Order
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RESEARCH

Forbidden Subposets in the Cycle Poset

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Abstract
The cycle poset consists of the intervals of the cyclic permutation of the elements **1, 2, ...**, n, ordered by inclusion. Suppose that $\boldsymbol{F}$ is a set of such intervals, none of them is a less than $\boldsymbol{s}$ others. The maximum size of $\boldsymbol{F}$ is determined under this condition. It is also shown that if the largest size of a set in this poset without containing a small subposet $\boldsymbol{P}$ is known, it solves the same problem, up to an additive constant, in the grid poset consisting of the pairs $(i, j)(1 \leq i, j \leq n)$ and ordered coordinate-wise.

Keywords Forbidden subposet · Cycle poset

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1 Introduction

Let $[n] = \{1, 2, \ldots, n\}$ be a finite set, $2^{[n]}$ the family of its subsets. The classical theorem of Sperner [9] states that if $\mathcal{F} \subset 2^{[n]}$ is a family of subsets without inclusion then the family cannot have more members than the one consisting of all sets of size $\lfloor \frac{n}{2} \rfloor$. More formally, if $A, B \in \mathcal{F}(A \neq B)$ implies $A \not\subset B$ then $|\mathcal{F}| \leq \binom{n}{\lfloor \frac{n}{2} \rfloor}$. There are hundreds of generalizations of this theorem, but we concentrate here only on the ones where the forbidden configuration can be expressed in terms of inclusion. Let P be a fixed "small" partially ordered set. Then $\mathrm{La}(n, P)$ denotes the size of the largest family $\mathcal{F} \subset 2^{[n]}$ satisfying the condition that none of its members form a copy of P with respect to the inclusion relation. Sperner's theorem claims for instance that $\mathrm{La}(n, I) = \binom{n}{\lfloor \frac{n}{2} \rfloor}$, where I is the poset consisting of two comparable elements. Another example is the following one. Let V be the poset consisting of 3 elements

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a, b, c satisfying the relations $a < c, b < c$. Then $\mathrm{La}(n, V)$ is the size of the largest family on n elements containing no 3 distinct sets A, B, C such that $A \subset C, B \subset C$. Its size was asymptotically determined in [8]. There are many results for particular small posets P and more general ones where $\mathrm{La}(n, P)$ is bounded in terms of parameters of P. Some older surveys are [7] and [5]. The recent book [3] devotes a chapter to the "Forbidden subposet problems".

Now consider the underlying set $[n]$ to be cyclically ordered: after the natural order, n is followed by 1. The family of all intervals (that is subsets consisting of consecutive elements) in this cyclic ordering is denoted by I_n. The poset defined on I_n by inclusion is called the *cycle poset*. Let us denote the cycle poset by C_n that means $C_n := (I_n, \subset)$. Paper [6] showed first the strong connection between the cycle and the Boolean posets. The solution of the extremal problem in the former one led to the solution in the latter one by easy double counting. Later several other works (among others [2] and [1]) exhibited this connection.

Let us sketch how Sperner's theorem can be proved using the cycle poset. The maximum number of intervals in I_n without inclusion is obviously n since the intervals "starting" at an element x are pairwise comparable. This is true for every cyclic ordering σ therefore the number of pairs (σ, A) where $A \in \mathcal{F}$ and $A \in I_n$ according to σ is at most $(n-1)!n = n!$. On the other hand for a fixed member $A \in \mathcal{F}$ the number of cyclic permutations where A is an interval is at least $\lfloor \frac{n}{2} \rfloor! \lceil \frac{n}{2} \rceil!$, therefore the number of pairs is at least $|\mathcal{F}| \lfloor \frac{n}{2} \rfloor! \lceil \frac{n}{2} \rceil!$. We obtained an inequality equivalent to the statement of the theorem.

Let B be the poset consisting of 4 elements a, b, c, d satisfying the relations $a < c, b < c, a < d, b < d$. Then $\mathrm{La}(n, B)$ is the size of the largest family on n elements containing no 4 distinct sets A, B, C, D such that $A \cup B \subseteq C \cap D$. Its size was determined in [1] by a similar method. It was first proved for the cycle poset that one cannot choose more than $2n$ intervals without a copy of B. Again, double counting leads to the solution in the Boolean lattice: the two largest levels give the largest possible family. But unfortunately, this is not true in general only very seldom, and in many other cases, the solution of the extremal problem for the cycle poset does not lead to the solution in the Boolean lattice, but still encourages us to study the cycle poset which seems to be interesting in itself.

Let P be again a fixed "small" partially ordered set. We will study the largest size of a subset G of I_n such that it does not contain a copy of P, for certain posets P. Let this maximum be denoted by $\mathrm{La}(C_n, P)$.

The following poset is closely related to C_n. Consider the set $[n]^2 = \{1, 2, ..., n\}^2$ which is ordered coordinate-wise. Elements will be denoted by lower case letters $a, b, x, y, ...,$ etc., and their ith $(1 \le i \le 2)$ coordinate by $a_i, b_i, x_i, y_i, ...$ etc. The order $(\leqslant)$ on $[n]^2$ is defined as follows: for $x, y \in [n]^2$ we have $x \leqslant y$ if and only if $x_i \le y_i$ for $i = 1, 2$. We define the *grid poset* as the ordering above on the set $[n]^2$ and denote it briefly by $[n]^2$. Here $\mathrm{La}([n]^2, P)$ denotes the size of the largest subset G of $[n]^2$ such that it does not contain a copy of P. Our Theorem 2 will show a close relationship between $\mathrm{La}(C_n, P)$ and $\mathrm{La}([n]^2, P)$.

We denote by V_s the poset on $s+1$ distinct elements $a, b_1, b_2, .., b_s$ with $a < b_1, b_2, ..., b_s$. Let us define k_s to be the maximum k such that $\binom{k+1}{2} \le s$ for $s \ge 1$, $c_s = s - \binom{k+1}{2}$. These numbers were originally defined in [4] in a slightly different context and in a slightly different way.

D. Gerbner, D.T. Nagy, B. Patkós and M. Vizer proved the following theorem.

Theorem A [4] $La([n]^2, V_s) = \left(k_s + \frac{c_s}{k_s+1} + o(1)\right) n.$

Our Theorem 1 claims that $La(C_n, V_s) \le \left(k_s + \frac{c_s}{k_s+1}\right) n$ *with equality if* $k_s + 1$ *divides* c_s. *The combination of our Theorems 1 and 2 gives an easier proof and a more precise estimate on* $La([n]^2, V_s)$. *This will be called Theorem 3.*

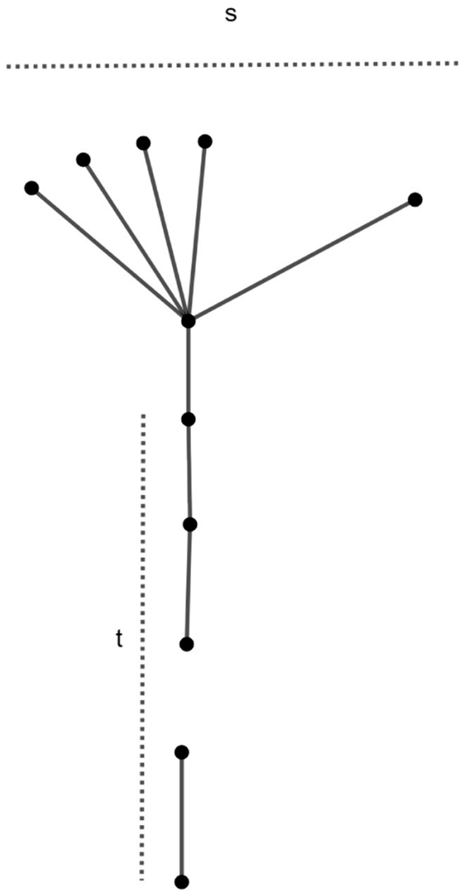
Theorem 4 determines $La(C_n, Y_{s,t})$ *where* $Y_{s,t}$ *is a* V_s *extended with a handle of length* t *(see* Fig. 1*).*

2 Results

If $A \subset I_n$ then min A denotes the set of all elements $x \in A$ such that there is no $y \in A$, $y < x$.

Definition 1 Let C_n be the cycle poset and $A \subseteq I_n$. The *upper set* of A is defined to be the set that consists of all elements of I_n which are greater than or equal to some element of A and is denoted by $A^{\uparrow} = \{y \in I_n : x \le y \quad \text{for} \quad \text{some} \quad x \in A\}$. Let us note that $(\min A)^{\uparrow} = A^{\uparrow}$. In Fig. 2, a part of the Hasse diagram of the cycle poset is shown; of course, the proper picture should be in 3 dimensions, then it is on a cylinder. But when we represent it in 2 dimensions, we either connect the leftmost and rightmost elements between any two neighboring levels or repeat them as in Fig. 2. Let A consist of the bold points and square points. Then min A is the set of square points and the checkered part is $A^{\uparrow}$.

Fig. 1 $Y_{s,t}$



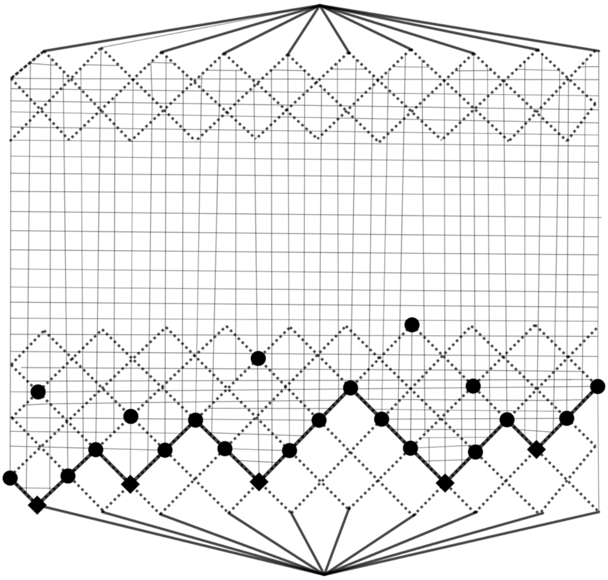


Fig. 2 A part of the Hasse diagram for $n = 9$

Definition 2 Let C_n be the cycle poset, we define *right-i-lines*, denoted by R_i and *left-i-lines* denoted by L_i to be the following sets

$$\begin{aligned} R_i &= \{\{i\}, \{i, i+1\}, \{i, i+1, i+2\}, ..., \{i, i+1, .., n+i-2\}\}, \\ L_i &= \{\{i\}, \{i-1, i\}, \{i-2, i-1, i\}, ..., \{n-i+2, n-i+3, .., i-1, i\}\}, \end{aligned}$$

where the values are modulo n. In Fig. 3, right-i-lines are shown by the hatched lines, and left-i-lines are shown by the bold lines. Here we didn't show the lines crossing the border where the"cylinder" was cut.

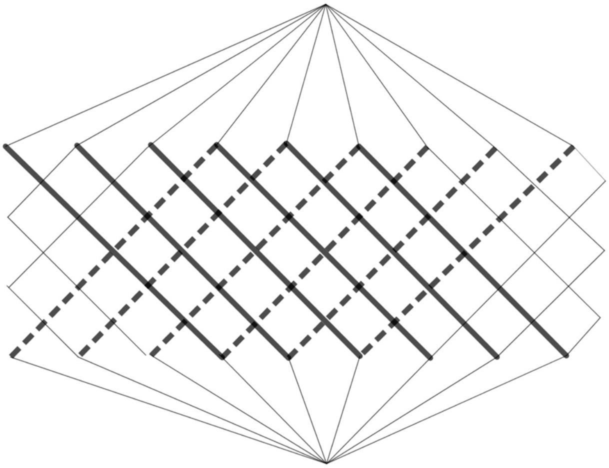


Fig. 3 Right and left-i-lines

Theorem 1 *Suppose that* $s \geq 1$ *and* $n \geq k_s + 1$. *Then* $La(C_n, V_s) \leqslant \left(k_s + \frac{c_s}{k_s+1}\right) n$ *holds, with equality if* $k_s + 1 \mid n$.

Proof First, if $k_s + 1 \mid n$, we need a construction for the lower bound. We take k_s consecutive full levels (note that each level has n elements) and on the $(k_s + 1)$th (as it is shown in Fig. 4, where the bold points give the best construction for $n = 12$ and $s = 8$, where $k_s = 3$ and $c_s = 2$), we choose c_s elements for every $k_s + 1$ consecutive elements in cyclic order in the $(k_s + 1)$st level. It is easy to see that this construction contains no copy of V_s and the number of elements is $k_s n + \left(\frac{c_s}{k_s+1}\right) n$.

Suppose now that F is V_s-free, $\emptyset \not\in F$ and prove $|F| \leqslant \left(k_s + \frac{c_s}{k_s+1}\right) n$. We count the number of pairs (m, f) where $m \in \bigcup_i \{\min(R_i \cap F^{\uparrow})\}$, $f \in F$ and $m \leq f$ in two different ways (Fig. 5 shows the elements of the set F with the bold circle points, $\min F$ with the square points, $R_i \bigcap F^{\uparrow}$ with the cross points, $\min(R_i \bigcap F^{\uparrow})$ with the empty circles. Right-i-lines are shown by hatched lines). First fix $m \in \bigcup_i \{\min\, (R_i \cap F^{\uparrow})\}$. Since m is either in F or above an element of F there can be at most $s - 1$ elements of F above m otherwise a V_s would be formed. Since f can be m itself, the number of elements f that are greater than or equal to m is at most s. Since we have n right-i-lines and every line has at most one element m, the element m can be chosen in at most n different ways. Therefore the number of pairs (m, f) is at most sn.

Secondly, let us fix $f \in F$ and let $\mu(f)$ denote the number of possible m's that are less than or equal to f. From the first part of the proof, we have $\sum \mu(f) \leqslant ns$. Now we try to find the minimum of the left-hand side. The assumption $\emptyset \notin F$ ensures that $\mu(f)$ cannot be 0. The first question is how many of the values $\mu(f)$ can be 1? $\mu(f) = 1$ holds if and only if

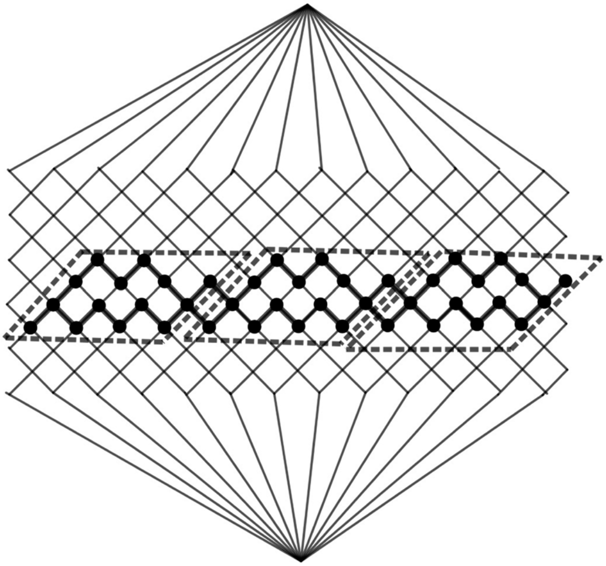


Fig. 4 The best construction without v_8 for $n = 12$

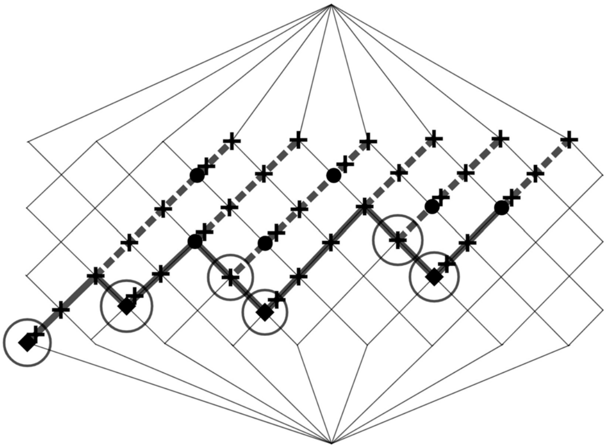


Fig. 5 min F, $R_i \bigcap^{F\uparrow}$, $\min(R_i \bigcap^{F\uparrow})$ and right-i-lines are shown

f is the smallest element of $L_i \bigcap F^{\uparrow}$ for some i. Hence the number of elements f satisfying $\mu(f) = 1$ is at most n. Now the general question is how many of the values $\mu(f)$ can be r? $\mu(f) = r$ holds if and only if f is the rth smallest element of $L_i \bigcap F^{\uparrow}$ for some i. Hence the number of elements f satisfying $\mu(f) = r$ is at most n. Then we have

$$\begin{aligned} n + 2n + 3n + \cdots + rn + (r+1)y &= (1 + 2 + 3 + \cdots + r)n + (r+1)y \\ &= \binom{r+1}{2} n + (r+1)y \leq \sum \mu(f) \leqslant ns, \end{aligned} \quad (1)$$

where

$$| F | = rn + y, \qquad 0 \leqslant y < n. \quad (2)$$

Here (1) implies that $\binom{r+1}{2} \leq s$ and $r \leq k_s$ by definition. If $r \leq k_s - 1$, then (2) leads to a contradiction: $| F | < (k_s - 1)n + n = k_s n$. Hence $r = k_s$ can be supposed. Then we have

$$\binom{k_s+1}{2} n + (k_s+1)y \leq sn \Longrightarrow y \leq \frac{(s - \binom{k_s+1}{2})n}{k_s+1}.$$

But according to the definition $s - \binom{k_s+1}{2} = c_s$, so

$$y \leq \frac{c_s n}{k_s + 1}. \quad (3)$$

From (2) we obtain $| F | \leq k_s n + \frac{c_s n}{k_s+1}$.

On the other hand, if $\emptyset \in F$ (and $s > 1$) then we have the following upper bound,

$$|F| \leq 1 + \mathrm{La}(C_n, V_{s-1}).$$

Here $c_s > 0$ implies $k_{s-1} = k_s$, $c_{s-1} = c_s - 1$ and we obtain

$$|F| \leq 1 + \left(k_s + \frac{c_s - 1}{k_s + 1}\right) n$$

which is really at most $(k_s + \frac{c_s}{k_s+1})n$ if $n \geq k_s + 1$. The case when $c_s = 0$ can be settled similarly.

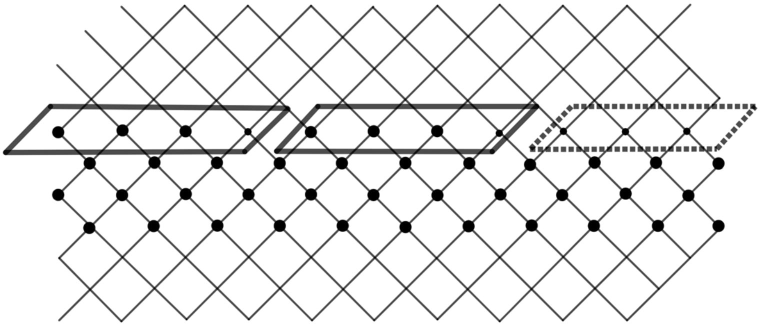


Fig. 6 The construction for the lower bound when $k_s + 1$ does not divide n, $s = 9$ and $n = 11$

Remark Our construction giving the lower bound also works when $k_s + 1$ does not divide n. Let $n = q(k_s + 1) + x$, $(0 \leq x \leq k_s)$. Take k_s consecutive full levels, divide the $(k_s + 1)$st level into parts of length $k_s + 1$, choose the first c_s elements from each part but do not choose any from the last (shorter) part. (See Fig. 6 where $n = 11$, $s = 9$, $k_9 = 3$, $c_9 = 3$, $x = 3$.) This gives the lower bound $(k_s + \frac{c_s}{k_s + 1})n - \frac{xc_s}{k_s + 1}$, where $\frac{xc_s}{k_s + 1} \leq c_s$.

Theorem 2 *Suppose that there is an optimal construction for La(C_n, P) on $w(P) \leq n - 1$ levels, then we have*

$$\mathrm{La}(C_n, P) - \frac{w(P)^2}{4} \leq \mathrm{La}([n]^2, P) \leq \frac{\mathrm{La}(C_{2n}, P)}{2}.$$

Proof It is easy to see that $\mathrm{La}(C_{2n}, P) \geq 2\mathrm{La}([n]^2, P)$ (see Fig. 7, in which two incomparable copies of $[n]^2$ are embedded in C_{2n} for $n = 6$). Hence the right-hand side inequality is clear.

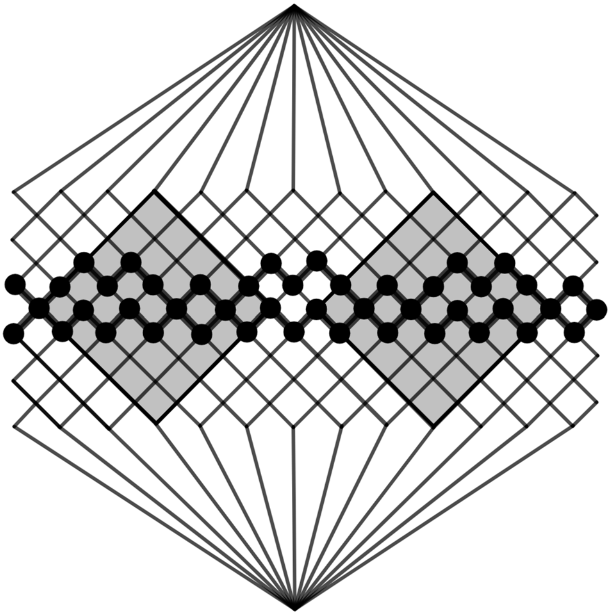


Fig. 7 Two copies of $[6]^2$ in C_{12}

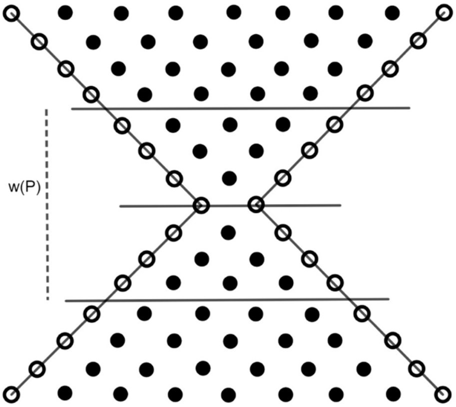


Fig. 8 The difference between the cycle and the grid posets

Now we prove the left-hand side. Observe the construction that is suitable for the cycle poset, is suitable for the grid poset except for the ends (in Fig. 7, these two Hasse diagrams for $P = V_8$ and $n = 6$ are compared).

In Fig. 8, the difference between the cycle poset and the grid poset is shown. The construction on $w(P)$ middle levels in C_n is the basis of the construction for $[n]^2$: We simply delete the elements in the difference of C_n and $[n]^2$. In the worst case all elements in the $w(P)$ middle levels are deleted. Therefore their number is an upper bound for the number of deleted elements.

In the case when $w(P)$ is odd, this is calculated below,

$$2\left(1+2+3+...+\left(\frac{w(P)-1}{2}\right)\right)=\left(\frac{w(P)-1}{2}\right)\cdot\left(\frac{w(P)+1}{2}\right)=\frac{w(P)^2-1}{4}.$$

and in the case when $w(P)$ is even, this is calculated below.

$$\begin{aligned}&\left(1+2+3+\cdots+\left(\frac{w(P)}{2}-1\right)\right)+\left(1+2+3+...+\frac{w(P)}{2}\right)\\ &=\left(\frac{\left(\frac{w(P)}{2}-1\right)\cdot\left(\frac{w(P)}{2}\right)}{2}\right)+\left(\frac{\left(\frac{w(P)}{2}\right)\cdot\left(\frac{w(P)}{2}+1\right)}{2}\right)=\frac{w(P)^2}{4}\end{aligned}$$

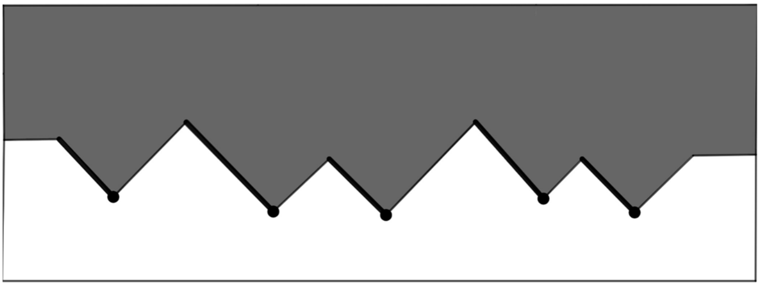


Fig. 9 A part of the set M which is used in the proof of the Theorem 4

The combination of Theorems 1, 2, and the Remark lead to the following sharpening of Theorem A.

Theorem 3 *Suppose that* $s \geq 1$ *and* $n \geq k_s + 1$. *Then*

$$\left(k_s + \frac{c_s}{k_s+1}\right)n - \frac{(k_s+1)^2}{4} - c_s \leq \mathrm{La}([n]^2, V_s) \leq \left(k_s + \frac{c_s}{k_s+1}\right)n.$$

Theorem 4 *Suppose that* $s \geq 1, t \geq 0$ *and* $n \geq k_s + 1$. *Then we have* $La(C_n, Y_{s,t}) \leqslant \left(t + k_s + \frac{c_s}{k_s+1}\right)n$, *with equality if* $k_s + 1 \mid n$.

Proof If $k_s + 1 \mid n$, we need a construction for the lower bound. Take $t + k_s$ full levels (note that each level has n elements) and divide the $(t + k_s + 1)$th level into segments of length $k_s + 1$, choose the first c_s elements from every such segment. It is easy to see that this construction contains no copy of $Y_{s,t}$ and the number of elements is $(t+k_s)n + \left(\frac{c_s}{k_s+1}\right)n$.

Now we prove the upper bound by using induction on t. Let F be a $Y_{s,t}$-free set of elements. Suppose first that $\emptyset \notin F$. For the basic value, $t = 0$ we have the statement from Theorem 1. Now we suppose that the statement is proved for t, and prove it for $t + 1$.

Let us denote the unique element of $\min(R_i \bigcap F^{\uparrow})$ with m_i (unless the set is empty) and collect all m_i's in the set $M = \{m_i \text{ for all } 1 \leqslant i \leqslant n, R_i \bigcap F^{\uparrow} \neq \emptyset\}$ (in Fig. 9 the set M is shown by bold lines, the shaded part denotes $R_i \bigcap F^{\uparrow}$ and minimal elements of F are shown by bold circle points).

Suppose that F is a $Y_{s,t+1}$-free set of elements. Then if we omit all points of M from F, the set $F - M$ will be a $Y_{s,t}$-free set of elements. Then using the induction's assumption we have

$$\mid F - M \mid \leqslant \left(t + k_s + \frac{c_s}{k_s+1}\right)n,$$

Knowing that the size of M is at most n, we are done for $t + 1$ and the proof is completed for this case.

The case when $\emptyset \in F$ is even easier, since then $F - \emptyset$ cannot contain a copy of $Y_{s,t}$.

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Declarations

Conflicts of Interest We declare that the authors have no competing interests as defined by Springer, or other interests that might be perceived to influence the results and/or discussion reported in this paper.

Ethical approval Not applicable.

References

1. Bonis, A.D., Katona, G.O.H., Swanepoel, K.: Largest family without $A \cup B \subset C \cap D$, J. Combin. Theory Ser. A **111**, 331–336 (2005)
2. Erdős, P.L., Frankl, P., Katona, G.O.H.: Extremal hypergraph problems and convex hulls, Combinatorica. **5**, 011-026 (1985)
3. Dániel, G., Patkós, B.: Etremal Finite Set Theory (2019) CRC Press
4. Dániel, G., Nagy, D. T., Patkós, B., Vizer, M.: Forbidden subposet problems in the grid. Discret. Math. **345**(3),(2022)
5. Griggs, J. R., Li W-T .:Progress on poset-free families of subsets, Recent Trends in Combinatorics pp. 317–338 (2016) Springer
6. Katona, G.O.H.: A simple proof of the Erdős–Chao Ko–Rado theorem. J. Combin. Theory Ser. B **13**, 183–184 (1972)
7. Katona, G.O.H.: Forbidden inclusion patterns in the families of subsets (introducing a method), in: Horizons of Combinatorics (Ervin Győri, Gyula O.H. Katona, László Lovász Eds.) Bolyai Society Mathematical Studies, 17, Bolyai János Mathematical Society, pp. 119-140. Budapest and Springer-Verlag, (2008)
8. Katona, G.O.H., Tarján, T.: Extremal problems with excluded subgraphs in the n-cube, Graph Theory, Lagów, 1981, Lecture Notes in Math. 1018 (Springer-Verlag, Berlin,) 084-093 (1983)
9. Sperner, E.: Ein Satz über Untermengen einer endlichen Menge. Math. Z. **27**, 544–548 (1928)