Annales Mathematicae et Informaticae
60 (2024) pp. 44–53
DOI: 10.33039/ami.2024.03.005
URL: https://ami.uni-eszterhazy.hu

A note on the exponential Diophantine equation $(a^x - 1)(b^y - 1) = az^2$

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Abstract. Let a, b be fixed positive integers such that $(a \bmod 8, b \bmod 8) \in \{(0,3),(0,5),(2,3),(2,5),(4,3),(6,5)\}$. In this paper, using elementary methods with some classical results for Diophantine equations, we prove the following three results: (i) The equation $(*)$ $(a^x - 1)(b^y - 1) = az^2$ has no positive integer solutions (x, y, z) with $2 \nmid x$ and $x > 1$. (ii) If $a = 2$ and $b \equiv 5 \pmod 8$, then $(*)$ has no positive integer solutions (x, y, z) with $2 \nmid x$. (iii) If $a = 2$ and $b \equiv 3 \pmod 8$, then the positive integer solutions (x, y, z) of $(*)$ with $2 \nmid x$ are determined. These results improve the recent results of R.-Z. Tong: On the Diophantine equation $(2^x - 1)(p^y - 1) = 2z^2$, Czech. Math. J. 71 (2021), 689–696. Moreover, under the assumption that a is a square, we prove that $(*)$ has no positive integer solutions (x, y, z) even with $2 \mid x$ in some cases.

Keywords: polynomial-exponential Diophantine equation, Pell's equation, generalized Ramanujan-Nagell equation

AMS Subject Classification: 11D61

1. Introduction

Let $\mathbb{N}$ be the set of all positive integers. Let a, b be fixed positive integers with $\min\{a, b\} > 1$. In 2000, L. Szalay [7] completely solved the equation

$$(2^x - 1)(3^x - 1) = z^2, \quad x, z \in \mathbb{N}. \tag{1.1}$$

Submitted: November 25, 2021
Accepted: March 22, 2024
Published online: March 24, 2024

He proved that (1.1) has no solutions (x, z). Since then, this result has led to a series of related studies for the equation

$$(a^x - 1)(b^x - 1) = z^2, \quad x, z \in \mathbb{N} \tag{1.2}$$

(see [3]). Obviously, the solution of (1.2) involves a system of generalized Ramanujan-Nagell equations. Recently, R.-Z. Tong [8] discussed the equation

$$(2^x - 1)(p^y - 1) = 2z^2, \quad x, y, z \in \mathbb{N}, \tag{1.3}$$

where p is an odd prime with $p \equiv \pm 3 \pmod 8$. He proved the following two results: (i) (1.3) has no solutions (x, y, z) with $2 \nmid x$, $2 \mid y$ and $y > 4$. (ii) If $p \neq 2g^2 + 1$, where g is an odd positive integer, then (1.3) has no solutions (x, y, z) with $2 \nmid x$. In this paper, we will discuss the generalized form of (1.3) as follows:

$$(a^x - 1)(b^y - 1) = az^2, \quad x, y, z \in \mathbb{N}. \tag{1.4}$$

For any positive integer n, let r_n, s_n be the positive integers satisfying

$$r_n + s_n\sqrt{2} = \left(3 + 2\sqrt{2}\right)^n. \tag{1.5}$$

For any odd positive integer m, let R_m, S_m be the positive integers satisfying

$$R_m + S_m\sqrt{2} = \left(1 + \sqrt{2}\right)^m. \tag{1.6}$$

Using elementary methods with some classical results for Diophantine equations, we prove the following results:

Theorem 1.1. *If*

$$(a \bmod 8, b \bmod 8) \in \{(0,3), (0,5), (2,3), (2,5), (4,3), (6,5)\}, \tag{1.7}$$

then (1.4) *has no solutions* (x, y, z) *with* $2 \nmid x$ *and* $x > 1$.

Theorem 1.2. *If* $a = 2$ *and* $b \equiv 5 \pmod 8$, *then* (1.4) *has no solutions* (x, y, z) *with* $2 \nmid x$. *If* $a = 2$ *and* $b \equiv 3 \pmod 8$, *then* (1.4) *has only the following solutions* (x, y, z) *with* $2 \nmid x$:

(i) $b = 3$, $(x, y, z) = (1, 1, 1)$, $(1, 2, 2)$ *and* $(1, 5, 11)$.

(ii) $b = 2g^2 + 1$, $(x, y, z) = (1, 1, g)$, *where* g *is an odd positive integer with* $g > 1$.

(iii) $b = r_m$, $(x, y, z) = (1, 2, s_m)$, *where* m *is an odd positive integer with* $m > 1$.

Theorem 1.3. *Let* $N(a, b)$ *denote the number of solutions* (x, y, z) *of* (1.4) *with* $2 \nmid x$. *If* $a = 2$ *and* $b \equiv 3 \pmod 8$, *then*

$$N(2, b) = \begin{cases} 3, & \text{if } b = 3, \\ 2, & \text{if } b = 2g^2 + 1 \text{ and } g = R_m \text{ with } m > 1, \\ 1, & \text{if } b = 2g^2 + 1 \text{ and } g \neq R_m, \\ 0, & \text{otherwise.} \end{cases}$$

Obviously, the above theorems improve the result of [8].

The following results concern the solvability of (1.4) including even the case where $2 \mid x$.

Theorem 1.4. *If* $(a \bmod 8, b \bmod 8) \in \{(0,3),(0,5),(4,3)\}$ *and* a *is a square, then* (1.4) *has no solutions* (x,y,z) *with* $x>1$.

Theorem 1.5. *Assume that one of the following conditions holds*:

(i) $a=4$ *and either* $b=3$ *or* b *has a prime divisor* p *with* $p \equiv 11 \pmod{24}$.

(ii) $a=16$ *and either* $b \in \{3,5\}$ *or* b *has a prime divisor* p *with*

$$p \equiv 11, 13, 29, 37, 43, 59, 67 \text{ or } 101 \pmod{120}.$$

Then, (1.4) *has no solutions.*

2. Preliminaries

Let D be a nonsquare positive integer, and let D_1, D_2 be positive integers such that $D_1>1$, $D_1D_2=D$ and $\gcd(D_1,D_2)=1$. By the basic properties of Pell's equation (see [5, 10] and [4, Lemma 1]), we obtain the following two lemmas immediately.

Lemma 2.1. *The equation*

$$u^2 - Dv^2 = 1, \quad u, v \in \mathbb{N} \tag{2.1}$$

has solutions (u,v), *and it has a unique solution* (u_1,v_1) *such that* $u_1+v_1\sqrt{D} \le u+v\sqrt{D}$, *where* (u,v) *runs through all solutions of* (2.1). *The solution* (u_1,v_1) *is called the least solution of* (2.1). *For any positive integer* n, *let* $u_n+v_n\sqrt{D} = \left(u_1+v_1\sqrt{D}\right)^n$. *Then we have*

(i) $(u,v)=(u_n,v_n)$ $(n=1,2,\dots)$ *are all solutions of* (2.1).

(ii) *If* $2 \mid n$, *then each prime divisor* p *of* u_n *satisfies* $p \equiv \pm 1 \pmod 8$.

(iii) *If* $2 \nmid n$, *then* $u_1 \mid u_n$.

Lemma 2.2. *If the equation*

$$D_1U^2 - D_2V^2 = 1, \quad U, V \in \mathbb{N} \tag{2.2}$$

has solutions (U,V), *then it has a unique solution* (U_1,V_1) *such that* $U_1\sqrt{D_1} + V_1\sqrt{D_2} \le U\sqrt{D_1}+V\sqrt{D_2}$, *where* (U,V) *runs through all solutions of* (2.2). *The solution* (U_1,V_1) *is called the least solution of* (2.2). *For any odd positive integer* m, *let* $U_m\sqrt{D_1}+V_m\sqrt{D_2} = \left(U_1\sqrt{D_1}+V_1\sqrt{D_2}\right)^m$. *Then we have*

(i) $(U,V)=(U_m,V_m)$ $(m=1,3,\dots)$ *are all solutions of* (2.2).

(ii) $u_1 + v_1\sqrt{D} = \left(U_1\sqrt{D_1} + V_1\sqrt{D_2}\right)^2$, *where* (u_1, v_1) *is the least solution of* (2.1).

For any positive integer l, let $\mathrm{ord}_2(l)$ denote the order of 2 in the factorization of l.

Lemma 2.3. *If* (2.2) *has solutions* (U, V), *then every solution* (U, V) *of* (2.2) *satisfies* $\mathrm{ord}_2(D_1U^2) = \mathrm{ord}_2(D_1U_1^2)$, *where* (U_1, V_1) *is the least solution of* (2.2).

Proof. By (i) of Lemma 2.2, there exists an odd positive integer m which makes $U\sqrt{D_1} + V\sqrt{D_2} = \left(U_1\sqrt{D_1} + V_1\sqrt{D_2}\right)^m$, whence we get

$$U = U_1 \sum_{i=0}^{(m-1)/2} \binom{m}{2i} \left(D_1U_1^2\right)^{(m-1)/2-i} \left(D_2V_1^2\right)^i. \tag{2.3}$$

Since $D_1U_1^2 - D_2V_1^2 = 1$ implies that $D_1U_1^2$ and $D_2V_1^2$ have opposite parity, we have

$$2 \nmid \sum_{i=0}^{(m-1)/2} \binom{m}{2i} \left(D_1U_1^2\right)^{(m-1)/2-i} \left(D_2V_1^2\right)^i. \tag{2.4}$$

Hence, by (2.3) and (2.4), we get $\mathrm{ord}_2(U) = \mathrm{ord}_2(U_1)$. It implies that $\mathrm{ord}_2(D_1U^2) = \mathrm{ord}_2(D_1U_1^2)$. The lemma is proved. □

Lemma 2.4. *Let* r_n, s_n *be defined as in* (1.5). *Then* $(u, v) = (r_n, s_n)$ $(n = 1, 2, \dots)$ *are all solutions of the equation*

$$u^2 - 2v^2 = 1, \quad u, v \in \mathbb{N}, \tag{2.5}$$

and

$$r_n \equiv \begin{cases} 1 \pmod 8, & \text{if } 2 \mid n, \\ 3 \pmod 8, & \text{if } 2 \nmid n. \end{cases} \tag{2.6}$$

Proof. Since $(u_1, v_1) = (3, 2)$ is the least solution of (2.5), by (i) of Lemma 2.1, we see from (1.5) that $(u, v) = (r_n, s_n)$ $(n = 1, 2, \dots)$ are all solutions of (2.5). By (1.5) we have

$$r_n = \sum_{i=0}^{[n/2]} \binom{m}{2i} 3^{n-2i} \cdot 8^i,$$

where $[n/2]$ is the integer part of $n/2$. It follows that

$$r_n \equiv 3^n \pmod 8,$$

whence we obtain (2.6). The lemma is proved. □

Lemma 2.5. *For any odd positive integer* m, *we have* $r_m = 2R_m^2 + 1$, *where* r_m, R_m *are defined as in* (1.5) *and* (1.6) *respectively.*

Proof. Since $3+2\sqrt{2}=\left(1+\sqrt{2}\right)^2$ and $3-2\sqrt{2}=\left(1-\sqrt{2}\right)^2$, by (1.5) and (1.6), we have

$$\begin{aligned} r_m &= \frac{1}{2}\left(\left(3+2\sqrt{2}\right)^m+\left(3-2\sqrt{2}\right)^m\right)=\frac{1}{2}\left(\left(1+\sqrt{2}\right)^{2m}+\left(1-\sqrt{2}\right)^{2m}\right)\\ &= \frac{1}{2}\left(\left(\left(1+\sqrt{2}\right)^m+\left(1-\sqrt{2}\right)^m\right)^2-2\left(1+\sqrt{2}\right)^m\left(1-\sqrt{2}\right)^m\right)\\ &= \frac{1}{2}\left((2R_m)^2+2\right)=2R_m^2+1. \end{aligned}$$

The lemma is proved. □

Lemma 2.6 ([9]). *The equation*

$$2X^2+1=Y^3,\quad X,Y\in\mathbb{N}$$

has no solutions (X,Y).

Lemma 2.7 ([6]). *The equation*

$$2X^2+1=Y^q,\quad X,Y\in\mathbb{N},\ q\ \textit{is an odd prime with}\ q>3$$

has only the solution $(X,Y,q)=(11,3,5)$.

Lemma 2.8 ([1, 2]). *The equation*

$$X^4-DY^2=1,\quad X,Y\in\mathbb{N}$$

has solutions (X,Y) *if and only if either* $X^2=u_1$ *or* $X^2=2u_1^2-1$.

Lemma 2.9. *The equation*

$$2X^2+1=Y^t,\quad X,Y,t\in\mathbb{N},\ t>2 \tag{2.7}$$

has only the solution $(X,Y,t)=(11,3,5)$.

Proof. Let (X,Y,t) be a solution of (2.7), and let q be the largest prime divisor of t. By Lemmas 2.6 and 2.7, (2.7) has only the solution $(X,Y,t)=(11,3,5)$ with $q\geq 3$. Since $t>2$, if $q=2$, then $4\mid t$ and the equation

$$(X')^4-2(Y')^2=1,\quad X',Y'\in\mathbb{N} \tag{2.8}$$

has a solution $(X',Y')=(Y^{t/4},X)$. However, since the least solution of (2.5) is $(u_1,v_1)=(3,2)$, neither $u_1=3$ nor $2u_1^2-1=17$ is a square. By Lemma 2.8, (2.8) has no solutions (X',Y'). Therefore, (2.7) has no solutions (X,Y,t) with $q=2$. The lemma is proved. □

3. Proof of Theorem 1.1

In this section, we assume that (1.7) holds and that (x, y, z) is a solution of (1.4) with $2 \nmid x$ and $x > 1$. Then we have

$$x \geq 3. \tag{3.1}$$

Since $\gcd(a, a^x - 1) = 1$, by (1.4), we get

$$a^x - 1 = df^2, \;\; b^y - 1 = adg^2, \;\; z = dfg, \quad d, f, g \in \mathbb{N}. \tag{3.2}$$

By the first equality of (3.2), we have

$$\gcd(a, d) = 1. \tag{3.3}$$

Since $2 \mid a$, by (3.1) and the first equality of (3.2), we get $2 \nmid f$ and

$$d \equiv df^2 \equiv a^x - 1 \equiv 0 - 1 \equiv 7 \pmod 8. \tag{3.4}$$

Hence, we see from (3.4) that

$$d \text{ is not a square.} \tag{3.5}$$

On the other hand, substituting (3.4) into the second equality of (3.2), we have

$$b^y \equiv 1 + 7ag^2 \equiv \begin{cases} 1 \pmod 8, & \text{if } a \equiv 0 \pmod 8 \text{ or } 2 \mid g, \\ 7 \pmod 8, & \text{if } a \equiv 2 \pmod 8 \text{ and } 2 \nmid g, \\ 5 \pmod 8, & \text{if } a \equiv 4 \pmod 8 \text{ and } 2 \nmid g, \\ 3 \pmod 8, & \text{if } a \equiv 6 \pmod 8 \text{ and } 2 \nmid g. \end{cases} \tag{3.6}$$

Further, since $b \equiv \pm 3 \pmod 8$, we get

$$b^y \equiv \begin{cases} 1 \pmod 8, & \text{if } 2 \mid y, \\ \pm 3 \pmod 8, & \text{if } 2 \nmid y. \end{cases} \tag{3.7}$$

Therefore, in view of (1.7), comparing (3.6) and (3.7), we obtain

$$2 \mid y. \tag{3.8}$$

We see from (3.8) and the second equality of (3.2) that the equation

$$u^2 - adv^2 = 1, \quad u, v \in \mathbb{N} \tag{3.9}$$

has a solution

$$(u, v) = (b^{y/2}, g). \tag{3.10}$$

By (3.3) and (3.5), ad is a nonsquare positive integer. Hence, applying (i) of Lemma 2.1 to (3.10), there exists a positive integer n' which makes

$$b^{y/2} + g\sqrt{ad} = \left(u_1 + v_1\sqrt{ad}\right)^{n'}, \tag{3.11}$$

where (u_1, v_1) is the least solution of (3.9).

For any positive integer n, let

$$u_n + v_n\sqrt{ad} = \left(u_1 + v_1\sqrt{ad}\right)^n. \tag{3.12}$$

If $2 \mid n'$, then from (3.11) and (3.12) we get $b^{y/2} = u_{n'}$ and, by (ii) of Lemma 2.1, $b \equiv \pm 1 \pmod 8$, which contradicts the assumption. So we get

$$2 \nmid n'. \tag{3.13}$$

Since $2 \nmid x$, we see from the first equality of (3.2) that the equation

$$aU^2 - dV^2 = 1, \quad U, V \in \mathbb{N} \tag{3.14}$$

has a solution

$$(U, V) = \left(a^{(x-1)/2}, f\right). \tag{3.15}$$

Let (U_1, V_1) be the least solution of (3.14). For any odd positive integer m, let

$$U_m\sqrt{a} + V_m\sqrt{d} = \left(U_1\sqrt{a} + V_1\sqrt{d}\right)^m. \tag{3.16}$$

Applying (i) of Lemma 2.2 to (3.15), by (3.16), there exists an odd positive integer m' which makes

$$\left(a^{(x-1)/2}, f\right) = (U_{m'}, V_{m'}). \tag{3.17}$$

Hence, by Lemma 2.3, we get from (3.1) and (3.17) that

$$\operatorname{ord}_2(aU_1^2) = \operatorname{ord}_2(aU_{m'}^2) = \operatorname{ord}_2(a^x) \geq x \geq 3. \tag{3.18}$$

By (ii) of Lemma 2.2, we find from (3.11), (3.13) and (3.16) that

$$\begin{aligned} b^{y/2} + g\sqrt{ad} &= \left(U_1\sqrt{a} + V_1\sqrt{d}\right)^{2n'} = \left(\left(U_1\sqrt{a} + V_1\sqrt{d}\right)^{n'}\right)^2 \\ &= \left(U_{n'}\sqrt{a} + V_{n'}\sqrt{d}\right)^2. \end{aligned} \tag{3.19}$$

Since $aU_{n'}^2 - dV_{n'}^2 = 1$, by (3.19), we have

$$b^{y/2} = aU_{n'}^2 + dV_{n'}^2 = 2aU_{n'}^2 - 1. \tag{3.20}$$

Further, by Lemma 2.3, we have $\operatorname{ord}_2(aU_{n'}^2) = \operatorname{ord}_2(aU_1^2)$. Hence, by (3.18), we get $\operatorname{ord}_2(aU_{n'}^2) \geq 3$ and $aU_{n'}^2 \equiv 0 \pmod 8$. Therefore, by (3.20), we obtain $b^{y/2} \equiv 7 \pmod 8$. But, since $b \equiv \pm 3 \pmod 8$, it is impossible. Thus, the theorem is proved.

4. Proof of Theorem 1.2

In this section, we assume that $a = 2$, $b \equiv \pm 3 \pmod 8$ and (x, y, z) is a solution of (1.4) with $2 \nmid x$. By Theorem 1.1, we have

$$x = 1. \tag{4.1}$$

Since $a = 2$, substituting (4.1) into (3.2), we get

$$d = f = 1 \tag{4.2}$$

and

$$b^y - 1 = 2g^2, \ z = g, \quad g \in \mathbb{N}. \tag{4.3}$$

If $b \equiv 5 \pmod 8$, then from the first equality of (4.3) we get $1 = (-2/b) = (2/b) = -1$, a contradiction, where $(*/b)$ is the Jacobi symbol. Therefore, if $a = 2$ and $b \equiv 5 \pmod 8$, then (1.4) has no solutions (x, y, z) with $2 \nmid x$.

We just need to consider the case $b \equiv 3 \pmod 8$. Applying Lemma 2.9 to the first equality of (4.3), by (4.1) and (4.3), equation (1.4) has only the solution

$$b = 3, \quad (x, y, z) = (1, 5, 11) \tag{4.4}$$

with $y > 2$.

When $y = 2$, by the first equality of (4.3), $(u, v) = (b, g)$ is a solution of (2.5) Since $(u_1, v_1) = (3, 2)$ is the least solution of (2.5), by (i) of Lemma 2.1, we get from (1.5) that

$$(b, g) = (r_{n'}, s_{n'}), \quad n' \in \mathbb{N}. \tag{4.5}$$

Further, since $b \equiv 3 \pmod 8$, by Lemma 2.4, we see from (4.5) that $2 \nmid n'$. Hence, by (4.1), (4.2), (4.3) and (4.5), we obtain

$$b = r_m, \ (x, y, z) = (1, 2, s_m), \quad m \in \mathbb{N}, \ 2 \nmid m. \tag{4.6}$$

When $y = 1$, by (4.1), (4.2) and (4.3), we have

$$b = 2g^2 + 1, \ (x, y, z) = (1, 1, g), \quad g \in \mathbb{N}, \ 2 \nmid g. \tag{4.7}$$

Thus, since $r_1 = 2 \cdot 1^2 + 1 = 3$, the combination of (4.4), (4.6) and (4.7) yields the solutions (i), (ii) and (iii). The theorem is proved.

5. Proof of Theorem 1.3

By Theorem 1.2, we get $N(2, 3) = 3$ immediately. By Lemma 2.5, if $b = 2g^2 + 1$ and $g = R_m$ with $m > 1$, then $b = r_m > 3$. Hence, by Theorem 1.2, we have $N(2, b) = 2$. In addition, if $b = 2g^2 + 1$ with $g \neq R_m$ or $b \neq 2g^2 + 1$, then $N(2, b) = 1$ or 0. The theorem is proved.

6. Proof of Theorems 1.4 and 1.5

Proof of Theorem 1.4. By Theorem 1.1, we may assume that $x = 2x_0$ for some $x_0 \in \mathbb{N}$. In addition, since a is a square, we may write $a = a_0^2$ for some $a_0 \in \mathbb{N}$. Then, by the first equality of (3.2), we get

$$(a_0^{x_0})^4 - df^2 = 1. \tag{6.1}$$

It is clear from (6.1) that

$$d \text{ is not a square.} \tag{6.2}$$

Applying Lemma 2.8 to (6.1), we see that either $a^{x_0} = u'_1$ or $a^{x_0} = 2(u'_1)^2 - 1$, where (u'_1, v'_1) is the least solution of (2.1) with $D = d$. Since $2 \mid a$, we must have

$$a^{x_0} = u'_1. \tag{6.3}$$

On the other hand, we know by $4 \mid a$ and $2 \mid x$ that (3.4) holds, which together with (3.6) and (3.7) yields $2 \mid y$. Since $a = a_0^2$, we see from the second equality of (3.2) that (2.1) with $D = d$ has a solution $(u, v) = (b^{y/2}, a_0 g)$. By (i) of Lemma 2.1 and (6.2), we have

$$(u'_n, v'_n) = \left(b^{y/2}, a_0 g\right), \quad n \in \mathbb{N}, \tag{6.4}$$

where $u'_n + v'_n\sqrt{d} = \left(u'_1 + v'_1\sqrt{d}\right)^n$. If $2 \mid n$, then, by (ii) of Lemma 2.1, $b \equiv \pm 1 \pmod 8$, which contradicts the assumption. If $2 \nmid n$, then, by (iii) of Lemma 2.1, $u'_1 \mid u'_n$. However, by (6.3) and (6.4), we have $a \mid b^{y/2}$, which contradicts $2 \mid a$ and $b \equiv \pm 3 \pmod 8$. The theorem is proved. □

Proof of Theorem 1.5. By Theorem 1.4, we have

$$x = 1. \tag{6.5}$$

(i) Substituting $a = 4$ and (6.5) into (3.2), we get

$$d = 3, \ f = 1$$

and

$$b^y - 1 = 12g^2, \ z = 3g, \quad g \in \mathbb{N}. \tag{6.6}$$

Obviously, we have $b \neq 3$. If b has a prime divisor p with $p \equiv 11 \pmod{24}$, then by (6.6) we have

$$-1 = \left(\frac{-1}{p}\right) = \left(\frac{12g^2}{p}\right) = \left(\frac{3}{p}\right) = 1,$$

a contradiction. Thus, (i) is proved.

(ii) Substituting $a = 16$ and (6.5) into (3.2), we get

$$d = 15, \ f = 1$$

and

$$b^y - 1 = 15 \cdot 16g^2, \; z = 15g, \quad g \in \mathbb{N}. \tag{6.7}$$

Obviously, we have $b \notin \{3, 5\}$. If b has a prime divisor p with $p \equiv 11, 43, 59$ or $67 \pmod{120}$, then, by (6.7),

$$-1 = \left(\frac{-1}{p}\right) = \left(\frac{15}{p}\right) = 1,$$

a contradiction. If b has a prime divisor p with $p \equiv 13, 29, 37$ or $101 \pmod{120}$, then, by (6.7),

$$1 = \left(\frac{-1}{p}\right) = \left(\frac{15}{p}\right) = -1,$$

a contradiction. Thus, the theorem is proved. □

Acknowledgements. The authors thank the referee for careful reading and helpful comments.

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