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A DYNAMIC THEORY OF MATERIAL INSTABILITY

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Dedicated to my father, Professor Gyula Béda, on the occasion of his seventieth birthday

Abstract. In continuum mechanics dynamical effects are usually connected to waves. Such studies lead to the wave dynamical theory of constitutive equations. In stability analysis, the terms of the theory of dynamical systems are valuable tools. Most of the material instability investigations published deal with small deformations and static or quasi-static loading conditions. To study dynamical effects and finite deformations as well we need first of all appropriate constitutive equations. In order to take second gradient dependent materials widely used for numeric investigations of post-localization into account we shall assume that a jump exists in the derivative of the acceleration field and this singular surface propagates with finite velocities (generalized wave). From that assumption conditions are obtained for example for the second order derivatives of the variables of the constitutive equations. In material instability problems we prescribe that the loss of stability should be a generic one in terms of the theory of dynamical systems. There are two main points. Firstly, in the generic case the multiplicity of the critical eigenvalue should be one or at least finite, moreover the two basic types of stability loss should not be coexistent. Secondly, we would require a finite dimensional critical eigenspace. These lead to further conditions for the constitutive equations.

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1. Introduction

In recent years several new results of the theory of dynamical systems [1, 19] have already been successfully used in various fields of mechanics [7, 18]. This paper is to analyze material instability by considering solid continua as dynamical systems [6, 8]. This kind of investigation is closely related to perturbation analysis [8, 20]. In the theory of dynamical systems the definition of material stability/instability is based on the Lyapunov stability concept of the theory of dynamical systems (see [12] for details), for this reason we call it ”dynamic material instability”. The linear concept of the stability loss of a state of the system means that the real part of certain eigenvalues of the linear operator describing its behavior changes its sign.

The eigenvectors connected to them are called in applications the critical eigenmodes [9].

Unfortunately, for the classical setting [14, 15] it is not possible to obtain specific critical eigenmodes at the onset of material instability. On the other hand, in the finite element calculation of material instability problems the classical formulation of the basic equations of solid continua results in a definite mesh dependence [11, 16, 17]. These are very similar phenomena. In those papers mesh dependence was eliminated by the inclusion of rate dependence or nonlocality (second gradient effects) into the constitutive equations. Most of the investigations published deal with small deformation and static or quasi-static loading conditions. If dynamical effects are to be taken into account, we need appropriate constitutive equations. Such materials were studied by postulating the existence of a (second order) acceleration wave with finite wave speed [2]. This approach is called the wave dynamical theory of constitutive equations [3, 4, 6]. However, such constitutive theory based on second order waves cannot treat the cases of non-locality like second gradient materials [11, 20].

The aim of the paper is to study non-local material instability problems in case of finite deformation. We assume for the solid body that a generalized wave exists in the derivative of the acceleration field and this singular surface propagates forwards and backwards with finite velocities. From that assumption conditions are obtained for the second order derivatives of the variables of the constitutive equations [5]. Additionally we prescribe that the loss of stability should be a generic [1] one in terms of the theory of dynamical systems [19], which is essential in dealing with instability problems. There are two main points here. One is quite practical: a numerical solution of the material instability problems in the non-generic case may suffer serious technical difficulties (loss of convergence, mesh sensitivity [11] etc.). The other is of theoretical significance. By modelling physical phenomena we should have a set of equations which is typical (or generic), that is, differs only a little from the "exact unknown mathematical model".

The second section presents the set of the fundamental equations of the solid continuum assuming large deformations. It consists of the Cauchy equations of motion, the kinematic equation (for large displacements) and the constitutive equations. Such physically objective quantities are the Lie derivative of the stress gradient tensor, the Lie derivative of the (Euler) strain gradient tensor and the second covariant derivative of the stress and strain tensors.

In the next section we perform a material instability investigation for finite displacements with an appropriate constitutive equation in a uniaxial case. In this section the wave speed equation is a scalar third order algebraic one and should have real nonzero solutions [2]. By using the dynamical systems theory we should have a generic behavior (as it is defined in the theory of dynamical systems [8]) at the loss of stability because of the aforementioned general modelling concept of physical phenomena. There are two different ways for the loss of stability of a dynamical system [18]. These are the so-called static and dynamic bifurcations and should be completely different phenomena.

2. The set of basic equations for finite deformations

First of all we need the equations of motion

$$t^{kp}{}_{;p} + q^k = \rho \dot{v}^k, \qquad t^{kp} = t^{pk}\,. \tag{2.1}$$

Here, and in all further equations and expressions Roman indices run from 1 to 3.

For finite deformation

$$v_{ij} = L_v\left(a_{ij}\right), \tag{2.2}$$

where

$$L_v a_{ij} \equiv \dot{a}_{ij} + a_{ip}v^p_{;j} + a_{pj}v^p_{;i}\,.$$

The notations are: q^k denotes body force, ρ is mass density, $X^K_{,p}$ is the deformation gradient, g_{pq}, G_{KL} are metric tensors in the current and the initial configurations, v^i and $v^i_{;j}$ are velocity and velocity gradient, v_{ij} is the deformation rate tensor. Cauchy stress tensor is denoted by t^{pk} and

$$a_{ik} = \frac{1}{2}\left(g_{ik} - X^K_{,i}X^L_{,k}G_{KL}\right)$$

denotes Euler strain tensor, respectively. A semicolon means covariant derivative and an overdot indicates material time derivative:

$$\dot{v}^i = \frac{\partial v^i}{\partial \tau} + v^k v^i_{;k}$$

where τ denotes time. Note that the brackets used to distinguish Lie derivative can have upper and lower indices as in (2.2), we use them to show clearly for which variable it is applied. (For example $L_v\left(t^{kp}_{\ \ \ell}\right)$ is the Lie derivative of the covariant derivative of the stress tensor t^{kp} and not the covariant derivative of the Lie derivative.) Assume that the constitutive equation has the form

$$f_\alpha\left(L_v\left(t^{kp}{}_{;\ell}\right), L_v\left(a_{ij;\ell}\right), t^{kp}{}_{;\ell m}, a_{ij;\ell m}\right) = 0, \tag{2.3}$$

where $\alpha = 1, 2, \ldots, 6$. We use physically objective quantities such as

- the Lie derivative of the stress gradient tensor

$$L_v\left(t^{kp}{}_{;\ell}\right) = \left(t^{kp}{}_{;\ell}\right)^{\cdot} - t^{qp}{}_{;\ell}v^k_{;q} - t^{kq}{}_{;\ell}v^p_{;q} + t^{kp}{}_{;q}v^q_{;\ell}$$

- the Lie derivative of the (Euler) strain gradient tensor

$$L_v\left(a_{ij;k}\right) = \left(a_{ij;k}\right)^{\cdot} + a_{qj;k}v^q_{;i} + a_{iq;k}v^q_{;j} + a_{ij;q}v^q_{;k}$$

- the second covariant derivative of the stress tensor $t^{kp}{}_{;\ell m}$,
- the second covariant derivative of the strain $a_{ij;\ell m}$.

The set of equations (2.1), (2.2) and (2.3) has as many scalar variables as the number of equations in the set thus it can be considered to be the set of fundamental equations. We remark that the continuity equation for ρ can also be taken into account, but it is not necessary for the following calculations.

In the next section we perform several simplifications. One is the assumption that a uniaxial case is considered. Then, instead of the tensorial variables t^{pk}, a^{ij}, v^i, the scalar variables a, t, v can be used. They denote the first component (the x component) of the corresponding tensorial variables and depend obviously on x only. Additionally, we restrict the form of the constitutive equation to a quasi-linear one.

3. Material instability in the uniaxial case

Now we perform a material instability investigation of state S^0 of the solid body by considering finite displacements in the uniaxial case with an appropriate constitutive equation of type (2.3)

$$L_v\left(t_{,x}\right)+K_1 L_v\left(a_{,x}\right)+K_2 t_{,xx}+K_3 a_{,xx}=0 \tag{3.1}$$

where partial derivatives of a function g are denoted by $g_{,x}=\frac{\partial g}{\partial x}$, or $g_{,\tau}=\frac{\partial g}{\partial \tau}$ and coefficients K_1, K_2, K_3 are considered to be piecewise constants. Let us substitute the uniaxial forms of the Lie derivatives into (3.1). After some rearrangements

$$\dot{t}_{,x}=t_{,x}v_{,x}-K_1\left(\dot{a}_{,x}+3a_{,x}v_{,x}\right)-K_2 t_{,xx}-K_3 a_{,xx} \tag{3.2}$$

where the uniaxial material time derivatives are $\dot{v}=v_{,\tau}+vv_{,x}$ and $\dot{a}=a_{,\tau}+va_{,x}$.

The wave dynamical theory of constitutive equations [4] leads to the following third degree polynomial equation

$$\rho c^3-\rho K_2 c^2-K_1(2a-1)c+K_3(2a-1)=0, \tag{3.3}$$

which should have real nonzero wave-speed solutions c [2]. Assume that S^0 is described by values a_0, t_0, v_0 of the field variables. Then these values should satisfy the nonlinear system of fundamental equations formed by (3.2) and the uniaxial forms of (2.1) and (2.2):

$$\dot{v}=\frac{1}{\rho}t_{,x}, \quad \dot{a}=v_{,x}-2av_{,x}\,. \tag{3.4}$$

Lyapunov stability investigates the response of a mechanical system to arbitrary small perturbations, thus the perturbed quantities $a_0+\Delta a, t_0+\Delta t, v_0+\Delta v$ should be substituted into (3.2) and (3.4). Note that the use of small perturbations is not a restriction in the sense of stability because of its local nature [12, 18, 19]. Having done the necessary calculations and by linearizing the set of equations (3.2) and (3.4) at S^0 a system of differential equations is obtained for the perturbations

$$\begin{aligned} v_{,\tau\tau} &= C_1 v+C_2 a_{,x}+C_3 a+C_4 v_{,x}+ \\ &\qquad +C_5 v_{,xx}+C_6 a_{,xx}+C_7 v_{,x\tau}\ , \\ a_{,\tau} &= D_1 v+D_2 a_{,x}+D_3 a+D_4 v_{,x} \end{aligned} \tag{3.5}$$

where Δ is omitted for the sake of simplicity and the following notations are used:

$$C_1 = -2v_{0,x\tau} - 2v_{0,xx}v_0 \ , \qquad C_2 = \frac{2K_1}{\rho}v_{0,x} \ ,$$
$$C_3 = \frac{2K_1}{\rho}v_{0,xx} \ , \qquad C_4 = \frac{2K_1}{\rho}a_{0,x} - K_2 \ ,$$
$$C_5 = v_0^2 - \frac{K_1}{\rho} + \frac{2K_1}{\rho}a_0 \ , \qquad C_6 = -\frac{K_3}{\rho}, C_7 = 2v_0 \ ,$$
$$D_1 = -a_{0,x} \ , \qquad D_2 = -v_0 \ , \qquad D_3 = -2v_{0,x} \ , \qquad D_4 = -2a_0 + 1 \ .$$

Let us introduce new variables $y_1 = a, y_2 = v, y_3 = v_{,\tau}$ a vector $y = [y_1, y_2, y_3]$ and an operator

$$\mathbf{H} := \begin{bmatrix} H_1 & H_2 & 0 \\ 0 & 0 & 1 \\ H_3 & H_4 & H_5 \end{bmatrix},$$

where the elements

$$H_1 = D_2\frac{\partial}{\partial x} + D_3, \qquad H_2 = D_4\frac{\partial}{\partial x} + D_1, \qquad H_3 = C_6\frac{\partial^2}{\partial x^2} + C_2\frac{\partial}{\partial x} + C_3,$$
$$H_4 = C_5\frac{\partial^2}{\partial x^2} + C_4\frac{\partial}{\partial x} + C_1, \qquad H_5 = C_7\frac{\partial}{\partial x}$$

are differential operators. Then a dynamical system

$$\frac{\partial}{\partial\tau}y = \mathbf{H}y. \tag{3.6}$$

can be attached to (3.5) [8]. The characteristic equation of (3.6) reads

$$\lambda y = \mathbf{H}y. \tag{3.7}$$

and the linear Lyapunov stability condition of state S^0 is: $\mathrm{Re}\lambda \leq 0$ for all eigenvalues of (3.7). Stability boundary is at $\mathrm{Re}\lambda = 0$. The loss of stability can be classified as a static bifurcation (or divergence) type instability ($\mathrm{Re}\lambda = 0$, $\mathrm{Im}\lambda = 0$) or a dynamic one ($\mathrm{Re}\lambda = 0$, $\mathrm{Im}\lambda \neq 0$) [12]. To find the eigenvalues of equation (3.7) requires the solution of a boundary value problem, which may cause serious difficulties and needs numerical computations.

To continue using analytic methods we should perform simplifications: the use of small periodic perturbations. While stability is considered here as a local property of a state the small perturbation technique is quite obvious, but not its periodicity. It is really a restriction, but used widely in the engineering literature of the linear case [20]. (A detailed study on that restriction is presented in [8].) While perturbations are small, $a_\tau = v_x$ and then equations (3.5) can be transformed into the velocity field,

$$\begin{aligned} v_{,\tau\tau\tau} &= C_1 v_{,\tau} + C_2 v_{,xx} + C_3 v_{,x} + C_4 v_{,x\tau} + C_5 v_{,xx\tau} + C_6 v_{,xxx} + C_7 v_{,x\tau\tau} \ , \\ v_{x\tau} &= D_1 v_{,\tau} + D_2 v_{,xx} + D_3 v_{,x} + D_4 v_{,x\tau} \ . \end{aligned} \tag{3.8}$$

By assuming periodic perturbations

$$v = \exp{(i\omega x)} \tag{3.9}$$

in a similar way as it was done in the general case with (3.7) the characteristic equation results in a set of algebraic equations

$$\begin{aligned} \lambda^3 &= C_1\lambda - C_2\omega^2 - C_5\omega^2\lambda^2\,, \\ 0 &= C_3 + C_4\lambda - C_6\omega^2 + C_7\lambda^2\,, \\ 0 &= D_1\lambda - D_2\omega^2\,, \\ \lambda &= D_3 + D_4\lambda\,, \end{aligned} \tag{3.10}$$

and the static bifurcation condition is the existence of a $\lambda = 0$ solution of (3.10). Then we obtain the following relations

$$D_3 = 0, \iff \frac{\partial v_0}{\partial x} = 0, \tag{3.11}$$

$$D_2 = 0, \iff v_0 = 0, \tag{3.12}$$

$$C_2 = 0, \iff K_2\frac{\partial v_0}{\partial x} = 0, \tag{3.13}$$

and finally equations

$$C_3 = 0, \iff K_1\frac{\partial^2 v_0}{\partial x^2} = 0, \tag{3.14}$$

and

$$C_6 = 0, \iff K_3 = 0, \tag{3.15}$$

or

$$C_3 - C_6\omega^2 = 0, \iff 2K_1\frac{\partial^2 v_0}{\partial x^2} + K_3\omega^2 = 0, \tag{3.16}$$

should be satisfied. Obviously (3.11) implies (3.13), thus there is a static bifurcation if

A: (3.11), (3.12), (3.14) and (3.15), or
B: (3.11), (3.12), and (3.16) are valid.

Case **A** does not meet the conditions originated by wave dynamics: there is a zero wave speed solution c of (3.3). If $K_3 = 0$ from (3.15) is substituted into (3.3)

$$\left(\rho c^2 - \rho K_2 c - K_1(2a-1)\right)c = 0$$

is obtained thus $c = 0$ is a solution. In the classical material instability concept [14, 15] it means localization. On the other hand, if (3.15) holds, the constitutive equation (3.1) has no second strain gradient dependent term, which corresponds to the fact that there is a stationary singular surface (a localization zone of zero width). Thus we have exactly the classical result of Rice [15]. However, in case **B** from equation (3.16)

$$\omega^2 = -\frac{2K_1}{K_3}\frac{\partial^2 v_0}{\partial x^2},$$

if $\frac{2K_1}{K_3}\frac{\partial^2 v_0}{\partial x^2} < 0$. This means that there is a critical eigenfunction to the zero eigenvalue, that is, we have a critical periodic perturbation (3.9)

$$v_{cr} = \exp\left(ix\sqrt{-\frac{2K_1}{K_3}\frac{\partial^2 v_0}{\partial x^2}} \right)$$

at which state S^0 undergoes a static bifurcation.

Let us now study the dynamic bifurcation case. Then we need $\lambda^2 < 0$ solution of equation (3.10). The conditions are (3.11), (3.12) and

$$C_5 = 0 \iff v_0^2 - \frac{K_1}{\rho} + \frac{2K_1}{\rho} a_0 = 0 \ , \tag{3.17}$$

$$C_4 = 0 \iff \frac{2K_1}{\rho} a_{0,x} - K_2 = 0 \ , \tag{3.18}$$

$$D_1 = 0 \iff a_{0,x} = 0 \ , \tag{3.19}$$

$$D_4 = 1 \iff a_0 = 0 \ . \tag{3.20}$$

Then from (3.17), (3.12) and (3.20)

$$K_1 = 0, \tag{3.21}$$

and from (3.18) and (3.19)

$$K_2 = 0. \tag{3.22}$$

Moreover, the second equation of (3.10) and (3.18) with (3.12) imply (3.15)

$$K_3 = 0.$$

Finally from the first equation of (3.10) substituting (3.12), (3.11) and (3.17) we have

$$\lambda^2 = -2\frac{\partial^2 v_0}{\partial x \partial \tau} \tag{3.23}$$

thus there is a dynamical bifurcation if conditions (3.11), (3.12), (3.15), (3.19), (3.20), (3.21), (3.22) are satisfied and

$$\frac{\partial^2 v_0}{\partial x \partial \tau} > 0. \tag{3.24}$$

Unfortunately this is not a generic dynamical bifurcation. We can easily see that (3.21) implies (3.14), consequently a dynamical bifurcation is coexistent with a static bifurcation of case **A**. Moreover, if (3.15), (3.21) and (3.22) are valid, equation (3.3) has a zero solution $c = 0$, that is, if at least one of conditions (3.19), (3.20), (3.22) or (3.24) fails (because then no coexistent dynamical bifurcation is present), we may speak about a stationary discontinuity as a static bifurcation type instability phenomenon. We remark that this result forms a bridge between the dynamical systems approach [8] and the wave dynamical theory because it can be obtained from both of them [10].

4. Conclusions

By using a second order constitutive equation of form (3.1) for finite deformations both types **A** and **B** of the static bifurcation instability are generic in the sense of dynamical systems theory because there is no coexistent dynamical bifurcation. Moreover, a nice (and useful [7]) property of the small deformation case of second strain gradient dependent materials was preserved: the dimension of the critical eigenspace at static bifurcation remains finite (case **B**). When this term (case **A**) is neglected, we cannot find a unique critical eigenfunction but a "stationary discontinuity": the jump (discontinuity surface in the higher derivatives of the field variables) stops at the conditions of instability.

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