)|^\nu \mathrm{d}x - \lambda a_2 - \mu a_4.
\end{aligned}$$

Since $\nu > 2$, this condition guarantees that I_λ is unbounded from below. Thus, all hypotheses of Theorem 2.2 are satisfied. Therefore, for each

$$\lambda \in \left(0, \frac{\min\{m, \zeta_0\}\gamma^2}{8\int_0^1 \sup_{|t|\leq\gamma} F(x,t)\mathrm{d}x}\right),$$

the functional I_λ admits two distinct critical points that are solutions of the problem $(P^{f,g}_{\lambda,\mu})$. □

Remark 4.2. Theorem 1.1 immediately follows from Theorem 4.1.

Remark 4.3. In Theorem 2.1, it is observed that if either $f(x,0) \neq 0$ for some $x \in [0,1]$ or $g(x,0) \neq 0$ for some $x \in [0,1]$, or both conditions hold true, then Theorem 4.1 guarantees the existence of two nontrivial solutions for the problem $(P^{f,g}_{\lambda,\mu})$. However, if the condition $f(x,0) \neq 0$ for some $x \in [0,1]$ and $g(x,0) \neq 0$ for some $x \in [0,1]$ does not hold, the second solution u_2 of the problem $(P^{f,g}_{\lambda,\mu})$ may be trivial, but the problem still has at least one nontrivial solution.

Remark 4.4. Using similar arguments as those provided in the proof of [7, Theorem 3.5], the non-triviality of the second solution guaranteed by Theorem 4.1 can also be achieved in the case where $f(x,0) = 0$ for all $x \in [0,1]$, provided that an extra condition at zero is imposed.

Specifically, this condition entails the existence of a non-empty open set $D \subseteq [0,1]$ and a set $B \subset D$ of positive Lebesgue measure such that

$$\limsup_{\xi \to 0^+} \frac{\operatorname{ess\,inf}_{x\in B} F(x,\xi)}{|\xi|^2} = \infty \quad \text{and} \quad \liminf_{\xi \to 0^+} \frac{\operatorname{ess\,inf}_{x\in D} F(x,\xi)}{|\xi|^2} > -\infty.$$

See [18, Theorem 3.1] for more details.

5 Another multiplicity result for the case $\mu = 0$

In this section, we focus on establishing the existence of at least two and three solutions for the problem ($P^{f,g}_{\lambda,\mu}$) in the case $\mu = 0$. To achieve this, we define

$$F^c = \int_0^1 \sup_{|t|\le c} F(x,t)\mathrm{d}x \quad \text{and} \quad F_c = \inf_{x\in[0,1]} F(x,c)$$

for every $c > 0$.

Theorem 5.1. *Assume that there exist two positive constants $\bar{\gamma}$ and $\bar{\eta}$ such that*

$$\sqrt{\frac{2h(2\bar{\eta}) + 2h(-2\bar{\eta}) + 8\zeta_1\bar{\eta}^2}{\min\{m,\zeta_0\}}} < \bar{\gamma} \tag{5.1}$$

and suppose that the assumptions (A_1) and (A_3) in Theorem 3.1 hold. Moreover, assume that

(A_5) $\frac{F^{\bar{\gamma}}}{\min\{m,\zeta_0\}\bar{\gamma}^2} < \frac{\frac{1}{2}F_{\bar{\eta}} - F^{\bar{\gamma}}}{2h(2\bar{\eta})+2h(-2\bar{\eta})+8\zeta_1\bar{\eta}^2}$.

Then, for each

$$\lambda \in \left(\frac{h(2\bar{\eta}) + h(-2\bar{\eta}) + 4\zeta_1\bar{\eta}^2}{2F_{\bar{\eta}} - 4F^{\bar{\gamma}}}, \frac{\min\{m,\zeta_0\}\bar{\gamma}^2}{8F^{\bar{\gamma}}} \right),$$

the problem ($P^{f,g}_{\lambda,\mu}$) in the case $\mu = 0$ admits at least three solutions in X.

Proof. Put $I_\lambda = \Phi + \lambda\Psi$, where

$$\Phi(u) = \int_0^1 h(u'(x))\mathrm{d}x + \frac{1}{2}\int_0^1 \zeta(x)|u(x)|^2\mathrm{d}x \tag{5.2}$$

and

$$\Psi(u) = -\int_0^1 F(x,u(x))\mathrm{d}x$$

for all $u \in X$. Standard arguments demonstrate that Φ and Ψ are Gâteaux differentiable functionals, and their Gâteaux derivatives at the point $u \in X$ are given by

$$\Phi'(u)(v) = \int_0^1 h'(u'(x))v'(x)\mathrm{d}x + \int_0^1 \zeta(x)u(x)v(x)\mathrm{d}x$$

and

$$\Psi'(u)(v) = -\int_0^1 f(x,u(x))v(x)\mathrm{d}x$$

for all $u, v \in X$, respectively. Hence, a critical point of the functional $\Phi + \lambda\Psi$, gives us a solution of $(P^{f,g}_{\lambda,\mu})$ in the case $\mu = 0$. Our goal is to apply Theorem 2.3 to Φ and Ψ. By sequentially weakly lower semicontinuity of the norm and continuity of h, the functional Φ is sequentially weakly lower semicontinuous. Moreover, from Section 3, Φ is continuously Gâteaux differentiable while Proposition 2.7 gives that its Gâteaux derivative admits a continuous inverse on X^*. The functional $\Psi : X \to \mathbb{R}$ is well defined and is continuously Gâteaux differentiable whose Gâteaux derivative is compact. Then, it is enough to show that Φ and Ψ satisfy (c_1) and (c_2) in Theorem 2.3. Now, we fix $0 < \epsilon < \frac{\min\{m,\zeta_0\}}{2\lambda C_1^2}$. From the assumption (A_3) there is a function $\rho_\varepsilon : [0,1] \to \mathbb{R}$ with $\rho_\epsilon(x) < \infty$ for all $x \in [0,1]$ such that

$$F(x,t) \leq \varepsilon t^2 + \rho(x)$$

for every $(x,t) \in [0,1] \times \mathbb{R}$. It follows that for each $u \in X$,

$$\begin{aligned}\Phi(u) - \lambda\Psi(u) &\geq \frac{\min\{m,\zeta_0\}}{2}\|u\|^2 - \lambda\int_0^1 F(x,u(x))\mathrm{d}x\\ &\geq \frac{\min\{m,\zeta_0\}}{2}\|u\|^2 - \lambda\epsilon\int_0^1 u^2(x)\mathrm{d}x - \lambda\int_0^1 \rho(x)\mathrm{d}x\\ &\geq \left(\frac{\min\{m,\zeta_0\}}{2} - \lambda C_1^2\epsilon\right)\|u\|^2\mathrm{d}x - \lambda\int_0^1 \rho(x)\mathrm{d}x\end{aligned}$$

and thus

$$\lim_{\|u\|\to\infty}(\Phi(u) + \lambda\Psi(u)) = \infty,$$

which means the functional $I_\lambda = \Phi + \lambda\Psi$ is coercive. Now it remains to show that (c_2) of Theorem 2.3 is fulfilled. Let $\bar{r} = \frac{\min\{m,\zeta_0\}}{8}\bar{\gamma}^2$ and

$$w(x) = \begin{cases} \bar{\eta}, & x \in \left[0, \frac{1}{4}\right], \\ 2\bar{\eta}x + \frac{\bar{\eta}}{2}, & x \in \left[\frac{1}{4}, \frac{1}{2}\right], \\ -2\bar{\eta}x + \frac{5\bar{\eta}}{2}, & x \in \left[\frac{1}{2}, \frac{3}{4}\right], \\ \bar{\eta}, & x \in \left[\frac{3}{4}, 1\right]. \end{cases}$$

Clearly, $w \in X$. Then, we have $\Phi(0) = \Psi(0) = 0$,

$$\Phi(w) < \frac{1}{4}h(2\bar{\eta}) + \frac{1}{4}h(-2\bar{\eta}) + \zeta_1\bar{\eta}^2$$

and

$$\Phi(w) > \frac{1}{4}h(2\bar{\eta}) + \frac{1}{4}h(-2\bar{\eta}) + \frac{\zeta_0\bar{\eta}^2}{8}.$$

Thus by (5.1), $\Phi(w) > \bar{r}$. Moreover

$$\Psi(w) = -\int_0^1 F(x,w(x))\mathrm{d}x \leq -\int_{\frac{1}{4}}^{\frac{3}{4}} F(x,\bar{\eta})\mathrm{d}x \leq -\frac{1}{2}F_{\bar{\eta}}.$$

Taking (3.8) into account, for every $u \in X$ such that $\Phi(u) < \bar{r}$, we have

$$\sup_{x\in[0,1]} |u(x)| \leq \bar{\gamma}.$$

Thus,

$$\sup_{\Phi(u)<\bar{r}} \Psi(u) = \sup_{u\in\Phi^{-1}(-\infty,\bar{r})} \int_0^1 F(x,u(x))\mathrm{d}x \le \int_0^1 \sup_{|t|\le\bar{\gamma}} F(x,t)\mathrm{d}x = F^{\bar{\gamma}}. \tag{5.3}$$

By simple calculations and from the definition of $\varphi(\bar{r})$, since $\Phi(0) = \Psi(0) = 0$ and $\overline{\Phi^{-1}(-\infty,\bar{r})}^{w} = \Phi^{-1}(-\infty,\bar{r})$, one has

$$\begin{aligned}\varphi_1(\bar{r}) &= \inf_{u\in\Phi^{-1}(]-\infty,\bar{r}[)} \frac{\Psi(u) - \inf_{\overline{\Phi^{-1}(-\infty,\bar{r})}^{w}} \Psi}{\bar{r} - \Phi(u)} \le \frac{-\inf_{\overline{\Phi^{-1}(-\infty,\bar{r})}^{w}} \Psi}{\bar{r}} \\ &\le \frac{8}{\min\{m,\zeta_0\}} \frac{\int_0^1 \sup_{|t|\le\bar{\gamma}} F(x,t)\mathrm{d}x}{\bar{\gamma}^2} = \frac{8F^{\bar{\gamma}}}{\min\{m,\zeta_0\}\bar{\gamma}^2}.\end{aligned}$$

On the other hand, by (5.3), one has

$$\begin{aligned}\varphi_2(\bar{r}) &= \inf_{u\in\Phi^{-1}(-\infty,\bar{r})} \sup_{v\in\Phi^{-1}[\bar{r},\infty)} \frac{\Psi(u)-\Psi(v)}{\Phi(u)-\Phi(v)} \ge \inf_{u\in\Phi^{-1}(-\infty,\bar{r})} \frac{\Psi(u)-\Psi(w)}{\Phi(w)-\Phi(u)} \\ &\ge \frac{\inf_{u\in\Phi^{-1}(-\infty,\bar{r})} \Psi(u)-\Psi(w)}{\Phi(w)-\Phi(u)} \\ &\ge \frac{-\int_0^1 \sup_{|t|\le\bar{\gamma}} F(x,t)\mathrm{d}x + \int_{\frac{1}{4}}^{\frac{3}{4}} F(x,\bar{\eta})\mathrm{d}x}{\Phi(w)-\Phi(u)} \\ &\ge \frac{2F_{\bar{\eta}} - 4F^{\bar{\gamma}}}{h(2\bar{\eta}) + h(-2\bar{\eta}) + 4\zeta_1\bar{\eta}^2}.\end{aligned}$$

Hence, from (A_5), one has

$$\varphi_1(\bar{r}) < \varphi_2(\bar{r}).$$

Therefore, from Theorem 2.3, taking also into account that

$$\frac{1}{\varphi_2(\bar{r})} \le \frac{h(2\bar{\eta}) + h(-2\bar{\eta}) + 4\zeta_1\bar{\eta}^2}{2F_{\bar{\eta}} - 4F^{\bar{\gamma}}}$$

and

$$\frac{1}{\varphi_1(\bar{r})} \ge \frac{\min\{m,\zeta_0\}\bar{\gamma}^2}{8F^{\bar{\gamma}}},$$

we obtain the desired conclusion. □

Remark 5.2. When the assumption (A_5) of Theorem 5.1 holds, simple calculations show that the condition

(A_6) $\frac{F^{\bar{\gamma}}}{\min\{m,\zeta_0\}\bar{\gamma}^2} < \frac{F_{\bar{\eta}}}{4h(2\bar{\eta})+4h(-2\bar{\eta})+16\zeta_1\bar{\eta}^2}$

implies (A_5) of Theorem 5.1. Hence, if the assumptions (5.1) and (A_6) hold, then for each

$$\lambda \in \left(\frac{h(2\bar{\eta}) + h(-2\bar{\eta}) + 4\zeta_1\bar{\eta}^2}{2F_{\bar{\eta}}}, \frac{\min\{m,\zeta_0\}\bar{\gamma}^2}{8F^{\bar{\gamma}}} \right),$$

the problem $(P^{f,g}_{\lambda,\mu})$ in the case $\mu = 0$ admits at least three solutions.

Now, we present an application of Theorem 2.4, which will be utilized later to derive multiple solutions for the problem $(P^{f,g}_{\lambda,\mu})$ in the case $\mu = 0$, without the need for assumption (A_3).

Theorem 5.3. *Assume that there exist three positive constants $\bar{\gamma}_1$, $\bar{\eta}$ and $\bar{\gamma}_2$ with*

$$\bar{\gamma}_1 < \sqrt{\frac{8}{\min\{m,\zeta_0\}}\left(\frac{1}{4}g(2\bar{\eta}) + \frac{1}{4}g(-2\bar{\eta}) + \frac{\zeta_0\bar{\eta}^2}{8}\right)} \tag{5.4}$$

and

$$\sqrt{\frac{8}{\min\{m,\zeta_0\}}\left(\frac{1}{4}g(2\bar{\eta}) + \frac{1}{4}g(-2\bar{\eta}) + \zeta_1\bar{\eta}^2\right)} < \bar{\gamma}_2 \tag{5.5}$$

such that the assumption (A_5) in Theorem 5.1 holds and

(A_7) $\frac{1}{\min\{m,\zeta_0\}}\max\left\{\frac{F^{\bar{\gamma}_1}}{\bar{\gamma}_1^2}, \frac{F^{\bar{\gamma}_2}}{\bar{\gamma}_2^2}\right\} < \frac{F_{\bar{\eta}}}{4h(2\bar{\eta})+4h(-2\bar{\eta})+16\zeta_1\bar{\eta}^2}$.

Then, for each

$$\lambda \in \Lambda = \left(\frac{h(2\bar{\eta}) + h(-2\bar{\eta}) + 4\zeta_1\bar{\eta}^2}{2F_{\bar{\eta}}}, \min\left\{\frac{\min\{m,\zeta_0\}\bar{\gamma}_1^2}{8F^{\bar{\gamma}_1}}, \frac{\min\{m,\zeta_0\}\bar{\gamma}_2^2}{8F^{\bar{\gamma}_2}}\right\}\right),$$

the problem $(P^{f,g}_{\lambda,\mu})$ in the case $\mu = 0$ admits at least two solutions $u_{1,\lambda}$ and $u_{2,\lambda}$ such that $\max_{x\in[0,1]}|u_{1,\lambda}(x)| < \bar{\gamma}_1$ *and* $\max_{x\in[0,1]}|u_{2,\lambda}(x)| < \bar{\gamma}_2$.

Proof. Put

$$\overline{f}(x,t) = \begin{cases} f(x,-\bar{\gamma}_2), & \text{if } (x,t) \in [0,1]\times(-\infty,\bar{\gamma}_2), \\ f(x,t), & \text{if } (x,t) \in [0,1]\times[-\bar{\gamma}_2,\bar{\gamma}_2], \\ f(x,\bar{\gamma}_2), & \text{if } (x,t) \in [0,1]\times(\bar{\gamma}_2,\infty). \end{cases}$$

Clearly, $\overline{f} : [0,1]\times\mathbb{R} \to \mathbb{R}$ is a Carathéodory function. Now put $\overline{F}(x,\xi) = \int_0^\xi \overline{f}(x,t)\mathrm{d}x$ for all $(x,\xi) \in [0,1]\times\mathbb{R}$ and take X and Φ as (2.1) and (5.2), respectively, and

$$\Psi(u) = -\int_0^1 \overline{F}(x,u(x))\mathrm{d}x$$

for all $u \in X$. Our goal is to apply Theorem 2.4 to Φ and Ψ. It is well known that $\lim_{\|u\|\to\infty}\Phi(u) = \infty$ and Ψ is a differentiable functional whose differential at the point $u \in X$ is

$$\Psi'(u)(v) = -\int_0^1 \overline{f}(x,u(x))v(x)\mathrm{d}x$$

for any $v \in X$ as well as it is sequentially weakly lower semicontinuous. Furthermore $\Psi' : X \to X^*$ is a compact operator. Thus, it is enough to show that Φ and Ψ satisfy the conditions (c_1), (c_2) and (c_3) in Theorem 2.4. Let

$$\bar{r}_1 = \frac{\min\{m,\zeta_0\}}{8}\bar{\gamma}_1^2, \qquad \bar{r}_2 = \frac{\min\{m,\zeta_0\}}{8}\bar{\gamma}_2^2$$

and w as in the proof of Theorem 5.1. Due to the assumptions (3.4), (5.4) and (5.5) we have $\bar{r}_1 < \Phi(w) < \bar{r}_2$ and $\inf_X \Phi < \bar{r}_1 < \bar{r}_2$. Moreover, arguing as in the proof of Theorem 5.1 and taking also into account Remark 5.2 we obtain

$$\varphi_1(\bar{r}_1) \le \frac{8}{\min\{m,\zeta_0\}}\frac{\int_0^1 \sup_{|t|\le\bar{\gamma}_1} F(x,t)\mathrm{d}x}{\bar{\gamma}_1^2} = \frac{8F^{\bar{\gamma}_1}}{\min\{m,\zeta_0\}\bar{\gamma}_1^2},$$

$$\varphi_1(\bar{r}_2) \leq \frac{8}{\min\{m,\zeta_0\}} \frac{\int_0^1 \sup_{|t|\leq\bar{\gamma}_2} F(x,t)\mathrm{d}x}{\bar{\gamma}_2^2} = \frac{8F^{\bar{\gamma}_2}}{\min\{m,\zeta_0\}\bar{\gamma}_2^2}$$

and

$$\varphi_2^*(\bar{r}_1,\bar{r}_2) \geq \frac{2F_{\bar{\eta}}}{h(2\bar{\eta}) + h(-2\bar{\eta}) + 4\zeta_1\bar{\eta}^2}.$$

Hence, from (A_7), the conditions (c_2) and (c_3) of Theorem 2.4 hold. Therefore, from Theorem 2.4 we obtain that, for each $\lambda \in \Lambda$, the problem

$$\begin{cases} -p(u')u'' + \zeta(x)u = \lambda\overline{f}(x,u(x)), & \text{a.e. } x \in [0,1], \\ u(1) - u(0) = u'(1) - u'(0) = 0 \end{cases}$$

admits at least two solutions $u_{1,\lambda}$ and $u_{2,\lambda}$ such that $\max_{x\in[0,1]} |u_{1,\lambda}(x)| < \bar{\gamma}_1$ and $\max_{x\in[0,1]} |u_{2,\lambda}(x)| < \bar{\gamma}_2$. Observing that these solutions are also solutions for the problem $(P^{f,g}_{\lambda,\mu})$ in the case $\mu = 0$, the conclusion follows. □

Now, we provide some remarks on our results in this section.

Remark 5.4. In Theorems 5.1 and 5.3, we investigated the critical points of the functional I_λ naturally associated with the problem $(P^{f,g}_{\lambda,\mu})$ in the case $\mu = 0$. It is worth noting that, in general, I_λ can be unbounded from below in X. For instance, when $f(t) = 1 + |t|^{\vartheta-2}t$ for all $t \in \mathbb{R}$ with $\vartheta > 2$, for any fixed $u \in X\backslash 0$ and $\iota \in \mathbb{R}$, we obtain

$$\begin{aligned} I_\lambda(\iota u) &\leq \frac{\max\{m,\zeta_1\}}{2}\|\iota u\|^2 - \lambda\int_0^1 F(x,\iota u(x))\mathrm{d}x \\ &\leq \frac{\max\{M,\zeta_1\}\iota^2}{2}\|u\|^2 - \lambda\iota C_4\|u\| - \lambda C_5\frac{\iota^\vartheta}{\vartheta}\|u\|^\vartheta \to -\infty \end{aligned}$$

where C_4 and C_5 are positive constants, as $\iota \to \infty$. Hence, we can not use direct minimization to find critical points of the functional I_λ.

Remark 5.5. We observe that if f is non-negative, Theorem 5.3 represents a bifurcation result, indicating that the pair $(0,0)$ belongs to the closure of the set

$$\left\{(u_\lambda,\lambda) \in X \times (0,\infty) : u_\lambda \text{ is a non-trivial solution of } (P^{f,g}_{\lambda,\mu}),\ \mu = 0\right\} \subset X \times \mathbb{R}.$$

Indeed, if λ goes to zero, by Theorem 5.3 we have that $\bar{\gamma}_i \to 0$, $i = 1,2$ and since $\max_{x\in[0,1]} |u_{i,\lambda}(x)| < \bar{\gamma}_i$, $i = 1,2$, there exist two sequences $\{u_j\}$ in X and $\{\lambda_j\}$ in $\mathbb{R}^+$ (here $u_j = u_{\lambda_j}$) such that

$$\lambda_j \to 0^+ \quad \text{and} \quad \|u_j\| \to 0,$$

as $j \to \infty$. Moreover, since f is nonnegative, $\Psi(u) < 0$ for all $u \in \mathbb{R}$ and thus

$$(0,\lambda^*) \ni \lambda \mapsto I_\lambda(u_\lambda)$$

is strictly decreasing. Hence, for every $\lambda_1,\lambda_2 \in (0,\lambda^*)$, with $\lambda_1 \neq \lambda_2$, solutions u_{λ_1} and u_{λ_2} ensured by Theorem 2.4 are different.

Remark 5.6. If f is non-negative, then the solutions guaranteed by Theorems 5.1 and 5.3 are also non-negative.

Now, we highlight some results where the function f has separated variables. Specifically, consider the following problem

$$\begin{cases} -p(u')u'' + \zeta(x)u = \lambda\theta(x)f(u(x)), & \text{a.e. } x \in [0,1], \\ u(1) - u(0) = u'(1) - u'(0) = 0 \end{cases} \qquad (P_\lambda^{f,\theta})$$

where $\theta : [0,1] \to \mathbb{R}$ is a non-negative and non-zero function such that $\theta(x) < \infty$ for all $x \in [0,1]$ and $f : \mathbb{R} \to \mathbb{R}$ is a non-negative and continuous function. Put

$$F(\xi) = \int_0^{\xi} f(s)ds$$

for all $\xi \in \mathbb{R}$.

The following existence results are consequences of Theorems 5.1 and 5.3, respectively, by setting $f(x,t) = \theta(x)f(x)$ for every $(x,t) \in [0,1] \times \mathbb{R}$.

Theorem 5.7. *Assume that there exist two positive constants $\bar{\gamma}$ and $\bar{\eta}$, with*

$$\sqrt{\frac{2h(2\bar{\eta}) + 2h(-2\bar{\eta}) + 8\zeta_1\bar{\eta}^2}{\min\{m,\zeta_0\}}} < \bar{\gamma}$$

such that

(A_8) $\theta(x) \geq 0$ *for every* $x \in [0,1]$ *and* $f(t) \geq 0$ *for every* $t \in [0, \frac{1}{4}] \bigcup [\frac{3}{4}, 1]$,

(A_9) $\frac{1}{\min\{m,\zeta_0\}} \frac{\int_0^1 \theta(x)\mathrm{d}xF(\bar{\gamma})}{\bar{\gamma}^2} < \frac{\int_{\frac{1}{4}}^{\frac{3}{4}} \theta(x)\mathrm{d}xF(\bar{\eta})}{2h(2\bar{\eta})+2h(-2\bar{\eta})+8\zeta_1\bar{\eta}^2}$,

(A_{10}) $\limsup_{|\xi|\to\infty} \frac{F(\xi)}{|\xi|^2} \in (-\infty, 0]$.

Then, for each

$$\lambda \in \left(\frac{1}{4\int_{\frac{1}{4}}^{\frac{3}{4}} \theta(x)\mathrm{d}x} \frac{h(2\bar{\eta}) + h(-2\bar{\eta}) + 4\zeta_1\bar{\eta}^2}{F(\bar{\eta})}, \frac{\min\{m,\zeta_0\}}{8\int_0^1 \theta(x)\mathrm{d}x} \frac{\bar{\gamma}^2}{F(\bar{\gamma})} \right),$$

the problem $(P_\lambda^{f,\theta})$ admits at least three solutions in X.

Theorem 5.8. *Assume that there exist three positive constants $\bar{\gamma}_1$, $\bar{\eta}$ and $\bar{\gamma}_2$ with*

$$\bar{\gamma}_1 < \sqrt{\frac{8}{\min\{m,\zeta_0\}} \left(\frac{1}{4}g(2\bar{\eta}) + \frac{1}{4}g(-2\bar{\eta}) + \frac{\zeta_0\bar{\eta}^2}{8} \right)}$$

and

$$\sqrt{\frac{8}{\min\{m,\zeta_0\}} \left(\frac{1}{4}g(2\bar{\eta}) + \frac{1}{4}g(-2\bar{\eta}) + \zeta_1\eta^2 \right)} < \bar{\gamma}_2$$

such that

(A_{11}) $\theta(x) \geq 0$ *for every* $x \in [0,1]$ *and* $f(t) \geq 0$ *for every* $t \in [0, \frac{1}{4}] \cup [\frac{3}{4}, 1]$,

(A_{12}) $\frac{\int_0^1 \theta(x)\mathrm{d}x}{\min\{m,\zeta_0\}} \max\left\{ \frac{F(\bar{\gamma}_1)}{\bar{\gamma}_1^2}, \frac{F(\bar{\gamma}_2)}{\bar{\gamma}_2^2} \right\} < \frac{\int_{\frac{1}{4}}^{\frac{3}{4}} \theta(x)\mathrm{d}xF(\bar{\eta})}{2h(2\bar{\eta})+2h(-2\bar{\eta})+8\zeta_1\bar{\eta}^2}$.

Then, for each

$$\lambda \in \Lambda = \left(\frac{1}{4\int_{\frac{1}{4}}^{\frac{3}{4}} \theta(x)\mathrm{d}x} \frac{h(2\bar{\eta}) + h(-2\bar{\eta}) + 4\zeta_1\bar{\eta}^2}{F(\bar{\eta})}, \frac{\min\{m, \zeta_0\}}{8\int_0^1 \theta(x)\mathrm{d}x} \min\left\{ \frac{\bar{\gamma}_1^2}{F(\bar{\gamma}_1)}, \frac{\bar{\gamma}_2^2}{F(\bar{\gamma}_2)} \right\} \right),$$

the problem $(P_\lambda^{f,\theta})$ *admits at least two solutions* $u_{1,\lambda}$ *and* $u_{2,\lambda}$ *such that* $\max_{x\in[0,1]} |u_{1,\lambda}(x)| < \gamma_1$ *and* $\max_{x\in[0,1]} |u_{2,\lambda}(x)| < \gamma_2$.

Now, we point out a special case of Theorem 5.7.

Theorem 5.9. *Assume that*

$$\lim_{\xi\to 0^+} \frac{f(\xi)}{\xi} = \lim_{|\xi|\to\infty} \frac{f(\xi)}{|\xi|} = 0 \tag{5.6}$$

and there exists a positive constant $\bar{\eta}$ *such that* $F(\bar{\eta}) > 0$. *Then, for each* $\lambda > \lambda^*$, *where*

$$\lambda^* = \frac{1}{4\int_{\frac{1}{4}}^{\frac{3}{4}} \theta(x)\mathrm{d}x} \inf_{\bar{\eta}>0} \frac{h(2\bar{\eta}) + h(-2\bar{\eta}) + 4\zeta_1\bar{\eta}^2}{F(\bar{\eta})},$$

the problem $(P_\lambda^{f,\theta})$ *admits at least one nonnegative and one non zero solution in* X.

Proof. Let $\lambda > \lambda^*$. Then, there is $\bar{\eta} > 0$ such that

$$\lambda > \frac{1}{4\int_{\frac{1}{4}}^{\frac{3}{4}} \theta(x)\mathrm{d}x} \inf_{\bar{\eta}>0} \frac{h(2\bar{\eta}) + h(-2\bar{\eta}) + 4\zeta_1\bar{\eta}^2}{F(\bar{\eta})}.$$

From (5.6) we obtain

$$\lim_{u\to 0^+} \frac{\sup_{|\xi|\le u} f(\xi)}{u} = \lim_{u\to\infty} \frac{\sup_{|\xi|\le u} f(\xi)}{u} = 0.$$

So we can pick $\bar{\gamma}_1$ and $\bar{\gamma}_2$ such that

$$\bar{\gamma}_1 < \sqrt{\frac{8}{\min\{m, \zeta_0\}} \left(\frac{1}{4} g(2\bar{\eta}) + \frac{1}{4} g(-2\bar{\eta}) + \frac{\zeta_0 \bar{\eta}^2}{8} \right)}$$

and

$$\sqrt{\frac{8}{\min\{m, \zeta_0\}} \left(\frac{1}{4} g(2\bar{\eta}) + \frac{1}{4} g(-2\bar{\eta}) + \zeta_1 \eta^2 \right)} < \bar{\gamma}_2,$$

$\frac{\sup_{|\xi|\le\bar{\gamma}_1} f(\xi)}{\bar{\gamma}_1} < \frac{\min\{m,\zeta_0\}}{8\int_0^1 \theta(x)\mathrm{d}x} \frac{\bar{\gamma}_1^2}{F(\bar{\gamma}_1)}$ and $\frac{\sup_{|\xi|\le\bar{\gamma}_2} f(\xi)}{\bar{\gamma}_2} < \frac{\min\{m,\zeta_0\}}{8\int_0^1 \theta(x)\mathrm{d}x} \frac{\bar{\gamma}_2^2}{F(\bar{\gamma}_2)}$. Hence, from Theorem 5.8 we obtain the conclusion. □

Remark 5.10. Theorem 1.2 immediately follows from Theorem 5.9.

6 Ethical statement

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Declarations

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References

[1] R. A. Adams, J. J. F. Fournier, *Sobolev spaces*, Elsevier/Academic Press, Amsterdam, 2003. MR2424078; Zbl 1098.46001

[2] G. A. Afrouzi, S. Heidarkhani, Three solutions for a quasilinear boundary value problem, *Nonlinear Anal.* **69**(2008), 3330–3336. https://doi.org/10.1016/j.na.2007.09.022; MR2450542; Zbl 1158.34311

[3] A. Ambrosetti, P. H. Rabinowitz, Dual variational methods in critical point theory and applications, *J. Funct. Anal.* **14**(1973), 349–381. https://doi.org/10.1016/0022-1236(73)90051-7; MR0370183; Zbl 0273.49063

[4] D. Averna, G. Bonanno, A three critical point theorem and its applications to the ordinary Dirichlet problem, *Topol. Methods Nonlinear Anal.* **22**(2003), 93–103. https://doi.org/; MR2037268; Zbl 1048.58005

[5] G. Bonanno, A critical point theorem via the Ekeland variational principle, *Nonlinear Anal.* **75**(2012), 2992–3007. https://doi.org/10.1016/j.na.2011.12.003; MR2878492; Zbl 1239.58011

[6] G. Bonanno, A critical points theorem and nonlinear differential problems, *J. Global Optim.* **28**(2004), 249–258. https://doi.org/10.1023/B:JOGO.0000026447.51988.f6; MR2074785; Zbl 1087.58007

[7] G. Bonanno, Relations between the mountain pass theorem and local minima, *Adv. Nonlinear Anal.* **1**(2012) 205–220. https://doi.org/10.1515/anona-2012-0003; MR3034869; Zbl 1277.35170

[8] G. Bonanno, A. Chinnì, Existence and multiplicity of weak solutions for elliptic Dirichlet problems with variable exponent, *J. Math. Anal. Appl.* **418**(2014), 812–827. https://doi.org/10.1016/j.jmaa.2014.04.016; MR3206681; Zbl 1312.35111

[9] G. Bonanno, R. Livrea, Periodic solutions for a class of second order Hamiltonian systems, *Electron. J. Differential Equations* **2005**, No. 115, 13 pp. MR2174547; Zbl 1096.34027

[10] G. Bonanno, D. O'Regan, A boundary value problem on the half-line via critical point methods, *Dynam. Systems Appl.* **15**(2006), 395–408. MR2367653

[11] G. Caristi, S. Heidarkhani, A. Salari, Variational approaches to Kirchhoff-type second-order impulsive differential equations on the half-line, *Results Math.* **73**(2018), 1–31. https://doi.org/10.1007/s00025-018-0772-2; MR3764544; Zbl 1390.34053

[12] M. Ferrara, S. Heidarkhani, S. Moradi, G. Caristi, A sequence of solutions for impulsive quasilinear periodic boundary value problems, preprint.

[13] M. Ferrara, S. Heidarkhani, S. Moradi, G. Caristi, Energy estimates and existence results for a quasilinear periodic boundary value problem, *Mediterr. J. Math.* **21**(2024), No. 137, 1–21. https://doi.org/10.1007/s00009-024-02669-2; MR4756237; Zbl 1540.34057

[14] J. R. Graef, S. Heidarkhani, L. Kong, A critical points approach for the existence of multiple solutions of a Dirichlet quasilinear system, *J. Math. Anal. Appl.* **388**(2012), 1268–1278. https://doi.org/10.1016/j.jmaa.2011.11.019; MR3356852; Zbl 1244.34024

[15] J. R. Graef, S. Heidarkhani, L. Kong, S. Moradi, Multiple solutions to a quasilinear periodic boundary value problem with impulsive effects, *Electron. J. Qual. Theory Differ. Equ.* **2024**, No. 12, 1–18. https://doi.org/10.14232/ejqtde.2024.1.12; MR4710534; Zbl 07911172

[16] A. Ghobadi, S. Heidarkhani, Variational approach for an elastic beam equation with local nonlinearities, *Ann. Polon. Math.* **130**(2023), 201–222. https://doi.org/10.4064/ap220519-20-10; MR4638662; Zbl 1531.34031

[17] S. Heidarkhani, Multiple solutions for a quasilinear second order differential equation depending on a parameter, *Acta Math. Appl. Sin. Engl. Ser.* **32**(2016), 199–208. https://doi.org/10.1007/s10255-016-0548-y; MR3482433; Zbl 1338.34052

[18] S. Heidarkhani, S. Moradi, Existence results for a quasilinear periodic boundary value problem, preprint.

[19] S. Heidarkhani, S. Moradi, M. Ferrara, D. Barilla, G. Caristi, A variational approach for quasilinear periodic boundary value problems, preprint.

[20] R. Livrea, Existence of three solutions for a quasilinear two point boundary value problem, *Arch. Math.* **79**(2002), 288–298. https://doi.org/10.1007/s00013-002-8315-0; MR1944953; Zbl 1015.34012

[21] N. Matzakos, N. S. Papageorgiou, Existence of periodic solutions for quasilinear ordinary differential equations with discontinuities, *Demonstratio Math.* **33**(2000), 753–770. https://doi.org/10.1515/dema-2000-0408; MR1807641; Zbl 0985.34012

[22] D. Motreanu, V. Rădulescu, *Variational and non-variational methods in nonlinear analysis and boundary value problems*, Nonconvex Optimization and Applications, Vol. 67, Kluwer Academic Publishers, Dordrecht, 2003. https://doi.org/10.1007/978-1-4757-6921-0; MR1985870; Zbl 1040.49001

[23] N. S. Papageorgiou, F. Papalini, Existence of two solutions for quasilinear periodic differential equations with discontinuities, *Arch. Math. (Brno)* **38**(2002), 285–296. https://doi.org/eudml.org/doc/248927; MR1942658; Zbl 1090.34013

[24] P. H. Rabinowitz, *Minimax methods in critical point theory with applications to differential equations*, CBMS Reg. Conf. Ser. Math., Vol. 65, Amer. Math. Soc. Providence, RI, 1986. https://doi.org/10.1090/cbms/065; MR0845785; Zbl 0609.58002

[25] B. Ricceri, A general variational principle and some of its applications, *J. Comput. Appl. Math.* **113**(2000), 401–410. https://doi.org/10.1016/S0377-0427(99)00269-1; MR1735837; Zbl 0946.49001

[26] T. Shen, W. Liu, Infinitely many solutions for second-order quasilinear periodic boundary value problems with impulsive effects, *Mediterranean J. Math.* **14**(2017), 1–12. https://doi.org/10.1007/s00009-017-0931-9; MR3646944; Zbl 1376.34026

[27] Z. Wang, D. Liu, M. Song, Existence of three periodic solutions for a quasilinear periodic boundary value problem, *AIMS Math.* **5**(2020), 6061–6072. https://doi.org/10.3934/math.2020389; MR4148933; Zbl 1484.34088

[28] E. Zeidler, *Nonlinear functional analysis and its applications, Vol. II/B*, Springer, Berlin, Heidelberg, New York, 1990. https://doi.org/10.1007/978-1-4612-0981-2; MR1033498; Zbl 0684.47029