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Uniform approximation of a class of impulsive delayed Hopfield neural networks on the half-line

This paper is dedicated to the memory of István Győri

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Abstract. In this work, we investigate a uniform approximation of a nonautonomous delayed CNN-Hopfield-type impulsive system with an associated impulsive differential system where a partial discretization is introduced with the help of piecewise constant arguments. Sufficient conditions are formulated, which imply that the error estimate decays exponentially with time on the half-line $[0, \infty)$. A critical step for the proof of this estimate is to show that, under the assumed conditions, the solutions of the Hopfield impulsive system are exponentially bounded and exponentially stable. A bounded coefficients case is also analyzed under simplified conditions. An example is presented and simulated in order to show the applicability of our conditions.

Keywords: Hopfield neural networks, hybrid equations, impulsive differential equations, numerical approximation of solutions, piecewise constant arguments.

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1 Introduction

Cellular Neural Networks (CNNs) are widely used as mathematical models of the interactions of the neurons in the human brain. For its construction, electrical and chemical properties have been considered. The synapses correspond to the connections of the neurons (excitatory and inhibitory) and are modeled by positive and negative weights. The weighted neural inputs are added up. Then, the so-called activation function defines the amplitude of the response signal of the neuron.

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In [16], John Hopfield proposed a novel type of CNN in order to find how human memory works. They were called *Hopfield cellular neural network*, and it is represented by the following nonlinear system:

$$x_i'(t) = -a_i(t)x_i(t) + \sum_{j=1}^{m} b_{ij}g_j(x_j(t)) + c_i(t), \quad i = 1, \dots, m, \tag{1.1}$$

This model corresponds to a mesh of linked neurons, where every neuron is connected to all other neurons without self-connection. The states of the neurons are of binary type, and it depends on whether the neuron's input exceeds a fixed value. This type of CNN has been applied in psychology and combinatorics, among others (see [22]). Since the signals travel at a finite speed between the neurons, time delays are natural to introduce in the models. Without completeness, we refer to [3, 6, 7, 18, 20, 23, 27] for investigations of different classes of delayed CNN models.

In [19], A. D. Myshkis introduced differential equations of the form

$$x'(t) = f(t, x(t), x(\tau(t))),$$

where $\tau(t)$ corresponds to a deviated argument (a discontinuous piecewise constant function). These type of equations are called *Differential Equations with Piecewise Constant Arguments* (*DEPCA*). The research in this new field started in the 80's with the works of S. Busenberg and K. L. Cooke with a model of vertically transmitted diseases (see [8, 26]). There are many fields where this type of equations have been applied (see [5, 10, 17]).

In [2], M. U. Akhmet investigated systems of the form

$$y'(t) = f(t, y(t), y(\gamma(t))), \tag{1.2}$$

where $\gamma(t)$ is a *piecewise constant argument of generalized type*. More precisely, given $(t_n)_{n\in\mathbb{Z}}$ and $(\zeta_n)_{n\in\mathbb{Z}}$ such that $t_n < t_{n+1}, \forall n \in \mathbb{Z}$ with $\lim_{n\to\pm\infty} t_n = \pm\infty$ and $t_n \le \zeta_n \le t_{n+1}$, then

$$\gamma(t) = \zeta_n, \quad \text{if } t \in I_n = [t_n, t_{n+1}).$$

When such a function γ is introduced, it generates advanced and delayed arguments in the equation, dividing the interval I_n into two pieces $I_n = I_n^+ \cup I_n^-$, where $I_n^+ = [t_n, \zeta_n]$ corresponds to the advanced, and $I_n^- = [\zeta_n, t_{n+1}]$ to the delayed interval. These equations are known as *Differential Equations with Piecewise Constant Argument of Generalized Type* (*DEPCAG*). In this class of differential equations, the solutions are continuous functions, although γ is a discontinuous function. Integrating (1.2) from t_n to t_{n+1} we obtain a difference equation, giving the character of hybrid to this kind of equations (see also [21]).

The following example will be important for the rest of the work (when $k = 0$). Consider $\gamma(t) = \left[\frac{t+k}{h}\right]h$ with $0 \le k < h$, where $[\cdot]$ is the greatest integer function. We have

$$\left[\frac{t+k}{h}\right]h = nh, \text{ when } t \in I_n = [nh - k, (n+1)h - k).$$

Hence, $\gamma(t) - t \ge 0 \;\Leftrightarrow\; t \le nh$ and $\gamma(t) - t \le 0 \;\Leftrightarrow\; t \ge nh$, that implies

$$I_n^+ = [nh - k, nh], \quad I_n^- = [nh, (n+1)h - k].$$

Now, if additionally a jump condition is applied at the endpoints of the intervals $I_n = [t_n, t_{n+1})$, it defines the class of *Impulsive differential equations with piecewise constant argument of generalized type*, (*IDEPCAG*) (see [1]),

$$\begin{aligned} y'(t) &= f(t, y(t), y(\gamma(t))), \qquad t \ne t_n \\ \Delta y(t_n) := y(t_n) - y(t_n^-) &= J_n(y(t_n^-)), \qquad t = t_n, \quad n \in \mathbb{N}. \end{aligned} \tag{1.3}$$

Definition 1.1 (IDEPCAG solution)**.** A piecewise continuous function $y(t)$ is a solution of (1.3) if:

(i) $y(t)$ is continuous on $I_n = [t_n, t_{n+1})$ with discontinuities of the first kind at t_n with $n \in \mathbb{Z}$, where $y'(t)$ exists at each $t \in \mathbb{R}$ with the possible exception of the points t_n, where the lateral derivatives exist.

(ii) On each interval I_n, the ordinary differential equation

$$y'(t) = f(t, y(t), y(\zeta_n))$$

holds, with $\gamma(t) = \zeta_n$.

(iii) For $t = t_n$, the following impulsive condition holds:

$$\Delta y(t_n) = y(t_n) - y(t_n^-) = J_n(y(t_n^-)),$$

i.e., $y(t_n) = y(t_n^-) + J_n(y(t_n^-))$, where $y(t_n^-)$ denotes the left-hand limit of the function y at t_n.

I. Győri used first DEPCAG to approximate linear delay equations with constant delays in [12]. He defined three variants of approximating DEPCAG and proved the convergence of each method on compact time intervals. See also [14] for further generalization of this approach for other classes of differential equations.

In [9], Cooke and Győri proposed an approximation of a linear delay differential equation

$$x'(t) = \sum_{i=1}^{N} q_i x(t - \tau_i), \qquad t \geq 0, \tag{1.4}$$

$$x(t) = \phi(t), \qquad t \in [-\tau, 0], \tag{1.5}$$

where $q_i \in \mathbb{R}, \tau_i > 0$, and $\phi \in C([-\tau, 0], \mathbb{R})$. Here $C([-\tau, 0], \mathbb{R})$ denotes the space of real-valued continuous functions defined on $[-\tau, 0]$. In order to approximate (1.4)–(1.5), they proposed the following DEPCAG

$$y'(t) = \sum_{i=1}^{N} q_i y([t/h - [\tau_i/h]]h), \qquad t \geq 0, \tag{1.6}$$

$$y(nh) = \phi(nh), \qquad n = k, \dots, 0. \tag{1.7}$$

In this case, the approximation considered was uniform over the non-compact interval $[0, \infty)$. The main assumption is a condition of asymptotic stability of the trivial solution of (1.4). Note that in [13] I. Győri and F. Hartung extended this result for linear neutral differential equations.

Recently, in [15] F. Hartung investigated the numerical approximation of the following scalar delay differential equation with impulsive self-support condition

$$\begin{aligned} x'(t) &= \alpha x(t) + \beta x(t - \tau), \qquad \text{a.e } \ t \geq 0 \\ x(t) &= c + d, \qquad \text{if } x(t^-) = c \end{aligned} \tag{1.8}$$

with the initial condition

$$x(t) = \varphi(t), \qquad \text{if } t \in [-\tau, 0],$$

where $c, d > 0, \alpha + |\beta| < 0$, $\tau > 0$, $c < \varphi(t)$, for $t \in [-\tau, 0]$, and $\varphi : [-\tau, 0] \to \mathbb{R}$ a Lipschitz continuous function. The approximating equation is an associated DEPCA with a self-support condition

$$\begin{aligned} y'(t) &= \alpha y([t/h]h) + \beta y([t/h]h - [\tau/h]h), \qquad \text{a.e } t \geq 0 \\ y(kh) &= c + d, \qquad \text{if } y(kh^-) \leq c \end{aligned} \tag{1.9}$$

with the initial condition

$$y(t) = \varphi(t), \qquad \text{if } t \in [-\tau, 0].$$

The convergence of (1.9) was proved at every point except the impulsive time moments.

In [24], R. Torres et al. considered the following impulsive Hopfield-type CNN system with impulses

$$\begin{aligned} x_i'(t) &= -a_i(t)x_i(t) + \sum_{j=1}^{m} b_{ij}(t)g_j(x_j(t)) + c_i(t), && t \geq 0, \quad t \neq t_k, \\ \Delta x_i(t_k) &= -p_{i,k}x_i(t_k^-) + e_{i,k} + J_{i,k}(x_i(t_k^-)), && t = t_k, \\ x_i(t_0) &= x_i^0, \end{aligned} \tag{1.10}$$

and the following IDEPCA system

$$\begin{aligned} y_i'(t) &= -a_i(t)y_i(t) + \sum_{j=1}^{m} b_{ij}(t)g_j(y_j(\gamma(t))) + c_i(t), && t \geq 0, \quad t \neq \gamma(t_k) \\ \Delta y_i(\gamma(t_k)) &= -p_{i,k}y_i(\gamma(t_k)^-) + e_{i,k} + J_{i,k}(y_i(\gamma(t_k)^-)), && t = \gamma(t_k), \\ y_i(t_0) &= y_i^0, \end{aligned} \tag{1.11}$$

where $\gamma(t) = [t/h]h$. Assuming an ergodic stability condition over the corresponding linear homogeneous system associated with (1.10), the uniform approximation of (1.10) by the IDEPCA (1.11) was concluded over $[0, \infty)$, where the error of approximation was given by

$$|x_i(t) - y_i(t)| \leq \frac{|x_i^0 - y_i^0|}{1 - \theta_c} + \frac{o_i(h)}{1 - \theta_c},$$

with $o_i(h) \to 0$ as $h \to 0$, and $0 < \theta_c < 1$ were defined in [24].

In [11], M. Elghandouri and K. Ezzinbi, using resolvent operators theory, obtained an approximation of the mild solutions of the delayed semilinear integro-differential equation

$$\begin{aligned} x'(t) &= A(t)x(t) + \int_0^t G(t-s)x(s)ds + f(t, x(t-r)), && t \geq 0, \\ x(t) &= \varphi(t), && t \in [-r, 0], \end{aligned} \tag{1.12}$$

using an integro-differential equation with piecewise constant arguments

$$\begin{aligned} x_h'(t) &= A(t)x_h(t) + \int_0^t G(t-s)x_h(s)ds + f(t, x_h(\gamma_h(t-r))), && t \geq 0, \\ x_h(0) &= \varphi(0),\ x_h(t) = \varphi(kh), && t \in [kh, (k+1)h), \end{aligned} \tag{1.13}$$

with $k = -l, \ldots, -1$, and $\gamma_h(t) = [t/h]h$, on the Banach space $(X, \|\cdot\|)$. The approximation was done over compact and unbounded intervals. They also obtained an exponential error decay by using the stability of the resolvent operator and the Halanay's Inequality.

The interested reader in approximation of solutions of differential equations by using piecewise constant argument can see [25] for an elementary and simple introduction to the subject.

1.1 Aim of the work

In this paper, we use $\gamma(t) = [t/h]\,h$ as the piecewise constant argument function, where $[\cdot]$ is the greatest integer part function, and $h > 0$ is a fixed discretization parameter. We note that γ depends on the selection of h, but for simplicity, this dependence is not indicated explicitly in the notation, but it always should be kept in mind.

We consider a delayed CNN system with impulses

$$\begin{aligned} x_i'(t) &= -a_i(t)x_i(t) + \sum_{j=1}^{m} b_{ij}(t)g_j(x_j(t-\tau_j)) + c_i(t), && t \ge 0, \quad t \ne t_k, \\ \Delta x_i(t_k) &= -p_{i,k}x_i(t_k^-) + e_{i,k} + J_{i,k}(x_i(t_k^-)), && k \in \mathbb{N}, \\ x_i(t) &= \varphi_i(t), && t \in [-\tau, 0]. \end{aligned} \tag{1.14}$$

Similar delayed CNN systems (without impulses) were investigated, e.g., in [7, 18, 20].

For a fixed discretization parameter $h > 0$ we associate to (1.14) the IDEPCA system

$$\begin{aligned} y_i'(t) &= -a_i(t)y_i(t) + \sum_{j=1}^{m} b_{ij}(t)g_j(y_j(\gamma(t)-\gamma(\tau_j))) + c_i(t), && t \ge 0, \quad t \ne \gamma(t_k), \\ \Delta y_i(\gamma(t_k)) &= -p_{i,k}y_i(\gamma(t_k)^-) + e_{i,k} + J_{i,k}(y_i(\gamma(t_k)^-)), && k \in \mathbb{N}, \\ y_i(t) &= \psi_i(t), && t \in [-\tau, 0], \end{aligned} \tag{1.15}$$

where $i = 1, 2, \dots, m$, $k \in \mathbb{N} = \{1, 2, 3 \dots\}$, t_k, $p_{i,k}$, $e_{i,k}$ are real sequences, a_i, b_{ij}, c_i are real-valued locally integrable functions on $[0, \infty)$, $J_{i,k} \in C(\mathbb{R}, \mathbb{R})$ and $g_j \in C(\mathbb{R}, \mathbb{R})$ for all $i, j = 1, \dots, m$ and $k \in \mathbb{N}$; the constant delays satisfy $\tau_i \ge 0$ and $\tau = \max\{\tau_1, \dots, \tau_m\} > 0$, and the initial functions $\varphi_i, \psi_i : [-\tau, 0] \to \mathbb{R}$ are continuous for $i = 1, \dots, m$.

We note that the initial time in system (1.14) is fixed to be 0. This does not affect the generality of the problem, but it simplifies the definition of the approximation in (1.15), since, in this way, the initial time is a member of the mesh points of the piecewise constant approximation, and $\gamma(t) \ge 0$, and hence $\gamma(t) - \gamma(\tau_j) \ge -\tau_j \ge -\tau$ for $t \ge 0$ and $j = 1, \dots, m$. Also, the impulse times t_k form a strictly monotone increasing sequence of positive reals, and they are approximated by $\gamma(t_k)$ for $k \in \mathbb{N}$ for the sake of easier computation of the numerical scheme.

For simplicity of the notation we introduce $t_0 = 0$, so the sequence t_k is defined for $k \in \mathbb{N}_0 = \{0, 1, 2, \dots\}$.

The main goal of this manuscript is to show that the solutions of (1.15) approximate that of (1.14) uniformly on $[0, \infty)$, i.e.,

$$\sup_{t \in [0,\infty)} |x_i(t) - y_i(t)| \to 0, \quad \text{as } h \to 0+, \quad i = 1, \dots, m,$$

assuming also $\varphi_i = \psi_i$ for $i = 1, \dots, m$, and we show that, under certain conditions, the error estimate goes to 0 as $t \to \infty$ with an exponential speed.

Remark 1.2. This paper extends the work of [24] for the case when $\tau_i > 0$ in (1.14). Moreover, in this work, we assume a different set of conditions and we use M-matrix technique to get our main results. Another improvement corresponds to the exponential error decay of the approximation, see Theorem 3.1 below. A key step to obtain the main result is to show that, under the assumed conditions, the solutions of (1.14) are exponentially bounded (see Lemma 2.2) and are exponentially stable (see Lemma 2.4, below).

1.2 Hypotheses and main assumptions

In this manuscript, we will use the following assumptions on the parameters of problem (1.14):

(H1) Let $g_j \in C(\mathbb{R}, \mathbb{R})$ be such that $g_j(0) = 0$, and there exist constants $L_j \geq 0$ such that

$$\left|g_j(u) - g_j(v)\right| \leq L_j \left|u - v\right|, \qquad u, v \in \mathbb{R}, \quad j = 1, 2, \ldots, m.$$

(H2) Let $J_{i,k} \in C(\mathbb{R}, \mathbb{R})$ be such that $J_{i,k}(0) = 0$, and there exist constants $l_{i,k} \geq 0$ such that

$$|J_{i,k}(u) - J_{i,k}(v)| \leq l_{i,k} \left|u - v\right|, \qquad u, v \in \mathbb{R}, \quad i = 1, \ldots, m, \quad k \in \mathbb{N}.$$

(H3) There exist positive constants $p_i^*, l_i^*, e_i^*, \underline{\delta}$ and real constants $\underline{p}_i$ for $i = 1, \ldots, m$ such that

(i) $\underline{p}_i \leq p_{i,k} \leq p_i^* < 1, \qquad k \in \mathbb{N}, \quad i = 1, \ldots, m;$

(ii) $0 \leq l_{ik} \leq l_i^*, \qquad k \in \mathbb{N}, \quad i = 1, \ldots, m;$

(iii) $|e_{i,k}| \leq e_i^*, \qquad k \in \mathbb{N}, \quad i = 1, \ldots, m;$

(iv) $0 < \underline{\delta} \leq t_{k+1} - t_k, \qquad k \in \mathbb{N}_0.$

(H4) There exist positive constants $\sigma_i, \Lambda_{ij}, c_i^*$ for $i, j = 1, \ldots, m$ and ε_0 such that $0 < \varepsilon_0 < \sigma_i$ for $i = 1, \ldots, m$, and

(i) $\sigma_i(t-s) \leq \int_s^t a_i(u)\,du - \sum_{j \in J(s,t)} \ln(1 - p_{i,j}), \quad 0 \leq s < t, \quad i = 1, \ldots, m,$ where $J(s,t) = \{j \in \mathbb{N} : s \leq t_j < t\};$

(ii) $\int_0^t e^{-(\sigma_i - \varepsilon_0)(t-s)} |b_{ij}(s)|\,ds \leq \Lambda_{ij}, \qquad t \geq 0, \quad i, j = 1, \ldots, m;$

(iii) $|c_i(t)| \leq c_i^*, \qquad t \geq 0, \quad i = 1, \ldots, m.$

(H5) $\sum_{j=1}^m (1 - \underline{p}_i^-)\Lambda_{ij} L_j + \dfrac{(1 - \underline{p}_i^-) l_i^*}{(1 - p_i^*)(1 - e^{-\sigma_i \underline{\delta}})} < 1, \qquad i = 1, \ldots, m,$ where $\underline{p}_i^- = \min\{0, \underline{p}_i\}$.

(H6) There exist positive constants β_1 and β_2 such that

$$|e_{i,k}| \leq e^{-\beta_1 t_k} e_i^*, \qquad k \in \mathbb{N}, \qquad \text{and} \qquad |c_i(t)| \leq e^{-\beta_2 t} c_i^*, \quad t \geq 0, \quad i = 1, \ldots, m.$$

(H7) There exist positive constants a_i^* for $i = 1, \ldots, m$ such that $a_i(t) \leq a_i^*$, $t \geq 0$, $i = 1, \ldots, m$.

(H8) There exist positive constants b_{ij}^* for $i, j = 1, \ldots, m$ and L_φ such that

$$|b_{ij}(t)| \leq b_{ij}^*, \qquad t \geq 0, \qquad \text{and} \qquad |\varphi_i(t) - \varphi_i(\bar{t})| \leq L_\varphi |t - \bar{t}|, \qquad t, \bar{t} \in [-\tau, 0]$$

for $i, j = 1, \ldots, m$.

Remark 1.3. We comment that (H6) and (H8) yield (H3) (iii), (H4) (iii) and (H4) (ii) with $\Lambda_{ij} = \frac{b_{ij}^*}{\sigma_i - \varepsilon_0}$, but they are not assumed in Lemmas 2.2 and 2.4.

2 Auxiliary results

Recall that t_k is a strictly monotone increasing sequence which tends to $+\infty$ as $k \to \infty$. We denote the set of time moments by $\mathcal{T} = \{t_k : k \in \mathbb{N}\}$. Throughout this manuscript, we use the notation $\ell(t)$ for the uniquely defined nonnegative integer with the property that

$$t \in [t_{\ell(t)}, t_{\ell(t)+1}), \qquad t \geq 0. \tag{2.1}$$

Note that if $t \notin \mathcal{T}$, then $t_{\ell(t)} < t$, otherwise $t_{\ell(t)} = t$.

We use the vector notation $\mathbf{x}(t) = (x_1(t), \dots, x_m(t))^T$ throughout the manuscript. For a norm of vector $\mathbf{x} = (x_1, \dots, x_m)^T \in \mathbb{R}^m$ we use the infinity norm $|\mathbf{x}|_\infty = \max\{|x_1|, \dots, |x_m|\}$. The corresponding induced matrix norm is denoted by $\|A\|_\infty$ for $A \in \mathbb{R}^{m\times m}$. For continuous functions $\psi : [-\tau, 0] \to \mathbb{R}$ and $\boldsymbol{\psi} : [-\tau, 0] \to \mathbb{R}^m$ we use the supremum norm $|\psi|_C = \max_{-\tau\leq t\leq 0} |\psi(t)|$ and $|\boldsymbol{\psi}|_C = \max_{-\tau\leq t\leq 0} |\boldsymbol{\psi}(t)|_\infty$, respectively.

The notation $\mathbf{x} \leq \mathbf{y}$ is used for $\mathbf{x} = (x_1, \dots, x_m)^T \in \mathbb{R}^m$ and $\mathbf{y} = (y_1, \dots, y_m)^T \in \mathbb{R}^m$ if the componentwise comparisons $x_i \leq y_i$ hold for all $i = 1, \dots, m$. We note that $\mathbf{0} \leq \mathbf{x} \leq \mathbf{y}$ implies $|\mathbf{x}|_\infty \leq |\mathbf{y}|_\infty$. We say that a matrix $A \in \mathbb{R}^{m\times m}$ is *monotone* if $A\mathbf{x} \leq A\mathbf{y}$ yields $\mathbf{x} \leq \mathbf{y}$ for every $\mathbf{x}, \mathbf{y} \in \mathbb{R}^m$. Let $I \in \mathbb{R}^{m\times m}$ denote the identity matrix. We say that the matrix $I - A \in \mathbb{R}^{m\times m}$ is a *nonsingular M-matrix* if $\rho(A) < 1$, where $\rho(A)$ is the spectral radius of A. We refer to [4] for 50 equivalent definitions of a nonsingular M-matrix.

The following variation of constants formula was formulated in [24] for the system (1.14) without delays. It is straightforward to extend it for (1.14).

Lemma 2.1. *The solution* $\mathbf{x}(t) = (x_1(t), \dots, x_m(t))^T$ *of* (1.14) *satisfies*

$$\begin{aligned} x_i(t) &= e^{-\int_0^t a_i(u)du} \left(\prod_{j=1}^{\ell(t)} (1 - p_{i,j}) \right) \varphi_i(0) \\ &+ \sum_{r=1}^{\ell(t)} \left(\prod_{j=r}^{k(t)} (1 - p_{i,j}) \right) \int_{t_{r-1}}^{t_r} e^{-\int_s^t a_i(u)du} G_i(s, \mathbf{x}(s)) ds \\ &+ \sum_{r=1}^{\ell(t)} \left(\prod_{j=r+1}^{k(t)} (1 - p_{i,j}) \right) e^{-\int_{t_r}^t a_i(u)du} (J_{i,r}(x_i(t_r^-)) + e_{i,r}) \\ &+ \int_{t_{\ell(t)}}^t e^{-\int_s^t a_i(u)du} G_i(s, \mathbf{x}(s)) ds, \qquad i = 1, \dots, m, \quad t \geq 0, \end{aligned} \tag{2.2}$$

where

$$G_i(t, \mathbf{x}(t)) = \sum_{j=1}^m b_{ij}(t) g_j(x_j(t - \tau_j)) + c_i(t). \tag{2.3}$$

Next we show that under conditions (H1)–(H5) the solutions of (1.14) are bounded on $[0, \infty)$. Moreover, if assumption (H6) holds, the solutions are exponentially bounded.

Lemma 2.2. *Suppose (H1)–(H5) hold. Then all solutions of* (1.14) *are bounded on* $[0, \infty)$*. Moreover, if (H6) holds too, then for every solution* $\mathbf{x}(t)$ *of* (1.14) *there exist positive constants* α_0 *and* K_0 *such that*

$$|\mathbf{x}(t)|_\infty \leq K_0 e^{-\alpha_0 t}, \qquad t \geq -\tau, \tag{2.4}$$

i.e., every solution of (1.14) *is exponentially bounded.*

Proof. 1. First, we prove the boundedness of the solutions. Let $\mathbf{x}(t) = (x_1(t), \ldots, x_m(t))^T$ be the solution of (1.14) corresponding to initial condition $\boldsymbol{\varphi} = (\varphi_1, \ldots, \varphi_m)^T$. Then x_i satisfies the variation of constant formula (2.2), where H_i is defined by (2.3).

Now suppose $s \in [t_{r-1}, t_r)$ for some $r \in \mathbb{N}$, $s < t$ and $t \notin \mathcal{T}$. Then we get $s < t_r < t_{\ell(t)} < t < t_{\ell(t)+1}$, $J(s,t) = \{r, r+1, \ldots, \ell(t)\}$ if $t > t_r$, and $J(s,t) = \emptyset$ if $t \leq t_r$, therefore (H3) (i) and (H4) (i) yield

$$e^{-\int_s^t a_i(u)\,du} \left(\prod_{j=r}^{\ell(t)} (1 - p_{i,j}) \right) = \exp\left(-\int_s^t a_i(u)\,du + \sum_{j \in J(s,t)} \ln(1 - p_{i,j}) \right) \leq e^{-\sigma_i(t-s)}.$$

If $t = t_{r_0}$ for some $r_0 \in \mathbb{N}$, $s \in [t_{r-1}, t_r)$, $s < t$, then $J(s, t_{r_0}) = \{r, r+1, \ldots, r_0 - 1\}$, and

$$e^{-\int_s^{t_{r_0}} a_i(u)\,du} \left(\prod_{j=r}^{\ell(t)} (1 - p_{i,j}) \right) = (1 - p_{i,r_0}) \exp\left(-\int_s^{t_{r_0}} a_i(u)\,du + \sum_{j=r}^{r_0-1} \ln(1 - p_{i,j}) \right) \leq (1 - \underline{p}_i) e^{-\sigma_i(t_{r_0}-s)}.$$

Hence

$$e^{-\int_s^t a_i(u)\,du} \left(\prod_{j=r}^{\ell(t)} (1 - p_{i,j}) \right) \leq (1 - \underline{p}_i^-) e^{-\sigma_i(t-s)}, \quad s \in [t_{r-1}, t_r), \; s \leq t, \; i = 1, \ldots, m, \tag{2.5}$$

where $\underline{p}_i^- = \min\{0, \underline{p}_i\}$.

For $s \in (t_{\ell(t)}, t)$ it follows $J(s,t) = \emptyset$, and

$$e^{-\int_s^t a_i(u)\,du} = \exp\left(-\int_s^t a_i(u)\,du + \sum_{j \in J(s,t)} \ln(1 - p_{i,j}) \right) \leq e^{-\sigma_i(t-s)}. \tag{2.6}$$

For $t_r < t$, $t \notin \mathcal{T}$ we get $t_r < t_{r+1} \leq t_{\ell(t)} < t$, so $J(t_r, t) = \{r, r+1, \ldots, \ell(t)\}$, and therefore (H3) (i) and (H4) (i) imply

$$\begin{aligned} e^{-\int_{t_r}^t a_i(u)\,du} \left(\prod_{j=r+1}^{\ell(t)} (1 - p_{i,j}) \right) &= \frac{1}{1 - p_{i,r}} e^{-\int_{t_r}^t a_i(u)\,du} \left(\prod_{j=r}^{\ell(t)} (1 - p_{i,j}) \right) \\ &\leq \frac{1}{1 - p_i^*} \exp\left(-\int_{t_r}^t a_i(u)\,du + \sum_{j \in J(t_r,t)} \ln(1 - p_{i,j}) \right) \\ &\leq \frac{1}{1 - p_i^*} e^{-\sigma_i(t - t_r)}. \end{aligned}$$

Finally, if $t = t_{r_0}$ for some $r_0 \in \mathbb{N}$ and $r < r_0$, then it follows $J(t_r, t_{r_0}) = \{r, r+1, \ldots, r_0 - 1\}$, $\ell(t) = r_0$, and so

$$e^{-\int_{t_r}^{t_{r_0}} a_i(u)\,du} \left(\prod_{j=r+1}^{\ell(t)} (1 - p_{i,j}) \right) = \frac{1 - p_{i,r_0}}{1 - p_{i,r}} \exp\left(-\int_{t_r}^{t_{r_0}} a_i(u)\,du + \sum_{j=r}^{r_0-1} \ln(1 - p_{i,j}) \right) \leq \frac{1 - \underline{p}_i}{1 - p_i^*} e^{-\sigma_i(t_{r_0} - t_r)}.$$

Combining the above two cases, we get

$$e^{-\int_{t_r}^{t} a_i(u)\,du}\left(\prod_{j=r+1}^{\ell(t)}(1-p_{i,j})\right)\le\frac{1-\underline{p}_i^-}{1-p_i^*}e^{-\sigma_i(t-t_r)},\qquad t\ge t_r,\;\; r\in\mathbb{N},\;\; i=1,\dots,m. \tag{2.7}$$

Then (2.2), (2.5), (2.6) and (2.7) imply for $t\ge 0$

$$\begin{aligned}|x_i(t)|&\le(1-\underline{p}_i^-)e^{-\sigma_i t}|\varphi_i(0)|+\sum_{r=1}^{\ell(t)}\int_{t_{r-1}}^{t_r}(1-\underline{p}_i^-)e^{-\sigma_i(t-s)}|G_i(s,\mathbf{x}(s))|\,ds\\&\quad+\sum_{r=1}^{\ell(t)}\frac{1-\underline{p}_i^-}{1-p_i^*}e^{-\sigma_i(t-t_r)}(|J_{i,r}(x_i(t_r^-))|+|e_{i,r}|)+\int_{t_{\ell(t)}}^{t}e^{-\sigma_i(t-s)}|G_i(s,\mathbf{x}(s))|\,ds\\&=(1-\underline{p}_i^-)e^{-\sigma_i t}|\varphi_i(0)|+\int_0^t(1-\underline{p}_i^-)e^{-\sigma_i(t-s)}|G_i(s,\mathbf{x}(s))|\,ds\\&\quad+\sum_{r=1}^{\ell(t)}\frac{1-\underline{p}_i^-}{1-p_i^*}e^{-\sigma_i(t-t_r)}(|J_{i,r}(x_i(t_r^-))|+|e_{i,r}|).\end{aligned}\tag{2.8}$$

The assumed relations (H1)–(H4), (2.8), $t\in[t_{\ell(t)},t_{\ell(t)+1})$,

$$|J_{i,r}(x_i(t_r^-))|\le l_{i,r}|x_i(t_r^-)|\le l_i^*|x_i(t_r^-)|,\qquad i=1,\dots,m,\quad r\in\mathbb{N}$$

and

$$|G_i(s,\mathbf{x}(s))|\le\sum_{j=1}^{m}|b_{ij}(s)|L_j|x_j(s-\tau_j)|+|c_i(s)|,\qquad i=1,\dots,m,\quad s\ge 0$$

yield

$$\begin{aligned}|x_i(t)|&\le(1-\underline{p}_i^-)e^{-\sigma_i t}|\varphi_i(0)|\\&\quad+\int_0^t(1-\underline{p}_i^-)e^{-\sigma_i(t-s)}\Big(\sum_{j=1}^{m}|b_{ij}(s)|L_j|x_j(s-\tau_j)|+|c_i(s)|\Big)\,ds\\&\quad+\frac{1-\underline{p}_i^-}{1-p_i^*}\sum_{r=1}^{\ell(t)}e^{-\sigma_i(t-t_r)}(l_i^*|x_i(t_r^-)|+|e_{i,r}|),\qquad i=1,\dots,m,\quad t\ge 0.\end{aligned}\tag{2.9}$$

Using relation $\underline{\delta}\le t_{r+1}-t_r$ for $r\in\mathbb{N}_0$ from (H4) and $t\in[t_{\ell(t)},t_{\ell(t)+1})$, we obtain

$$\begin{aligned}\sum_{r=1}^{\ell(t)}e^{-\sigma_i(t-t_r)}&=\sum_{r=1}^{\ell(t)}e^{-\sigma_i(t-t_{\ell(t)}+(t_{\ell(t)}-t_{\ell(t)-1})+\cdots+(t_{r+1}-t_r))}\\&\le\sum_{r=1}^{\ell(t)}e^{-\sigma_i(t-t_{\ell(t)})}e^{-\sigma_i(\ell(t)-r)\underline{\delta}}\\&\le\sum_{r=1}^{\ell(t)}\left(e^{-\sigma_i\underline{\delta}}\right)^{\ell(t)-r}\\&\le\frac{1}{1-e^{-\sigma_i\underline{\delta}}},\qquad t\ge 0.\end{aligned}\tag{2.10}$$

Combining (2.9) with assumptions (H3), (H4), relation (2.10), and the estimate

$$\int_0^t e^{-\sigma_i(t-s)}\,ds\le\frac{1}{\sigma_i},\qquad t\ge 0,\tag{2.11}$$

we get for $t \geq 0$ and $i = 1, \dots, m$

$$\begin{aligned}
|x_i(t)| &\leq (1-\underline{p}_i^-)e^{-\sigma_i t}|\varphi_i(0)| \\
&\quad + \int_0^t (1-\underline{p}_i^-)e^{-\sigma_i(t-s)}\Big(\sum_{j=1}^m |b_{ij}(s)|L_j \sup_{-\tau\leq u\leq s}|x_j(u)| + c_i^*\Big)\,ds \\
&\quad + \frac{1-\underline{p}_i^-}{1-p_i^*}\sum_{r=1}^{\ell(t)} e^{-\sigma_i(t-t_r)}\big(l_i^* \sup_{0\leq u\leq t}|x_i(u)| + e_i^*\big) \\
&\leq (1-\underline{p}_i^-)|\varphi_i(0)| + \sum_{j=1}^m (1-\underline{p}_i^-)\Lambda_{ij}L_j \sup_{-\tau\leq u\leq t}|x_j(u)| + \frac{(1-\underline{p}_i^-)c_i^*}{\sigma_i} \\
&\quad + \frac{1-\underline{p}_i^-}{(1-p_i^*)(1-e^{-\sigma_i\underline{\delta}})}\Big(l_i^* \sup_{-\tau\leq u\leq t}|x_i(u)| + e_i^*\Big). \qquad (2.12)
\end{aligned}$$

Since the right-hand side of (2.12) is monotone increasing in t, and $|x_i(u)| \leq |\varphi_i|_C \leq (1-\underline{p}_i^-)|\varphi_i|_C$ for $u \in [-\tau, 0]$, (2.12) yields

$$\begin{aligned}
\sup_{-\tau\leq u\leq t}|x_i(u)| &\leq (1-\underline{p}_i^-)|\varphi_i|_C + \sum_{j=1}^m (1-\underline{p}_i^-)\Lambda_{ij}L_j \sup_{-\tau\leq u\leq t}|x_j(u)| + \frac{(1-\underline{p}_i^-)c_i^*}{\sigma_i} \\
&\quad + \frac{1-\underline{p}_i^-}{(1-p_i^*)(1-e^{-\sigma_i\underline{\delta}})}\Big(l_i^* \sup_{-\tau\leq u\leq t}|x_i(u)| + e_i^*\Big), \qquad (2.13)
\end{aligned}$$

for $i = 1, \dots, m$ and $t \geq 0$. Fix a nonnegative parameter α. Then we introduce the corresponding notations

$$\begin{aligned}
&\mathbf{v}^{(\alpha)}(t) = \Big(\sup_{-\tau\leq u\leq t} e^{\alpha u}|x_1(u)|, \dots, \sup_{-\tau\leq u\leq t} e^{\alpha u}|x_m(u)|\Big)^T \in \mathbb{R}^m, \qquad t \geq -\tau, \\
&\mathbf{a}^{(\alpha)} = (a_1^{(\alpha)}, \dots, a_m^{(\alpha)})^T \in \mathbb{R}^m, \quad \text{where} \\
&a_i^{(\alpha)} = (1-\underline{p}_i^-)\left(|\varphi_i|_C + \frac{c_i^*}{\sigma_i-\alpha} + \frac{e_i^*}{(1-p_i^*)(1-e^{-(\sigma_i-\alpha)\underline{\delta}})}\right), \\
&A^{(\alpha)} = (a_{ij}) \in \mathbb{R}^{m\times m}, \quad a_{ij}^{(\alpha)} = \begin{cases} (1-\underline{p}_i^-)\Lambda_{ij}L_i e^{\alpha\tau_i} + \frac{(1-\underline{p}_i^-)l_i^*}{(1-p_i^*)(1-e^{-(\sigma_i-\alpha)\underline{\delta}})}, & i = j, \\ (1-\underline{p}_i^-)\Lambda_{ij}L_j e^{\alpha\tau_j}, & i \neq j. \end{cases} \qquad (2.14)
\end{aligned}$$

Hence (2.13) implies the vector inequality

$$\mathbf{v}^{(0)}(t) \leq \mathbf{a}^{(0)} + A^{(0)}\mathbf{v}^{(0)}(t), \qquad t \geq 0.$$

The definition of $\mathbf{a}^{(0)}$ yields $\mathbf{v}^{(0)}(t) \leq \mathbf{a}^{(0)}$ for $t \in [-\tau, 0]$, so

$$\mathbf{v}^{(0)}(t) \leq \mathbf{a}^{(0)} + A^{(0)}\mathbf{v}^{(0)}(t), \qquad t \geq -\tau.$$

Assumption (H5) implies $\|A^{(0)}\|_\infty < 1$, so $I - A^{(0)}$ is a nonsingular M-matrix. Therefore Theorem 6.2.3 in [4] yields that $I - A^{(0)}$ is monotone, and

$$(|x_1(t)|, \dots, |x_m(t)|)^T \leq \mathbf{v}^{(0)}(t) \leq (I - A^{(0)})^{-1}\mathbf{a}^{(0)}, \qquad t \geq -\tau.$$

It follows

$$|\mathbf{x}(t)|_\infty \leq |(I - A^{(0)})^{-1}\mathbf{a}^{(0)}|_\infty, \qquad t \geq -\tau,$$

i.e., $\mathbf{x}(t)$ is bounded on $[-\tau,\infty)$.

2. Next, we show the exponential boundedness of the solutions under the additional assumption (H6).

We select a positive constant α_0 such that

$$\alpha_0 < \min\{\varepsilon_0,\beta_1,\beta_2\} \quad\text{and}\quad \sum_{j=1}^m (1-\underline{p}_i^-)\Lambda_{ij}L_j e^{\alpha_0\tau_j} + \frac{(1-\underline{p}_i^-)l_i^*}{(1-p_i^*)(1-e^{-(\sigma_i-\alpha_0)\underline{\delta}})} < 1 \tag{2.15}$$

for $i=1,\dots,m$. Note that such α_0 exists since (H5) holds. Multiplying both sides of (2.9) by $e^{\alpha_0 t}$ we get

$$\begin{aligned} e^{\alpha_0 t}|x_i(t)| &\le (1-\underline{p}_i^-)e^{-(\sigma_i-\alpha_0)t}|\varphi_i(0)| \\ &\quad + \int_0^t (1-\underline{p}_i^-)e^{-(\sigma_i-\alpha_0)(t-s)}\Big(\sum_{j=1}^m |b_{ij}(s)|L_j e^{\alpha_0\tau_j}e^{\alpha_0(s-\tau_j)}|x_j(s-\tau_j)| + e^{\alpha_0 s}|c_i(s)|\Big)ds \\ &\quad + \frac{1-\underline{p}_i^-}{1-p_i^*}\sum_{r=1}^{\ell(t)} e^{-(\sigma_i-\alpha_0)(t-t_r)}(l_i^* e^{\alpha_0 t_r}|x_i(t_r^-)| + e^{\alpha_0 t_r}|e_{i,r}|) \end{aligned}$$

for $i=1,\dots,m$ and $t\ge 0$. Then (H3), (H4), (H6), $\alpha_0<\min\{\varepsilon_0,\beta_1,\beta_2\}$, and (2.10) and (2.11) where σ_i is replaced by $\sigma_i-\alpha_0$ imply

$$\begin{aligned} e^{\alpha_0 t}|x_i(t)| &\le (1-\underline{p}_i^-)|\varphi_i|_C + \sum_{j=1}^m (1-\underline{p}_i^-)\Lambda_{ij}L_j e^{\alpha_0\tau_j}\sup_{-\tau\le u\le t} e^{\alpha_0 u}|x_j(u)| + \frac{(1-\underline{p}_i^-)c_i^*}{\sigma_i-\alpha_0} \\ &\quad + \frac{(1-\underline{p}_i^-)l_i^*}{(1-p_i^*)(1-e^{-(\sigma_i-\alpha_0)\underline{\delta}})}\sup_{-\tau\le u\le t} e^{\alpha_0 u}|x_i(u)| + \frac{(1-\underline{p}_i^-)e_i^*}{(1-p_i^*)(1-e^{-(\sigma_i-\alpha_0)\underline{\delta}})} \end{aligned}$$

for $i=1,\dots,m$ and $t\ge 0$. Then the monotonicity of the right-hand side and $e^{\alpha_0 t}|x_i(t)| \le |\varphi_i|_C \le (1-\underline{p}_i^-)|\varphi_i|_C$ for $-\tau\le t\le 0$ imply the vector inequality

$$\mathbf{v}^{(\alpha_0)}(t) \le \mathbf{a}^{(\alpha_0)} + A^{(\alpha_0)}\mathbf{v}^{(\alpha_0)}(t), \qquad t\ge -\tau. \tag{2.16}$$

Relation (2.15) yields $\|A^{(\alpha_0)}\|_\infty < 1$, so $I-A^{(\alpha_0)}$ is a nonsingular M-matrix, hence $I-A^{(\alpha_0)}$ is monotone. Therefore

$$(e^{\alpha_0 t}|y_1(t)|,\dots,e^{\alpha_0 t}|y_m(t)|)^T \le \mathbf{v}^{(\alpha_0)}(t) \le (I-A^{(\alpha_0)})^{-1}\mathbf{a}^{(\alpha_0)}, \qquad t\ge-\tau,$$

so (2.4) holds with

$$K_0 = |(I-A^{(\alpha_0)})^{-1}\mathbf{a}^{(\alpha_0)}|_\infty,$$

i.e., $\mathbf{x}(t)$ is exponentially bounded on $[-\tau,\infty)$. □

Remark 2.3. Let $A^{(0)}$ be defined by (2.14) with $\alpha=0$. We remark that (H5) can be replaced by the weaker condition $\rho(A^{(0)})<1$, and the statement of Lemma 2.2 remains true.

Our next result shows that every solution of (1.14) is exponentially stable.

Lemma 2.4. *Suppose (H1)–(H5) hold. Then there exist positive constants α_0 and K_1 such that*

$$|\mathbf{x}(t)-\bar{\mathbf{x}}(t)|_\infty \le K_1 e^{-\alpha_0 t}|\boldsymbol{\varphi}-\bar{\boldsymbol{\varphi}}|_C, \qquad t\ge 0, \tag{2.17}$$

where $\boldsymbol{\varphi}(t)=(\varphi_1,\dots,\varphi_m)^T$ and $\bar{\boldsymbol{\varphi}}=(\bar\varphi_1,\dots,\bar\varphi_m)^T$ are two initial functions in (1.14), and $\mathbf{x}(t)=(x_1(t),\dots,x_m(t))^T$ and $\bar{\mathbf{x}}(t)=(\bar x_1(t),\dots,\bar x_m(t))^T$ are the corresponding solutions of (1.14), respectively, i.e., every solution of (1.14) is exponentially stable.

Proof. Let $(\varphi_1,\dots,\varphi_m)^T$ and $(\bar\varphi_1,\dots,\bar\varphi_m)^T$ be the vectors of two initial functions in (1.14), and $(x_1(t),\dots,x_m(t))^T$ and $(\bar x_1(t),\dots,\bar x_m(t))^T$ be the corresponding solutions of (1.14). Then the variation of constant formula (2.2) yields

$$\begin{aligned}
x_i(t)-\bar x_i(t) &= e^{-\int_0^t a_i(u)\,du}\left(\prod_{j=1}^{\ell(t)}(1-p_{i,j})\right)(\varphi_i(0)-\bar\varphi_i(0))\\
&\quad+\sum_{r=1}^{\ell(t)}\left(\prod_{j=r}^{\ell(t)}(1-p_{i,j})\right)\int_{t_{r-1}}^{t_r} e^{-\int_s^t a_i(u)\,du}\Big(G_i(s,\mathbf{x}(s))-G_i(s,\bar{\mathbf{x}}(s))\Big)\,ds\\
&\quad+\sum_{r=1}^{\ell(t)}\left(\prod_{j=r+1}^{\ell(t)}(1-p_{i,j})\right)e^{-\int_{t_r}^t a_i(u)\,du}\Big(J_{i,r}(x_i(t_r^-))-J_{i,r}(\bar x_i(t_r^-))\Big)\\
&\quad+\int_{t_{\ell(t)}}^t e^{-\int_s^t a_i(u)\,du}\Big(G_i(s,\mathbf{x}(s))-G_i(s,\bar{\mathbf{x}}(s))\Big)\,ds,\quad i=1,\dots,m,\quad t\ge 0.
\end{aligned}$$

We have

$$|G_i(s,\mathbf{x}(s))-G_i(s,\bar{\mathbf{x}}(s))|\le\sum_{j=1}^m|b_{ij}(s)|L_j|x_j(s-\tau_j)-\bar x_i(s-\tau_j)|,$$

hence, similarly to the derivation of (2.9), we get

$$\begin{aligned}
|x_i(t)-\bar x_i(t)| &\le (1-\underline{p}_i^-)e^{-\sigma_i t}|\varphi_i(0)-\bar\varphi_i(0)|\\
&\quad+\sum_{j=1}^m\int_0^t(1-\underline{p}_i^-)e^{-\sigma_i(t-s)}|b_{ij}(s)|L_j|x_j(s-\tau_j)-\bar x_i(s-\tau_j)|\,ds\\
&\quad+\frac{1-\underline{p}_i^-}{1-p_i^*}\sum_{r=1}^{\ell(t)}e^{-\sigma_i(t-t_r)}l_i^*|x_i(t_r^-)-\bar x_i(t_r^-)|,\qquad i=1,\dots,m,\quad t\ge 0.
\end{aligned}\tag{2.18}$$

We select a positive constant α_0 such that (2.15) is satisfied. Note that such α_0 exists since (H5) holds. Multiplying both sides of (2.18) by $e^{\alpha_0 t}$ we obtain

$$\begin{aligned}
&e^{\alpha_0 t}|x_i(t)-\bar x_i(t)|\\
&\quad\le(1-\underline{p}_i^-)e^{(\alpha_0-\sigma_i)t}|\varphi_i(0)-\bar\varphi_i(0)|\\
&\qquad+\sum_{j=1}^m\int_0^t(1-\underline{p}_i^-)e^{-(\sigma_i-\alpha_0)(t-s)}|b_{ij}(s)|L_je^{\alpha_0\tau_j}e^{\alpha_0(s-\tau_j)}|x_j(s-\tau_j)-\bar x_i(s-\tau_j)|\,ds\\
&\qquad+\frac{1-\underline{p}_i^-}{1-p_i^*}\sum_{r=1}^{\ell(t)}e^{-(\sigma_i-\alpha_0)(t-t_r)}l_i^*e^{\alpha_0t_r}|x_i(t_r^-)-\bar x_i(t_r^-)|
\end{aligned}\tag{2.19}$$

for $i=1,\dots,m$ and $t\ge 0$. Introduce the functions

$$v_i(t)=\sup_{-\tau\le u\le t}e^{\alpha_0 u}|x_i(u)-\bar x_i(u)|,\qquad i=1,\dots,m,\quad t\ge-\tau.$$

Then (2.19) combined with (2.10) where σ_i is replaced by $\sigma_i-\alpha_0$ and

$$e^{\alpha_0 u}|x_i(u)-\bar x_i(u)|\le e^{\alpha_0 u}|\mathbf{x}(u)-\bar{\mathbf{x}}(u)|_\infty\le|\boldsymbol{\varphi}-\bar{\boldsymbol{\varphi}}|_C\le(1-\underline{p}_i^-)|\boldsymbol{\varphi}-\bar{\boldsymbol{\varphi}}|_C,\quad -\tau\le u\le 0$$

imply

$$
\begin{aligned}
v_i(t) &\le (1-\underline{p}_i^-)|\boldsymbol{\varphi}-\bar{\boldsymbol{\varphi}}|_C + \sum_{j=1}^m \int_0^t (1-\underline{p}_i^-)e^{-(\sigma_i-\alpha_0)(t-s)}|b_{ij}(s)|L_j e^{\alpha_0\tau_j} v_j(s)\,ds \\
&\quad + \frac{1-\underline{p}_i^-}{1-p_i^*}\sum_{r=1}^{\ell(t)} e^{-(\sigma_i-\alpha_0)(t-t_r)} l_i^* v_i(t) \\
&\le (1-\underline{p}_i^-)|\boldsymbol{\varphi}-\bar{\boldsymbol{\varphi}}|_C + \sum_{j=1}^m (1-\underline{p}_i^-)\Lambda_{ij}L_j e^{\alpha_0\tau_j} v_j(t) \\
&\quad + \frac{(1-\underline{p}_i^-)l_i^*}{(1-p_i^*)(1-e^{-(\sigma_i-\alpha_0)\underline{\delta}})} v_i(t)
\end{aligned}
$$

for $i = 1,\dots,m$ and $t \ge 0$. Therefore, the vector inequality

$$
\mathbf{v}(t) \le \mathbf{b} + A^{(\alpha_0)}\mathbf{v}(t), \qquad t \ge -\tau, \tag{2.20}
$$

holds, where

$$
\begin{aligned}
\mathbf{v}(t) &= (v_1(t),\dots,v_m(t))^T \in \mathbb{R}^m, \qquad t \ge -\tau, \\
\mathbf{b} &= \Big((1-\underline{q}_1^-)|\boldsymbol{\varphi}-\bar{\boldsymbol{\varphi}}|_C, \dots, (1-\underline{q}_m^-)|\boldsymbol{\varphi}-\bar{\boldsymbol{\varphi}}|_C\Big)^T \in \mathbb{R}^m,
\end{aligned} \tag{2.21}
$$

and $A^{(\alpha_0)}$ is defined by (2.14). Relation (2.15) yields $\|A^{(\alpha_0)}\|_\infty < 1$, hence $I - A^{(\alpha_0)}$ is a nonsingular M-matrix, so $I - A^{(\alpha_0)}$ is monotone. Therefore (2.20) gives

$$
\mathbf{v}(t) \le (I - A^{(\alpha_0)})^{-1}\mathbf{b}, \qquad t \ge -\tau,
$$

and hence

$$
e^{\alpha_0 t}|\mathbf{x}(t)-\bar{\mathbf{x}}(t)|_\infty \le |\mathbf{v}(t)|_\infty \le \|(I-A^{(\alpha_0)})^{-1}\|_\infty |\mathbf{b}|_\infty = K_1|\boldsymbol{\varphi}-\bar{\boldsymbol{\varphi}}|_C,
$$

where $K_1 = \|(I-A^{(\alpha_0)})^{-1}\|_\infty \max\{1-\underline{q}_1^-,\dots,1-\underline{q}_m^-\}$. This completes the proof of (2.17). □

Remark 2.5. Let $A^{(\alpha)}$ be the matrix defined by (2.14). We note that $\rho(A^{(0)}) < 1$ implies $\rho(A^{(\alpha_0)}) < 1$ for sufficiently small $\alpha_0 > 0$, so assumption (H5) in Lemma 2.4 can be replaced by the weaker condition $\rho(A^{(0)}) < 1$.

Next, we prove an estimate which will be important in the proof of our main result in the next section.

Lemma 2.6. *Suppose (H1)–(H5) hold. Let $\mathbf{x} = (x_1,\dots,x_m)^T$ be a solution of (1.14), and let $\alpha_0 > 0$ be the corresponding constant from Lemma 2.2. For $0 < \alpha < \alpha_0$ and $u > 0$ define*

$$
\begin{aligned}
\omega_\alpha(u) = \sup\Big\{ e^{\alpha t}|\mathbf{x}(t)-\mathbf{x}(\bar{t})|_\infty \;:\; \Big(t,\bar{t} \in [t_r,t_{r+1}),\ r\in\mathbb{N} \quad \textit{or} \quad t,\bar{t}\in[-\tau,t_1)\Big), \\
\textit{and} \quad |\bar{t}-t| \le u \Big\}.
\end{aligned} \tag{2.22}
$$

(i) Then

$$
\lim_{u\to 0+} \omega_\alpha(u) = 0, \qquad 0 < \alpha < \alpha_0. \tag{2.23}
$$

(ii) Assume further (H6), (H7) and (H8). Then there exist $M_0 > 0$ and $u_0 > 0$ such that

$$\omega_\alpha(u) \le M_0 u, \qquad 0 < u \le u_0, \quad 0 < \alpha < \alpha_0. \tag{2.24}$$

Proof. (i) It follows from Lemma 2.2 that $\mathbf{x}$ satisfies (2.4). Fix $0 < \alpha < \alpha_0$, $\varepsilon > 0$ and $\bar{u} > 0$. Since $t_k \to \infty$ as $k \to \infty$, there exists k_0 such that

$$K_0 e^{\alpha_0 \bar{u}} e^{(\alpha-\alpha_0)t} < \frac{\varepsilon}{2}, \qquad t \ge t_{k_0}.$$

Then, using (2.4) and the triangle inequality, we get

$$e^{\alpha t}|\mathbf{x}(t) - \mathbf{x}(\bar{t})|_\infty < e^{\alpha t}\left(K_0 e^{-\alpha_0 t} + K_0 e^{-\alpha_0 \bar{t}}\right) \le K_0 e^{(\alpha-\alpha_0)t} + K_0 e^{\alpha_0 u} e^{(\alpha-\alpha_0)t} < \varepsilon,$$

for $t, \bar{t} \ge t_{k_0}$, $|\bar{t} - t| \le u$ and $0 < u < \bar{u}$.

The function $e^{\alpha t} x_i(t)$ is uniformly continuous on the intervals $[t_k, t_{k+1})$ for $k = 1, \dots, k_0 - 1$ and $i = 1, \dots, m$, and on the interval $[-\tau, t_1)$ since it has continuous extension to the closed intervals $[t_k, t_{k+1}]$ and $[-\tau, t_1]$. Therefore, there exists $\delta > 0$ such that

$$|e^{\alpha t}\mathbf{x}(t) - e^{\alpha \bar{t}}\mathbf{x}(\bar{t})|_\infty < \frac{\varepsilon}{2} \qquad \text{and} \qquad \delta < \min\left\{\bar{u}, \frac{\varepsilon}{2(e-1)K_0\alpha}, \frac{1}{\alpha}\right\}$$

if $t, \bar{t} \in [t_r, t_{r+1})$ for some $r \in \{1, \dots, k_0 - 1\}$ or $t, \bar{t} \in [-\tau, t_1)$, and $|\bar{t} - t| \le \delta$. Then (2.4) and the estimate

$$|e^s - 1| \le (e-1)|s|, \qquad |s| \le 1 \tag{2.25}$$

imply

$$\begin{aligned} e^{\alpha t}|\mathbf{x}(t) - \mathbf{x}(\bar{t})|_\infty &\le |e^{\alpha t}\mathbf{x}(t) - e^{\alpha \bar{t}}\mathbf{x}(\bar{t})|_\infty + |e^{\alpha t} - e^{\alpha \bar{t}}||\mathbf{x}(\bar{t})|_\infty \\ &< \frac{\varepsilon}{2} + |e^{\alpha(t-\bar{t})} - 1|e^{\alpha \bar{t}}|\mathbf{x}(\bar{t})|_\infty \\ &< \frac{\varepsilon}{2} + (e-1)\alpha\delta K_0 \\ &< \varepsilon, \end{aligned}$$

if $t, \bar{t} \in [t_r, t_{r+1})$ for some $r \in \{1, \dots, k_0 - 1\}$ or $t, \bar{t} \in [-\tau, t_1)$, and $|\bar{t} - t| \le \delta$. Hence $\omega_\alpha(u) \le \varepsilon$ for $0 < \alpha < \alpha_0$ and $0 < u < \delta$, which completes the proof of (2.23).

(ii) Note that it follows from the proof of Lemma 2.2 that $\alpha < \alpha_0 < \beta_2$. Since $x_i(t)$ is continuously differentiable on $[t_r, t_{r+1})$, we get from (1.14) that for $t, \bar{t} \in [t_r, t_{r+1})$ for some $r \in \mathbb{N}_0$

$$e^{\alpha t}(x_i(t) - x_i(\bar{t})) = \int_{\bar{t}}^t e^{\alpha t}\left(-a_i(s)x_i(s) + \sum_{j=1}^m b_{ij}(s) g_j(x_j(s-\tau_j)) + c_i(s)\right) ds.$$

Define the constant $M = a_i^* K_0 + \sum_{j=1}^m b_{ij}^* L_j e^{\alpha_0 \tau_j} K_0 + c_i^*$. Then, using $e^{\alpha s}|x_i(s)| \le e^{\alpha_0 s}|x_i(s)| \le K_0$ from (2.4) and $e^{\alpha s}|c_i(s)| \le e^{\beta_2 s}|c_i(s)| \le c_i^*$ from (H6) and (2.25), we get

$$\begin{aligned} e^{\alpha t}|x_i(t) - x_i(\bar{t})| &\le \int_{\bar{t}}^t e^{\alpha(t-s)}\left(a_i^* e^{\alpha s}|x_i(s)| + \sum_{j=1}^m b_{ij}^* L_j e^{\alpha_0 \tau_j} e^{\alpha(s-\tau_j)}|x_j(s-\tau_j)| + e^{\alpha s}|c_i(s)|\right) ds \\ &\le M \int_{\bar{t}}^t e^{\alpha(t-s)}\, ds \\ &= M\left(\frac{e^{\alpha(t-\bar{t})} - 1}{\alpha}\right) \\ &\le M(e-1)u, \qquad t, \bar{t} \in [t_r, t_{r+1}), \quad r \in \mathbb{N}_0, \end{aligned}$$

for $|t-\bar{t}| \le u \le u_0 = \frac{1}{\alpha_0}$.

Assumption (H8) yields

$$e^{\alpha t}|x_i(t)-x_i(\bar{t})| \le |\varphi_i(t)-\varphi_i(\bar{t})| \le L_\varphi |t-\bar{t}|, \qquad t,\bar{t} \in [-\tau,0].$$

Suppose $-\tau \le \bar{t} \le 0 \le t < t_1$ and $|t-\bar{t}| \le u < u_0$. Then combining the above two cases and $u_0 = \frac{1}{\alpha_0}$ we obtain

$$\begin{aligned} e^{\alpha t}|x_i(t)-x_i(\bar{t})| &\le e^{\alpha t}\Big(|x_i(t)-x_i(0)| + |x_i(0)-x_i(\bar{t})|\Big) \\ &\le M(e-1)t + e^{\alpha u_0} L_\varphi(-\bar{t}) \\ &\le M_0 u, \end{aligned}$$

where $M_0 = \max\{M(e-1), eL_\varphi\}$.

For $-\tau \le t \le 0 \le \bar{t} < t_1$ and $|t-\bar{t}| \le u < u_0$ we get

$$e^{\alpha t}|x_i(t)-x_i(\bar{t})| \le |x_i(t)-x_i(0)| + e^{\alpha \bar{t}}|x_i(0)-x_i(\bar{t})| \le L_\varphi(-t) + M(e-1)\bar{t} \le M_0 u.$$

The proof of (2.24) is completed. □

3 Main results

In this section, we prove that the solutions of (1.15) approximate that of (1.14) uniformly on $[0,\infty)$.

Theorem 3.1. *Suppose (H1)–(H7) hold. Let $\mathbf{x} = (x_1,\dots,x_m)^T$ be the solution of (1.14) corresponding to initial function $\boldsymbol{\varphi} = (\varphi_1,\dots,\varphi_m)^T$, and $\mathbf{y} = (y_1,\dots,y_m)^T$ be the solution of (1.15) corresponding to $h > 0$ and an initial function $\boldsymbol{\psi} = (\psi_1,\dots,\psi_m)^T$, and let $\alpha_0 > 0$ be the corresponding constant from Lemma 2.2.*

(i) Then for every $0 < \alpha < \alpha_0$ and $\varepsilon > 0$ there exist constants $K_2 > 0$ and $h^* > 0$, and a function $\theta(h)$ such that $\theta(h) \to 0$ as $h \to 0+$, and

$$|\mathbf{x}(t)-\mathbf{y}(t)|_\infty \le e^{-\alpha t} K_2 \Big(|\boldsymbol{\varphi}-\boldsymbol{\psi}|_C + \theta(h) + \varepsilon\Big), \qquad t \in [-\tau,\infty), \quad 0 < h < h^*. \tag{3.1}$$

(ii) Assume further (H8). Then for every $0 < \alpha < \alpha_0$ there exist constants $K_2 > 0$, $M > 0$ and $\bar{h} > 0$ such that

$$|\mathbf{x}(t)-\mathbf{y}(t)|_\infty \le e^{-\alpha t} K_2 \Big(|\boldsymbol{\varphi}-\boldsymbol{\psi}|_C + Mh\Big), \qquad t \in [-\tau,\infty), \quad 0 < h < \bar{h}. \tag{3.2}$$

Proof. The variation of constants formula (2.2) applied for problem (1.15) gives

$$\begin{aligned} y_i(t) &= e^{-\int_0^t a_i(u)\,du} \left(\prod_{j=1}^{\ell(t)} (1-p_{i,j}) \right) \psi_i(0) \\ &\quad + \sum_{r=1}^{\ell(t)} \left(\prod_{j=r}^{\ell(t)} (1-p_{i,j}) \right) \int_{t_{r-1}}^{t_r} e^{-\int_s^t a_i(u)\,du} G_i(s,\mathbf{y}(\gamma(s)))\,ds \\ &\quad + \sum_{r=1}^{\ell(t)} \left(\prod_{j=r+1}^{\ell(t)} (1-p_{i,j}) \right) e^{-\int_{\gamma(t_r)}^t a_i(u)\,du} (J_{i,r}(y_i(\gamma(t_r)^-)) + e_{i,r}) \\ &\quad + \int_{t_{\ell(t)}}^t e^{-\int_s^t a_i(u)\,du} G_i(s,\mathbf{y}(\gamma(s)))\,ds, \qquad i = 1,\dots,m, \quad t \ge 0, \end{aligned}$$

where

$$G_i(t, \mathbf{y}(\gamma(t))) = \sum_{j=1}^{m} b_{ij}(t) g_j(y_j(\gamma(t) - \gamma(\tau_j))) + c_i(t), \qquad i = 1, \dots, m.$$

Combining it with (2.2) we get

$$\begin{aligned} x_i(t) - y_i(t) &= e^{-\int_0^t a_i(u)\,du} \left(\prod_{j=1}^{\ell(t)} (1 - p_{i,j}) \right) (\varphi_i(0) - \psi_i(0)) \\ &+ \sum_{r=1}^{\ell(t)} \left(\prod_{j=r}^{\ell(t)} (1 - p_{i,j}) \right) \int_{t_{r-1}}^{t_r} e^{-\int_s^t a_i(u)\,du} \Big(G_i(s, \mathbf{x}(s)) - G_i(s, \mathbf{y}(\gamma(s))) \Big)\, ds \\ &+ \sum_{r=1}^{\ell(t)} \left(\prod_{j=r+1}^{\ell(t)} (1 - p_{i,j}) \right) \Big(e^{-\int_{t_r}^t a_i(u)\,du} (J_{i,r}(x_i(t_r^-)) + e_{i,r}) \\ &\quad - e^{-\int_{\gamma(t_r)}^{\mathbf{t}} a_i(u)\,du} (J_{i,r}(y_i(\gamma(t_r)^-)) + e_{i,r}) \Big) \\ &+ \int_{t_{\ell(t)}}^t e^{-\int_s^t a_i(u)\,du} \Big(G_i(s, \mathbf{x}(s)) - G_i(s, \mathbf{y}(\gamma(s))) \Big)\, ds \end{aligned}$$

for $i = 1, \dots, m$ and $t \geq 0$. Therefore, (2.5), (2.6) and $|\varphi_i(0) - \psi_i(0)| \leq |\boldsymbol{\varphi} - \boldsymbol{\psi}|_C$ yield

$$\begin{aligned} |x_i(t) - y_i(t)| &\leq (1 - \underline{p}_i^-) e^{-\sigma_i t} |\boldsymbol{\varphi} - \boldsymbol{\psi}|_C \\ &\quad + \int_0^t (1 - \underline{p}_i^-) e^{-\sigma_i (t-s)} \Big| G_i(s, \mathbf{x}(s)) - G_i(s, \mathbf{y}(\gamma(s))) \Big|\, ds + A_i(t) \end{aligned} \tag{3.3}$$

for $i = 1, \dots, m$ and $t \geq 0$, where

$$\begin{aligned} A_i(t) = \sum_{r=1}^{\ell(t)} \left(\prod_{j=r+1}^{\ell(t)} (1 - p_{i,j}) \right) \Big| & e^{-\int_{t_r}^t a_i(u)\,du} (J_{i,r}(x_i(t_r^-)) + e_{i,r}) \\ & - e^{-\int_{\gamma(t_r)}^t a_i(u)\,du} (J_{i,r}(y_i(\gamma(t_r)^-)) + e_{i,r}) \Big|. \end{aligned}$$

Let α_0 and K_0 be the constants from (2.4). We select a positive constant α such that

$$0 < \alpha < \min\{\varepsilon_0, \beta_1, \beta_2, \alpha_0\}$$

and

$$e^{\alpha \underline{\delta}} \sum_{j=1}^{m} (1 - \underline{p}_i^-) \Lambda_{ij} L_j e^{\alpha \tau_j} + \frac{(1 - \underline{p}_i^-) l_i^* e^{\alpha \underline{\delta}}}{(1 - p_i^*)(1 - e^{-(\sigma_i - \alpha)\underline{\delta}})} < 1, \qquad i = 1, \dots, m. \tag{3.4}$$

Note that such α exists since (H5) holds. Multiplying (3.3) with $e^{\alpha t}$ and using (H1) we get

$$\begin{aligned}
& e^{\alpha t}|x_i(t)-y_i(t)| \\
&\quad \le (1-\underline{p}_i^-)e^{-(\sigma_i-\alpha)t}|\boldsymbol{\varphi}-\boldsymbol{\psi}|_C \\
&\qquad + \int_0^t (1-\underline{p}_i^-)e^{\alpha t-\sigma_i(t-s)}\Big|G_i(s,\mathbf{x}(s))-G_i(s,\mathbf{y}(\gamma(s)))\Big|\,ds + e^{\alpha t}A_i(t) \\
&\quad \le (1-\underline{p}_i^-)|\boldsymbol{\varphi}-\boldsymbol{\psi}|_C \\
&\qquad + \sum_{j=1}^m \int_0^t (1-\underline{p}_i^-)e^{-(\sigma_i-\alpha)(t-s)}|b_{ij}(s)|L_j e^{\alpha s}\Big|x_j(s-\tau_j)-y_j(\gamma(s)-\gamma(\tau_j))\Big|\,ds \\
&\qquad + e^{\alpha t}A_i(t) \\
&\quad \le (1-\underline{p}_i^-)|\boldsymbol{\varphi}-\boldsymbol{\psi}|_C \\
&\qquad + \sum_{j=1}^m \int_0^t (1-\underline{p}_i^-)e^{-(\sigma_i-\alpha)(t-s)}|b_{ij}(s)|L_j e^{\alpha s}\Big(\Big|x_j(s-\tau_j)-x_j(\gamma(s)-\gamma(\tau_j))\Big| \\
&\qquad\quad + \Big|x_j(\gamma(s)-\gamma(\tau_j))-y_j(\gamma(s)-\gamma(\tau_j))\Big|\Big)ds + e^{\alpha t}A_i(t)
\end{aligned} \tag{3.5}$$

for $i=1,\dots,m$ and $t\ge 0$. We have

$$\begin{aligned}
A_i(t) \le{}& \sum_{r=1}^{\ell(t)}\left(\prod_{j=r+1}^{\ell(t)}(1-p_{i,j})\right)\left|e^{-\int_{t_r}^t a_i(u)\,du}-e^{-\int_{\gamma(t_r)}^t a_i(u)\,du}\right||J_{i,r}(x_i(t_r^-))| \\
&+ \sum_{r=1}^{\ell(t)}\left(\prod_{j=r+1}^{\ell(t)}(1-p_{i,j})\right)e^{-\int_{t_r}^t a_i(u)\,du}e^{-\int_{\gamma(t_r)}^{t_r} a_i(u)\,du} \\
&\times\Big|J_{i,r}(x_i(t_r^-))-J_{i,r}(y_i(\gamma(t_r)^-))\Big| \\
&+ \sum_{r=1}^{\ell(t)}\left(\prod_{j=r+1}^{\ell(t)}(1-p_{i,j})\right)\left|e^{-\int_{t_r}^t a_i(u)\,du}-e^{-\int_{\gamma(t_r)}^t a_i(u)\,du}\right||e_{i,r}|
\end{aligned} \tag{3.6}$$

for $i=1,\dots,m$ and $t\ge 0$. Assumption (H7), estimates $1-e^{-t}<t$ for $t>0$, the following direct consequence of $|t-\gamma(t)|<h$:

$$t-h<\gamma(t)\le t, \qquad t\in\mathbb{R}, \tag{3.7}$$

and (2.7) imply

$$\begin{aligned}
&\left(\prod_{j=r+1}^{\ell(t)}(1-p_{i,j})\right)\left|e^{-\int_{t_r}^t a_i(u)\,du}-e^{-\int_{\gamma(t_r)}^t a_i(u)\,du}\right| \\
&\qquad = e^{-\int_{t_r}^t a_i(u)\,du}\left(\prod_{j=r+1}^{\ell(t)}(1-p_{i,j})\right)\left(1-e^{-\int_{\gamma(t_r)}^{t_r} a_i(u)\,du}\right) \\
&\qquad \le \frac{1-\underline{p}_i^-}{1-p_i^*}e^{-\sigma_i(t-t_r)}(1-e^{-a_i^*h}) \\
&\qquad \le \frac{1-\underline{p}_i^-}{1-p_i^*}e^{-\sigma_i(t-t_r)}a_i^*h, \qquad t\ge t_r.
\end{aligned} \tag{3.8}$$

Then, combining (H2), (H3) (ii), (3.6), (3.7) and (3.8), we obtain

$$
\begin{aligned}
e^{\alpha t}A_i(t) \le & \sum_{r=1}^{\ell(t)} e^{-(\sigma_i-\alpha)(t-t_r)} \frac{(1-\underline{p}_i^-)a_i^* h l_i^*}{1-p_i^*} e^{\alpha t_r}|x_i(t_r^-)| \\
& + \sum_{r=1}^{\ell(t)} e^{-(\sigma_i-\alpha)(t-t_r)} \frac{(1-\underline{p}_i^-)l_i^*}{1-p_i^*} e^{\alpha t_r} \\
& \times \Big(|x_i(t_r^-) - x_i(\gamma(t_r)^-)| + |x_i(\gamma(t_r)^-) - y_i(\gamma(t_r)^-)| \Big) \\
& + \sum_{r=1}^{\ell(t)} e^{-(\sigma_i-\alpha)(t-t_r)} \frac{(1-\underline{p}_i^-)a_i^* h}{1-p_i^*} e^{\alpha t_r}|e_{i,r}|, \qquad i=1,\dots,m, \quad t\ge 0. \qquad (3.9)
\end{aligned}
$$

For $0 < h < \underline{\delta}$ we have $t_{r-1} < \gamma(t_r) \le t_r$, hence (2.22) and (3.7) yield

$$
e^{\alpha t_r}|x_i(t_r^-) - x_i(\gamma(t_r)^-)| = \lim_{n\to\infty} e^{\alpha\left(t_r - \frac{1}{n}\right)} \left| x_i\left(t_r - \frac{1}{n}\right) - x_i\left(\gamma(t_r) - \frac{1}{n}\right) \right| \le \omega_\alpha(h)
$$

for $h < \underline{\delta}$. Therefore (3.9), (2.10) with σ_i replaced by $\sigma_i - \alpha$, $e^{\alpha t_r}|x_i(t_r^-)| \le e^{\alpha_0 t_r}|x_i(t_r^-)| \le K_0$, $e^{\alpha t_r}|e_{i,r}| \le e_i^*$ and $e^{\alpha t_r} = e^{\alpha(t_r - \gamma(t_r))}e^{\alpha\gamma(t_r)} \le e^{\alpha h}e^{\alpha\gamma(t_r)}$ imply

$$
\begin{aligned}
e^{\alpha t}A_i(t) \le & \sum_{r=1}^{\ell(t)} e^{-(\sigma_i-\alpha)(t-t_r)} \frac{(1-\underline{p}_i^-)a_i^* h l_i^*}{1-p_i^*} K_0 \\
& + \sum_{r=1}^{\ell(t)} e^{-(\sigma_i-\alpha)(t-t_r)} \frac{(1-\underline{p}_i^-)l_i^*}{1-p_i^*} \Big(\omega_\alpha(h) + e^{\alpha h} \sup_{0\le u\le t} e^{\alpha u}|x_i(u) - y_i(u)| \Big) \\
& + \sum_{r=1}^{\ell(t)} e^{-(\sigma_i-\alpha)(t-t_r)} \frac{(1-\underline{p}_i^-)a_i^* h e_i^*}{1-p_i^*} \\
\le & \frac{(1-\underline{p}_i^-)(a_i^* h l_i^* K_0)}{(1-p_i^*)(1-e^{-(\sigma_i-\alpha)\underline{\delta}})} + \frac{(1-\underline{p}_i^-)l_i^*\omega_\alpha(h)}{(1-p_i^*)(1-e^{-(\sigma_i-\alpha)\underline{\delta}})} \\
& + \frac{(1-\underline{p}_i^-)a_i^* h e_i^*}{(1-p_i^*)(1-e^{-(\sigma_i-\alpha)\underline{\delta}})} \\
& + \frac{(1-\underline{p}_i^-)l_i^* e^{\alpha h}}{(1-p_i^*)(1-e^{-(\sigma_i-\alpha)\underline{\delta}})} \sup_{0\le u\le t} e^{\alpha u}|x_i(u) - y_i(u)| \\
\le & d_i^* h + \bar{d}_i\omega_\alpha(h) + \hat{d}_i \sup_{0\le u\le t} e^{\alpha u}|x_i(u) - y_i(u)| \qquad (3.10)
\end{aligned}
$$

for $t \ge 0$, $i = 1,\dots,m$ and $0 < h < \underline{\delta}$, where

$$
d_i^* = \frac{(1-\underline{p}_i^-)(a_i^* l_i^* K_0 + a_i^* e_i^*)}{(1-p_i^*)(1-e^{-(\sigma_i-\alpha)\underline{\delta}})}, \qquad \bar{d}_i = \frac{(1-\underline{p}_i^-)l_i^*}{(1-p_i^*)(1-e^{-(\sigma_i-\alpha)\underline{\delta}})},
$$

and

$$
\hat{d}_i = \frac{(1-\underline{p}_i^-)l_i^* e^{\alpha\underline{\delta}}}{(1-p_i^*)(1-e^{-(\sigma_i-\alpha)\underline{\delta}})}.
$$

We introduce the functions

$$
\eta_i(t,h) = \sum_{j=1}^{m} \int_0^t e^{-(\sigma_i-\alpha)(t-s)}|b_{ij}(s)|L_j e^{\alpha s}\Big| x_j(s-\tau_j) - x_j(\gamma(s) - \gamma(\tau_j)) \Big| ds \qquad (3.11)
$$

for $t \geq 0,\ h > 0$, and

$$w_i(t) = \max_{-\tau \leq u \leq t} e^{\alpha u}|x_i(u) - y_i(u)|, \qquad t \geq -\tau, \quad h > 0$$

for $i = 1, \ldots, m$. Then estimate (3.5) together with (3.10), (3.11) and

$$e^{\alpha s} = e^{\alpha(s-\gamma(s)+\gamma(\tau_j))} e^{\alpha(\gamma(s)-\gamma(\tau_j))} \leq e^{\alpha(h+\tau_j)} e^{\alpha(\gamma(s)-\gamma(\tau_j))} \tag{3.12}$$

yields

$$\begin{aligned}
e^{\alpha t}|x_i(t) - y_i(t)| &\leq (1 - \underline{p}_i^-)|\boldsymbol{\varphi} - \boldsymbol{\psi}|_C + (1 - \underline{p}_i^-)\eta_i(t,h) + e^{\alpha t} A_i(t) \\
&\quad + \sum_{j=1}^{m} \int_0^t (1 - \underline{p}_i^-) e^{-(\sigma_i - \alpha)(t-s)} |b_{ij}(s)| L_j e^{\alpha s} \\
&\qquad \times \Big| x_j(\gamma(s) - \gamma(\tau_j)) - y_j(\gamma(s) - \gamma(\tau_j)) \Big| ds \\
&\leq (1 - \underline{p}_i^-)|\boldsymbol{\varphi} - \boldsymbol{\psi}|_C + (1 - \underline{p}_i^-)\eta_i(t,h) + d_i^* h + \bar{d}_i \omega_\alpha(h) + \hat{d}_i w_i(t) \\
&\quad + \sum_{j=1}^{m} \int_0^t (1 - \underline{p}_i^-) e^{-(\sigma_i - \alpha)(t-s)} |b_{ij}(s)| L_j e^{\alpha(h+\tau_j)} w_j(s)\, ds \\
&\leq (1 - \underline{p}_i^-)|\boldsymbol{\varphi} - \boldsymbol{\psi}|_C + (1 - \underline{p}_i^-)\eta_i(t,h) + d_i^* h + \bar{d}_i \omega_\alpha(h) + \hat{d}_i w_i(t) \\
&\quad + \sum_{j=1}^{m} (1 - \underline{p}_i^-) \Lambda_{ij} L_j e^{\alpha(h_0+\tau_j)} w_j(t)
\end{aligned} \tag{3.13}$$

for $0 < h < \underline{\delta}$, $t \geq 0$ and $i = 1, \ldots, m$.

Define $h_1 = \underline{\delta}/2$, and next we suppose that $0 < h < h_1$, $j \in \{1, \ldots, m\}$ and $r \in \mathbb{N}$. Relation (3.7) implies

$$t_r \leq (s-h) - \tau_j \leq \gamma(s) - \gamma(\tau_j) \leq s - (\tau_j - h) < t_{r+1}, \qquad s \in [t_r + \tau_j + h, t_{r+1} + \tau_j - h).$$

Therefore

$$s - \tau_j \in [t_r, t_{r+1}) \quad \text{and} \quad \gamma(s) - \gamma(\tau_j) \in [t_r, t_{r+1}), \qquad s \in [t_r + \tau_j + h, t_{r+1} + \tau_j - h). \tag{3.14}$$

Moreover, (3.7) yields

$$|s - \tau_j - (\gamma(s) - \gamma(\tau_j))| = |s - \gamma(s) - (\tau_j - \gamma(\tau_j))| \leq h, \qquad s \in [t_r + \tau_j + h, t_{r+1} + \tau_j - h). \tag{3.15}$$

Hence it follows from (2.22), (3.14) and (3.15) that

$$\begin{aligned}
e^{\alpha s}|x_j(s - \tau_j) - x_j(\gamma(s) - \gamma(\tau_j))| &\leq e^{\alpha \tau_j} e^{\alpha(s-\tau_j)} |x_j(s - \tau_j) - x_j(\gamma(s) - \gamma(\tau_j))| \\
&\leq e^{\alpha \tau_j} \omega_\alpha(h), \qquad s \in [t_r + \tau_j + h, t_{r+1} + \tau_j - h).
\end{aligned} \tag{3.16}$$

Similarly, it is easy to check that

$$\begin{aligned}
e^{\alpha s}|x_j(s - \tau_j) - x_j(\gamma(s) - \gamma(\tau_j))| &\leq e^{\alpha \tau_j} e^{\alpha(s-\tau_j)} |x_j(s - \tau_j) - x_j(\gamma(s) - \gamma(\tau_j))| \\
&\leq e^{\alpha \tau_j} \omega_\alpha(h), \qquad s \in [0, t_1 + \tau_j - h).
\end{aligned} \tag{3.17}$$

We define the sets

$$\mathcal{A}_{j,h} = [0, t_1 + \tau_j - h] \cup \bigcup_{r=1}^{\infty} [t_r + \tau_j + h, t_{r+1} + \tau_j - h], \qquad \mathcal{B}_{j,h} = [0, \infty) \setminus \mathcal{A}_{j,h}$$

for $j = 1, \ldots, m$ and $0 < h < h_1$. We use relation (3.16) to estimate the function $\eta_i(t,h)$ defined by (3.11) for $i = 1, \ldots, m$, $t \geq 0$, $0 < h < h_1$, and for $s \in \mathcal{A}_{j,h}$:

$$\begin{aligned}
\eta_i(t,h) &= \sum_{j=1}^{m} \Big(\int_{\mathcal{A}_{j,h}\cap[0,t]} e^{-(\sigma_i-\alpha)(t-s)} |b_{ij}(s)| L_j e^{\alpha s} \Big| x_j(s-\tau_j) - x_j(\gamma(s)-\gamma(\tau_j)) \Big| \, ds \\
&\quad + \int_{\mathcal{B}_{j,h}\cap[0,t]} e^{-(\sigma_i-\alpha)(t-s)} |b_{ij}(s)| L_j e^{\alpha s} \Big| x_j(s-\tau_j) - x_j(\gamma(s)-\gamma(\tau_j)) \Big| \, ds \Big) \\
&\leq \sum_{j=1}^{m} \Big(\int_0^t e^{-(\sigma_i-\alpha)(t-s)} |b_{ij}(s)| L_j e^{\alpha\tau_j} \omega_\alpha(h) \, ds \\
&\quad + \int_{\mathcal{B}_{j,h}\cap[0,t]} e^{-(\sigma_i-\alpha)(t-s)} |b_{ij}(s)| L_j e^{\alpha s} \Big| x_j(s-\tau_j) - x_j(\gamma(s)-\gamma(\tau_j)) \Big| \, ds \Big). \qquad (3.18)
\end{aligned}$$

For $s \in \mathcal{B}_{j,h}$ we use a different estimate. Let $0 < h < h_2 = \min\{h_1, 1/\alpha_0\}$. We have from Lemma 2.2

$$\begin{aligned}
& e^{\alpha s} |x_j(s-\tau_j) - x_j(\gamma(s)-\gamma(\tau_j))| \\
&\quad \leq e^{\alpha\tau_j} e^{\alpha(s-\tau_j)} |x_j(s-\tau_j)| + e^{\alpha(s-\gamma(s)+\gamma(\tau_j))} e^{\alpha(\gamma(s)-\gamma(\tau_j))} |x_j(\gamma(s)-\gamma(\tau_j))| \\
&\quad \leq e^{\alpha\tau_j} K_0 e^{-(\alpha_0-\alpha)(s-\tau_j)} + e^{\alpha(s-\gamma(s)+\gamma(\tau_j))} K_0 e^{-(\alpha_0-\alpha)(\gamma(s)-\gamma(\tau_j))} \\
&\quad \leq e^{\alpha\tau_j} K_0 e^{-(\alpha_0-\alpha)(s-\tau_j)} + e^{\alpha(h+\tau_j)} K_0 e^{-(\alpha_0-\alpha)(s-h-\tau_j)} \\
&\quad \leq e^{\alpha\tau_j} (1 + e^{\alpha_0 h}) K_0 e^{-(\alpha_0-\alpha)(s-\tau_j)} \\
&\quad \leq e^{\alpha\tau_j} (1+e) K_0 e^{-(\alpha_0-\alpha)(s-\tau)}, \qquad j = 1, \ldots, m, \quad s \geq 0. \qquad (3.19)
\end{aligned}$$

Fix $\varepsilon > 0$. Then it follows from (3.19) that there exists $T = T(\varepsilon, \alpha)$ such that

$$e^{\alpha s} |x_j(s-\tau_j) - x_j(\gamma(s)-\gamma(\tau_j))| \leq e^{\alpha\tau_j} \frac{\varepsilon}{M_2}, \qquad s \geq T, \quad j = 1, \ldots, m, \qquad (3.20)$$

where $M_2 = \max_{i=1,\ldots,m} \sum_{j=1}^{m} \Lambda_{ij} L_j e^{\alpha\tau_j}$. Let $k_0 \in \mathbb{N}$ be the smallest index such that $t_{k_0} \geq T$. We recall that $h_2 \leq \underline{\delta}/2$. So if $1 \leq r \leq k_0$, then for $s \in [t_r + \tau_j - h, t_r + \tau_j + h]$ and $0 < h < h_2$ we have

$$s - \tau_j \leq t_{k_0} + h < t_{k_0+1} \quad \text{and} \quad \gamma(s) - \gamma(\tau_j) \leq s - \tau_j + h \leq t_{k_0} + 2h < t_{k_0+1}. \qquad (3.21)$$

Similarly, for $r > k_0$, $s \in [t_r + \tau_j - h, t_r + \tau_j + h]$ and $0 < h < h_2$ we get

$$s - \tau_j \geq t_r - h > t_{r-1} \geq t_{k_0} \geq T, \quad \text{and} \quad \gamma(s) - \gamma(\tau_j) \geq s - h - \tau_j \geq t_r - 2h > t_{r-1} \geq t_{k_0}. \qquad (3.22)$$

Define the sets

$$\mathcal{C}_{j,h} = \bigcup_{r=1}^{k_0} [t_r + \tau_j - h, t_r + \tau_j + h] \quad \text{and} \quad \mathcal{D}_{j,h} = \bigcup_{r=k_0+1}^{\infty} [t_r + \tau_j - h, t_r + \tau_j + h].$$

Then, clearly, $\mathcal{B}_{j,h} = \mathcal{C}_{j,h} \cup \mathcal{D}_{j,h}$, $j = 1, \ldots, m$. Define the constants $\tilde{b}_{ij} = \tilde{b}_{ij}(\varepsilon)$ by

$$\tilde{b}_{ij} = \max_{0 \leq u \leq t_{k_0+1}} |b_{ij}(u)|, \qquad i, j = 1, \ldots, m.$$

Then (H3), (3.18), (3.19), (3.20), (3.21) and (3.22) yield

$$\begin{aligned}\eta_i(t,h) &\le \sum_{j=1}^{m}\Big(\beta^*_{ij}L_j e^{\alpha\tau_j}\omega_\alpha(h)\\&\quad+\int_{\mathcal{C}_{j,h}\cap[0,t]} e^{-(\sigma_i-\alpha)(t-s)}|b_{ij}(s)|L_j e^{\alpha s}\Big|x_j(s-\tau_j)-x_j(\gamma(s)-\gamma(\tau_j))\Big|\,ds\\&\quad+\int_{\mathcal{D}_{j,h}\cap[0,t]} e^{-(\sigma_i-\alpha)(t-s)}|b_{ij}(s)|L_j e^{\alpha s}\Big|x_j(s-\tau_j)-x_j(\gamma(s)-\gamma(\tau_j))\Big|\,ds\Big)\\&\le M_2\omega_\alpha(h)+\sum_{j=1}^{m}\sum_{r=1}^{\ell(t)}\int_{t_r+\tau_j-h}^{t_r+\tau_j+h} e^{-(\sigma_i-\alpha)(t-s)}\tilde{b}_{ij}L_j e^{\alpha\tau_j}(1+e)K_0\,ds\\&\quad+\sum_{j=1}^{m}\int_0^t e^{-(\sigma_i-\alpha)(t-s)}|b_{ij}(s)|L_j e^{\alpha\tau_j}\frac{\varepsilon}{M_2}\,ds\\&\le M_2\omega_\alpha(h)+\sum_{j=1}^{m}\sum_{r=1}^{\ell(t)}\int_{t_r+\tau_j-h}^{t_r+\tau_j+h} e^{-(\sigma_i-\alpha)(t-s)}\tilde{b}_{ij}L_j e^{\alpha\tau_j}(1+e)K_0\,ds+\varepsilon\end{aligned}\tag{3.23}$$

for $t \ge 0$ and $0 < h < h_2$. Relation (2.25) gives

$$e^t - e^{-t} = e^{-t}(e^{2t}-1) \le (e-1)2t, \qquad t \in [0,1],$$

hence, using (H7), (2.10) with σ_i is replaced by $\sigma_i - \alpha$, and (3.23), we get for $i = 1,\dots,m$

$$\begin{aligned}\eta_i(t,h) &\le M_2\omega_\alpha(h)+\varepsilon\\&\quad+\sum_{j=1}^{m}\tilde{b}_{ij}L_j e^{\alpha\tau}(1+e)K_0\sum_{r=1}^{\ell(t)}\left(\frac{e^{-(\sigma_i-\alpha)(t-t_r-\tau_j-h)}-e^{-(\sigma_i-\alpha)(t-t_r-\tau_j+h)}}{\sigma_i-\alpha}\right)\\&= M_2\omega_\alpha(h)+\varepsilon\\&\quad+\sum_{j=1}^{m}\tilde{b}_{ij}L_j e^{\alpha\tau}\left(\frac{(1+e)K_0}{\sigma_i-\alpha}\right)\left(e^{(\sigma_i-\alpha)(\tau_j+h)}-e^{(\sigma_i-\alpha)(\tau_j-h)}\right)\sum_{\ell=1}^{\ell(t)}e^{-(\sigma_i-\alpha)(t-t_\ell)}\\&\le M_2\omega_\alpha(h)+\varepsilon+\sum_{j=1}^{m}\tilde{b}_{ij}L_j e^{\alpha\tau}\left(\frac{(1+e)K_0}{\sigma_i-\alpha}\right)\left(\frac{e^{(\sigma_i-\alpha)h}-e^{-(\sigma_i-\alpha)h}}{1-e^{-(\sigma_i-\alpha)\underline{\delta}}}\right)\\&\le M_2\omega_\alpha(h)+\varepsilon+\tilde{g}_i h, \qquad 0<h<h^*,\end{aligned}\tag{3.24}$$

where $\tilde{g}_i = \tilde{g}_i(\varepsilon)$ is defined by

$$\tilde{g}_i = \sum_{j=1}^{m}\tilde{b}_{ij}L_j e^{\sigma_i\tau}\frac{(1+e)K_0 2(e-1)}{1-e^{-(\sigma_i-\alpha)\underline{\delta}}} \quad\text{and}\quad h^* = \min\left\{h_2, \frac{1}{\sigma_1-\alpha},\dots,\frac{1}{\sigma_m-\alpha}\right\}.$$

Then (3.13) yields for $i = 1,\dots,m$, $t \ge 0$ and $0 < h < h^*$

$$\begin{aligned}w_i(t) &\le (1-\underline{p}_i^-)\Big(|\boldsymbol{\varphi}-\boldsymbol{\psi}|_C + M_2\omega_\alpha(h)+\tilde{g}_i h+\varepsilon\Big)+d_i^* h+\bar{d}_i\omega_\alpha(h)\\&\quad+\sum_{j=1}^{m}(1-\underline{p}_i^-)\Lambda_{ij}L_j e^{\alpha(h_0+\tau_j)}w_j(t)+\hat{d}_i w_i(t).\end{aligned}\tag{3.25}$$

Relation (3.25) gives the vector inequality

$$\mathbf{w}(t) \le \mathbf{d}(h) + C\mathbf{w}(t), \qquad t \ge 0, \quad 0 < h < h^*, \tag{3.26}$$

where

$$\begin{aligned}
\mathbf{w}(t) &= (w_1(t),\dots,w_m(t))^T,\\
\mathbf{d}(h) &= (d_1(h),\dots,d_m(h))^T, \quad \text{where}\\
d_i(h) &= (1-\underline{p}_i^-)\Big(|\boldsymbol{\varphi}-\boldsymbol{\psi}|_C + M_2\omega_\alpha(h) + \tilde{g}_i h + \varepsilon\Big) + d_i^* h + \bar{d}_i\omega_\alpha(h),\\
C &= (c_{ij}) \in \mathbb{R}^{m\times m}, \qquad c_{ij} = \begin{cases} (1-\underline{p}_i^-)\Lambda_{i,i}L_i e^{\alpha(h_0+\tau_i)} + \hat{d}_i, & i=j,\\ (1-\underline{p}_i^-)\Lambda_{ij}L_j e^{\alpha(h_0+\tau_j)}, & i\neq j. \end{cases}
\end{aligned} \tag{3.27}$$

Relation (3.4) yields that $\|C\|_\infty < 1$ for $0 < h < h^*$. Then $I - C$ is an M-matrix, hence (3.26) implies

$$\mathbf{w}(t) \le (I-C)^{-1}\mathbf{d}(h), \qquad t \ge 0, \quad 0 < h < h^*.$$

Therefore, the definitions of $\mathbf{w}(t)$, $\mathbf{d}(h)$ and

$$w_i(t) \le \max_{-\tau\le u\le t} |x_i(u) - y_i(u)| \le |\boldsymbol{\varphi}-\boldsymbol{\psi}|_C \le d_i(h), \qquad t\in[-\tau,0], \quad i=1,\dots,m$$

gives

$$\mathbf{w}(t) \le (I-C)^{-1}\mathbf{d}(h), \qquad t \ge -\tau, \quad 0 < h < h^*,$$

which yields (3.1) with

$$K_2 = \|(I-C)^{-1}\|_\infty \max_{i=1,\dots,m}(1-\underline{p}_i^-) \quad \text{and} \quad \theta(h) = (M_2 + \max_{i=1,\dots,m}\bar{d}_i)\omega_\alpha(h) + h\max_{i=1,\dots,m}(\tilde{g}_i + d_i^*).$$

(ii) To prove (3.2) we now assume (H8) too.

Let M_0 and u_0 be the constants defined by Lemma 2.6 (ii). We consider estimate (3.18) of the proof of part (i). Now, since $b_{ij}(s)$ is bounded by b_{ij}^* for all $s \ge 0$, we estimate the last integral similarly to steps used for the set $C_{j,h}$ in the proof of part (i), but using b_{ij}^* instead of $\tilde{b}_{ij}$:

$$\eta_i(t,h) \le M_2\omega_\alpha(h) + \sum_{j=1}^m \sum_{r=1}^{\ell(t)} \int_{t_r+\tau_j-h}^{t_r+\tau_j+h} e^{-(\sigma_i-\alpha)(t-s)} b_{ij}^* L_j e^{\alpha\tau_j}(1+e)K_0\,ds.$$

Then a calculation similar to that used in (3.24) and (2.24) gives

$$\eta_i(t,h) \le M_2\omega_\alpha(h) + g_i^* h \le M_2 M_0 h + g_i^* h, \qquad 0 < h < \bar{h}, \tag{3.28}$$

where

$$g_i^* = \sum_{j=1}^m b_{ij}^* L_j e^{\sigma_i\tau}\left(\frac{(1+e)K_0 2(e-1)}{1-e^{-(\sigma_i-\alpha)\underline{\delta}}}\right) \quad \text{and} \quad \bar{h} = \min\{u_0, h^*\}.$$

Combining (2.24), (3.13) and (3.28) we get for $i = 1,\dots,m$, $t \ge 0$ and $0 < h < \bar{h}$

$$\begin{aligned}
w_i(t) \le{}& (1-\underline{p}_i^-)\Big(|\boldsymbol{\varphi}-\boldsymbol{\psi}|_C + M_2M_0h + g_i^*h\Big) + d_i^*h + \bar{d}_iM_0h\\
&+ \sum_{j=1}^m (1-\underline{p}_i^-)\Lambda_{ij}L_je^{\alpha(h_0+\tau_j)}w_j(t) + \hat{d}_iw_i(t),
\end{aligned}$$

hence, the vector inequality

$$\mathbf{w}(t) \le \hat{\mathbf{d}}(h) + C\mathbf{w}(t), \qquad t \ge 0, \quad 0 < h < \bar{h}$$

and therefore

$$\mathbf{w}(t) \le \hat{\mathbf{d}}(h) + C\mathbf{w}(t), \qquad t \ge -\tau, \quad 0 < h < \bar{h} \tag{3.29}$$

holds, where

$$\hat{\mathbf{d}}(h) = (\hat{d}_1(h), \dots, \hat{d}_m(h))^T,$$
$$\hat{d}_i(h) = (1 - \underline{p}_i^-)|\boldsymbol{\varphi} - \boldsymbol{\psi}|_C + \Big((1 - \underline{p}_i^-)(M_2 M_0 + g_i^*) + d_i^* + \bar{d}_i M_0\Big)h.$$

Then (3.29) implies

$$\mathbf{w}(t) \le (I - C)^{-1}\hat{\mathbf{d}}(h), \qquad t \ge -\tau, \quad 0 < h < \bar{h},$$

which proves (3.2) with

$$K_2 = \|(I - C)^{-1}\|_\infty \max_{i=1,\dots,m} (1 - \underline{p}_i^-) \quad \text{and} \quad M = M_2 M_0 + \max_{i=1,\dots,m} (g_i^* + d_i^* + \bar{d}_i M_0). \qquad \square$$

Remark 3.2. We note that relation (3.1) gives for $\boldsymbol{\varphi} = \boldsymbol{\psi}$ that

$$\sup_{t\in[-\tau,\infty)} |\mathbf{x}(t) - \mathbf{y}(t)|_\infty \le K_2(\theta(h) + \varepsilon),$$

which yields that the solutions of (1.15) approximate that of (1.14) uniformly on $[0, \infty)$.

Remark 3.3. Let C be the matrix defined by (3.27). Assumption (H5) in Theorem 3.1 can be replaced by the weaker condition $\rho(C) < 1$, and the statement of the theorem remains true.

Thanks to our main Theorem 3.1 and Lemma 2.2 we can give the following result concerning the *transference of the exponential estimate* of the solutions of (1.14) to the approximate solution.

Proposition 3.4. *Suppose (H1)–(H7) hold. Let* $\mathbf{x} = (x_1, \dots, x_m)^T$ *be the solution of* (1.14) *corresponding to initial function* $\boldsymbol{\varphi} = (\varphi_1, \dots, \varphi_m)^T$*, and* $\mathbf{y} = (y_1, \dots, y_m)^T$ *be the solution of* (1.15) *corresponding to* $h > 0$ *and an initial function* $\boldsymbol{\psi} = (\psi_1, \dots, \psi_m)^T$*, and let* $\alpha_0 > 0$ *be the corresponding constant from Lemma* 2.2.

(i) Then for every $0 < \alpha < \alpha_0$ *and* $\varepsilon > 0$ *there exist constants* $K_3 > 0$ *and* $h^* > 0$*, and a function* $\theta(h)$ *such that* $\theta(h) \to 0$ *as* $h \to 0+$*, and*

$$|\mathbf{y}(t)|_\infty \le e^{-\alpha t} K_3 \Big(1 + |\boldsymbol{\varphi} - \boldsymbol{\psi}|_C + \theta(h) + \varepsilon\Big), \qquad t \in [-\tau, \infty), \quad 0 < h < h^*. \tag{3.30}$$

(ii) Assume further (H8). Then for every $0 < \alpha < \alpha_0$ *there exist constants* $K_3 > 0$*,* $M > 0$ *and* $\bar{h} > 0$ *such that*

$$|\mathbf{y}(t)|_\infty \le e^{-\alpha t} K_3 \Big(1 + |\boldsymbol{\varphi} - \boldsymbol{\psi}|_C + Mh\Big), \qquad t \in [-\tau, \infty), \quad 0 < h < \bar{h}. \tag{3.31}$$

Proof. The proof follows immediately from $|\mathbf{z}(t)|_\infty \le |\mathbf{z}(t) - \mathbf{x}(t)|_\infty + |\mathbf{x}(t)|_\infty$, (2.4), (3.1) and (3.2) with $K_3 = \max\{K_0, K_2\}$. $\square$

4 The bounded coefficients case

In the following, we give a practical result concerning the bounded coefficients case for a CNN delayed impulsive system as a simple consequence of Theorem 3.1. In addition to our assumptions (H1)–(H3), we suppose that the coefficient functions $a_i(t)$ are bounded below too, and $b_{ij}(t)$ are bounded. This will allow to simplify conditions (H4)–(H8), as follows:

(H4′) There exist positive constants $\underline{a}_i, a_i^*, c_i^*, \sigma_i$ for $i = 1, \ldots, m$ such that

(i) $\underline{a}_i \le a_i(t) \le a_i^*$ for $t \ge 0$ and $i = 1, \ldots, m$;

(ii) $\underline{a}_i - \frac{1}{\underline{\delta}} \ln(1 - \underline{p}_i^-) \ge \sigma_i$, $i = 1, \ldots, m$, where $\underline{p}_i^- = \min\{0, \underline{p}_i\}$;

(iii) $|b_{ij}(t)| \le b_{ij}^*$, $t \ge 0$, $i = 1, \ldots, m$.

(H5′) There exists ε_0 such that $0 < \varepsilon_0 < \sigma_i$ for $i = 1, \ldots, m$, and

$$\sum_{j=1}^{m} \frac{(1 - \underline{p}_i^-) b_{ij}^*}{\sigma_i - \varepsilon_0} L_j + \frac{(1 - \underline{p}_i^-) l_i^*}{(1 - p_i^*)(1 - e^{-\sigma_i \underline{\delta}})} < 1, \qquad i = 1, \ldots, m.$$

(H6′) There exist positive constants β_1, β_2 and c_i^* $(i = 1, \ldots, m)$ such that

$$|e_{i,k}| \le e^{-\beta_1 t_k} e_i^*, \qquad k \in \mathbb{N}, \qquad \text{and} \qquad |c_i(t)| \le e^{-\beta_2 t} c_i^*, \qquad t \ge 0, \quad i = 1, \ldots, m.$$

(H7′) There exists a positive constant L_φ such that

$$|\varphi_i(t) - \varphi_i(\bar{t})| \le L_\varphi |t - \bar{t}|, \qquad t, \bar{t} \in [-\tau, 0], \quad i = 1, \ldots, m.$$

Note that (H4) (i) was used in the proofs of Lemma 2.2, Lemma 2.4 and Theorem 3.1 to prove estimates (2.5), (2.6) and (2.7). Now we show that our boundedness assumptions (H4′) imply the same estimates (2.5), (2.6) and (2.7).

Suppose $\underline{p}_i < 0$, $s \in [t_{r-1}, t_r)$ for some $r \in \mathbb{N}$, and $t \ge t_r$. Then (H3) (iv), (H4′) and the estimates $(\ell(t) - r)\underline{\delta} \le t_{\ell(t)} - t_r \le t - s$ yield

$$\begin{aligned} e^{-\int_s^t a_i(u)\,du} \left(\prod_{j=r}^{\ell(t)} (1 - p_{i,j}) \right) &= \exp\left(-\int_s^t a_i(u)\,du + \sum_{j=r}^{\ell(t)} \ln(1 - p_{i,j}) \right) \\ &\le \exp\left(-\underline{a}_i (t - s) + (\ell(t) - r + 1) \ln(1 - \underline{p}_i) \right) \\ &\le (1 - \underline{p}_i) \exp\left(-\underline{a}_i (t - s) + \frac{t - s}{\underline{\delta}} \ln(1 - \underline{p}_i) \right) \\ &\le (1 - \underline{p}_i) e^{-\sigma_i (t - s)}. \end{aligned}$$

If $s < t < t_r$, then $\ell(t) < r$, so $\prod_{j=r}^{\ell(t)} (1 - p_{i,j}) = 1$. In the case $\underline{p}_i \ge 0$ we have $\sigma_i \le \underline{a}_i$, hence

$$e^{-\int_s^t a_i(u)\,du} \left(\prod_{j=r}^{\ell(t)} (1 - p_{i,j}) \right) \le e^{-\int_s^t a_i(u)\,du} \le e^{-\sigma_i (t - s)}.$$

Therefore (2.5) holds. (2.6) and (2.7) can be proved similarly under assumption (H4′).

Using Remark 1.3, we get immediately that (H5′) implies (H5). Hence Theorem 3.1 has the following corollary.

Corollary 4.1. *Assume (H1)–(H3),(H4′)–(H7′) hold. Let* $\mathbf{x} = (x_1, \dots, x_m)^T$ *be the solution of* (1.14) *corresponding to initial function* $\boldsymbol{\varphi} = (\varphi_1, \dots, \varphi_m)^T$*, and* $\mathbf{y} = (y_1, \dots, y_m)^T$ *be the solution of* (1.15) *corresponding to* $h > 0$ *and an initial function* $\boldsymbol{\psi} = (\psi_1, \dots, \psi_m)^T$*, and let* $\alpha_0 > 0$ *be the corresponding constant from Lemma 2.2. Then for every* $0 < \alpha < \alpha_0$ *there exist constants* $K_2 > 0$*,* $M > 0$ *and* $\bar{h} > 0$ *such that* (3.2) *holds. Hence, the solutions of* (1.14) *are approximated by that of* (1.15) *uniformly over* $[0, \infty)$*.*

5 An example

Now, we present an example to illustrate the applicability of our conditions.

Example 5.1. Consider the system

$$\begin{aligned}
x_1'(t) &= -a_1(t)x_1(t) + \sum_{j=1}^{2} b_{1j}(t)g_j(x_j(t-\tau_j)) + c_1(t), \quad && t \neq t_n \\
x_2'(t) &= -a_2(t)x_2(t) + \sum_{j=1}^{2} b_{2j}(t)g_j(x_j(t-\tau_j)) + c_2(t), \quad && t \neq t_n \\
\Delta x_1(t_n) &= -p_{1,n}x_1(t_n^-) + e_{1,n} + J_{1,n}(x_1(t_n^-)), && n \in \mathbb{N} \\
\Delta x_2(t_n) &= -p_{2,n}x_2(t_n^-) + e_{2,n} + J_{2,n}(x_2(t_n^-)), && n \in \mathbb{N},
\end{aligned} \tag{5.1}$$

where

$$\begin{aligned}
a_1(t) &= 2 + \sin(\sqrt{3}t), & a_2(t) &= 4 + \cos(t); \\
b_{11}(t) &= 0.5\sin(t), & b_{21}(t) &= 0.2\sin(t); \\
b_{12}(t) &= 0.3\cos(t), & b_{22}(t) &= 0.3\sin(t); \\
c_1(t) &= \exp(-t), & c_2(t) &= \exp(-2t); \\
p_{1,n} &= -0.15,\ n \in \mathbb{N}, & p_{2,n} &= -0.4,\ n \in \mathbb{N}; \\
e_{1,n} &= \exp(-3t_n),\ n \in \mathbb{N}, & e_{2,n} &= \exp(-4t_n),\ n \in \mathbb{N}; \\
g_1(x) &= \tanh(x), & g_2(x) &= \tanh(x); \\
J_{1,n}(x) &= \tfrac{1}{10}\tanh(x),\ n \in \mathbb{N}, & J_{2,n}(x) &= \tfrac{1}{10}\tanh(x),\ n \in \mathbb{N}.
\end{aligned} \tag{5.2}$$

We suppose $\tau_1 = 1$, $\tau_2 = 2$, $\tau = \max\{\tau_1, \tau_2\} = 2$, the initial functions $\varphi_1(t), \varphi_2(t), \psi_1(t), \psi_2(t) : [-2, 0] \to \mathbb{R}$, defined as $\varphi_1(t) = \psi_1(t) = 0.5\sin(t)$ and $\varphi_2(t) = \psi_2(t) = \cos(t)$. Also, we consider $t_n = n$ for $n \in \mathbb{N}$.

System (5.1) is approximated by the following IDEPCA system

$$\begin{aligned}
y_1'(t) &= -a_1(t)y_1(t) + \sum_{j=1}^{2} b_{1j}(t)g_j(y_j(\gamma(t) - \gamma(\tau_j))) + c_1(t), \quad && t \neq \gamma(t_n) \\
y_2'(t) &= -a_2(t)y_2(t) + \sum_{j=1}^{2} b_{2j}(t)g_j(y_j(\gamma(t) - \gamma(\tau_j))) + c_2(t), \quad && t \neq \gamma(t_n) \\
\Delta y_1(\gamma(t_n)) &= -p_{1,n}y_1(\gamma(t_n)^-) + e_{1,n} + I_{1,n}(y_1(\gamma(t_n)^-)), && n \in \mathbb{N} \\
\Delta y_2(\gamma(t_n)) &= -p_{2,n}y_2(\gamma(t_n)^-) + e_{2,n} + I_{2,n}(y_2(\gamma(t_n)^-)), && n \in \mathbb{N},
\end{aligned} \tag{5.3}$$

where $\gamma(t) = [t/h]h$ and all the coefficients are given in (5.2). Because $\tanh(x)$ is a Lipschitz-type function, with Lipschitz constant 1, we can conclude that $L_i = 1$ and $l_i^* = l_{i,n} = \frac{1}{10}$, $i = 1, 2,\ n \in \mathbb{N}$. We have $\underline{\delta} = 1$, $\underline{a}_1 = 1$, $\underline{a}_2 = 3$, $\underline{p}_1 = p_1^* = -0.15$ and $\underline{p}_2 = p_2^* = -0.4$, hence we get

$$\underline{a}_1 - \frac{1}{\underline{\delta}} \ln(1 - \underline{p}_1^-) \approx 0.86024, \qquad \underline{a}_2 - \frac{1}{\underline{\delta}} \ln(1 - \underline{p}_2^-) \approx 2.66353,$$

so $\sigma_1 = 0.86$ and $\sigma_2 = 2.66$ satisfy (H4′) (ii). Use $\varepsilon_0 = 0.01$, $b_{11}^* = 0.5$, $b_{12}^* = 0.3$, $b_{21}^* = 0.2$ and $b_{22}^* = 0.3$. Then

$$\frac{1 - \underline{q}_1^-}{\sigma_1 - \varepsilon_0} \sum_{j=1}^{m} b_{1j}^* L_j + \frac{(1 - \underline{q}_1^-) l_1^*}{(1 - q_1^*)(1 - e^{-\sigma_1 \underline{\delta}})} = \frac{1.15}{0.85}(0.5 + 0.3) + \frac{0.1}{1 - e^{-0.86}} \approx 0.91748,$$

and

$$\frac{1 - \underline{q}_2^-}{\sigma_2 - \varepsilon_0} \sum_{j=1}^{m} b_{2j}^* L_j + \frac{(1 - \underline{q}_2^-) l_1^*}{(1 - q_2^*)(1 - e^{-\sigma_2 \underline{\delta}})} = \frac{1.4}{0.85}(0.2 + 0.3) + \frac{0.1}{1 - e^{-2.66}} \approx 0.31884,$$

therefore (H5′) is satisfied. Therefore Corollary 4.1 yields that the solutions of (5.3) approximate that of (5.1) uniformly on $[0, \infty)$ as h goes to 0.

Figures 5.1–5.2 illustrate the solution of (5.1) and its approximation by the solution of (5.3) corresponding to the discretization parameter $h = 0.1$. Note that for this value of h and the definition of t_n, we have $\gamma(t_n) = t_n$ for all $n \in \mathbb{N}$. Both initial value problems are solved numerically using the function `ddesd` in *Matlab* on the consecutive intervals $[t_n, t_{n+1}]$. The blue curves are the graphs of $x_i(t)$, and the red dots are the values of the function $y_i(t)$ at the time values $t = 0.1n$, $n \in \mathbb{N}_0$. At the impulse time points, the left-hand limits of $y_i(t)$ are also displayed. Even though the discretization parameter is relatively large for this numerical experience, we see that the approximation error becomes smaller as time increases. This is a consequence of estimate (3.2).

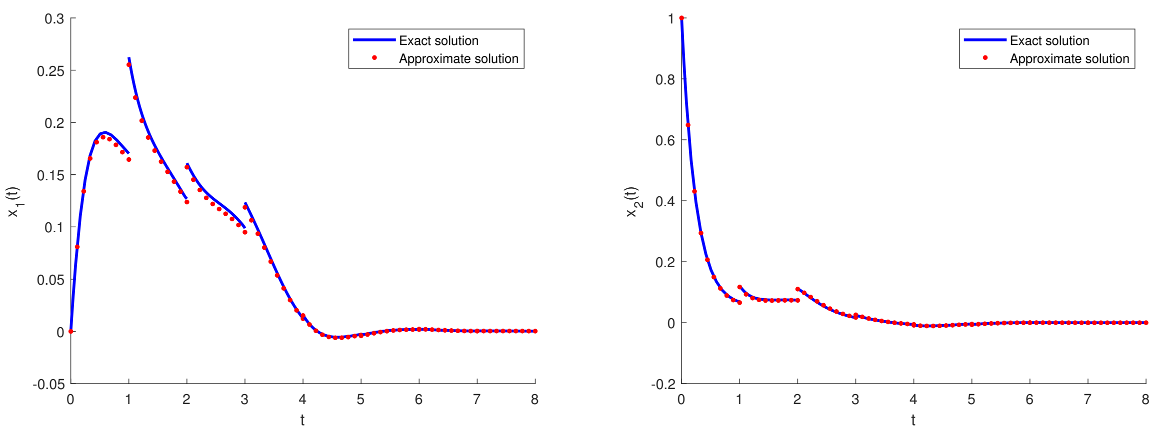

Figure 5.1: Graphs of $x_1(t), y_1(0.1n)$ (on the left) and $x_2(t), y_2(0.1n)$ (on the right) with $h = 0.1$

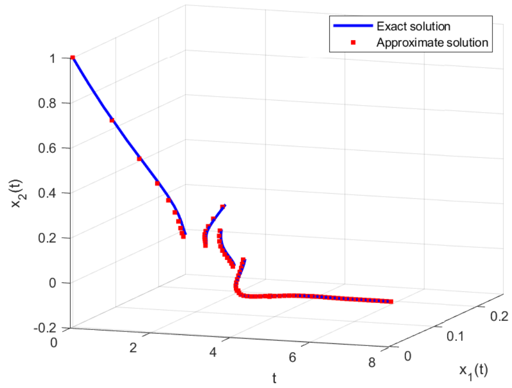

Figure 5.2: The graphs of $(t, x_1(t), x_2(t))$ and $(0.1n, z_1(0.1n), z_2(0.1n))$ with $h = 0.1$.

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