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On elementary representations of $\cos 75°$ and $\cos 15°$*

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Abstract. In this paper we present geometric and algebraic representation of values of $\cos 75°$ and $\sin 75°$ without using trigonometric identities or iterated applications of taking roots.

Keywords: values of sine and cosine, representation of real numbers, roots

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1. Introduction

Trigonometry undoubtedly plays an important role in higher mathematics, physics, engineering and various other fields of science. Nevertheless, students fail to understand and apply trigonometry when it is needed, and the reason for this is quite complex. A semantic analysis shows that only fifty-three per cent of students can solve problems where the unknowns are a distance or an angle and for which knowledge it is necessary to calculate the sine or cosine of an angle [6]. Even preservice teacher students with prior knowledge do not manage to connect the visual representation and the symbolic or verbal representation of sine and cosine, although they are successful in solving problems in all three ways [5]. The rea-

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son for this could be that they either do not have enough experience in solving problems or that they do not have enough visual experience with sine and cosine values. Understanding trigonometric functions is not a straightforward process and can be affected by many different variables – such as understanding different units for angle measures, the ability to associate triangles with numerical relationships, or understanding functions for which there is no explicit algebraic formula to determine their values[4, 7, 14]. However, several studies suggest that appropriately designed learning environments can improve students' geometric understanding. In particular, various visual representations, including dynamic ones, are especially important to support the learning process [4, 8]. The effectiveness of processing is improved when we work on multiple levels of representation at each stage [2, 3, 9]:

- we physically construct, fold, etc. the corresponding construction;
- we record our observations in drawings, tables, etc.
- we also express our experiences, assumptions and arguments in words

"Consequently, there is a need for more activities that can be used to assess students' understanding of concepts that are integral to the learning of trigonometry." (Arsalan Wares) ([11] pp. 141.)

In this note we present geometric ways to find the values of $\cos 75^\circ$ and $\sin 75^\circ$ without using trigonometric identities or iterated applications of taking roots. The topic is particularly topical as trigonometric identities are no longer included in the 2020 national curriculum. The introduction of trigonometric functions as functions and the associated calculations are not supported on the intermediate level. Trigonometric identities, like expressing $\sin(\alpha + \beta)$ or $\cos 2\alpha$ are totally missing.

The current National Core Curriculum from 2020 specifies the following:
Grades 9–10.: No trigonometry.
Grades 11–12.:

- Sine, cosine, tangent of acute angles.
- Calculations in right triangles using angle functions in practical situations.
- Sine, cosine, tangent of obtuse angles.
- Understanding relationships between different angle functions of a given angle: Pythagorean identity, co-function identities, and supplementary angles.
- Determining an angle using a calculator given the value of an angle function.
- Calculating the area of a triangle knowing two sides and the included angle.
- Knowledge and application of the sine and cosine rules.
- Proof of the sine rule.
- Calculations in quadrilaterals and polygons using angle functions.

We can see that the curriculum does not extend angle functions to arbitrary rotation angles, so they are essentially not treated as functions. There is no need to know how to graph them, transform them, and certainly not to understand trigonometric identities, addition formulas, and other operations with them. Thus, we are spared from interesting problems like finding $\sin 75°$ by expanding $\sin(45° + 30°)$. However, the National Core Curriculum does not set an upper limit on the material that can be taught to students. Therefore, those studying mathematics on an advanced level can still solve problems requiring more complex algebraic and trigonometric knowledge, and the advanced-level textbook does indeed include such content. Still, this knowledge is no longer commonly expected.

Values of $\cos 75°$ and $\sin 75°$ can be calculated with the Ailles-square [1], or with paper-folding [12]. In [13] a tricky way is applied to find these values with the use of two diagonals of a regular dodecagon. Another construction can be found in [11], but it is using iterated square roots.

In this paper we give different ways to find the values of $\cos 75°$ and $\sin 75°$ on a high-school level. Besides trigonometric identities we avoid iterated roots, as well. This work is a continuation of a previous study in which we presented and examined several geometric constructions for upper elementary and high school levels [10].

2. Elementary calculation methods

In this chapter, we show methods of finding the algebraic form of

$$a = 2\cos 15° \quad \text{and} \quad b = 2\cos 75° = 2\sin 15°$$

via geometric construction. We shall always aim at finding a degree two polynomial (equation) such that $a = 2\cos 15°$ is a root (solution) of it. The most convinient polynomials are

$$\begin{aligned}
&x^2 + \sqrt{3} - 2 \quad &&\text{with roots} \quad a = \sqrt{2-\sqrt{3}} \quad &&\text{and} \quad -a = -\sqrt{2-\sqrt{3}},\\
&x^2 - \sqrt{6}x + 1 \quad &&\text{with roots} \quad a = \frac{\sqrt{6}+\sqrt{2}}{2} \quad &&\text{and} \quad b = \frac{\sqrt{6}-\sqrt{2}}{2},\\
&x^2 - \sqrt{2}x - 1 \quad &&\text{with roots} \quad a = \frac{\sqrt{6}+\sqrt{2}}{2} \quad &&\text{and} \quad -b = \frac{\sqrt{2}-\sqrt{6}}{2}.
\end{aligned}$$

We would like to avoid nested roots so that we will arrive at either $x^2 - \sqrt{2}x - 1$ or $x^2 - \sqrt{6}x + 1$. We can see that

$$a + b = \sqrt{6} \quad \text{and} \quad a - b = \sqrt{2}$$

If we also consider

$$a^2 + b^2 = 4,$$

then after substituting $a = b + \sqrt{2}$ or $a = \sqrt{6} - b$ we obtain the desired degree 2 polynomials. We show a method to find $a - b$ and two constructions to find $a + b$.

2.1. Aiming for $a = b + \sqrt{2}$: The house-shaped construction

If we fit isosceles triangles, $AEH\Delta$, $BHC\Delta$ and $DCE\Delta$ with sides 1 and with angles 90° and 30° and 150°, then we obtain Figure 1. Then $HAB\Delta$ is an isosceles triangle with $AHB\sphericalangle = 60°$, hence $HAB\Delta$ is equilateral. To reverse the construction, let's take a square with unit-length sides and construct an equilateral triangle with unit-length sides on one of the square's sides. Then $|EC| = a$. If we draw the $ABH\Delta$ equilateral triangle, point H will be on the segment EC, and the notations and values in Figure 1 will hold.

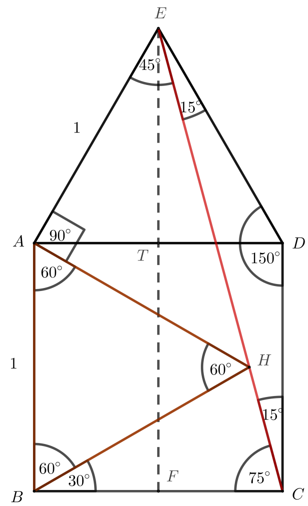

Figure 1. With additional segments $2\sin 15°$ can be represented.

If we draw the common axis of symmetry for the square $ABCD$ and the equilateral triangle $ADE\Delta$ we obtain points T and F. The segment ET is the height of an equilateral triangle with unit sides, thus its length is $\frac{\sqrt{3}}{2}$. Then

$$FC = \frac{1}{2}BC = \frac{1}{2},$$
$$EF = ET + TF = \frac{\sqrt{3}}{2} + 1.$$

Using the Pythagorean theorem, we can calculate the length of the hypotenuse CE:

$$\left(\frac{\sqrt{3}}{2} + 1\right)^2 + \left(\frac{1}{2}\right)^2 = (CE)^2,$$
$$\frac{\left(\sqrt{3}+2\right)^2}{4} + \frac{1}{4} = \frac{8 + 4\sqrt{3}}{4} = 2 + \sqrt{3} = (CE)^2,$$

$$CE = \sqrt{2+\sqrt{3}}.$$

This is an elementary method to calculate $2\cos 15°$

$$2\cos 15° = \sqrt{2+\sqrt{3}}.$$

We obtain the value with doubly nested square roots, so this is a dead end.

We want to calculate the lengths of segments EH and CH in Figure 1. We know that the angles of the original triangle $CDE\Delta$, which is isosceles, are $DEC\sphericalangle = ECD\sphericalangle = 15°$ and $EDC\sphericalangle = 150°$. Triangle $AHE\Delta$ is an isosceles right triangle, and triangle $BCH\Delta$ is an isosceles triangle with base angles $BCH\sphericalangle = CHB\sphericalangle = 75°$ and sides of unit-lengths. Hence

$$EH = \sqrt{2}, \quad CH = b, \quad EH = a.$$

Segment EH can be given as the sum of segments CH and EH:

$$b = a + \sqrt{2}. \tag{2.1}$$

Using

$$a^2 + b^2 = 4 \tag{2.2}$$

Substituting Equation (2.1) into Equation (2.2) we get

$$x^2 - \sqrt{2}a - 1 = 0$$

and the quadratic formula gives

$$\frac{\sqrt{2} \pm \sqrt{2 - 4\cdot 1\cdot(-1)}}{2} = \frac{\sqrt{2}\pm\sqrt{6}}{2}.$$

Thus, without resorting to algebraic manipulations, we can determine $2\cos 15°$.

2.2. Aming for $a + b$: Cyclic trapezoid-shaped construction

The first idea of constructing $\sqrt{6}$ is to construct an isosceles triangle $CKD\Delta$ with angle $120°$ and sides $\sqrt{2}$. Let it complete with two isosceles right triangles with sides of unit length to a trapezoid. Or, from another point of view with an isosceles triangle $AKF\Delta$ with angle $150°$ and sides 1 as in Figure 2. The angles on the longer base of the trapezoid $CDFA$ are $75°$, and the angles on the shorter base are $105°$. Let CD be parallel to DF. Then

$$AF = \sqrt{6}, \quad AB = b, \quad CD = BF = a. \tag{2.3}$$

Thus

$$\sqrt{6} = AF = AB + CD = a + b.$$

Substituting Equation (2.3) into $a^2+b^2=4$ we get the following quadratic equation:

$$a^2-\sqrt{6}a+1=0$$

from which the segment lengths can be obtained without the use of nested roots form the quadratic formula

$$a=\frac{\sqrt{6}\pm\sqrt{6-4\cdot 1\cdot 1}}{2}=\frac{\sqrt{6}\pm\sqrt{2}}{2}.$$

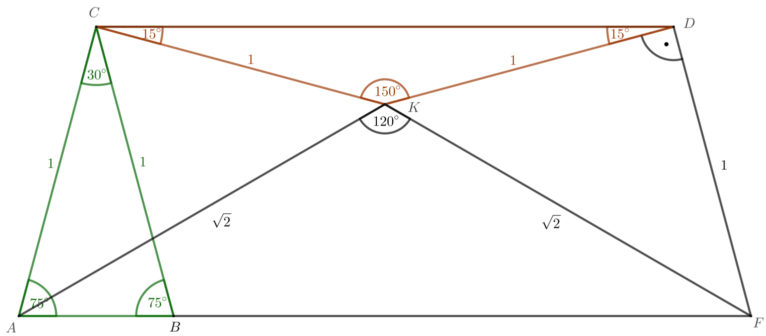

Figure 2. Symmetrical trapezoid-shaped construction.

2.3. Finding $a+b$: Pentagon shaped construction

Another way to construct $\sqrt{6}$ is by drawing an isosceles triangle with leg $\sqrt{3}$, triangle $EAF\Delta$ in Figure 3. Draw the isosceles triangles $ECA\Delta$ and $EDF\Delta$ with angle 120° of unit size lengths. Again we obtain the trapezoid of Figure 2, putting the $ECD\Delta$ on the "top". From here the same calculations as in the previous section lead to a.

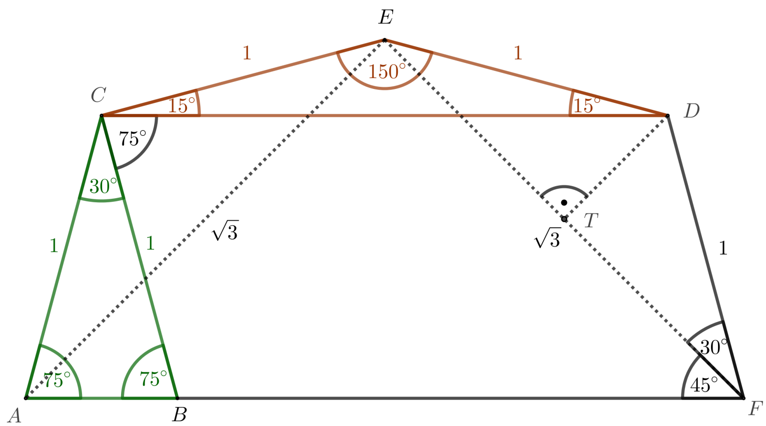

Figure 3. Symmetric pentagon-shaped construction.

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