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ORIGINAL PAPER

A New Construction for the Planar Turán Number of Cycles

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Abstract

The planar Turán number $\mathrm{ex}_{\mathcal{P}}(n, C_k)$ is the maximum number of edges in an n-vertex planar graph not containing a cycle of length k. Let $k \geq 11$ and c, d be constants. Cranston et al., and independently Lan and Song showed that $\mathrm{ex}_{\mathcal{P}}(n, C_k) \geq 3n - 6 - cn/k$ holds for large n. Moreover, Cranston et al. conjectured that $\mathrm{ex}_{\mathcal{P}}(n, C_k) \leq 3n-6-dn/k^{\log_2 3}$ when n is large. In this note, we prove that $\mathrm{ex}_{\mathcal{P}}(n, C_k) \geq 3n-6-6 \cdot 3^{\log_2 3} n/k^{\log_2 3}$ holds for every $k \geq 7$. This implies that the conjecture of Cranston et al. is essentially best possible.

Keywords Planar Turán number · Extremal graphs

Mathematics Subject Classification MSC 05C35 · MSC 05C38

1 Introduction

In this article, the cycle and complete graph on k vertices are denoted by C_k and K_k, respectively. For $k \in \{4, 5\}$, let Θ_k denote the graph obtained from C_k by adding a chord.

The Turán number $\mathrm{ex}(n, H)$ for a graph H is the maximum number of edges in an n-vertex graph containing no copy of H as a subgraph. The first result on this topic was obtained by Turán [12], who proved that the balanced complete r-partite graph is the unique extremal graph for $\mathrm{ex}(n, K_{r+1})$. The Erdős–Stone–Simonovits Theorem [4, 5] generalizes this result and asymptotically determines $\mathrm{ex}(n, H)$ for all non-bipartite graphs H as follows: $\mathrm{ex}(n, H) = \left(1 - \frac{1}{\chi(H)-1}\right)\binom{n}{2} + o(n^2)$, where $\chi(H)$ denotes the chromatic number of H.

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In 2016, Dowden [3] initiated the study of planar Turán problems, in which we want to determine the maximum number of edges of an n-vertex planar graph containing no copy of H as a subgraph. This number is denoted by $\mathrm{ex}_{\mathcal{P}}(n, H)$. Dowden [3] proved that $\mathrm{ex}_{\mathcal{P}}(n, C_4) \leq (15n - 30)/7$ for all $n \geq 4$, and $\mathrm{ex}_{\mathcal{P}}(n, C_5) \leq (12n - 33)/5$ for all $n \geq 11$, and showed that these upper bounds are tight for infinitely many n. Lan, Shi, and Song [8] showed that $\mathrm{ex}_{\mathcal{P}}(n, \Theta_4) \leq (12n - 24)/5$ for all $n \geq 4$, and $\mathrm{ex}_{\mathcal{P}}(n, \Theta_5) \leq (5n - 10)/2$ for all $n \geq 5$, and $\mathrm{ex}_{\mathcal{P}}(n, C_6) \leq (18n - 36)/7$, and the bounds on $\mathrm{ex}_{\mathcal{P}}(n, \Theta_4)$ and $\mathrm{ex}_{\mathcal{P}}(n, \Theta_5)$ are tight for infinitely many n. The upper bound on $\mathrm{ex}_{\mathcal{P}}(n, C_6)$ was improved by Ghosh, Győri, Martin, Paulos, and Xiao [6]. They proved $\mathrm{ex}_{\mathcal{P}}(n, C_6) \leq (5n - 14)/2$ for all $n \geq 18$, showed that this bound is tight for infinitely many n, and they also proposed the following conjecture.

Conjecture 1 (Ghosh et al. [6]) For each $k \geq 7$ and sufficiently large n, we have

$$\mathrm{ex}_{\mathcal{P}}(n, C_k) \leq 3n - 6 - \frac{3n + 6}{k}.$$

Recently, Conjecture 1 was disproved by Cranston, Lidický, Liu, and Shantanam [2] and independently by Lan and Song [9] for $k \geq 11$ and sufficiently large n.

Theorem 1 (Lan and Song [9]) *Let $k \geq 11$ and $n \geq k - 4 + \lfloor (k-1)/2 \rfloor$. Then there exists a constant c_k such that*

$$\mathrm{ex}_{\mathcal{P}}(n, C_k) \geq \left(3 - \frac{3 - \frac{2}{k-1}}{k - 6 + \lfloor (k-1)/2 \rfloor} \right) n + c_k.$$

Furthermore, Cranston et al. proposed a revised conjecture.

Conjecture 2 (Cranston et al. [2]) There exists a constant d such that for all $k \geq 3$ and all sufficiently large n, we have

$$\mathrm{ex}_{\mathcal{P}}(n, C_k) \leq 3n - 6 - \frac{dn}{k^{\log_2 3}}.$$

Independently of each other, Shi, Walsh, Yu [11] and Győri, Li, Zhou [7] proved that $\mathrm{ex}_{\mathcal{P}}(n, C_7) \leq (18n - 48)/7$, thereby affirming Conjecture 1 for $k = 7$.

In this note, we give a new construction for the lower bound of $\mathrm{ex}_{\mathcal{P}}(n, C_k)$ and obtain the following theorem.

Theorem 2 *For all $k \geq 7$ and sufficiently large n, we have*

$$\mathrm{ex}_{\mathcal{P}}(n, C_k) \geq 3n - 6 - \frac{6 \cdot 3^{\log_2 3} n}{k^{\log_2 3}}.$$

Note that Theorem 2 implies Conjecture 2 is essentially best possible.

2 Our construction

In this section, we show our construction and prove Theorem 2. We first define a sequence of planar graphs T_i and use them in our construction. This sequence was first introduced by Moon and Moser [10].

Let T_1 be a copy of K_4 and xyz be the outer cycle. Assume that T_{i-1} is defined for some $i \geq 2$. Let T_i be the graph obtained from T_{i-1} as follows: in each inner face of T_{i-1}, add a new vertex and join the new vertex to the three vertices incident to this face.

By the above construction, each T_i is a triangulation, i.e., a planar graph whose each face is a triangle. The outer cycle of T_i is xyz and

$$|V(T_i)| = 4 + 3 + 3^2 + \cdots + 3^{i-1} = \frac{3^i + 5}{2}.$$

Furthermore, Chen and Yu [1] showed that T_i has the following properties.

Lemma 1 (Chen–Yu [1]) *For any integer $i \geq 2$, we have the following.*

(i) *The length of the longest path between x and y in T_i is $3 \cdot 2^{i-1}$.*
(ii) *The length of the longest cycle in T_i is $7 \cdot 2^{i-2}$.*

After defining the graphs T_i, we show our construction. Let $k \geq 7$ and let i be the maximum integer such that $3 \cdot 2^{i-1} < k/2$, i.e., let

$$i = \lceil \log_2 k/3 \rceil - 1. \tag{1}$$

Let $n \geq \frac{3^i+5}{2}$ and $s = \left\lceil \frac{n-2}{(3^i+5)/2-2} \right\rceil$. Let $H_1, \ldots, H_{s-1}$ be $s-1$ copies of T_i and let H_s be a subgraph of T_i such that H_s is a triangulation and has $n - (s-1)\left(\frac{3^i+5}{2} - 2\right)$ vertices. Such a graph H_s exists because of the process of the construction of T_i. For each $1 \leq j \leq s$, we may assume that $x_j y_j z_j$ is the outer cycle of H_j, and let the other triangular face containing $x_j y_j$ be $x_j y_j w_j$. Let H_j^- be the subgraph obtained from H_j by deleting the edge $x_j y_j$. Then each face of H_j^- is a triangle except the outer face whose boundary is the cycle $x_j w_j y_j z_j$. Let us identify the vertices x_j of H_j^- for all $1 \leq j \leq s$ as a new vertex x, and identify the vertices y_j of H_j^- for all $1 \leq j \leq s$ as a new vertex y, and add the new edge xy to this graph. Let H denote the resulting graph (see Fig. 1).

We call each $H_j^- + xy$ a gadget of H. Note that each gadget is a copy of T_i except the last gadget $H_s^- + xy$, which is a subgraph of T_i. Now we show that H contains no C_k. Let C be a longest cycle in H. Since $\{x, y\}$ is a vertex-cut of H, the cycle C passes through at most two gadgets. If C passes through only one gadget, then by (ii) of Lemma 1 and by (1), we have

$$|V(C)| \leq 7 \cdot 2^{i-2} < \frac{7k}{12}.$$

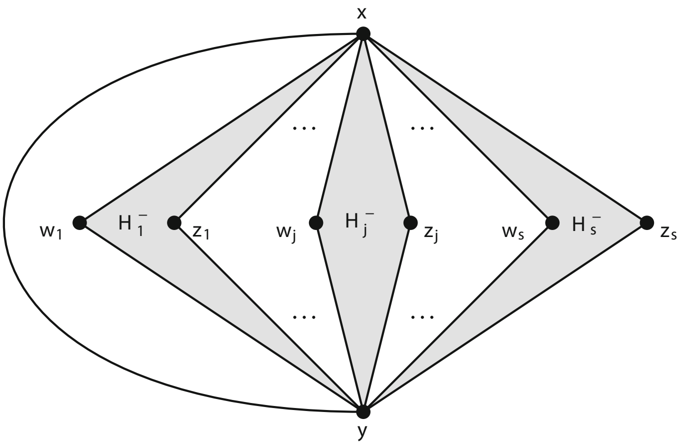

Fig. 1 The graph H constructed in Sect. 2

If C passes through two gadgets, then clearly $x, y \in V(C)$, thus by (i) of Lemma 1 and by (1), we have

$$|V(C)| \leq 2(3 \cdot 2^{i-1}) < k.$$

Hence, H contains no C_k.

Next we calculate the number of edges of H. Clearly, H is an n-vertex planar graph. By adding the edges $z_1w_2,\ z_2w_3, \dots, z_{s-1}w_s$ to H, it becomes a triangulation. Hence, we have

$$|E(H)| = 3n - 6 - (s - 1).$$

Then $s = \left\lceil \frac{n-2}{(3^i+5)/2-2} \right\rceil$ and (1) imply

$$\begin{aligned} |E(H)| &\geq 3n - 6 - \frac{2(n-2)}{3^i + 1} = 3n - 6 - \frac{2n-4}{3^{\lceil \log_2 k/3 \rceil - 1} + 1} \\ &\geq 3n - 6 - \frac{2n}{3^{\log_2 k/3 - 1}} = 3n - 6 - \frac{3^{\log_2 3 + 1} \cdot 2n}{3^{\log_2 k}} \\ &\geq 3n - 6 - \frac{6 \cdot 3^{\log_2 3} \cdot n}{k^{\log_2 3}}. \end{aligned}$$

Hence, $\mathrm{ex}_{\mathcal{P}}(n, C_k) \geq 3n - 6 - \frac{6 \cdot 3^{\log_2 3} n}{k^{\log_2 3}}$ and we are done.

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Declarations

Conflict of interest The authors declare that they have no known competing financial interests or personal relationships that could have appeared to influence the work reported in this paper.

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