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The matroid of a graphing ☆

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ABSTRACT

Graphings serve as limit objects for bounded-degree graphs. We define the "cycle matroid" of a graphing as a submodular setfunction, with values in $[0, 1]$, which generalizes (up to normalization) the cycle matroid of finite graphs. We prove that for a Benjamini–Schramm convergent sequence of graphs, the total rank, normalized by the number of nodes, converges to the total rank of the limit graphing.

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1. Introduction

Graphings serve as limit objects for bounded-degree graphs [1,15]. They can be defined as bounded-degree graphs on $(0, 1]$, whose edge set is a Borel set, and have a certain measure-preservation property (see Section 4.1 for details). Instead of $(0, 1]$, we could consider any other non-countable standard Borel probability space, since these are all Borel isomorphic by Kuratowski's Theorem (see e.g. Kechris [14], Section 15B). Our goal is to extend the definition of the cycle matroid of a graph to graphings. (For the theory of finite matroids, see Oxley [17] and Schrijver [18].)

Defining matroids on infinite sets is a nontrivial problem. A thorough treatment is given in [7], focusing on the notion of independent sets. When considering graphings, we have to take into account measurability conditions and the uniform measure on points; this makes things more complicated, but also provides important tools like choosing a uniform random point or a uniform random edge, which would not make sense on a countably infinite graph. Accordingly, our approach is more measure-theoretic, very roughly speaking, treating the rank function as a generalized measure.

In the cycle matroid of a finite graph $G = (V, E)$, the rank of a set $X \subseteq E$ is determined by the partition $\mathcal{P}$ of the subgraph (V, X) into connected components: we have $r(X) = |V| - |\mathcal{P}|$. This rank function is submodular:

$$r(X \cup Y) + r(X \cap Y) \le r(X) + r(Y) \qquad (X, Y \subseteq E). \tag{1}$$

Let us also remark that a set X is independent in this matroid if and only if the rank function r, restricted to X, is just the cardinality function.

One could introduce the rank function on the edge-set of a graphing by defining it for finite subsets X of the edge set as in the cycle matroid of the graph formed by X (ignoring isolated nodes). One could extend the rank function to infinite sets of edges by giving it the value ∞. We choose a perhaps more interesting approach, generalizing the *normalized rank function*, defined for a finite graph $G = (V, E)$ as $\rho(X) = r(X)/|V|$. Clearly this function is submodular, with values in $[0, 1]$.

The idea of the generalization is the following observation: Let $\mathcal{P}$ be the partition of the node set V into the connected components of subgraph (V, X). For $v \in V$, we denote by $\mathcal{P}_v$ the partition class containing v. Let $\mathbf{u}$ be a uniform random node of V, then

$$\mathsf{E}\Big(\frac{1}{|\mathcal{P}_{\mathbf{u}}|}\Big) = \frac{|\mathcal{P}|}{|V|} = 1 - \rho(X). \tag{2}$$

This definition can be extended almost verbatim to partitions of graphings into connected components; however, measurability issues and infinite classes make it technically involved to prove submodularity and other properties. We will be interested in partitions with many finite classes: If all, or almost all, partition classes are infinite, then the expectation above will be 0 and $\rho(X) = 1$. Using the appropriate generalization of (2), we can define the "cycle matroid" of a graphing as a submodular setfunction on the Borel subsets of the edge set E, with values in $[0, 1]$. The proof of submodularity is obtained through a more general result about a function defined on measurable partitions of a Borel space.

What are the "independent sets" in this generalized matroid? In the case of finite graphs, an important object is the *matroid polytope*, the convex hull of indicator vectors of subforests, i.e., independent sets in the cycle matroid. The points of this polytope are called *fractional independent sets*. The results of [16] suggest that fractional independent sets for a graphing $\mathbf{G} = (J, \mathcal{B}, E, \lambda)$ could be defined as measures α on Borel subsets of the edge set E, upper bounded by ρ. These measures, called *minorizing measures*, form a convex set $\mathrm{mp}(\mathbf{G})$ in the Banach space of all finite measures on E. The finite analogy suggests that the extremal points of $\mathrm{mp}(\mathbf{G})$ correspond to the subforests of $\mathbf{G}$, but we show by an example that not all subforests give rise to minorizing measures. Analogous questions can be raised for *maximal minorizing measures* $\alpha \in \mathrm{mp}(\mathbf{G})$ with $\alpha(E) = \rho(E)$. We show that certain subforests of the graphing do give rise to extremal points of $\mathrm{mp}(\mathbf{G})$, but we don't have a full characterization of extremal minorizing measures.

A next step should be to develop the limit theory for ρ. There has been interesting work on the convergence of sequences matroids [5,12,6,13], but these don't seem to apply to our case.

A sequence of finite graphs with all degrees bounded by D is said to *locally converge* to a graphing $\mathbf{G}$, if for every $r \geq 1$, the distribution of the r-ball centered at a random node of G_n converges to the distribution of the r-ball centered at a random point of $\mathbf{G}$. (The notion of convergence of a bounded-degree graph sequence is due to Benjamini and Schramm [1], with a different construction for the limit object; see [15] for convergence to graphings.) To develop a limit theory for cycle matroids, we want to show that if a sequence of bounded-degree graphs G_n converges to a graphing $\mathbf{G}$, then the setfunctions ρ_{G_n} converge to the setfunction $\rho_{\mathbf{G}}$ in some sense. This is not resolved in this paper; but we do prove that for a Benjamini–Schramm convergent sequence of graphs, the rank of the edge set, normalized by the number of nodes, converges to the rank of the limit graphing. See the Concluding Remarks for more recent information on this issue.

2. $\mathcal{B}$-partitions

2.1. The semilattice of $\mathcal{B}$-partitions

Let $(J, \mathcal{B})$ be a standard Borel space, let $\mathcal{P}$ be a partition of J, and for $X \subseteq J$, define

$$\mathcal{P}(X) = \cup\{P \in \mathcal{P} : \; P \cap X \neq \emptyset\}.$$

We say that $\mathcal{P}$ is a $\mathcal{B}$-*partition*, if $\mathcal{P}(A) \in \mathcal{B}$ whenever $A \in \mathcal{B}$. For a partition $\mathcal{P}$ we denote the class containing $x \in J$ by $\mathcal{P}_x$. Clearly $\mathcal{P}_x \in \mathcal{B}$ if $\mathcal{P}$ is a $\mathcal{B}$-partition, but this is not a sufficient condition: let $f:\ [0,1) \to [1/2,1)$ be a non-measurable map, then the partition of $[0,1)$ into the pairs $\{x, f(x)\}$ is a partition into sets in $\mathcal{B}$, but not a $\mathcal{B}$-partition. We denote by $\mathcal{P}_{\mathsf{fin}}$ the family of finite partition classes of $\mathcal{P}$. Then $\cup\mathcal{P}_{\mathsf{fin}}$ is the union of finite classes of $\mathcal{P}$.

If $\mathcal{P}$ and $\mathcal{Q}$ are partitions of J, we denote by $\mathcal{P} \wedge \mathcal{Q}$ and $\mathcal{P} \vee \mathcal{Q}$ their meet and join in the partition lattice. We write $\mathcal{P} \le \mathcal{Q}$ if $\mathcal{P}$ is finer than $\mathcal{Q}$, which means that every class of $\mathcal{P}$ is contained in a class of $\mathcal{Q}$. We also say in this case that $\mathcal{Q}$ is coarser than $\mathcal{P}$.

Lemma 2.1. (a) *If $\mathcal{P}$ and $\mathcal{Q}$ are $\mathcal{B}$-partitions, then so is $\mathcal{P} \vee \mathcal{Q}$.* (b) *If, in addition, $\mathcal{Q}$ has a countable number of classes, then $\mathcal{P} \wedge \mathcal{Q}$ is a $\mathcal{B}$-partition.*

Proof. (a) Let $A \in \mathcal{B}$, and define $U_0 = A$, and recursively

$$U_{k+1} = \begin{cases} \mathcal{P}(U_k), & \text{if } k \text{ is even,} \\ \mathcal{Q}(U_k), & \text{if } k \text{ is odd.} \end{cases}$$

Then $U_k \in \mathcal{B}$ for all k, and so $(\mathcal{P} \vee \mathcal{Q})(A) = \cup_k U_k \in \mathcal{B}$.

(b) Let $U = \bigcup_{Q \in \mathcal{Q}} Q \cap \mathcal{P}(Q \cap A)$. We claim that

$$(\mathcal{P} \wedge \mathcal{Q})(A) = \{x \in J :\ \mathcal{P}_x \cap \mathcal{Q}_x \cap A \neq \emptyset\} = U. \tag{3}$$

The first equation is trivial. To verify the second, assume that $x \in J$ has the property that $\mathcal{P}_x \cap \mathcal{Q}_x \cap A \neq \emptyset$. Then there is a $y \in \mathcal{P}_x \cap \mathcal{Q}_x \cap A$. Hence $y \in \mathcal{P}_x$, which implies that $\mathcal{P}_x = \mathcal{P}_y$. Furthermore, $y \in \mathcal{Q}_x \cap A$ implies that $\mathcal{P}_y \subseteq \mathcal{P}(\mathcal{Q}_x \cap A)$, and hence $x \in \mathcal{P}(\mathcal{Q}_x \cap A)$. Since trivially $x \in \mathcal{Q}_x$, we have $x \in \mathcal{Q}_x \cap \mathcal{P}(\mathcal{Q}_x \cap A)$, so $x \in U$. Conversely, if $x \in Q \cap \mathcal{P}(Q \cap A)$ for some $Q \in \mathcal{Q}$, then clearly $Q = \mathcal{Q}_x$, and $x \in \mathcal{P}(\mathcal{Q}_x \cap A)$ means that $\mathcal{P}_x$ intersects $\mathcal{Q}_x \cap A$, so $\mathcal{P}_x \cap \mathcal{Q}_x \cap A \neq \emptyset$.

Clearly $Q \cap \mathcal{P}(Q \cap A) \in \mathcal{B}$, and since $\mathcal{Q}$ has a countable number of classes, (3) implies that $U = (\mathcal{P} \wedge \mathcal{Q})(A) \in \mathcal{B}$. □

Lemma 2.2. *Let $\mathcal{P}$ be a $\mathcal{B}$-partition, $A \in \mathcal{B}$, and let $A_k = \cup\{P \in \mathcal{P} :\ |P \cap A| = k\}$. Then $A_k \in \mathcal{B}$ for $k = 0, 1, \dots, \infty$.*

This lemma implies that $|\mathcal{P}_x|$ is a measurable function of x (which may have infinite values). In particular, the union of k-element partition classes is in $\mathcal{B}$, and hence the union $\cup\mathcal{P}_{\text{fin}}$ of finite classes of $\mathcal{P}$ is in $\mathcal{B}$.

Proof. We may assume that $(J,\mathcal{B})$ is the sigma-algebra of Borel sets in $(0,1)$. Let

$$C_k=\bigcup_{\substack{0=r_0<r_1<\ldots<r_k=1\\ r_i\in\mathbb{Q}}}\bigcap_{i=0}^{k-1}\mathcal{P}\big(A\cap(r_i,r_{i+1})\big).$$

Then C_k is a Borel set, which is the union of partition classes with at least k elements in A. So $A_k=C_k\setminus C_{k+1}$ is also Borel. $\square$

Let us call a set S a *finite-class representative* of a $\mathcal{B}$-partition $\mathcal{P}$, if it is a Borel set, $|S\cap P|=1$ for every finite $P\in\mathcal{P}$, and $|S\cap P|=0$ for every infinite $P\in\mathcal{P}$.

Lemma 2.3. *Every $\mathcal{B}$-partition has a finite-class representative.*

Proof. Again, assume that $(J,\mathcal{B})=(0,1)$, and for every finite class $P\in\mathcal{P}$, let s_P be its minimal element. It suffices to show that $S=\{s_P:\ P\in\mathcal{P}, P \text{ finite}\}$ is a Borel set. Indeed, the set $U_r=\cup\{P\in\mathcal{P}:\ |P\cap(0,r)|=1\}$ is Borel by Lemma 2.2, and then so is $U_r\cap(\cup\mathcal{P}_{\mathsf{fin}})$. Thus the set

$$V_r=U_r\cap(\cup\mathcal{P}_{\mathsf{fin}})\cap(0,r)=\big\{s_P:\ P \text{ finite}, P\cap(0,r)=\{s_P\}\big\},$$

is Borel, and hence $S=\cup_{r\in\mathbb{Q}\cap(0,1)}V_r$ is a Borel set. $\square$

2.2. Re-randomization

Now let π be a probability measure on $(J,\mathcal{B})$, and let $\mathbf{u}$ be a random point from π. If $\mathcal{P}_{\mathbf{u}}$ is finite, then let $\mathbf{v}$ be a uniform random point of P; else, let $\mathbf{v}=\mathbf{u}$. We say that $\mathbf{v}$ is obtained by *re-randomizing* $\mathbf{u}$ *along* $\mathcal{P}$. We say that $\mathcal{P}$ has the *re-randomizing property*, if re-randomizing results in a point distributed according to π.

Another way to express this property is that if μ_x is the uniform distribution on $\mathcal{P}_x$ if $\mathcal{P}_x$ is finite, and $\mu_x=\delta_x$ otherwise, then the mixture of the measurable family $M=(\mu_x:\ x\in J)$ of measures by π is π again. It is easy to check that every partition of a finite set endowed with the uniform distribution has the re-randomizing property.

The significance of the re-randomizing property for us is expressed by the following easy fact:

Lemma 2.4. *Let $\mathcal{P}$ be a $\mathcal{B}$-partition of J with the re-randomizing property. Let $f:\ J\to\mathbb{R}$ be an integrable function such that $f(x)=0$ if $\mathcal{P}_x$ is infinite, and $\sum_{x\in P}f(x)=0$ for every finite class $P\in\mathcal{P}$. Let $\mathbf{x}$ be a random point from π. Then $\mathsf{E}f(\mathbf{x})=0$.*

Proof. Let $\mathbf{y}$ be obtained from $\mathbf{x}$ by re-randomizing along $\mathcal{P}$. Then $\mathsf{E}f(\mathbf{y})=0$, since this holds when conditioning on any choice of $\mathbf{x}$. Since $\mathbf{y}$ has the same distribution as $\mathbf{x}$, the lemma follows. $\square$

Not every $\mathcal{B}$-partition has the re-randomizing property.

Example 2.5. Let $(J, \mathcal{B}, \pi)$ be the $[0,1]$ interval with the Borel sets and the Lebesgue measure. Partition the interval $[0,1)$ into pairs $\{x, 2x+1/3\}$, where $0 \le x < 1/3$. Then for a random point $\mathbf{u}$ from π, we have $\mathsf{P}(\mathbf{u} \in [0,1/3)) = 1/3$, but if we re-randomize to get $\mathbf{v}$, then $\mathsf{P}(\mathbf{v} \in [0,1/3)) = 1/2$.

There are two trivial operations on partitions that preserve the re-randomizing property: we can merge all infinite classes into one, or resolve all infinite classes into singletons. Let us state two less trivial constructions.

Lemma 2.6. *Let $\mathcal{P}$ and $\mathcal{Q}$ be $\mathcal{B}$-partitions of J such that $\mathcal{Q}$ is obtained from $\mathcal{P}$ by splitting some of its finite classes. If $\mathcal{P}$ has the re-randomizing property, then so does $\mathcal{Q}$.*

Proof. Let us do the following experiment: let $\mathbf{u}$ be a random point from π; let $\mathbf{v}$ be obtained from $\mathbf{u}$ by re-randomizing along $\mathcal{P}$; then let $\mathbf{w}$ be obtained by re-randomizing $\mathbf{v}$ along $\mathcal{Q}$. Then (by the re-randomizing property of finite partitions) $\mathbf{w}$ can be obtained by re-randomizing $\mathbf{u}$ along $\mathcal{P}$, so it has distribution π. But it can be obtained by re-randomizing $\mathbf{v}$ along $\mathcal{Q}$. Since $\mathbf{v}$ has distribution π, this proves the lemma. □

Lemma 2.7. *Let $\mathcal{P}$ and $\mathcal{Q}$ be $\mathcal{B}$-partitions with the re-randomizing property. Then so is $\mathcal{P} \vee \mathcal{Q}$.*

Proof. Let us construct a bipartite multigraph H_Z for every finite class Z of $\mathcal{P} \vee \mathcal{Q}$ as follows. Let $V(H_Z) = U \cup W$, where $U = \{P \in \mathcal{P} :\ P \subseteq Z\}$ and $W = \{Q \in \mathcal{Q} :\ Q \subseteq Z\}$. Let $P \in U$ be connected to $Q \in W$ by $|P \cap Q|$ edges. The edges of H_Z can be identified with the elements of Z, and the degree of $X \in V(H_Z)$ is $|X|$. It is also clear that H_Z is connected by the definition of $\mathcal{P} \vee \mathcal{Q}$.

Let $\mathbf{u}$ be a random point of J from the distribution π, and let Z be the partition class of $\mathcal{P} \vee \mathcal{Q}$ containing $\mathbf{u}$. Assume that Z is finite. Let us do a random walk on H_Z starting at $v^0 = \mathcal{P}_{\mathbf{u}}$. Since H_Z is bipartite, the distribution of the node v_k hit in step k will not tend to the stationary distribution, but the distribution of the edge used in step k will tend to the uniform distribution on $E(H_Z) = Z$, as it is easy to see.

Let $(v^0, v^1, \dots)$ be the sequence of edges of H_Z in this random walk; equivalently, $(v^0, v^1, \dots)$ is a sequence of points in Z. The distribution of v^k tends to the uniform distribution on Z as $k \to \infty$. On the other hand, it follows by the re-randomizing property of $\mathcal{P}$ and $\mathcal{Q}$ that for every k, the distribution of v^k is just π. So the limiting distribution of the v^k is also π. But this limiting distribution can be obtained from π by re-randomizing along Z. This proves that $\mathcal{P} \vee \mathcal{Q}$ has the re-randomizing property. □

3. A supermodular function on partitions

Let $\mathcal{P}$ be a $\mathcal{B}$-partition of J. Define

$$\psi(\mathcal{P}) = \mathsf{E}\Big(\frac{1}{|\mathcal{P}_{\mathbf{u}}|}\Big), \tag{4}$$

where $\mathbf{u}$ is chosen from the distribution π. Here $1/|\mathcal{P}_u| = 0$ if $\mathcal{P}_u$ is an infinite set. Clearly $0 \le \psi(\mathcal{P}) \le 1$. Let us note that if J is finite and π is the uniform distribution on J, then

$$\psi(\mathcal{P}) = \mathsf{E}\Big(\frac{1}{|\mathcal{P}_{\mathbf{u}}|}\Big) = \sum_{A\in\mathcal{P}} \frac{|A|}{|J|}\,\frac{1}{|A|} = \frac{|\mathcal{P}|}{|J|}. \tag{5}$$

It is also clear that ψ is a decreasing function on partitions (it is smaller on coarser partitions).

Our main lemma is the following.

Lemma 3.1. *Let $(J, \mathcal{B}, \pi)$ be a standard Borel probability space and let ψ be defined by (4). Let $\mathcal{P}$, $\mathcal{Q}$ and $\mathcal{R}$ be $\mathcal{B}$-partitions with the re-randomizing property such that $\mathcal{R} \le \mathcal{P}$ and $\mathcal{R} \le \mathcal{Q}$. Then*

$$\psi(\mathcal{R}) + \psi(\mathcal{P} \vee \mathcal{Q}) \ge \psi(\mathcal{P}) + \psi(\mathcal{Q}). \tag{6}$$

We see that $\mathcal{R}$ is a "proxy" for $\mathcal{P} \wedge \mathcal{Q}$, which may not be a $\mathcal{B}$-partition. In the next section, when applying the lemma to graphons, we can choose $\mathcal{R}$ to be a natural "meet" of the partitions $\mathcal{P}$ and $\mathcal{Q}$ (which will not be $\mathcal{P} \wedge \mathcal{Q}$).

Proof. $0°$ Let us prove the finite case first, when $\mathcal{B} = 2^J$. We don't have to worry about $\mathcal{P} \wedge \mathcal{Q}$ being a $\mathcal{B}$-partition, so we can replace $\mathcal{R}$ by $\mathcal{P} \wedge \mathcal{Q}$. Let $n = |J|$. We consider a partition $\mathcal{P}$ as a subgraph $G_{\mathcal{P}}$ of the complete graph K_n that is the disjoint union of complete graphs, corresponding to the partition classes. Then $G_{\mathcal{P}\wedge\mathcal{Q}} = G_{\mathcal{P}} \cap G_{\mathcal{Q}}$ and $G_{\mathcal{P}\vee\mathcal{Q}}$ is the transitive closure of $G_{\mathcal{P}} \cup G_{\mathcal{Q}}$. Furthermore, $n(1 - \psi(\mathcal{P}))$ is the rank of $G_{\mathcal{P}}$ in the cycle matroid of K_n. The inequality follows by the submodularity of the rank function of the cycle matroid.

$1°$ As a first step in the proof of the general case, we want to split those classes of $\mathcal{R}$ contained in $\cup\mathcal{Q}_{\mathsf{fin}}$ to singletons, so that the inequality to be proved remains equivalent. To achieve this, we will have to split some classes in $\mathcal{Q}_{\mathsf{fin}}$.

Let S be a finite-class representative for $\mathcal{R}$ (which exists by Lemma 2.3). We split each $Y \in \mathcal{Q}_{\mathsf{fin}}$ into $Y \cap S$ and singleton sets, and let $\mathcal{Q}'$ be the partition of J obtained. Let $\mathcal{R}'$ be obtained by splitting every class of $\mathcal{R}$ contained in $\cup\mathcal{Q}_{\mathsf{fin}}$ to singletons. It is easy to see that $\mathcal{P} \vee \mathcal{Q}' = \mathcal{P} \vee \mathcal{Q}$.

Let $\mathbf{u}$ be a random point from π, and let $\mathbf{v}$ be obtained by re-randomizing $\mathbf{u}$ along $\mathcal{Q}$. If $\mathcal{Q}_{\mathbf{u}}$ is finite, then

$$\frac{1}{|(\mathcal{P}\vee\mathcal{Q}')_{\mathbf{v}}|}=\frac{1}{|(\mathcal{P}\vee\mathcal{Q})_{\mathbf{v}}|},\qquad \text{and}\qquad \frac{1}{|\mathcal{R}'_{\mathbf{v}}|}=1.$$

Using (5), we have

$$\mathsf{E}_{\mathbf{v}}\Big(\frac{1}{|\mathcal{Q}'_{\mathbf{v}}|}\Big)=\frac{|\mathcal{Q}_{\mathbf{u}}\setminus S|+1}{|\mathcal{Q}_{\mathbf{u}}|},\qquad \text{and}\qquad \mathsf{E}_{\mathbf{v}}\Big(\frac{1}{|\mathcal{Q}_{\mathbf{v}}|}\Big)=\frac{1}{|\mathcal{Q}_{\mathbf{u}}|}.$$

Furthermore,

$$\mathsf{E}_{\mathbf{v}}\Big(\frac{1}{|\mathcal{R}_{\mathbf{v}}|}\Big)=\frac{\big|\mathcal{R}|_{\mathcal{Q}_{\mathbf{u}}}\big|}{|\mathcal{Q}_{\mathbf{u}}|}=\frac{|\mathcal{Q}_{\mathbf{u}}\cap S|}{|\mathcal{Q}_{\mathbf{u}}|}.$$

Combining these equations, we see that

$$\begin{aligned}
&\mathsf{E}_{\mathbf{v}}\Big(\frac{1}{|(\mathcal{P}\vee\mathcal{Q}')_{\mathbf{v}}|}\Big)+\mathsf{E}_{\mathbf{v}}\Big(\frac{1}{|\mathcal{R}'_{\mathbf{v}}|}\Big)-\mathsf{E}_{\mathbf{v}}\Big(\frac{1}{|\mathcal{P}_{\mathbf{v}}|}\Big)-\mathsf{E}_{\mathbf{v}}\Big(\frac{1}{|\mathcal{Q}'_{\mathbf{v}}|}\Big)\\
&\quad=\mathsf{E}_{\mathbf{v}}\Big(\frac{1}{|(\mathcal{P}\vee\mathcal{Q})_{\mathbf{v}}|}\Big)+1-\mathsf{E}_{\mathbf{v}}\Big(\frac{1}{|\mathcal{P}_{\mathbf{v}}|}\Big)-\frac{|\mathcal{Q}_{\mathbf{u}}\setminus S|+1}{|\mathcal{Q}_{\mathbf{u}}|}\\
&\quad=\mathsf{E}_{\mathbf{v}}\Big(\frac{1}{|(\mathcal{P}\vee\mathcal{Q})_{\mathbf{v}}|}\Big)+\frac{|\mathcal{Q}_{\mathbf{u}}\cap S|}{|\mathcal{Q}_{\mathbf{u}}|}-\mathsf{E}_{\mathbf{v}}\Big(\frac{1}{|\mathcal{P}_{\mathbf{v}}|}\Big)-\frac{1}{|\mathcal{Q}_{\mathbf{u}}|}\\
&\quad=\mathsf{E}_{\mathbf{v}}\Big(\frac{1}{|(\mathcal{P}\vee\mathcal{Q})_{\mathbf{v}}|}\Big)+\mathsf{E}_{\mathbf{v}}\Big(\frac{1}{|\mathcal{R}_{\mathbf{v}}|}\Big)-\mathsf{E}_{\mathbf{v}}\Big(\frac{1}{|\mathcal{P}_{\mathbf{v}}|}\Big)-\mathsf{E}_{\mathbf{v}}\Big(\frac{1}{|\mathcal{Q}_{\mathbf{v}}|}\Big).
\end{aligned}$$

If $\mathcal{Q}_{\mathbf{u}}$ is infinite, then $\mathbf{v}=\mathbf{u}$, $\mathcal{R}_{\mathbf{v}}=\mathcal{R}'_{\mathbf{v}}$ and $\mathcal{Q}_{\mathbf{v}}=\mathcal{Q}'_{\mathbf{v}}$, and the equation also holds. By Lemma 2.4, this implies that this equation holds when $\mathbf{v}$ is replaced by $\mathbf{u}$, and so

$$\psi(\mathcal{R}')+\psi(\mathcal{P}\vee\mathcal{Q}')-\psi(\mathcal{P})-\psi(\mathcal{Q}')=\psi(\mathcal{R})+\psi(\mathcal{P}\vee\mathcal{Q})-\psi(\mathcal{P})-\psi(\mathcal{Q}).$$

We may repeat this argument with the roles of $\mathcal{P}$ and $\mathcal{Q}$ interchanged. So we may assume that every class of $\mathcal{R}$ contained in a finite class of either $\mathcal{P}$ or $\mathcal{Q}$ is a singleton.

2° Next, we get rid of the non-singleton finite classes of $\mathcal{P}\vee\mathcal{Q}$. Let Z be the union of all finite classes of $\mathcal{P}\vee\mathcal{Q}$, and let $\mathcal{P}'$ and $\mathcal{Q}'$ be obtained by splitting each class of $\mathcal{P}$ and $\mathcal{Q}$ contained in Z to singletons. It is easy to see that $Z\in\mathcal{B}$, and that both $\mathcal{P}'$ and $\mathcal{Q}'$ are $\mathcal{B}$-partitions. By Lemma 2.6, both $\mathcal{P}'$ and $\mathcal{Q}'$ have the re-randomizing property.

We can write the inequality to be proved as

$$\int_J f_{\mathcal{P},\mathcal{Q}}(x)\,d\pi(x)\ge 0, \tag{7}$$

where

$$f(x)=f_{\mathcal{P},\mathcal{Q}}(x)=\frac{1}{|\mathcal{R}_x|}+\frac{1}{|(\mathcal{P}\vee\mathcal{Q})_x|}-\frac{1}{|\mathcal{P}_x|}-\frac{1}{|\mathcal{Q}_x|}.$$

Let $f_1 = f_{\mathcal{P}',\mathcal{Q}'}$. Then $f_1(x) = f(x)$ if $x \in J \setminus Z$, and $f_1(x) = 0$ if $x \in Z$. By the finite case treated above, $\int_Z f(x)\,d\pi(x) \geq 0$, and hence

$$\int_J f(x)\,d\pi(x) \geq \int_J f_1(x)\,d\pi(x).$$

So we may replace $(\mathcal{P}, \mathcal{Q})$ by $(\mathcal{P}', \mathcal{Q}')$, making the inequality to be proved tighter. In other words, we may assume that every class of $\mathcal{P} \vee \mathcal{Q}$ is either infinite or a singleton set.

3° Our next goal is to get rid of non-singleton finite classes of $\mathcal{P}$ containing a singleton class of $\mathcal{Q}$, or vice versa. Let S and T be the unions of singleton classes of $\mathcal{P}$ and $\mathcal{Q}$, respectively. For $Y \in \mathcal{Q}$, define

$$Y' = \begin{cases} Y \setminus S, & \text{if } Y \text{ is finite,} \\ Y, & \text{otherwise.} \end{cases}$$

If $|Y| > 1$, then the class of $\mathcal{P} \vee \mathcal{Q}$ containing Y is not a singleton, and hence it is infinite by 2°. This implies that Y must intersect at least one non-singleton class of $\mathcal{P}$, and so $Y' \neq \emptyset$. Let the partition $\mathcal{Q}'$ consist of the nonempty sets Y' ($Y \in \mathcal{Q}$), along with those singleton subsets of S that are contained in finite classes of $\mathcal{Q}$. We leave $\mathcal{R}$ unchanged.

We argue similarly as in 1°. Let $\mathbf{u}$ be a random point from π, and let $\mathbf{v}$ be obtained by re-randomizing $\mathbf{u}$ along $\mathcal{Q}$. Let $Y = \mathcal{Q}_{\mathbf{u}}$. If Y is finite, then by 1°,

$$\frac{1}{|\mathcal{R}_{\mathbf{v}}|} = 1, \qquad \text{and} \qquad \frac{1}{|\mathcal{Q}_{\mathbf{v}}|} = \frac{1}{|Y|},$$

and by 2°,

$$\mathsf{E}_{\mathbf{v}}\Big(\frac{1}{|(\mathcal{P} \vee \mathcal{Q})_{\mathbf{v}}|}\Big) = 0.$$

Furthermore, by (5),

$$\mathsf{E}_{\mathbf{v}}\Big(\frac{1}{|\mathcal{Q}'_{\mathbf{v}}|}\Big) = \frac{|Y \setminus Y'| + 1}{|Y|},$$

and

$$\mathsf{E}_{\mathbf{v}}\Big(\frac{1}{|(\mathcal{P} \vee \mathcal{Q}')_{\mathbf{v}}|}\Big) = \frac{|Y \setminus Y'|}{|Y|}.$$

Combining these equations, we get for finite Y,

$$
\begin{aligned}
&\mathsf{E}_{\mathbf{v}}\Big(\frac{1}{|(\mathcal{P}\vee\mathcal{Q}')_{\mathbf{v}}|}\Big)+\mathsf{E}_{\mathbf{v}}\Big(\frac{1}{|\mathcal{R}_{\mathbf{v}}|}\Big)-\mathsf{E}_{\mathbf{v}}\Big(\frac{1}{|\mathcal{P}_{\mathbf{v}}|}\Big)-\mathsf{E}_{\mathbf{v}}\Big(\frac{1}{|\mathcal{Q}'_{\mathbf{v}}|}\Big)\\
&\quad=\frac{|Y\setminus Y'|}{|Y|}+1-\mathsf{E}_{\mathbf{v}}\Big(\frac{1}{|\mathcal{P}_{\mathbf{v}}|}\Big)-\frac{|Y\setminus Y'|+1}{|Y|}=0+1-\mathsf{E}_{\mathbf{v}}\Big(\frac{1}{|\mathcal{P}_{\mathbf{v}}|}\Big)-\frac{1}{|Y|}\\
&\quad=\mathsf{E}_{\mathbf{v}}\Big(\frac{1}{|(\mathcal{P}\vee\mathcal{Q})_{\mathbf{v}}|}\Big)+\mathsf{E}_{\mathbf{v}}\Big(\frac{1}{|\mathcal{R}_{\mathbf{v}}|}\Big)-\mathsf{E}_{\mathbf{v}}\Big(\frac{1}{|\mathcal{P}_{\mathbf{v}}|}\Big)-\mathsf{E}_{\mathbf{v}}\Big(\frac{1}{|\mathcal{Q}_{\mathbf{v}}|}\Big).
\end{aligned}
$$

If Y is infinite, then $\mathbf{v}=\mathbf{u}$ and $\mathcal{Q}'_{\mathbf{u}}=\mathcal{Q}_{\mathbf{u}}$, so this equation also holds. As before, this shows that

$$
\begin{aligned}
&\psi(\mathcal{P}\wedge\mathcal{Q}')+\psi(\mathcal{P}\vee\mathcal{Q}')-\psi(\mathcal{P})-\psi(\mathcal{Q}')\\
&\qquad=\psi(\mathcal{P}\wedge\mathcal{Q})+\psi(\mathcal{P}\vee\mathcal{Q})-\psi(\mathcal{P})-\psi(\mathcal{Q}).
\end{aligned}
$$

Carrying out the same process with $\mathcal{P}$ and $\mathcal{Q}$ interchanged, we may assume that there is no point contained in a singleton class in one partition and in a non-singleton but finite class in the other.

4° To sum up, 3° implies that if a point x is contained in a singleton class of one of the partitions $\mathcal{P}$ or $\mathcal{Q}$, then it is contained either in a singleton class or in an infinite class of the other. So only four possibilities remain: the classes of $\mathcal{P}$ and $\mathcal{Q}$ containing x are (i) both singleton classes, or (ii) one singleton class and one infinite, or (iii) two infinite classes, or (iv) two non-singleton classes, at least one of which is finite. Note that in cases (i), (ii) and (iv), $|\mathcal{R}_x|=1$ by 1°.

In these four cases we have, respectively,

$$
\frac{1}{|\mathcal{R}_x|}+\frac{1}{|(\mathcal{P}\vee\mathcal{Q})_x|}-\frac{1}{|\mathcal{P}_x|}-\frac{1}{|\mathcal{Q}_x|}
\begin{cases}
=1+1-1-1=0, & \text{if (i) holds,}\\
=1+0-1-0=0, & \text{if (ii) holds,}\\
\ge 0+0-0-0=0, & \text{if (iii) holds,}\\
\ge 1+0-\frac12-\frac12=0 & \text{if (iv) holds.}
\end{cases}
$$

This proves the Lemma. □

4. Cycle matroids of graphings

4.1. Graphing basics

A *graphing* is a quadruple $\mathbf{G}=(J,\mathcal{B},E,\lambda)$, where $(J,\mathcal{B})$ is a standard Borel sigma-algebra, E is a Borel subset of $J\times J$ that is symmetric (invariant under interchanging the coordinates), λ is a probability measure on $(J,\mathcal{B})$, and a "measure-preservation" condition holds. To formulate this condition, we consider $\mathbf{G}$ as an undirected graph on vertex set J, where each edge $\{x,y\}$ is represented by two points (x,y) and (y,x) in E. It is assumed that all degrees of the graph (J,E) are finite and bounded by an integer

$D \geq 0$. Let $\deg(u) = \deg_{\mathbf{G}}(u)$ denote the degree of $u \in J$ in the graph $\mathbf{G}$; for $B \subseteq J$, let $\deg_B(u)$ denote the number of edges from the point $u \in J$ to B; for $X \subseteq E$, let $\deg_X(u)$ denote the number of edges in X incident with u. For $x \in J$, we denote by $\mathbf{G}_x$ the connected component of $\mathbf{G}$ containing x.

Now the measure preservation equation can be expressed as

$$\int_A \deg_B(x)\,d\lambda(x) = \int_B \deg_A(x)\,d\lambda(x) \qquad (A, B \in \mathcal{B}). \tag{8}$$

This condition is equivalent to saying that for a random point $\mathbf{x}$ from distribution λ, the (connected rooted) graph $(G_{\mathbf{x}}, \mathbf{x})$ is an involution-invariant random rooted graph as defined by Benjamini and Schramm [1].

The quantity

$$\overline{d} = \overline{d}_{\mathbf{G}} = \int_J \deg(u)\,d\lambda(u)$$

is the average degree of the graphing. The setfunction

$$\eta(A \times B) = \int_A \deg_B(x)\,d\lambda(x)$$

extends to a measure on $\mathcal{B} \times \mathcal{B}$, which we call the *edge measure* of $\mathbf{G}$. Clearly $\eta = \eta_{\mathbf{G}}$ is symmetric (invariant under interchanging the coordinates) and supported on E. We can express $\eta(X)$ for every Borel set $X \subseteq E$ as

$$\eta(X) = \int_J \deg_X(u)\,d\lambda(u). \tag{9}$$

Obviously, $(1/\overline{d})\eta$ is a probability measure on E, so we can talk about a "random edge" of $\mathbf{G}$.

In this paper, a *subgraphing* of a graphing $\mathbf{G} = (J, \mathcal{B}, E, \lambda)$ is a 4-tuple $\mathbf{H} = (J, \mathcal{B}, F, \lambda)$, where $F \subseteq E$ is a Borel set. It is not entirely obvious that $\mathbf{H}$ satisfies the graphing axioms; see Lemma 18.19 in [15]. Equation (9) implies that the edge measure of a subgraphing is the restriction of the edge measure of the graphing:

$$\eta_{\mathbf{H}}(X) = \eta(X) \qquad (X \in \mathcal{B} \times \mathcal{B}, X \subseteq F). \tag{10}$$

4.2. The rank function

An obvious way to define a matroid rank function of a graphing would be to consider finite subsets of edges, and consider the rank function $r(X)$ of the cycle matroid of the graph they form. Every infinite set will have infinite rank. We explore a more interesting possibility, generalizing the *normalized rank* of the cycle matroid of a graph $G = (V, E)$ on n nodes.

Let $c(G)$ denote the number of connected components of a graph G. Recalling that $r(X) = |V| - c(V, X)$, we define

$$\rho(X) = \frac{r(X)}{|V|} = 1 - \frac{c(V, X)}{|V|}.$$

This number is in $[0, 1]$. By (5), we can express it as

$$\rho(X) = 1 - \mathsf{E}_{\mathbf{u}}\Big(\frac{1}{|X_{\mathbf{u}}|}\Big), \tag{11}$$

where $X_{\mathbf{u}}$ is the set of nodes of the connected component of (V, X) containing $\mathbf{u}$.

Now this setfunction makes sense in the graphing case as well. Let $\mathbf{G} = (J, \mathcal{B}, E, \lambda)$ be a graphing. For every Borel set $X \subseteq E$, the quadruple $\mathbf{G}^X = (J, \mathcal{B}, X, \lambda)$ is a graphing. We denote by $\mathcal{P}^X$ the partition of J into the connected components of $\mathbf{G}^X$. Then we can define

$$\rho(X) = \rho_{\mathbf{G}}(X) = 1 - \mathsf{E}_{\mathbf{u}}\Big(\frac{1}{|V(\mathbf{G}^X_{\mathbf{u}}|}\Big) = 1 - \psi(\mathcal{P}^X), \tag{12}$$

where $\mathbf{u}$ is a random point from λ. We can see from this definition right away a nice property of $\rho(X)$, namely that it is an intrinsic parameter of the edge set X: if $\mathbf{H}$ is a subgraphing of $\mathbf{G}$, and X is a Borel subset of $E(\mathbf{H})$, then $\rho_{\mathbf{H}}(X) = \rho_{\mathbf{G}}(X)$.

Our goal is to apply the results of Section 3 to the partitions of J into the connected components of sub-graphings, showing that $1-\rho$ is supermodular, and so ρ is submodular. For this, we need a simple (but not quite trivial) lemma.

Lemma 4.1. *Let $\mathbf{G} = (J, \mathcal{B}, E, \lambda)$ be a graphing, and let $\mathcal{P}$ be the partition of J into the connected components of $\mathbf{G}$. Then $\mathcal{P}$ is a $\mathcal{B}$-partition with the re-randomizing property.*

Proof. The fact that $\mathcal{P}$ is a $\mathcal{B}$-partition, follows e.g. by Lemma 18.2 in [15], which asserts that if $A \in \mathcal{B}$, then the neighbors of A form a Borel set. Repeating this argument, we obtain that the set A_r of nodes that are at a distance at most r from A is a Borel set, and hence $\mathcal{P}(A) = \cup_r A_r$ is Borel.

To prove that $\mathcal{P}$ has the re-randomizing property, let $\mathbf{u}$ be a random point from λ, let $\mathbf{v}$ be obtained from $\mathbf{u}$ by re-randomizing along $\mathcal{P}$, and let λ' be the distribution of $\mathbf{v}$. Let $B = \cup\mathcal{P}_{\mathsf{fin}}$. Since $\lambda'|_{J\setminus B} = \lambda|_{J\setminus B}$ by definition, it suffices to show that $\lambda'(A) = \lambda(A)$ for all $A \in \mathcal{B}$, $A \subseteq B$. For $x, y \in J$, define

$$f(x, y) = \begin{cases} 1/|V(\mathbf{G}_x)|, & \text{if } x \in B \text{ and } y \in A \cap V(G_x), \\ 0, & \text{otherwise.} \end{cases}$$

Then

$$\sum_y f(x,y) = \begin{cases} \dfrac{|V(\mathbf{G}_x)\cap A|}{|V(\mathbf{G}_x)|}, & \text{if } x\in B,\\ 0, & \text{otherwise,}\end{cases}$$

and

$$\sum_x f(x,y) = \begin{cases} 1, & \text{if } y\in A,\\ 0, & \text{otherwise.}\end{cases}$$

By the Mass Transport Principle (in the form of [15], Proposition 18.49),

$$\begin{aligned}\lambda(A) &= \int\limits_J \sum_x f(x,y)\,d\lambda(y) = \int\limits_J \sum_y f(x,y)\,d\lambda(x)\\ &= \int\limits_B \frac{|V(\mathbf{G}_x)\cap A|}{|V(\mathbf{G}_x)|}\,d\lambda(x) = \mathsf{P}(u\in B, u'\in A) = \lambda'(A).\end{aligned}$$

This proves the re-randomizing property of $\mathcal{P}$. □

Theorem 4.2. *The setfunction ρ defined on the Borel subsets of E by (12) is increasing and submodular, and it satisfies the inequalities*

$$\frac{1}{1+D}\eta(X) \le \rho(X) \le \eta(X).$$

Proof. It is trivial that ρ is increasing. To prove submodularity, we use Lemma 3.1. We want to prove that $\psi(\mathcal{P}^X) = 1-\rho(X)$ (as defined by (4)) is a supermodular setfunction of X:

$$\psi(\mathcal{P}^{X\cup Y}) + \psi(\mathcal{P}^{X\cap Y}) \ge \psi(\mathcal{P}^X) + \psi(\mathcal{P}^Y). \tag{13}$$

Here $\mathcal{P}^{X\cup Y}$ etc. are $\mathcal{B}$-partitions with the re-randomizing property by Lemma 4.1. Furthermore, $\mathcal{P}^{X\cup Y} = \mathcal{P}^X \vee \mathcal{P}^Y$. In general, $\mathcal{P}^{X\cap Y} \neq \mathcal{P}^X \wedge \mathcal{P}^Y$, but $\mathcal{R} = \mathcal{P}^{X\cap Y}$ satisfies $\mathcal{R}\le\mathcal{P}^X$ and $\mathcal{R}\le\mathcal{P}^Y$, so Lemma 3.1 proves (13).

To prove the bounds on ρ, clearly $|V(\mathbf{G}_u^X)| \ge 1+\deg_X(u)$, and hence

$$1-\frac{1}{|V(\mathbf{G}_u^X)|} \ge 1-\frac{1}{1+\deg(u,X)} = \frac{\deg(u,X)}{1+\deg(u,X)} \ge \frac{\deg(u,X)}{1+D}.$$

On the other hand,

$$1-\frac{1}{|V(\mathbf{G}_u^X)|} \le \deg(u,X),$$

since both sides are zero if $\deg(u, X) = 0$, and the left hand side is at most 1 and the right hand side is at least 1 otherwise. Taking expectation, and using (9), we get the inequalities in the theorem. □

4.3. Bases and minorizing measures

Bases of the cycle matroid of a finite graph are spanning forests, so we expect that bases of the matroid of a graphing (if meaningful at all) will be connected to spanning forests. We say that a graphing is *acyclic*, or a *forest*, if it contains no (finite) cycle.

It is suggested in [16] that the "independent sets" for a general increasing submodular setfunction φ with $\varphi(\emptyset) = 0$ should be certain *minorizing charges*, i.e., (nonnegative) finitely additive measures α such that $0 \le \alpha \le \varphi$. In our case, Theorem 4.2 implies that $\alpha \le \rho \le \eta$. Since η is countably additive (σ-additive), it follows that α itself is countably additive, i.e., a measure in the usual sense (see e.g. [4], Proposition 2.3.2). We say that a minorizing measure α is *maximal*, if $\alpha(E) = \varphi(E)$.

One would expect that every Borel measurable spanning forest in a graphing gives rise to a maximal minorizing measure, analogously to the finite case. This does not hold for graphings in general (see Example 4.5), but we are going to prove this at least for certain special subforests (cf. also the Concluding Remarks).

We say that a graphing $\mathbf{G} = (J, \mathcal{B}, E, \lambda)$ is *hyperfinite*, if for every $\varepsilon > 0$ there is a Borel set $X \subseteq E$ with $\eta(X) \le \varepsilon$ such that all connected components of $\mathbf{G} \setminus X$ are finite. As an example, let $a \in (0, 1)$ be an irrational number, and consider the graphing $\mathbf{G}_a$ on the interval $[0, 1)$ in which every point x is connected to the two points $x \pm a \pmod 1$. Then every connected component of $\mathbf{G}_a$ is infinite, but $\mathbf{G}_a$ is hyperfinite, since for every $\varepsilon > 0$, deleting the points in the interval $(0, \varepsilon)$, every connected component will be finite.

Lemma 4.3. *Let $\mathbf{F} = (J, \mathcal{B}, E, \lambda)$ be a hyperfinite acyclic graphing. Then for all Borel sets $U \subseteq E$, we have $\rho(U) = \eta(U)/2$.*

In particular, ρ is not only submodular, but modular, and hence, a measure.

Proof. First, assume that every connected component Q of $\mathbf{F}$ is a finite tree. Then $U \cap E(Q)$ is independent in the cycle matroid of Q, and hence

$$\rho_Q(U \cap E(Q)) = \frac{|U \cap E(Q)|}{|V(Q)|}.$$

The quantities on both sides can be expressed using a random point $\mathbf{q}$ of $V(Q)$.

$$\rho_Q(U \cap E(Q)) = \mathsf{E}_{\mathbf{q}}\Big(1 - \frac{1}{|U_{\mathbf{q}}|}\Big),$$

and

$$|U \cap E(Q)| = \frac{1}{2} \sum_{q \in V(Q)} \deg_U(q) = \frac{|V(Q)|}{2} \mathsf{E}_{\mathbf{q}} \deg_U(\mathbf{q}).$$

Thus

$$\mathsf{E}_{\mathbf{q}} \Big(1 - \frac{1}{|U_{\mathbf{q}}|} - \frac{1}{2} \deg_U(\mathbf{q}) \Big) = 0.$$

Since this holds for every connected component Q of $\mathbf{F}$, Lemma 2.4 implies that the same relation holds for a random point $\mathbf{x}$ from λ:

$$\mathsf{E}_{\mathbf{x}} \Big(1 - \frac{1}{|U_{\mathbf{x}}|} - \frac{1}{2} \deg_U(\mathbf{x}) \Big) = 0.$$

But here

$$\mathsf{E}_{\mathbf{x}} \Big(1 - \frac{1}{|U_{\mathbf{x}}|} \Big) = \rho(U) \qquad \text{and} \qquad \frac{1}{2} \mathsf{E}_{\mathbf{x}} \big(\deg_U(\mathbf{x}) \big) = \frac{1}{2} \eta(U)$$

This settles the case when all components of $\mathbf{F}$ are finite.

Now let $\mathbf{F}$ be any hyperfinite forest. Fix an $\varepsilon > 0$. By hyperfiniteness, there is a Borel set $X \subseteq E$ such that $\eta(X) < \varepsilon$ and every connected component of $\mathbf{H} = \mathbf{F} \setminus X$ is finite. By Theorem 4.2, we have $\rho(U \cap X) \le \eta(U \cap X) \le D\varepsilon$, and so

$$\rho(U \setminus X) \le \rho(U) \le \rho(U \setminus X) + \rho(U \cap X) \le \rho(U \setminus X) + \varepsilon.$$

As noted before, we have $\rho(U \setminus X) = \rho_{\mathbf{H}}(U \setminus X)$. Hence by the special case proved above and by (10),

$$\rho(U \setminus X) = \frac{1}{2} \eta_{\mathbf{H}}(U \setminus X) = \frac{1}{2} \eta(U \setminus X),$$

and so

$$\rho(U) \le \rho(U \setminus X) + D\varepsilon = \frac{1}{2} \eta(U \setminus X) + \varepsilon \le \frac{1}{2} \eta(U) + \varepsilon,$$

while

$$\rho(U) \ge \rho(U \setminus X) = \frac{1}{2} \eta(U \setminus X) \ge \frac{1}{2} \eta(U) - \varepsilon.$$

Letting $\varepsilon \to 0$, the lemma follows. □

Recall that a minorizing measure is *extremal*, if it cannot be written as the average of two different minorizing measures. Equivalently, there is no nonzero signed measure β on $\mathcal{B}$ such that $-\alpha \le \beta \le \alpha$ and $\alpha - \rho \le \beta \le \rho - \alpha$.

Theorem 4.4. *Let* $\mathbf{G}$ *be a hyperfinite graphing and* $\mathbf{H} = (J, F)$*, a Borel measurable spanning subforest of* $\mathbf{G}$*. Then*

$$\alpha(X) = \frac{1}{2}\eta_{\mathbf{G}}(X \cap F)$$

defines an extremal maximal minorizing measure on the Borel subsets of E*.*

Proof. It is clear that α defines a measure. Also, trivially $\mathbf{H}$ is hyperfinite. Using (10) and Lemma 4.3, we get

$$\alpha(X) = \frac{1}{2}\eta(X \cap F) = \frac{1}{2}\,\eta_{\mathbf{H}}(X \cap F) = \rho(X \cap F),$$

and hence $\alpha(X) \le \rho(X)$. It is clear that $\alpha(E) = \alpha(F) = \rho(F) = \rho(E)$, so α is maximal. To show that it is extremal, consider a signed measure β on $\mathcal{B}$ such that $-\alpha \le \beta \le \alpha$ and $\alpha - \rho \le \beta \le \rho - \alpha$. For every Borel set $X \subseteq E$, we have

$$\alpha(X \setminus F) = \alpha(X) - \alpha(X \cap F) = 0,$$

and so β is zero on subsets of $E \setminus F$. Furthermore, $\alpha = \rho$ on subsets of F, so β is zero on subsets of F. Thus $\beta = 0$. □

We cannot drop the hyperfiniteness assumption in Lemma 4.3.

Example 4.5. Let $D = 2r - 1$ be an odd integer ($r \ge 3$), and let $\mathbf{G}$ be a graphing such that every connected component is a D-regular tree (such graphings exist, for example the Bernoulli graphing of a D-regular rooted tree, see e.g. [15]). By a theorem of Csóka, Lippner and Pikhurko [9], the edge set of $\mathbf{G}$ has a partition $E_0 \cup E_1 \cup \cdots \cup E_{D+1}$ such that $E_i \in \mathcal{B} \times \mathcal{B}$, $E_1, \dots, E_{D+1}$ are matchings, and E_0 covers a set S_0 of measure 0. Let $U = E_1 \cup \cdots \cup E_r$ and $W = E \setminus U$. Then every $u \in J \setminus S_0$ misses at most one of $E_1, \dots, E_{D+1}$. Hence the connected component of (J, U) containing u has at least r nodes, and the same holds for (J, W). This implies that

$$\rho(U) + \rho(W) \ge 2\big(1 - \frac{1}{r}\big) = 2 - \frac{2}{r} > 1 = \rho(U \cup W).$$

So ρ is not even additive.

4.4. Convergence of the total rank

A next step should be to develop the limit theory for ρ; in other words, to show that if a graph sequence $(G_n : n = 1, 2, \dots)$ with uniformly bounded degrees converges locally or locally-globally to a graphing $\mathbf{G}$, then the setfunctions ρ_{G_n} converge to the setfunction $\rho_{\mathbf{G}}$ in some sense. For local-global convergence, this is worked out in [2,3] (see the Concluding Remarks).

In this paper, we prove a weaker fact but under the more general notion of local (Benjamini–Schramm) convergence of bounded-degree graphs: the total rank converges. Given a graphing $\mathbf{G} = (J, \mathcal{B}, E, \lambda)$, we define its *total rank* as

$$\overline{\rho}(\mathbf{G}) = \rho_{\mathbf{G}}(E(\mathbf{G})) = 1 - \mathsf{E}\Big(\frac{1}{|V(\mathbf{G}_{\mathbf{u}})|}\Big), \tag{14}$$

where $\mathbf{u}$ is a random point from λ. The total rank of a finite graph is defined analogously.

Theorem 4.6. *If $G_1, G_2, \dots$ is a sequence of finite graphs with all degrees bounded by $D \ge 0$ locally converging to a graphing $\mathbf{G} = (J, \mathcal{B}, E, \lambda)$, then $\overline{\rho}(G_n) \to \overline{\rho}(\mathbf{G})$ as $n \to \infty$.*

Proof. Let $r \ge 1$. For $x \in V(G_n)$, let $B_r^n(x)$ denote the subgraph of G_n induced by the set of those nodes of G_n at graph-distance at most k from x, and let $V^n(x)$ denote the node set of the connected component of G_n containing x. For $x \in J$, we define $B_r(x)$ and $V(x)$ analogously. For every $x \in V(G_n)$ we have $B_r^n(x) \subseteq V^n(x)$ and

$$|B_r^n(x)| \begin{cases} = |V^n(x)|, & \text{if } B_r^n(x) = V^n(x), \\ \ge k, & \text{otherwise.} \end{cases}$$

Hence

$$\frac{1}{|B_r^n(x)|} - \frac{1}{k} \le \frac{1}{|V^n(x)|} \le \frac{1}{|B_r^n(x)|}$$

Taking expectation in x, and subtracting from 1, we get

$$1 - \mathsf{E}_x\Big(\frac{1}{|B_r^n(x)|}\Big) \le \overline{\rho}(G_n) \le 1 - \mathsf{E}_x\Big(\frac{1}{|B_r^n(x)|}\Big) + \frac{1}{k}. \tag{15}$$

Similarly,

$$1 - \mathsf{E}_x\Big(\frac{1}{|B_r(x)|}\Big) \le \overline{\rho}(\mathbf{G}) \le 1 - \mathsf{E}_x\Big(\frac{1}{|B_r(x)|}\Big) + \frac{1}{k}. \tag{16}$$

The distribution of $B_r^n(x)$ tends to the distribution of $B_r(x)$ by the definition of local convergence, and hence the upper and lower bounds in (15) tend to the upper and lower bounds in (16) as $n \to \infty$. Since this holds for every k, this implies that $\overline{\rho}(G_n) \to \overline{\rho}(\mathbf{G})$. □

Remark 4.7. The total rank is even easier to define for the Benjamini–Schramm representation of the local limit [1], namely an involution-invariant random rooted connected graph (G, o). Then

$$\overline{\rho}(\mathbf{G}) = 1 - \mathsf{E}\Big(\frac{1}{|V(G,o)|}\Big).$$

However, the submodularity over subgraphs would be more difficult to formulate.

Remark 4.8. A graphing parameter f is called *locally estimable*, if given an error bound $\varepsilon > 0$ and degree bound $D \geq 0$, there are integers $k, r \geq 1$ depending only on ε and D, such that for any graphing $\mathbf{G} = (J, \mathcal{B}, E, \lambda)$ with maximum degree D, selecting k independent random nodes $x_1, \dots, x_k$ from λ and considering their neighborhoods $B_r(x_1), \dots, B_r(x_k)$, we can compute an estimate $R = B_r(x_1), \dots, B_r(x_k)$ of $f(G)$ such that

$$\mathsf{P}\big(|R - f(G)| \geq \varepsilon\big) \leq \varepsilon. \tag{17}$$

This definition applies, in particular, to estimating f on finite graphs. By theorem of Elek [11], Theorem 4.6 implies that $\overline{\rho}$ is locally estimable.

5. Concluding remarks

In a couple of forthcoming joint papers of K. Bérczi, M. Borbényi, L.M. Tóth and this author [2,3], many of the questions raised in the Introduction are studied and several of them answered.

In infinite-dimensional spaces, there are various generalizations of "vertices"of a closed convex set K in a Banach space B. The most commonly used notion is that a point is *extremal*, if it cannot be written as the average of two different points of K. Other generalizations are also useful; for example, a point of K is *exposed*, if it is the unique maximizer of a continuous linear functional on B.

It was proved (under somewhat different conditions) by Choquet [8] and Šipoš [19] (see also [10], Chapter 10), that for every increasing submodular setfunction φ defined on Borel sets with $\varphi(\emptyset) = 0$ and every family $\mathcal{S}$ of Borel sets that is totally ordered by inclusion, there is a minorizing charge α with $\alpha(S) = \varphi(S)$ for all $S \in \mathcal{S}$. If the chain $\mathcal{S}$ is a maximal chain, then we call α a *greedy maximal minorizing measure*. To explain the name, this corresponds in the finite case to constructing a basis of the matroid by the Greedy Algorithm: the points are processed according to increasing values of the cost function, and at any time, the independent set already selected is as large as the rank of the set of points processed.

So we have different notions for the "bases" of a graphing: extremal measures, exposed measures, greedy measures (and more). We do not go into the discussion of this issue in this paper, as the relationship between these notions in the case of graphings is not clear at this time. But more on this can be found in [3]: It is shown that the exposed points of $\mathrm{mp}(\mathbf{G})$ are indeed obtain by restricting the edge measure to certain hyperfinite subforests (called "essential spanning forests" in the probability literature).

In [2], an appropriate notion of quotient-convergence of a sequence of matroids (even more generally, submodular setfunctions) is defined, and limit objects for convergent sequences are constructed. In [3], it is shown that if a sequence of bounded-degree graphs converges to a graphing in a stronger (local-global) sense, then the matroids of the graphs converge to the matroid of the graphing in the sense of quotient-convergence. It is also shown that local convergence is not enough for this conclusion.

Data availability

No data was used for the research described in the article.

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