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On exact overlaps of integrable matrix product states: inhomogeneities, twists and dressing formulas

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Abstract: Invoking a quantum dressing procedure as well as the representation theory of twisted Yangians we derive a number of summation formulas for the overlap between integrable matrix product states and Bethe eigenstates which involve only eigenvalues of fused transfer matrices and which are valid in the presence of inhomogeneities as well as twists. Although the method is general we specialize to the SO(6) spin chain for which integrable matrix product states corresponding to evaluation representations of the twisted Yangian $Y^+(4)$ encode the information about one-point functions of the D3-D5 domain wall version of $\mathcal{N} = 4$ SYM. Considering the untwisted and homogeneous limit of our summation formulas we finally fill the last gap in the analytical understanding of the overlap formula for the SO(6) sector of the D3-D5 domain wall system.

Keywords: Bethe Ansatz, Lattice Integrable Models, AdS-CFT Correspondence, Duality in Gauge Field Theories

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Contents

1 Introduction

Matrix product states of integrable spin chains have attracted interest as a means to represent boundary states within the framework of the AdS/CFT correspondence where the boundary state in the string theory language corresponds to a probe D-brane and in the gauge theory language to a Nahm pole defect [1–5]. Exploiting that the good conformal operators of the field theory are in one-to-one correspondence with the Bethe eigenstates of an underlying integrable spin chain [6, 7] the one point function of the defect field theory can be expressed as the overlap between a matrix product state and a Bethe eigenstate [1]. The technique can also be used to study one-point function on the Coulomb branch of $\mathcal{N}=4$ SYM [8] as well as certain three-point functions of the heavy-heavy-light type [9–11].

In numerous cases it was possible to find a closed expression for the above mentioned overlaps valid for all Bethe states of the integrable spin chain, see e.g. [12], and this fact was attributed to the matrix product state being a discrete version of Zamolodchikov's integrable boundary state [13, 14]. The overlap formulas for integrable matrix product states contain a universal factor which involves the Gaudin matrix of the Bethe eigenstate being considered. More precisely, the quantity which appears can be viewed as the super determinant of the Gaudin matrix [15]. The appearance of this structure was observed already in overlap formulas involving simple two-site product states such as the Néel state in the XXZ spin chain [16–18]. In addition to the superdeterminant of the Gaudin matrix overlap formulas for integrable matrix product states contain a non-universal pre-factor which encodes the properties of the boundary state. For particular types of matrix product states the pre-factor has a completely factorized form where the factors take the form of Baxter Q-functions. These are states built from repeated two-site states, i.e. matrix product states which generalize the Néel state. For the XXZ chain and for general $\mathrm{GL}(N)$ spin chains they were studied in respectively [19] and [20, 21] where closed overlap formulas were likewise found. Such states exist also for integrable super spin chains where the particular matrix product states correspond to field theoretical defects in the limit where the Nahm pole boundary conditions become standard Dirichlet boundary conditions [3, 15, 22].

For more generic, still integrable matrix product states such as the states describing the co-dimension one Nahm pole defect in $\mathcal{N}=4$ super Yang-Mills theory it has been observed that the pre-factor is in general not factorized but expressible as a sum over ratios of Baxter Q-functions, with the sum apparently being related to fused transfer matrices [23, 24]. For concreteness, let us present the matrix product state in question

$$\langle \mathrm{MPS}_k| = \sum_{i_1,\ldots i_{2J}} \mathrm{Tr}(\omega_{i_{2J}}\ldots\omega_{i_1})\langle e_{i_1}\ldots e_{i_{2J}}|. \tag{1.1}$$

This is a state of the integrable SO(6) spin chain in the fundamental representation, hence $i_1,\ldots,i_{2J}\in\{1,2,\ldots,6\}$. The six matrices involved, when collected in a list $\bar{\omega}=\{\omega_j\}_{j=1}^6$ take the form

$$\bar{\omega}=\{S_1,S_2,S_3,S_3,S_2,S_1\}, \tag{1.2}$$

where the S_i, $i=1,2,3$, constitute a k-dimensional irreducible representation of $\mathfrak{su}(2)$. The basis vectors e_i, $i=1,\ldots,6$ represent the three complex scalar fields of $\mathcal{N}=4$ SYM and their complex conjugates in the following way

$$\bar{e}=\{Z,Y,X,\bar{X},\bar{Y},\bar{Z}\}. \tag{1.3}$$

For a detailed description of the set-up in the AdS/CFT language we refer to [25–27].

There exists an easily applicable test to determine whether one can expect a closed overlap formula to exist for a given matrix product state, consisting in checking whether it is annihilated by all the parity odd conserved charges of its integrable spin chain host [14]. In practice, it is even sufficient to check if this criterion is fulfilled for the first in the series of odd charges [2, 23]. In order to actually derive the overlap formula a deeper understanding of the concept of boundary integrability is needed. In the same way as one needs the so-called RTT relation (to be made explicit in the next section) and not just the commutation relation for transfer matrices when one wants to explicitly determine the eigenstates and eigenvectors of an integrable spin chain, a relation which we will denote as the KT relation is needed to explicitly derive overlap formulas for the chain with a boundary. It turns out that the appropriate mathematical tool to deal with this type of relation is that of representation theory of twisted Yangians [28, 29]. The information about the boundary is encoded in a K-matrix which has to fulfil the former KT relation, and a K-matrix with this property precisely constitutes a representation of a twisted Yangian [30]. The relevance of twisted Yangians for integrable systems with boundaries was understood also in e.g. [31, 32]. Twisted Yangians have a co-module property which we will exploit to show that given one K-matrix which solves the KT relation, one can perform a dressing by the Yangian itself which gives rise to a novel solution of the KT relation. This dressing is a quantum analogue of the well-known dressing procedure of [33–35] by means of which one can generate novel solutions of a classically integrable equation in terms of already known solutions and which has earlier been used in the AdS/CFT context for generating novel solutions of the classical string equations of motion [36–38]. The overlap formula corresponding to the one-dimensional representation of the Yangian can be derived by existing methods (at least for vanishing twist) as the corresponding boundary state constitutes a special example of a two-site product state for which the overlap formula was determined for any GL(N) spin chain in [20, 21]. K-matrices which result from dressing the trivial representation give rise to overlaps which can directly be expressed in terms of eigenvalues of a single transfer matrix. In the present paper we will need to go beyond this type of K-matrices.

We have earlier pointed out the importance of the KT relation and the twisted Yangian for the computation of spin chain overlaps [19, 29] and we have made use of such considerations to determine the overlap formula for two specific matrix product states of relevance for the study of Nahm pole defects in $\mathcal{N}=4$ SYM. One of these was a matrix product state involving matrices whose commutators were the generators of an $\mathfrak{so}(5)$ algebra and whose overlap formula encoded the one-point functions in a defect version of $\mathcal{N}=4$ SYM dual to a D3-D7 probe brane model with flux [29, 39, 40]. The other one was a matrix product state of the type given in eq. (1.1), but restricted to an $\mathfrak{su}(3)$ subsector, meaning that one allows only $i_1,\ldots,i_{2J}\in\{1,2,3\}$ and considers the state to be a state in an integrable SU(3) spin chain [29, 41]. So far the overlap formula for the complete matrix product state in eq. (1.2) has resisted analytical treatment although a beautiful closed expression has been found numerically [24].

In this paper we sharpen and extend our twisted Yangian approach to matrix product states and fill this last gap in the analytical understanding of the D3-D5 overlap formula. In particular, we will allow for inhomogeneities and twists which were so far only considered in connection with overlaps for non-nested spin chains [19]. As we will explain the general matrix product state (1.1) corresponds to K-matrices that constitute so-called evaluation representations of the twisted Yangian $Y^+(4)$. These K-matrices can not be reached directly by dressing the trivial representation but requires one to go through a recursive dressing procedure by means of which one eventually gets an expression for the K-matrix as a linear combination of different, dressed versions of the trivial representation. The recursive dressing procedure is slightly different for integer and half-integer values of the spin and results in two different dressing formulas. These two dressing formulas constitute our main results and appear in eqs. (4.32) and (4.39). We stress that as will be explained in the text these formulas are valid also in the presence of twists and inhomogeneities. Considering the limit of vanishing twist and inhomogeneities we recover the overlap formula obtained numerically in [24] as a sum over eigenvalues of different transfer matrices.[1]

Our paper is organized as follows. We begin in section 2 by introducing the tools of group theory and of integrability needed to study the integrable $\mathfrak{so}(6)$ spin chain with twists and inhomogeneities in its fundamental representation. Subsequently, in section 3 we introduce the K-matrix and the crucial KT relation which defines the concept of an integrable boundary state. Making use of the fact that a K-matrix which solves the KT relation constitutes a representation of the twisted Yangian $Y^+(4)$ we devise a quantum dressing procedure which allows us to generate a series of integrable matrix product states starting from the trivial representation. In section 4 we turn to deriving, by means of examples, the generalized dressing formulas necessary to reach the two classes of integrable matrix product states which correspond to evaluation representations of $Y^+(4)$, the representations of relevance for the D3-D5 domain wall set-up. The details of this derivation are relegated to appendices B and C whereas appendix A contains the explicit K-matrices of the evaluation representations. Finally, in section 5 by considering the limit of vanishing twist and impurities we recover the overlap formula for the general D3-D5 matrix product state given in eq. (1.1), obtained numerically in [24]. We also show that our dressing formulas are in fact valid for any representation of the quantum space. Section 6 contains our conclusion.

2 Definitions for the SO(6) spin chains

We begin by giving a few basic definitions related to the algebra $\mathfrak{gl}_4$ and the integrability description of the SO(6) spin chain.

2.1 The $\mathfrak{gl}_4$ Lie-algebra and its representations

Let e_i be the usual basis in $\mathbb{C}^4$ and let $e_{i,j}$ be the unit matrices in $\mathbb{C}^4$ defined by $e_{i,j}e_k = \delta_{j,k}e_i$. Furthermore, let $e_{[i,j]}$ for $1 \leq i < j \leq 4$ be a basis of $\mathbb{C}^6$ coming from the anti-symmetrization of $\mathbb{C}^4 \otimes \mathbb{C}^4$ and let $e_{[i,j],[k,l]}$ be the unit matrices in $\mathbb{C}^6$ which fulfil $e_{[i,j],[k,l]}e_{[a,b]} = \delta_{k,a}\delta_{l,b}e_{[i,j]}$. We can identify the six dimensional basis with the three complex fields in $\mathcal{N} = 4$ SYM and

[1] It should be noted that one can also recover the numerical result by an argument which builds on assuming the full overlap to be a sum over products of two-particle overlaps at the various levels of nesting [42–44]. This argument, however, does not reveal the connection to fused transfer matrices.

their complex conjugates as follows

$$\begin{aligned} Z &\equiv e_{[1,2]}, & \bar{X} &\equiv e_{[2,3]}, \\ Y &\equiv e_{[1,3]}, & \bar{Y} &\equiv e_{[4,2]}, \\ X &\equiv e_{[1,4]}, & \bar{Z} &\equiv e_{[3,4]}, \end{aligned} \tag{2.1}$$

where $e_{[i,j]} = -e_{[j,i]}$.

We also define the $\mathfrak{gl}_4$ algebra as

$$[E_{i,j}, E_{k,l}] = \delta_{j,k} E_{i,l} - \delta_{i,l} E_{k,j}. \tag{2.2}$$

The matrices $e_{i,j}$ constitute the defining representation. Let us introduce the matrices $\mathcal{E}_{i,j}$

$$\mathcal{E}_{i,j} = \sum_{c=1}^{4} e_{[i,c],[j,c]}, \tag{2.3}$$

which generate the six-dimensional representation. For every 4-tuple $\lambda \equiv (\lambda_1, \lambda_2, \lambda_3, \lambda_4)$ we can define a highest weight irreducible representation (irrep) $E^{\lambda}_{i,j}$ of $\mathfrak{gl}_4$ by

$$\begin{aligned} E^{\lambda}_{i,i}|0_\lambda\rangle &= \lambda_i |0_\lambda\rangle, \\ E^{\lambda}_{i,j}|0_\lambda\rangle &= 0, \quad \text{for } i < j, \end{aligned} \tag{2.4}$$

where $|0_\lambda\rangle$ is the highest weight state. We will make use of the Gelfand-Tsetlin basis of the $\mathfrak{gl}_4$ irreps, see e.g. [45]. Every basis vector $|\Lambda\rangle$ of the irrep $E^{\lambda}_{i,j}$ corresponds to a GT-pattern

$$\Lambda = \begin{matrix} \lambda_{4,1} & & \lambda_{4,2} & & \lambda_{4,3} & & \lambda_{4,4} \\ & \lambda_{3,1} & & \lambda_{3,2} & & \lambda_{3,3} & \\ & & \lambda_{2,1} & & \lambda_{2,2} & & \\ & & & \lambda_{1,1} & & & \end{matrix} \tag{2.5}$$

consisting of non-negative integers for which $\lambda_{l+1,k} \geq \lambda_{l,k} \geq \lambda_{l+1,k+1}$. The first row contains the highest weights $\lambda_{4,k} \equiv \lambda_k$ of the representation. The weights of the basis vectors $|\Lambda\rangle$ can be expressed as

$$\begin{aligned} &E^{\lambda}_{i,i}|\Lambda\rangle = \omega_i |\Lambda\rangle, \\ &\omega_1 = \lambda_{1,1}, \quad \omega_j = \left(\sum_{k=1}^{j} \lambda_{j,k} - \sum_{k=1}^{j-1} \lambda_{j-1,k} \right). \end{aligned} \tag{2.6}$$

2.2 Transfer matrices

We define the following Lax-operators

$$\mathcal{L}^{\mathbf{4},\mathbf{6}}(u)=\sum_{i,j=1}^{4} e_{i,j}\otimes\Big(\delta_{i,j}+\frac{1}{u+1/2}\mathcal{E}_{j,i}\Big),\qquad \widehat{\mathcal{L}}^{\mathbf{4},\mathbf{6}}(u)=\sum_{i,j=1}^{4} e_{5-i,5-j}\otimes\Big(\delta_{i,j}+\frac{1}{-u+1/2}\mathcal{E}_{i,j}\Big),$$
$$\mathcal{L}^{\lambda,\mathbf{4}}(u)=\sum_{i,j=1}^{4}\Big(\delta_{i,j}+\frac{1}{u}E^{\lambda}_{j,i}\Big)\otimes e_{i,j},\tag{2.7}$$
$$\mathcal{L}^{\lambda,\mathbf{6}}(u)=\sum_{i<j,k<l}\Bigg((u+1/2)(\delta_{i,k}\delta_{j,l}-\delta_{k,j}\delta_{i,l})+\frac{u+1/2}{u-1/2}\Big(\delta_{j,l}E^{\lambda}_{k,i}-\delta_{i,l}E^{\lambda}_{k,j}\Big)+\Big(\delta_{i,k}E^{\lambda}_{l,j}-\delta_{k,j}E^{\lambda}_{l,i}\Big)$$
$$+\frac{1}{u-1/2}\Big(E^{\lambda}_{k,i}E^{\lambda}_{l,j}-E^{\lambda}_{k,j}E^{\lambda}_{l,i}\Big)\Bigg)\otimes e_{[i,j],[k,l]}.$$

The first two act on a product space of the four-dimensional space of the defining representation and the six-dimensional space carried by the $\mathcal{E}_{i,j}$. For the last two, one of the product spaces involved is the space carried by the representation $E^{\lambda}_{i,j}$ which is of dimension

$$d_\lambda=\frac{1}{12}\prod_{i<j}(l_i-l_j),\qquad l_i=\lambda_i+4-i,\qquad i=1,\dots,4.\tag{2.8}$$

The first factor in the product space is referred to as the auxiliary space and the second one as the quantum space. The Lax matrices above satisfy the following algebra

$$\begin{aligned}
R_{1,2}(u-v)\mathcal{L}^{\mathbf{4},\mathbf{6}}_{1,3}(u)\mathcal{L}^{\mathbf{4},\mathbf{6}}_{2,3}(v)&=\mathcal{L}^{\mathbf{4},\mathbf{6}}_{2,3}(v)\mathcal{L}^{\mathbf{4},\mathbf{6}}_{1,3}(u)R_{1,2}(u-v),\\
\bar{R}_{1,2}(u-v)\widehat{\mathcal{L}}^{\mathbf{4},\mathbf{6}}_{1,3}(u)\mathcal{L}^{\mathbf{4},\mathbf{6}}_{2,3}(v)&=\mathcal{L}^{\mathbf{4},\mathbf{6}}_{2,3}(v)\widehat{\mathcal{L}}^{\mathbf{4},\mathbf{6}}_{1,3}(u)\bar{R}_{1,2}(u-v),\\
R_{1,2}(u-v)\widehat{\mathcal{L}}^{\mathbf{4},\mathbf{6}}_{1,3}(u)\widehat{\mathcal{L}}^{\mathbf{4},\mathbf{6}}_{2,3}(v)&=\widehat{\mathcal{L}}^{\mathbf{4},\mathbf{6}}_{2,3}(v)\widehat{\mathcal{L}}^{\mathbf{4},\mathbf{6}}_{1,3}(u)R_{1,2}(u-v),\\
\mathcal{L}^{\lambda,\mathbf{4}}_{1,2}(u-v)\mathcal{L}^{\lambda,\mathbf{6}}_{1,3}(u)\mathcal{L}^{\mathbf{4},\mathbf{6}}_{2,3}(v)&=\mathcal{L}^{\mathbf{4},\mathbf{6}}_{2,3}(v)\mathcal{L}^{\lambda,\mathbf{6}}_{1,3}(u)\mathcal{L}^{\lambda,\mathbf{4}}_{1,2}(u-v),
\end{aligned}\tag{2.9}$$

where we introduced the R-matrices

$$\begin{aligned}
R_{1,2}(u)&=\mathbf{1}+\frac{1}{u}\mathbf{P},\quad \mathbf{P}=\sum_{i,j=1}^{4}e_{i,j}\otimes e_{j,i},\\
\bar{R}_{1,2}(u)&=\mathbf{1}-\frac{1}{u}\mathbf{Q},\quad \mathbf{Q}=\sum_{i,j=1}^{4}e_{i,j}\otimes e_{5-i,5-j}.
\end{aligned}\tag{2.10}$$

These R-matrices have the $\mathfrak{gl}_4$ symmetry

$$\begin{aligned}
R_{1,2}(u)G_1G_2&=G_1G_2R_{1,2}(u),\\
\bar{R}_{1,2}(u)G_1(G^t)_2^{-1}&=G_1(G^t)_2^{-1}\bar{R}_{1,2}(u),
\end{aligned}\tag{2.11}$$

where $G\in\mathrm{GL}(4)$ and the superscript t is a special transposition in the auxiliary space, for which $\left[G^t\right]_{i,j}=G_{5-j,5-i}$.

Monodromy matrices can be defined in the usual way as follows

$$\begin{aligned}
T_0(u)&=\mathcal{L}^{\mathbf{4},\mathbf{6}}_{0,2J}(u+\theta_J)\mathcal{L}^{\mathbf{4},\mathbf{6}}_{0,2J-1}(u-\theta_J)\dots\mathcal{L}^{\mathbf{4},\mathbf{6}}_{0,2}(u+\theta_1)\mathcal{L}^{\mathbf{4},\mathbf{6}}_{0,1}(u-\theta_1),\\
\widehat{T}_0(u)&=\widehat{\mathcal{L}}^{\mathbf{4},\mathbf{6}}_{0,2J}(u+\theta_J)\widehat{\mathcal{L}}^{\mathbf{4},\mathbf{6}}_{0,2J-1}(u-\theta_J)\dots\widehat{\mathcal{L}}^{\mathbf{4},\mathbf{6}}_{0,2}(u+\theta_1)\widehat{\mathcal{L}}^{\mathbf{4},\mathbf{6}}_{0,1}(u-\theta_1),\\
T^{\lambda}_0(u)&=\mathcal{L}^{\lambda,\mathbf{6}}_{0,2J}(u+\theta_J)\mathcal{L}^{\lambda,\mathbf{6}}_{0,2J-1}(u-\theta_J)\dots\mathcal{L}^{\lambda,\mathbf{6}}_{0,2}(u+\theta_1)\mathcal{L}^{\lambda,\mathbf{6}}_{0,1}(u-\theta_1),
\end{aligned}\tag{2.12}$$

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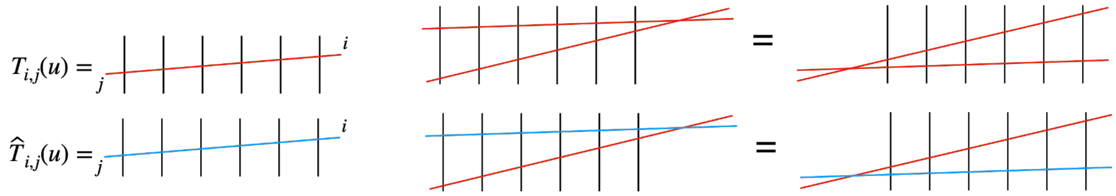

Figure 1. A graphical representation of the monodromy matrices and the RTT-relations. The red and blue lines correspond to the auxiliary space $(i, j = 1, \dots 4)$. The black lines are the six-dimensional representations. The intersection of two red lines represents the R-matrix, $R(u)$, and the intersection of a red and a blue line the crossed R-matrix, $\bar{R}(u)$.

where we note that we have introduced inhomogeneities in the form of the $\{\theta_i\}_{i=1}^{J}$ and that we consider a system of length $2J$. The monodromy matrices satisfy the RTT-algebra

$$\begin{aligned} R_{1,2}(u-v)T_1(u)T_2(v) &= T_2(v)T_1(u)R_{1,2}(u-v), \\ \bar{R}_{1,2}(u-v)\widehat{T}_1(u)T_2(v) &= T_2(v)\widehat{T}_1(u)\bar{R}_{1,2}(u-v), \\ R_{1,2}(u-v)\widehat{T}_1(u)\widehat{T}_2(v) &= \widehat{T}_2(v)\widehat{T}_1(u)R_{1,2}(u-v), \\ \mathcal{L}^{\lambda,\mathbf{4}}_{1,2}(u-v)T^{\lambda}_1(u)T_2(v) &= T_2(v)T^{\lambda}_1(u)\mathcal{L}^{\lambda,\mathbf{4}}_{1,2}(u-v). \end{aligned} \tag{2.13}$$

The $\widehat{T}$ is the "inverse" monodromy matrix

$$\widehat{T}^t(u)T(u) = \left[\prod_{j=1}^{J} \frac{(u-\theta_j)^2 - 9/4}{(u-\theta_j)^2 - 1/4} \frac{(u+\theta_j)^2 - 9/4}{(u+\theta_j)^2 - 1/4}\right] \mathbf{1}. \tag{2.14}$$

The graphical presentations of the monodromy matrices and the RTT-relations are shown in figure 1.

Next, we define the twisted transfer matrices

$$\mathcal{T}(u) = \mathrm{Tr}_0 T_0(u) G_0, \quad \widehat{\mathcal{T}}(u) = \mathrm{Tr}_0 \widehat{T}_0(u) \widehat{G}_0, \quad \mathcal{T}_\lambda(u) = \mathrm{Tr}_0 T^{\lambda}_0(u) G^{\lambda}_0, \tag{2.15}$$

where $G \in \mathrm{GL}(4)$ and $\widehat{G} = \left(G^{-1}\right)^t$, and we concentrate on diagonal twists, i.e.,

$$G = \mathrm{diag}(z_1, z_2, z_3, z_4), \qquad \widehat{G} = \mathrm{diag}(z_4^{-1}, z_3^{-1}, z_2^{-1}, z_1^{-1}). \tag{2.16}$$

The G can be also written as

$$G = \exp\left(\sum_{k=1}^{4} \phi_k e_{k,k}\right), \quad z_k = e^{\phi_k}, \tag{2.17}$$

and similarly, we can express the twist matrix G^λ as

$$G^\lambda = \exp\left(\sum_{k=1}^{4} \phi_k E^{\lambda}_{k,k}\right). \tag{2.18}$$

The transfer matrices generate commuting algebras

$$[\mathcal{T}_\lambda(u), \mathcal{T}_{\lambda'}(u')] = 0, \tag{2.19}$$

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which can be diagonalized simultaneously

$$\mathcal{T}_\lambda(u)|\bar{u}\rangle = \tau_\lambda(u|\bar{u})|\bar{u}\rangle, \tag{2.20}$$

where $|\bar{u}\rangle$ is the Bethe vector and $\bar{u} \equiv \{\bar{u}^1, \bar{u}^2, \bar{u}^3\}$ denotes the set of Bethe roots $\bar{u}^j = \left\{u_k^j\right\}_{k=1}^{n_j}$. The Bethe roots should satisfy the Bethe Ansatz equations

$$\begin{aligned}
\frac{z_1}{z_2} &= -\frac{Q_1(u_k^1+1)}{Q_1(u_k^1-1)}\frac{Q_2(u_k^1-1/2)}{Q_2(u_k^1+1/2)}, \\
\frac{z_2}{z_3}\frac{Q_\theta(u_k^2+1/2)}{Q_\theta(u_k^2-1/2)} &= -\frac{Q_2(u_k^2+1)}{Q_2(u_k^2-1)}\frac{Q_1(u_k^2-1/2)}{Q_1(u_k^2+1/2)}\frac{Q_3(u_k^2-1/2)}{Q_3(u_k^2+1/2)}, \\
\frac{z_3}{z_4} &= -\frac{Q_3(u_k^3+1)}{Q_3(u_k^3-1)}\frac{Q_2(u_k^3-1/2)}{Q_2(u_k^3+1/2)},
\end{aligned} \tag{2.21}$$

where we introduced the Q-functions

$$Q_\theta(u) = \prod_{j=1}^{J}(u-\theta_j)(u+\theta_j), \quad Q_k(u) = \prod_{j=1}^{n_k}(u-u_j^k), \tag{2.22}$$

including the inhomogeneities $\{\theta_j\}_{j=1}^J$. The transfer matrix eigenvalues have the form

$$\begin{aligned}
\tau_\lambda(u|\bar{u}) &= \sum_\Lambda \left[\prod_{j=1}^4 z_j^{\omega_j}\right] \frac{Q_\theta(u+1/2+\lambda_{2,1})Q_\theta(u-1/2+\lambda_{2,2})}{Q_\theta(u-1/2)} \prod_{k=1}^3 \mathcal{F}_\Lambda^{(k)}(u|\bar{u}), \\
\mathcal{F}_\Lambda^{(k)}(u|\bar{u}) &= \frac{Q_k(u+\lambda_{k+1,k+1}-k/2)}{Q_k(u+\lambda_{k,k}-k/2)} \prod_{j=1}^{k} \frac{Q_k(u+\lambda_{k+1,j}+k/2-j+1)}{Q_k(u+\lambda_{k,j}+k/2-j+1)} \prod_{j=1}^{k-1} \frac{Q_k(u+\lambda_{k-1,j}+k/2-j)}{Q_k(u+\lambda_{k,j}+k/2-j)}.
\end{aligned} \tag{2.23}$$

3 Integrable boundary states and twisted Yangians

Our goal is to calculate overlaps, $\langle \text{MPS}|\bar{u}\rangle$, between integrable matrix product states and on-shell Bethe eigenstates where the on-shell Bethe states were defined in the previous section. The purpose of this section is to define the integrable MPSs. We first define integrable boundary states and corresponding K-matrices using the KT-relation [20, 21]. The trace of the boundary state in the boundary space then gives the MPS. K-matrices which fulfil the KT relation constitute representations of the twisted Yangian $Y^+(4)$. We briefly review the properties of the evaluation representations of $Y^+(4)$ [30] and introduce the corresponding MPSs. These are the types of MPSs which appear in the D3-D5 domain wall problem.

3.1 Definition of the KT-relation

The integrable boundary states (which are considered in this paper) are defined by the crossed KT-relation

$$\sum_{k,\gamma} K_{i,k}^{\alpha,\gamma}(u)\langle\Psi_{\gamma,\beta}|T_{k,j}(u) = \sum_{k,\gamma}\langle\Psi_{\alpha,\gamma}|\widehat{T}_{i,k}(-u)K_{k,j}^{\gamma,\beta}(u), \tag{3.1}$$

where $i,j,k = 1,\dots,4$ and refer to the auxiliary space. The indices α,β,γ take values in the set $\{1,\dots,d_B\}$, and refer to a boundary vector space $\mathcal{V}_B \cong \mathbb{C}^{d_B}$ of dimension d_B,

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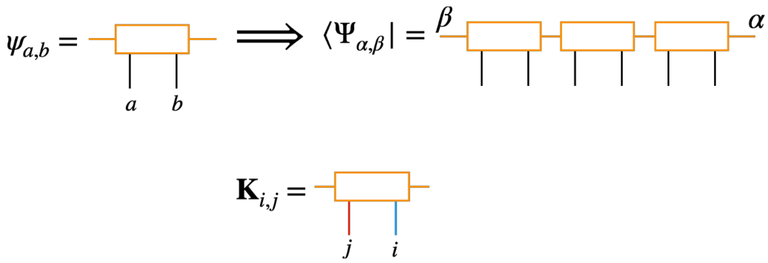

Figure 2. The graphical presentation of the K-matrix and the boundary state. The red and blue lines correspond to the auxiliary space ($i,j=1,\dots 4$). The black lines are the six-dimensional representations ($a,b=1,\dots 6$). The boundary space is denoted by orange lines ($\alpha,\beta=1,\dots,d_B$).

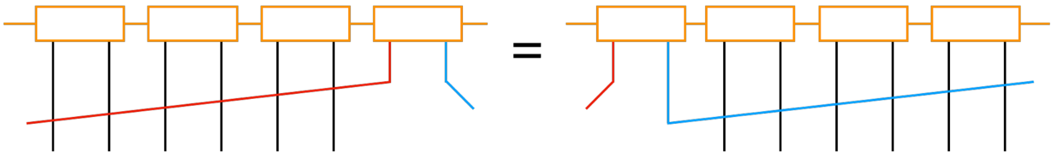

Figure 3. The graphical presentation of the KT-relation. The red and blue lines correspond to the auxiliary space. The black lines are the six-dimensional representations. The boundary space is denoted by orange lines.

denoted as the bond dimension. The states $\langle\Psi_{\alpha,\beta}|$ are covectors in the quantum space, $\mathcal{H}$, and $K_{i,k}^{\alpha,\gamma}(u)$'s are scalars in this space. We can thus define matrices in the boundary space (i.e. elements in $\mathrm{End}(\mathcal{V}_B)$) associated to the two latter objects in the following way $\mathbf{K}_{i,j}(u)=\left\{K_{i,k}^{\alpha,\beta}(u)\right\}_{\alpha,\beta=1}^{d_B}\in\mathrm{End}(\mathcal{V}_B)$ and $\langle\Psi|=\{\Psi_{\alpha,\beta}\}_{\alpha,\beta=1}^{d_B}\in\mathcal{H}^*\otimes\mathrm{End}(\mathcal{V}_B)$. We can also introduce the notation $\mathbf{K}(u)=\sum_{i,j}e_{i,j}\otimes\mathbf{K}_{i,j}(u)$ for which the KT-relation can be written in a more compact form

$$\mathbf{K}(u)\langle\Psi|T(u)=\langle\Psi|\widehat{T}(-u)\mathbf{K}(u). \tag{3.2}$$

The graphical presentations of the K-matrices, boundary states and the KT-relations are shown in figure 2 and 3.

3.2 Compatibility with the twist

Let us multiply the KT-relation with $G\otimes\mathbf{G}\in\mathrm{GL}(4)\otimes\mathrm{GL}(d_B)$, i.e. we allow for a twist in both the auxiliary and the boundary space

$$\mathbf{K}_{0,B}\langle\Psi_B|T_0\left(G_0\mathbf{G}_B\right)=\langle\Psi_B|\widehat{T}_0\mathbf{K}_{0,B}\left(G_0\mathbf{G}_B\right), \tag{3.3}$$

where the subscripts denote the vector spaces where the operators act, and for simplicity we do not write out the spectral parameter dependence. We can rearrange the l.h.s. and arrive at

$$\mathbf{K}_{0,B}\left(\langle\Psi_B|\mathbf{G}_B\right)T_0G_0=\langle\Psi_B|\widehat{T}_0\mathbf{K}_{0,B}\left(G_0\mathbf{G}_B\right). \tag{3.4}$$

Let us assume that G is a symmetry of the K-matrix, i.e.,

$$\mathbf{K}\left(G\otimes\mathbf{G}\right)=\left(\widehat{G}\otimes\mathbf{G}\right)\mathbf{K}. \tag{3.5}$$

Using this property the equation (3.4) simplifies as

$$\mathbf{K}_{0,B}\left(\langle\Psi_B|\mathbf{G}_B\right)T_0G_0 = \left(\langle\Psi_B|\mathbf{G}_B\right)\widehat{T}_0\widehat{G}_0\mathbf{K}_{0,B}.$$

Assuming $\mathbf{K}$ is invertible, we obtain the crossed integrability condition

$$\langle\mathrm{MPS}|\mathcal{T}(u) = \langle\mathrm{MPS}|\widehat{\mathcal{T}}(-u), \tag{3.6}$$

where we introduced the traced boundary state, i.e. the matrix product state

$$\langle\mathrm{MPS}| = \sum_{\alpha,\beta}\langle\psi_{\alpha,\beta}|\mathbf{G}_{\beta,\alpha} = \mathrm{Tr}_{\mathcal{V}_B}\left(\langle\Psi|\mathbf{G}\right). \tag{3.7}$$

3.3 Twisted Yangians

The compatibility of the crossed KT- and the RTT-relations requires that the K-matrix has to satisfy the crossed reflection equation

$$R_{1,2}(u-v)\mathbf{K}_1(-u)\bar{R}_{1,2}(u+v)\mathbf{K}_2(-v) = \mathbf{K}_2(-v)\bar{R}_{1,2}(u+v)\mathbf{K}_1(-u)R_{1,2}(u-v). \tag{3.8}$$

Let assume that the asymptotic expansion of the K-matrix starts as

$$\mathbf{K}(u) = \mathbf{1} + u^{-1}\sum e_{i,j}\otimes F_{i,j} + \mathcal{O}(u^{-2}). \tag{3.9}$$

The crossed reflection equation with this asymptotic expansion defines the twisted Yangian $Y^+(4)$ algebra. From the reflection equation we could derive that the operators $F_{i,j}$ generate an $\mathfrak{o}_4$ algebra

$$\begin{aligned}&[F_{i,j},F_{k,l}] = \delta_{j,k}F_{i,l} - \delta_{i,l}F_{k,j} - \delta_{j,5-l}F_{i,5-k} + \delta_{i,5-k}F_{5-l,j},\\ &F_{5-j,5-i} = -F_{i,j}.\end{aligned} \tag{3.10}$$

This means that the twisted Yangians have $\mathfrak{o}_4$ subalgebras. Therefore any $Y^+(4)$ representation (which gives a K-matrix) is an $\mathfrak{o}_4$ representation, too.

Let us continue with the symmetry algebra corresponding to the symmetry (3.5)

$$(h\otimes\mathbf{1} + 1\otimes\mathbf{h})\mathbf{K}(u) = \mathbf{K}(u)(-h^t\otimes\mathbf{1} + 1\otimes\mathbf{h}). \tag{3.11}$$

The solution of this equation can be obtained from the series expansion of the reflection equation [46] which leads to

$$h = f_{i,j} = e_{i,j} - e_{5-j,5-i}, \quad \mathbf{h} = -F_{j,i}, \tag{3.12}$$

for $i,j = 1,\dots,4$. Therefore the symmetry algebra is $\mathfrak{o}_4$. From the condition (3.5) we see that the twist is compatible with the boundary state only if the twist matrix is orthogonal, i.e. $G\in\mathrm{SO}(4)$. The most general diagonal twist is

$$G = \widehat{G} = \mathrm{diag}(z_1,z_2,z_2^{-1},z_1^{-1}), \tag{3.13}$$

and the corresponding boundary twist matrix is

$$\mathbf{G} = \exp(-\phi_1F_{1,1} - \phi_2F_{2,2}). \tag{3.14}$$

In the following we will need the highest weights of the twisted Yangian irreps. A vector $|0\rangle \in \mathcal{V}_B$ is highest weight vector if

$$\begin{aligned}\mathbf{K}_{i,j}(u)|0\rangle &= 0, \quad \text{for } i<j,\\ \mathbf{K}_{i,i}(u)|0\rangle &= \mu_i(u)|0\rangle,\end{aligned} \tag{3.15}$$

where the functions $\mu_i(u)$ are the highest weights. The twisted Yangian generators satisfy the relation [30]

$$\mathbf{K}_{5-j,5-i}(-u) = \mathbf{K}_{i,j}(u) + \frac{\mathbf{K}_{i,j}(u) - \mathbf{K}_{i,j}(-u)}{2u}, \tag{3.16}$$

therefore $\mu_1(u)$ and $\mu_2(u)$ are not independent of $\mu_3(u)$ and $\mu_4(u)$.

The highest weights can be used to identify irreps since two irreps are isomorphic iff the ratios

$$P_1(u) = \frac{\mu_4(u)}{\mu_3(u)}, \quad P_2(u) = \frac{\mu_3(-u)}{\mu_3(u)}, \tag{3.17}$$

are the same [30].

3.4 Selection rules for on-shell overlaps

We can apply the integrability condition (3.6) to an on-shell Bethe state and get

$$(\tau(v|\bar{u}) - \hat{\tau}(-v|\bar{u}))\langle \text{MPS}|\bar{u}\rangle = 0 \tag{3.18}$$

Non-vanishing on-shell overlaps $\langle \text{MPS}|\bar{u}\rangle \neq 0$ require the condition

$$\tau(v|\bar{u}) = \hat{\tau}(-v|\bar{u}). \tag{3.19}$$

The transfer matrix eigenvalues can be read off from eq. (2.23) and take the form

$$\begin{aligned}\tau(v|\bar{u}) &= z_1\frac{Q_\theta(v+3/2)}{Q_\theta(v+1/2)}\frac{Q_1(v-1/2)}{Q_1(v+1/2)} + z_2\frac{Q_\theta(v+3/2)}{Q_\theta(v+1/2)}\frac{Q_1(v+3/2)}{Q_1(v+1/2)}\frac{Q_2(v)}{Q_2(v+1)}+\\ &\quad + z_3\frac{Q_2(v+2)}{Q_2(v+1)}\frac{Q_3(v+1/2)}{Q_3(v+3/2)} + z_4\frac{Q_3(v+5/2)}{Q_3(v+3/2)},\\ \hat{\tau}(v|\bar{u}) &= z_4^{-1}\frac{Q_3(v-5/2)}{Q_3(v-3/2)} + z_3^{-1}\frac{Q_2(v-2)}{Q_2(v-1)}\frac{Q_3(v-1/2)}{Q_3(v-3/2)}+\\ &\quad + z_2^{-1}\frac{Q_\theta(v-3/2)}{Q_\theta(v-1/2)}\frac{Q_1(v-3/2)}{Q_1(v-1/2)}\frac{Q_2(v)}{Q_2(v-1)} + z_1^{-1}\frac{Q_\theta(v-3/2)}{Q_\theta(v-1/2)}\frac{Q_1(v+1/2)}{Q_1(v-1/2)}.\end{aligned} \tag{3.20}$$

For the untwisted case ($z_j = 1$) we can see that the condition (3.19) is satisfied when

$$Q_k(-v) = (-1)^{n_k} Q_k(v), \tag{3.21}$$

for $k = 1, 2, 3$ which is the well-known pairing condition for the roots [24, 42].

In the twisted case, however, it is not possible to formulate the selection rule in a nice way with these Bethe roots and Q-functions. To obtain the selection rules in a compact form we need to introduce the full system of $\mathcal{Q}$-functions [47] ($\mathcal{Q}_j, \mathcal{Q}_{jk}, \mathcal{Q}_{jkl}$ where $j,k,l \in \{1,2,3,4\}$), for reviews see [48–50]. The relevant $\mathcal{Q}\mathcal{Q}$-relations read

$$\begin{aligned}
\mathcal{Q}_j(v+1/2)\mathcal{Q}_k(v-1/2) - \mathcal{Q}_j(v-1/2)\mathcal{Q}_k(v+1/2) &\sim \mathcal{Q}_{jk}(v), \\
\mathcal{Q}_{jk}(v+1/2)\mathcal{Q}_{jl}(v-1/2) - \mathcal{Q}_{jk}(v-1/2)\mathcal{Q}_{jl}(v+1/2) &\sim Q_\theta(v)\mathcal{Q}_j(v)\mathcal{Q}_{jkl}(v), \\
\mathcal{Q}_{jkl}(v+1/2)\mathcal{Q}_{jkn}(v-1/2) - \mathcal{Q}_{jkl}(v-1/2)\mathcal{Q}_{jkn}(v+1/2) &\sim \mathcal{Q}_{jk}(v).
\end{aligned} \tag{3.22}$$

We can indentify the previous Q-functions in terms of the $\mathcal{Q}$-functions as $\mathcal{Q}_1(u) = z_1^{-u} Q_1(u)$, $\mathcal{Q}_{12}(u) = z_1^{-u} z_2^{-u} Q_2(u)$ and $\mathcal{Q}_{123}(u) = z_1^{-u} z_2^{-u} z_3^{-u} Q_3(u)$. The transfer matrix eigenvalues can be expressed as

$$\begin{aligned}
\tau(v|\bar{u}) &= \frac{Q_\theta(v+3/2)}{Q_\theta(v+1/2)} \frac{\mathcal{Q}_j(v-1/2)}{\mathcal{Q}_j(v+1/2)} + \frac{Q_\theta(v+3/2)}{Q_\theta(v+1/2)} \frac{\mathcal{Q}_j(v+3/2)}{\mathcal{Q}_j(v+1/2)} \frac{\mathcal{Q}_{jk}(v)}{\mathcal{Q}_{jk}(v+1)} + \\
&\quad + \frac{\mathcal{Q}_{jk}(v+2)}{\mathcal{Q}_{jk}(v+1)} \frac{\mathcal{Q}_{jkl}(v+1/2)}{\mathcal{Q}_{jkl}(v+3/2)} + \frac{\mathcal{Q}_{jkl}(v+5/2)}{\mathcal{Q}_{jkl}(v+3/2)}, \\
\hat{\tau}(v|\bar{u}) &= \frac{\mathcal{Q}_{jkl}(v-5/2)}{\mathcal{Q}_{jkl}(v-3/2)} + \frac{\mathcal{Q}_{jk}(v-2)}{\mathcal{Q}_{jk}(v-1)} \frac{\mathcal{Q}_{jkl}(v-1/2)}{\mathcal{Q}_{jkl}(v-3/2)} + \\
&\quad + \frac{Q_\theta(v-3/2)}{Q_\theta(v-1/2)} \frac{\mathcal{Q}_j(v-3/2)}{\mathcal{Q}_j(v-1/2)} \frac{\mathcal{Q}_{jk}(v)}{\mathcal{Q}_{jk}(v-1)} + \frac{Q_\theta(v-3/2)}{Q_\theta(v-1/2)} \frac{\mathcal{Q}_j(v+1/2)}{\mathcal{Q}_j(v-1/2)},
\end{aligned} \tag{3.23}$$

where we assumed that $z_1 z_2 z_3 z_4 = 1$. We already saw that the boundary state is compatible with the twist when $z_j^{-1} = z_{5-j}$. We see that the condition (3.19) leads to the selection rule

$$\mathcal{Q}_k(-v) = \pm \mathcal{Q}_{5-k}(v), \tag{3.24}$$

for $k = 1,2,3,4$.

3.5 Evaluation representations

Evaluation representations are the types of representations which are relevant for the D3-D5 domain wall in $\mathcal{N} = 4$ SYM and are introduced as follows. We have an evaluation homomorphism $\mathfrak{o}_4 \hookrightarrow Y^+(4)$:

$$F_{i,j} \hookrightarrow \mathbf{K}_{i,j}(u) = \delta_{i,j} + \frac{2}{2u+1} F_{i,j}, \tag{3.25}$$

therefore we can lift any $\mathfrak{o}_4$ irreps to $Y^+(4)$ irreps. Since $\mathfrak{o}_4 \cong \mathfrak{sl}_2 \oplus \mathfrak{sl}_2$, any 2-tuple (s_L, s_R) defines an $\mathfrak{o}_4$ highest weight irrep ($s_{L/R}$ are the spins of the $\mathfrak{sl}_2$ irreps). Let us denote by $V(s_L, s_R)$ the corresponding $Y^+(4)$ irrep.

Furthermore, let us parametrize the $\mathfrak{o}_4 = \mathfrak{sl}_2 \oplus \mathfrak{sl}_2$ generators as

$$\begin{aligned}
F_{1,1} &= S_3^R + S_3^L, & F_{2,2} &= S_3^R - S_3^L, \\
F_{1,2} &= \sqrt{2} S_+^L, & F_{2,1} &= \sqrt{2} S_-^L, \\
F_{1,3} &= \sqrt{2} S_+^R, & F_{3,1} &= \sqrt{2} S_-^R,
\end{aligned} \tag{3.26}$$

where

$$\left[S_j^{L/R}, S_k^{L/R}\right] = \sum_{l=1}^{3} i\epsilon_{j,k,l} S_l^{L/R}, \quad S_\pm^{L/R} = \frac{1}{\sqrt{2}} \left(S_1^{L/R} \pm i S_2^{L/R}\right). \tag{3.27}$$

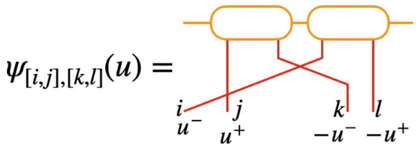

Figure 4. The graphical presentation fusion leading to the boundary state. The red lines correspond to the auxiliary space ($i, j = 1, \dots 4$). The boundary space is denoted by orange lines. The intersection of two red lines denote the R-matrix $R(u)$. Here, we used the shorthand notations $u^{\pm} = u \pm 1/2$ for the spectral parameter dependence. The rounded rectangles are the two-site boundary operators $\psi^{(0)}_{i,j}(u) = \mathbf{K}_{5-j,i}(u)$ for the defining representation of $\mathfrak{gl}_4$.

Using the relation (3.25), the explicit form of the K-matrix can easily be written down. We have listed it together with related quantities of relevance for the present section in appendix A.

The corresponding boundary state can be obtained using the fusion procedure sketched in figure 4. If the quantum space were the defining representation the boundary state would be constructed from the two-site operators $\psi^{(0)}_{i,j}(u) = \mathbf{K}_{5-j,i}(u)$. The fused two-site operator is

$$\begin{aligned} \psi_{[i,j],[k,l]}(u) = -u^2(\psi^{(0)}_{i,l}(u-1/2)\psi^{(0)}_{j,k}(u+1/2) - \psi^{(0)}_{j,l}(u-1/2)\psi^{(0)}_{i,k}(u+1/2))- \\ -\frac{u}{2}(\psi^{(0)}_{k,l}(u-1/2)\psi^{(0)}_{j,i}(u+1/2) - \psi^{(0)}_{k,l}(u-1/2)\psi^{(0)}_{i,j}(u+1/2)). \end{aligned} \tag{3.28}$$

The explicit form of the fused two-site operator $\psi_{a,b}(u)$, $a, b = 1, \dots 6$, is given in appendix A. From this two-site operator one builds the boundary state as

$$\langle \Psi | = \sum_{i_1,\dots,i_{2J}} \langle i_1, i_2, \dots, i_{2J-1}, i_{2J} | \otimes \psi_{i_{2J-1},i_{2J}}(\theta_J) \dots \psi_{i_1,i_2}(\theta_1). \tag{3.29}$$

The corresponding MPS can be written as

$$\langle \mathrm{MPS} | = \sum_{i_1,\dots,i_{2J}} \mathrm{Tr}\left[\psi_{i_{2J-1},i_{2J}}(\theta_J) \dots \psi_{i_1,i_2}(\theta_1)\mathbf{G}\right] \langle i_1, i_2, \dots, i_{2J-1}, i_{2J} |, \tag{3.30}$$

where the boundary twist matrix is defined as

$$\mathbf{G} = \exp(-\phi_1(S^R_3 + S^L_3) - \phi_2(S^R_3 - S^L_3)) = \exp((\phi_2 - \phi_1)S^L_3 - (\phi_1 + \phi_2)S^R_3). \tag{3.31}$$

For $u = 0$ the elementary building block $\psi_{a,b}(u)$ of the MPS is factorized as (cf. appendix A)

$$\psi_{a,b}(0) = \omega_b \omega_a, \tag{3.32}$$

where

$$\bar{\omega} = \{\omega_Z, \omega_Y, \omega_X, \omega_{\bar{X}}, \omega_{\bar{Y}}, \omega_{\bar{Z}}\} = \left\{\sqrt{2}S^R_-, \sqrt{2}S^L_-, S^R_3 + S^L_3, S^R_3 - S^L_3, -\sqrt{2}S^L_+, \sqrt{2}S^R_+\right\}. \tag{3.33}$$

In the homogenous limit the MPS simplifies as

$$\langle \mathrm{MPS} | = \sum_{i_1,\dots,i_{2J}} \mathrm{Tr}\left[\omega_{i_{2J}} \dots \omega_{i_2}\omega_{i_1}\mathbf{G}\right] \langle i_1, i_2, \dots, i_{2J} |. \tag{3.34}$$

We define two special classes of matrix product states. For the first one we set $s_L = 0$ and $s_R = s$ and denote the associated MPS as $\langle \mathrm{MPS}^R_{2s+1}|$. We give the associated K-matrix, $\mathbf{K}^R(u)$ in appendix A. The highest weights of the corresponding $Y^+(4)$ representation are

$$\mu^R_4(u) = \frac{(u + 1/2 - s)}{(u + 1/2)}, \qquad \mu^R_3(u) = \frac{(u + 1/2 - s)}{(u + 1/2)}, \tag{3.35}$$

therefore

$$P^R_1(u) = 1, \quad P^R_2(u) = \frac{u + 1/2}{u - 1/2}\frac{u - 1/2 + s}{u + 1/2 - s}. \tag{3.36}$$

We will denote this representation as $V_R(s) \equiv V(0, s)$.

We also define the special representations where $s_R = 0$ and $s_L = s$, for which we denote the associated MPS as $\langle \mathrm{MPS}^L_{2s+1}|$. Again we give the explicit form of the corresponding K-matrix, $\mathbf{K}^L(u)$ in appendix A. The highest weights of the corresponding $Y^+(4)$ representation are

$$\mu^L_4(u) = \frac{u + 1/2 - s}{u + 1/2}, \quad \mu^L_3(u) = \frac{u + 1/2 + s}{u + 1/2}. \tag{3.37}$$

Therefore

$$P^L_1(u) = \frac{u + 1/2 - s}{u + 1/2 + s}, \quad P^L_2(u) = \frac{u + 1/2}{u - 1/2}\frac{u - 1/2 - s}{u + 1/2 + s}. \tag{3.38}$$

We will denote this representation as $V_L(s) \equiv V(s, 0)$. The representations which are of relevance for the D3-D5 domain wall problem are the representations of type $V_L(s)$ and $V_R(s)$, cf. section 5.

4 Dressing

Let us assume a K-matrix which solves the KT relation is known. Starting from this solution one can obtain another solution be means of quantum dressing as we will now explain. The twisted Yangian $Y^+(4)$ has the following co-module property [30]:

$$\Delta : Y^+(4) \to Y(4) \otimes Y^+(4), \tag{4.1}$$

which is defined as

$$\Delta(\mathbf{K}_{i,j}(u)) = \sum_{k,l=1}^{4} T_{5-k,5-i}(u) T_{l,j}(-u) \otimes \mathbf{K}_{k,l}(u). \tag{4.2}$$

This can be used to "dress" representations of the twisted Yangians $Y^+(4)$ with the original Yangian $Y(4)$. For the original Yangian the evaluation homomorphism is

$$E_{i,j} \hookrightarrow T_{i,j}(u) = \delta_{i,j} + \frac{1}{u} E_{j,i}, \tag{4.3}$$

where $E_{i,j}$ are the $\mathfrak{gl}_4$ generators. For the 4-tuples $\lambda \equiv (\lambda_1, \lambda_2, \lambda_3, \lambda_4)$ we have the highest weight $\mathfrak{gl}_4$ representations $E^\lambda_{i,j}$ and the corresponding $Y(4)$ representations, $L(\lambda|\xi)$, which can be identified with the quantities denoted as $\mathcal{L}^{\lambda,\mathbf{4}}(u - \xi)$ in section 2.2, where ξ was an inhomogeneity. We can always choose $\lambda_4 = 0$ since

$$L(\lambda_1, \ldots, \lambda_4|\xi) \cong L(\lambda_1 - \xi, \ldots, \lambda_4 - \xi|0). \tag{4.4}$$

Let us pick a $Y^+(4)$ representation $\mathcal{V}_B = V$. After the dressing with $L(\lambda|\xi)$ we have a $Y^+(4)$ representation $\mathcal{V}_B = L(\lambda|\xi) \otimes V$ which can be written as

$$\widetilde{\mathbf{K}}_{0,L,V}(u) = \left[\mathcal{L}^{\lambda,\mathbf{4}}_{L,0}(u-\xi)\right]^{t_0} \mathbf{K}_{0,V}(u)\, \mathcal{L}^{\lambda,\mathbf{4}}_{L,0}(-u-\xi). \tag{4.5}$$

where the subscripts 0, L and V denote the vector spaces where the operators act. We can introduce the notation

$$\bar{\mathcal{L}}^{\lambda,\mathbf{4}}_{L,0}(u) \equiv \left[\mathcal{L}^{\lambda,\mathbf{4}}_{L,0}(-u)\right]^{t_0}, \tag{4.6}$$

for which the dressed K-matrix (4.5) can be written in a more compact form

$$\widetilde{\mathbf{K}}_{0,L,V}(u) = \bar{\mathcal{L}}^{\lambda,\mathbf{4}}_{L,0}(\xi-u)\mathbf{K}_{0,V}(u)\mathcal{L}^{\lambda,\mathbf{4}}_{L,0}(-u-\xi). \tag{4.7}$$

We can check that this matrix satisfies the reflection equation

$$R_{1,2}(u-v)\widetilde{\mathbf{K}}_{1,L,V}(-u)\bar{R}_{1,2}(u+v)\widetilde{\mathbf{K}}_{2,L,V}(-v) = \widetilde{\mathbf{K}}_{2,L,V}(-v)\bar{R}_{1,2}(u+v)\widetilde{\mathbf{K}}_{1,L,V}(-u)R_{1,2}(u-v), \tag{4.8}$$

using the bYB for the original $\mathbf{K}_{0,V}$ and the RLL-relations (2.9).

The natural question is: what is the integrable boundary state corresponding to the dressed K-matrix, cf. eq. (3.1). Let us introduce the dressed boundary state

$$\langle\tilde{\Psi}_{L,V}| = \langle\Psi_V|T^{\lambda}_L(-\xi), \tag{4.9}$$

where we used the monodromy matrix where the representation of the auxiliary space is λ. It is straightforward to show that this dressed boundary state $\langle\tilde{\Psi}_{L,V}|$ satisfies the KT-relation with the dressed K-matrix:

$$\widetilde{\mathbf{K}}_{0,L,V}(z)\langle\tilde{\Psi}_{L,V}|T_0(z) = \langle\tilde{\Psi}_{L,V}|\widehat{T}_0(-z)\widetilde{\mathbf{K}}_{0,L,V}(z). \tag{4.10}$$

One only needs the KT-relation for the original state $\langle\Psi_V|$ and the RTT-relations (2.13).

We can apply the dressing procedure to the one-dimensional representation, i.e. when the K-matrix is

$$\mathbf{K}_{i,j} = \delta_{i,j}, \tag{4.11}$$

and the boundary state is the simple two-site state

$$\langle\delta| = \langle\varphi|^{\otimes J}, \quad \langle\varphi| = \sum_{a,b}\varphi_{a,b}\langle a,b|, \tag{4.12}$$

where $\varphi_{i,j}$ is given in eq. (A.5). After the dressing the representation is $\mathcal{V}_B = L(\lambda|\xi)$ with the following dressed K-matrix and associated integrable boundary state

$$\mathbf{K}_{0,L}(z) = \bar{\mathcal{L}}^{\lambda,\mathbf{4}}_{L,0}(\xi-z)\mathcal{L}^{\lambda,\mathbf{4}}_{L,0}(-z-\xi), \quad \langle\Psi_L| = \langle\delta|T^{\lambda}_L(-\xi). \tag{4.13}$$

Taking the trace we get a twisted MPS which is simply the result of acting with a transfer matrix on the basic two-site state

$$\langle\mathrm{MPS}| = \langle\delta|\mathcal{T}_{\lambda}(-\xi). \tag{4.14}$$

We can check that the lowest weight state of the $\mathfrak{gl}_4$ is the highest weight state of the dressed K-matrix. The highest weights of the dressed representation $L(\lambda|\xi)$ are

$$\mu_4^D(u) = \frac{(u+\xi-\lambda_1)(u-\xi+\lambda_4)}{(u+\xi)(u-\xi)}, \quad \mu_3^D(u) = \frac{(u+\xi-\lambda_2)(u-\xi+\lambda_3)}{(u+\xi)(u-\xi)}, \tag{4.15}$$

therefore

$$P_1^D(u) = \frac{(u+\xi-\lambda_1)(u-\xi+\lambda_4)}{(u+\xi-\lambda_2)(u-\xi+\lambda_3)}, \quad P_2^D(u) = \frac{(u-\xi+\lambda_2)(u+\xi-\lambda_3)}{(u+\xi-\lambda_2)(u-\xi+\lambda_3)}. \tag{4.16}$$

Let us fix $\lambda_4 = 0$. The ratios of highest weights are same as $V_R(s)$ (3.36), if $\lambda_1 = \lambda_2 = s$, $\lambda_3 = \lambda_4 = 0$ and $\xi = 1/2$.

The advantage of the dressed representations is that the overlap has been reduced to the two-site state overlaps, since:

$$\frac{\langle \mathrm{MPS}|\bar{u}\rangle}{\langle \delta|\bar{u}\rangle} = \tau_\lambda(-\xi). \tag{4.17}$$

4.1 Generalized dressing formulas for MPSs

In the previous section we saw that for special MPSs obtained by dressing the K-matrix corresponding to the one-dimensional representation of $Y^+(4)$ by $L(\lambda|\xi)$ there exists a simple, closed formula for the overlap with Bethe eigenstates, because the MPS can be expressed as an action of a transfer matrix on the simple two site product state $|\delta\rangle$.

For general representations of $Y^+(4)$ such a simple relation to the state $|\delta\rangle$ does not exist. However, it is possible to express the general integrable MPS by a linear combination of the action of transfer matrices on this state, i.e.

$$\langle \mathrm{MPS}| = \langle \delta| \left[\sum_{j=1}^{k} F_j \mathcal{T}_{\lambda^{(j)}}(-\xi^{(j)}) \right], \tag{4.18}$$

where $F_j \in \mathbb{C}$. Thus, in the general case the overlap quotient is a linear combination of transfer matrix eigenvalues

$$\frac{\langle \mathrm{MPS}|\bar{u}\rangle}{\langle \delta|\bar{u}\rangle} = \sum_{j=1}^{k} F_j\, \tau_{\lambda^{(j)}}(-\xi^{(j)}), \tag{4.19}$$

i.e. the MPS overlap problem is reduced to the much simpler overlap problem for the basic two-site product state $|\delta\rangle$. For untwisted Bethe vectors this problem was already solved [20, 21]. For twisted Bethe states the functional SoV approach might provide a way forward [51–53].

Our goal is to find the generalized dressing formulas and the overlap ratios for the states $\langle \mathrm{MPS}_{2s+1}^{L/R}|$ which will allow us to fill the last gap in the proof of the expression for the tree level one-point functions of scalars in the D3-D5 domain wall version of $\mathcal{N}=4$ SYM. We follow the strategy of [29] where the overlap ratio was determined for simpler MPSs describing the tree-level one-point functions in the SU(3) sub-sector. In the first subsection we review the method of [29], next we apply it to the $\langle \mathrm{MPS}_{2s+1}^{L/R}|$.

4.2 Methodology

In the following we describe our methodology with a general example. Let us pick a $Y^+(4)$ irrep $V \equiv V^{(1)}$ with the corresponding K-matrix $\mathbf{K}_{i,j}(u) \equiv \mathbf{K}_{i,j}^{(1)}(u)$ and boundary state $\langle\Psi| \equiv \langle\Psi^{(1)}|$ which satisfy the KT-relation. The matrix product state $\langle\mathrm{MPS}| \equiv \langle\mathrm{MPS}^{(1)}|$ is defined from the boundary state by taking the twisted trace in the boundary space.

At first, we need to find a $Y(4)$ representation $L(\lambda|\xi) \equiv L^{(1)}(\lambda^{(1)}|\xi^{(1)})$ for which the dressed K-matrix $\mathbf{K}_{i,j}^{D}(u) \equiv \mathbf{K}_{i,j}^{D_1}(u)$ (given by (4.13)) has the same ratio of highest weights $P_k(u)$ as $\mathbf{K}_{i,j}^{(1)}(u)$. If the representation $L(\lambda|\xi) \equiv L^{(1)}(\lambda^{(1)}|\xi^{(1)})$ has the same dimension as $V^{(1)}$ then the two representations are isomorphic, i.e., $L^{(1)}(\lambda^{(1)}|\xi^{(1)}) = V^{(1)}$. Choosing a proper basis in the boundary space the two K-matrices are the same $\mathbf{K}_{i,j}^{D_1}(u) = \mathbf{K}_{i,j}^{(1)}(u)$, and therefore the boundary states are also the same

$$\langle\Psi^{D_1}| = \langle\delta|T^{\lambda^{(1)}}(-\xi^{(1)}) = \langle\Psi^{(1)}|. \tag{4.20}$$

Taking the twisted trace in the boundary space we obtain that

$$\langle\mathrm{MPS}^{(1)}| = \langle\delta|\mathcal{T}_{\lambda^{(1)}}(-\xi^{(1)}), \tag{4.21}$$

therefore the ratio of the overlaps is simply a transfer matrix eigenvalue

$$\frac{\langle\mathrm{MPS}^{(1)}|\bar{u}\rangle}{\langle\delta|\bar{u}\rangle} = \tau_{\lambda^{(1)}}(-\xi^{(1)}). \tag{4.22}$$

Let us continue with the other case when the dimension of $L^{(1)}(\lambda^{(1)}|\xi^{(1)})$ is bigger than the dimension of $V^{(1)}$. We know that two irreps with the same $P_k(u)$'s are isomorphic, therefore the representation $L^{(1)}(\lambda^{(1)}|\xi^{(1)})$ can not be irreducible, and it contains a $V^{(1)}$ irrep, i.e., we have the decomposition

$$L^{(1)}(\lambda^{(1)}|\xi^{(1)}) = V^{(1)} \oplus V^{(2)}. \tag{4.23}$$

This decomposition is not necessarily a direct sum, it can be semi-direct sum. It means that the dressed K-matrix has the following block form

$$\mathbf{K}_{i,j}^{D_1}(u) = \begin{pmatrix} \mathbf{K}_{i,j}^{(1)}(u) & X_{i,j}^{(1)} \\ 0 & \mathbf{K}_{i,j}^{(2)}(u) \end{pmatrix}, \quad \text{or} \quad \mathbf{K}_{i,j}^{D_1}(u) = \begin{pmatrix} \mathbf{K}_{i,j}^{(1)}(u) & 0 \\ X_{i,j}^{(1)} & \mathbf{K}_{i,j}^{(2)}(u) \end{pmatrix}, \tag{4.24}$$

where $\mathbf{K}_{i,j}^{(2)}(u)$ is the K-matrix of the representation $V^{(2)}$. For these K-matrices there are corresponding boundary states, and it follows that these should have the same block structure in the boundary space.

$$\langle\Psi^{D_1}| = \begin{pmatrix} \langle\Psi^{(1)}| & \langle X| \\ 0 & \langle\Psi^{(2)}| \end{pmatrix}, \quad \text{or} \quad \langle\Psi^{D_1}| = \begin{pmatrix} \langle\Psi^{(1)}| & 0 \\ \langle X| & \langle\Psi^{(2)}| \end{pmatrix}. \tag{4.25}$$

Taking the (twisted) trace in the boundary space we obtain the following relation

$$\langle\mathrm{MPS}^{(1)}| = \langle\delta|\mathcal{T}_{\lambda^{(1)}}(-\xi^{(1)}) - \langle\mathrm{MPS}^{(2)}|. \tag{4.26}$$

We have thus reduced the original MPS to a linear combination of the dressed state and a new MPS. Next, we repeat the steps above starting with $V^{(2)}, \mathbf{K}^{(2)}_{i,j}(u)$ and $\langle \text{MPS}^{(2)}|$. If $V^{(2)}$ is (isomorphic to) an irrep of $Y^+(4)$ we are basically done. If not we must continue the process above to obtain yet another representation $V^{(3)}, \mathbf{K}^{(3)}_{i,j}(u), \langle \text{MPS}^{(3)}|$. Assuming that the process stops after k steps, i.e. $V^{(k)} = L^{(k)}(\lambda^{(k)}|\xi^{(k)})$ the original MPS can be expressed as

$$\langle \text{MPS}^{(1)}| = F_1\langle\delta|\mathcal{T}_{\lambda^{(1)}}(-\xi^{(1)}) + F_2\langle\delta|\mathcal{T}_{\lambda^{(2)}}(-\xi^{(2)}) + \cdots + F_k\langle\delta|\mathcal{T}_{\lambda^{(k)}}(-\xi^{(k)}). \tag{4.27}$$

Since the KT-relation is linear in both the K-matrix and the boundary state the normalizations are not fixed, and therefore we have to introduce the scalars F_j. In sections 4.3 and appendices B and C we will show how the recursive procedure works in practice.

4.3 Generalized dressing for fixed spin

In this subsection we will analyze decompositions of the type (4.23) for fixed spin. We use the $Y^+(4) \to \mathfrak{o}_4$ embedding (3.9), i.e. the $\mathfrak{o}_4$ subalgebra of the twisted Yangian. The dressed K-matrix of the representation $L(\lambda|\xi)$ can be series expanded as

$$\mathbf{K}^{\lambda}_{i,j}(u) = \mathcal{L}^{\lambda,\mathbf{4}}_{5-k,5-i}(u-\xi)\mathcal{L}^{\lambda,\mathbf{4}}_{k,j}(-u-\xi) = \delta_{i,j} + u^{-1}\left(E^{\lambda}_{5-i,5-j} - E^{\lambda}_{j,i}\right) + \mathcal{O}(u^{-2}), \tag{4.28}$$

which follows from the definition of the $\mathcal{L}$'s, cf. section 2.2. Since the u^{-1} term defines the $\mathfrak{o}_4$ subalgebra and the dressed K-matrices are defined from a $\mathfrak{gl}_4$ representation we also have another algebra embedding $\mathfrak{gl}_4 \to \mathfrak{o}_4$ where the $\mathfrak{o}_4$ subalgebra of $\mathfrak{gl}_4$ is defined as

$$F_{i,j} = E_{5-i,5-j} - E_{j,i}. \tag{4.29}$$

The states $|\Lambda\rangle$ in $L(\lambda|\xi)$ have $\mathfrak{gl}_4$ weights $(\omega_1,\omega_2,\omega_3,\omega_4)$ but they also have $\mathfrak{o}_4$ spins (s_L, s_R) where these are defined as

$$\begin{aligned} S^R_3 &= \frac{1}{2}\left(F_{1,1} + F_{2,2}\right) = \frac{1}{2}\left(-E_{1,1} - E_{2,2} + E_{3,3} + E_{4,4}\right), \\ S^L_3 &= \frac{1}{2}\left(F_{1,1} - F_{2,2}\right) = \frac{1}{2}\left(-E_{1,1} + E_{2,2} - E_{3,3} + E_{4,4}\right), \end{aligned} \tag{4.30}$$

i.e. the $\mathfrak{o}_4$ spins (s_L, s_R) of the state $|\Lambda\rangle$ are

$$\begin{aligned} s_L &= \frac{1}{2}\left(-\omega_1 + \omega_2 - \omega_3 + \omega_4\right), \\ s_R &= \frac{1}{2}\left(-\omega_1 - \omega_2 + \omega_3 + \omega_4\right). \end{aligned} \tag{4.31}$$

Integer spins. We now apply the methodology of the previous section to the case of evaluation representations corresponding to integer spin. They key point is to realize that the decomposition of the $L((s,s,0,0)|-1/2)$ representation of $Y^+(4)$ into irreps always contains the direct sum $V_L(s) \oplus V_R(s)$ and that the remaining components can be found in closed form as well. In appendix B we show how the decomposition works for $s=1$, $s=2$ and $s=3$ and present the general branching rule in eq. (B.21). Using this rule we obtain the generalized dressing formula for integer spins

$$\begin{aligned} &\langle \text{MPS}^R_{2s+1}| + \langle \text{MPS}^L_{2s+1}| = \langle\delta|\mathcal{T}_{(s,s,0,0)}(-1/2) + \\ &+ \frac{Q_\theta(s)}{Q_\theta(1)}\langle\delta|\left(\mathcal{T}_{(s-1,s-3,0,0)}(-s+1/2) - \mathcal{T}_{(s,s-2,0,0)}(-s+1/2) - \mathcal{T}_{(s-3,s-3,0,0)}(-s+1/2)\right), \end{aligned} \tag{4.32}$$

where the normalization factor was fitted numerically.

Half integer spins. For half integer spins we cannot follow the previous strategy since there is no finite dimensional $\mathfrak{gl}_4$ irrep for which the ratio of highest weights (4.16) is the same as (3.36) for half integer spins. This suggests that the MPSs for half integer spins cannot be expressed through a two-site state. Therefore, in this case we look for another type of dressing formula. Our goal will be to relate the general half integer spin states $\langle \mathrm{MPS}^R_{2s+1}|$ to the 1/2 spin states $\langle \sigma^R| \equiv \langle \mathrm{MPS}^R_2|$ and $\langle \sigma^L| \equiv \langle \mathrm{MPS}^L_2|$.

In the following we denote the half integers as $s+1/2$ where s is integer. Using this convention, the highest weights of $V_R(s+1/2)$ is

$$P_1^R(u) = 1, \quad P_2^R(u) = \frac{(u+1/2)(u+s)}{(u-1/2)(u-s)}. \tag{4.33}$$

Now we dress the $V_R(1/2)$ and $V_L(1/2)$ K-matrices and boundary states. Therefore the dressed K-matrix is

$$\mathbf{K}^D_{i,j}(u) = \sum_{k,l=1}^{4} \mathcal{L}^{\lambda,\mathbf{4}}_{5-k,5-i}(\xi-u)\mathcal{L}^{\lambda,\mathbf{4}}_{l,j}(-u-\xi) \otimes \mathbf{K}^{L/R}_{k,l}(u) \in \mathrm{End}\left(L(\lambda|\xi)\otimes V_{L/R}(1/2)\right), \tag{4.34}$$

where $\mathbf{K}^{L/R}(u)$ are the K-matrices of the representations $V_{L/R}(1/2)$. We will denote the basis of the two dimensional space $V_{L/R}(1/2)$ as $|+\rangle$ and $|-\rangle$ for which

$$S_3^{L/R}|\pm\rangle = \pm\frac{1}{2}|\pm\rangle, \quad S_\pm^{L/R}|\pm\rangle = 0, \quad S_\mp^{L/R}|\pm\rangle = \frac{1}{\sqrt{2}}|\mp\rangle. \tag{4.35}$$

One can calculate the highest weights of $L(\lambda|\xi)\otimes V_R(1/2)$ (for which the K-matrix is given by (4.34)):

$$\begin{aligned}
\mu_4(u) &= \frac{u}{u+1/2}\frac{(u+\xi-\lambda_1)(u-\xi+\lambda_4)}{(u+\xi)(u-\xi)}, & \mu_3(u) &= \frac{u}{u+1/2}\frac{(u+\xi-\lambda_2)(u-\xi+\lambda_3)}{(u+\xi)(u-\xi)},\\
P_1(u) &= \frac{(u+\xi-\lambda_1)(u-\xi+\lambda_4)}{(u+\xi-\lambda_2)(u-\xi+\lambda_3)}, & P_2(u) &= \frac{u+1/2}{u-1/2}\frac{(u-\xi+\lambda_2)(u+\xi-\lambda_3)}{(u+\xi-\lambda_2)(u-\xi+\lambda_3)},
\end{aligned} \tag{4.36}$$

which are the same as (4.33) when $\lambda_3=\lambda_4=0$, $\lambda_1=\lambda_2=s$ and $\xi=0$. This means that the representation $L((s,s,0,0)|0)\otimes V_R(1/2)$ is not irreducible , i.e.,

$$L((s,s,0,0)|0)\otimes V_R(1/2) = V_R(s+1/2)\oplus V^{(2)}. \tag{4.37}$$

We also need highest weights of $L(\lambda|\xi)\otimes V_L(1/2)$:

$$\begin{aligned}
\mu_4(u) &= \frac{u}{u+1/2}\frac{(u+\xi-\lambda_1)(u-\xi+\lambda_4)}{(u+\xi)(u-\xi)}, & \mu_3(u) &= \frac{u+1}{u+1/2}\frac{(u+\xi-\lambda_2)(u-\xi+\lambda_3)}{(u+\xi)(u-\xi)},\\
P_1(u) &= \frac{u}{u+1}\frac{(u+\xi-\lambda_1)(u-\xi+\lambda_4)}{(u+\xi-\lambda_2)(u-\xi+\lambda_3)}, & P_2(u) &= \frac{(u-1)(u+1/2)}{(u+1)(u-1/2)}\frac{(u-\xi+\lambda_2)(u+\xi-\lambda_3)}{(u+\xi-\lambda_2)(u-\xi+\lambda_3)}.
\end{aligned} \tag{4.38}$$

We now apply the methodology of section 4.2 to relate $V^{(2)}$ to irreducible representations. In appendix C we show how the decomposition works for $s=1$ and $s=2$ and present the general branching rule in eq. (C.15). Using this rule we obtain the generalized dressing formula for half integer spins

$$\langle \mathrm{MPS}^R_{2s+2}| = \frac{1}{Q_\theta(1/2)} \langle \sigma^R | \mathcal{T}_{(s,s,0,0)}(0) \tag{4.39}$$
$$+ \frac{Q_\theta(s+1/2)}{Q_\theta(1/2)^2} \left(\langle \sigma^L | \mathcal{T}_{(s-1,s-2,0,0)}(-s) - \langle \sigma^L | \mathcal{T}_{(s,s-1,0,0)}(-s) - \langle \sigma^R | \mathcal{T}_{(s-2,s-2,0,0)}(-s) \right),$$

where the normalization factor was fitted numerically.

4.4 Universality of the generalized dressing formulas wrt. the quantum space

Since our calculation is based on the K-matrix (which is universal, i.e., does not depend on the representation of the quantum space) the generalized dressing formulas (4.32) and (4.39) are true for any representations of the quantum space (up to the normalization scalars). To demonstrate this universality let us pick another example. Let us change the quantum space to defining representation of $\mathfrak{gl}_4$ (usual SU(4) spin chain). Since the algebra is the same (RTT and the KT are the same), we only need to modify the explicit form the Lax-operators

$$\mathcal{L}^{\mathbf{4},\mathbf{4}}(u) = \sum_{i,j=1}^{4} e_{i,j} \otimes \left(\delta_{i,j} + \frac{1}{u} e_{j,i} \right), \qquad \widehat{\mathcal{L}}^{\mathbf{4},\mathbf{4}}(u) = \sum_{i,j=1}^{4} e_{5-i,5-j} \otimes \left(\delta_{i,j} + \frac{1}{-u} e_{i,j} \right), \tag{4.40}$$

$$\mathcal{L}^{\lambda,\mathbf{4}}(u) = \sum_{i,j=1}^{4} \left(\delta_{i,j} + \frac{1}{u} E^{\lambda}_{j,i} \right) \otimes e_{i,j}, \tag{4.41}$$

which gives the monodromy matrices

$$\begin{aligned} T_0(u) &= \mathcal{L}^{\mathbf{4},\mathbf{4}}_{0,2J}(u+\theta_J)\mathcal{L}^{\mathbf{4},\mathbf{4}}_{0,2J-1}(u-\theta_J)\dots\mathcal{L}^{\mathbf{4},\mathbf{4}}_{0,2}(u+\theta_1)\mathcal{L}^{\mathbf{4},\mathbf{4}}_{0,1}(u-\theta_1), \\ \widehat{T}_0(u) &= \widehat{\mathcal{L}}^{\mathbf{4},\mathbf{4}}_{0,2J}(u+\theta_J)\widehat{\mathcal{L}}^{\mathbf{4},\mathbf{4}}_{0,2J-1}(u-\theta_J)\dots\widehat{\mathcal{L}}^{\mathbf{4},\mathbf{4}}_{0,2}(u+\theta_1)\widehat{\mathcal{L}}^{\mathbf{4},\mathbf{4}}_{0,1}(u-\theta_1), \\ T^{\lambda}_0(u) &= \mathcal{L}^{\lambda,\mathbf{4}}_{0,2J}(u+\theta_J)\mathcal{L}^{\lambda,\mathbf{4}}_{0,2J-1}(u-\theta_J)\dots\mathcal{L}^{\lambda,\mathbf{4}}_{0,2}(u+\theta_1)\mathcal{L}^{\lambda,\mathbf{4}}_{0,1}(u-\theta_1). \end{aligned} \tag{4.42}$$

After this, the algebra of the monodromy matrix, the definition of the transfer matrices and the twists are the same. The forms of the boundary state and MPS are also the same as before, i.e. (3.29), (3.30) still hold, and we only need to update $\psi_{i,j}(u)$ as

$$\psi_{i,j}(u) = \mathbf{K}_{5-j,i}(u). \tag{4.43}$$

The MPS for integer spins can be expressed in the same way as before

$$\langle \mathrm{MPS}^R_{2s+1}| + \langle \mathrm{MPS}^L_{2s+1}| = \langle \delta | \mathcal{T}_{(s,s,0,0)}(-1/2) \tag{4.44}$$
$$+ \frac{Q_\theta(s-1/2)}{Q_\theta(1/2)} \langle \delta | \left(\mathcal{T}_{(s-1,s-3,0,0)}(-s+1/2) - \mathcal{T}_{(s,s-2,0,0)}(-s+1/2) - \mathcal{T}_{(s-3,s-3,0,0)}(-s+1/2) \right),$$

where we only needed to update the normalization factors. The MPS for half integer spins can similarly be expressed as

$$\langle \mathrm{MPS}^R_{2s+2}| = \langle \sigma^R | \mathcal{T}_{(s,s,0,0)}(0) \tag{4.45}$$
$$+ \frac{Q_\theta(s)}{Q_\theta(0)} \left(\langle \sigma^L | \mathcal{T}_{(s-1,s-2,0,0)}(-s) - \langle \sigma^L | \mathcal{T}_{(s,s-1,0,0)}(-s) - \langle \sigma^R | \mathcal{T}_{(s-2,s-2,0,0)}(-s) \right),$$

Note that when the taking the homogeneous limit of the above equation, one needs to be careful about the linear combination of monodromy matrices, which will give an extra $Q_\theta(0)$ that ensures a well defined MPS.

In summary, we learn that the generalized dressing formulas (4.32) and (4.39) are universal, i.e. they are true for any representations. This implies that, when searching for overlap formulas (even with brute force numerical methods), it is always advisable to use the simplest possible quantum space, since the formula is universal. For the given quantum space the formula is modified only by the normalization pre-factors which are easy to fit at the end.

4.5 Universality of the generalized dressing formulas wrt. the twist

Until now we focused on diagonal twists but in this section we will show that generalized dressing formulas can be derived for any twist.

The monodromy matrices have GL(4) symmetry

$$T^\lambda(u)\left[\mathcal{R}^\lambda \otimes \Delta(\mathcal{R})\right] = \left[\mathcal{R}^\lambda \otimes \Delta(\mathcal{R})\right] T^\lambda(u), \tag{4.46}$$

where $\mathcal{R} \in \mathrm{GL}(4)$, and $\Delta(\mathcal{R})$ and $\mathcal{R}^\lambda$ are its representations in the quantum and auxiliary spaces, respectively. More precisely

$$\mathcal{R} = \exp\left(\sum_{i,j} x_{i,j} e_{i,j}\right), \quad \mathcal{R}^{\mathbf{6}} = \exp\left(\sum_{i,j} x_{i,j} \mathcal{E}_{i,j}\right), \quad \mathcal{R}^\lambda = \exp\left(\sum_{i,j} x_{i,j} E^\lambda_{i,j}\right), \tag{4.47}$$

where $x_{i,j} \in \mathbb{C}$ for $i,j = 1,\dots,4$ and

$$\Delta(\mathcal{R}) = \mathcal{R}^{\mathbf{6}} \otimes \mathcal{R}^{\mathbf{6}} \otimes \cdots \otimes \mathcal{R}^{\mathbf{6}}. \tag{4.48}$$

From the symmetry equation of the monodromy matrix, eq. (4.46) we obtain

$$\mathrm{Tr}_0\left(T_0(u) G_0\right) \Delta(\mathcal{R}) = \Delta(\mathcal{R}) \mathrm{Tr}_0\left(T_0(u) \mathcal{R}_0^{-1} G_0 \mathcal{R}_0\right), \tag{4.49}$$

Therefore,

$$\mathcal{T}^G(u)\Delta(\mathcal{R}) = \Delta(\mathcal{R})\mathcal{T}^{G_\mathcal{R}}(u), \tag{4.50}$$

where $G_\mathcal{R} = \mathcal{R}^{-1} G \mathcal{R}$ and the superscript denotes the twist matrix. This relation can be also generalized to the other transfer matrices

$$\widehat{\mathcal{T}}^G(u)\Delta(\mathcal{R}) = \Delta(\mathcal{R})\widehat{\mathcal{T}}^{G_\mathcal{R}}(u), \quad \mathcal{T}^G_\lambda(u)\Delta(\mathcal{R}) = \Delta(\mathcal{R})\mathcal{T}^{G_\mathcal{R}}_\lambda(u), \tag{4.51}$$

where

$$\widehat{\mathcal{T}}^G(u) = \mathrm{Tr}_0\left(\widehat{T}_0(u)\widehat{G}_0\right), \quad \mathcal{T}^G_\lambda(u) = \mathrm{Tr}_0\left(T^\lambda_0(u) G^\lambda_0\right) \tag{4.52}$$

Let us introduce a rotated MPS by

$$\langle \mathrm{MPS}^\mathcal{R}| = \langle \mathrm{MPS}|\Delta(\mathcal{R}), \tag{4.53}$$

If an MPS is compatible with a certain twist matrix i.e.

$$\langle \mathrm{MPS}|\mathcal{T}^G(u) = \langle \mathrm{MPS}|\widehat{\mathcal{T}}^G(-u), \tag{4.54}$$

then the rotated MPS is compatible with the rotated twist

$$\langle \mathrm{MPS}^{\mathcal{R}}|\mathcal{T}^{G_{\mathcal{R}}}(u) = \langle \mathrm{MPS}^{\mathcal{R}}|\widehat{\mathcal{T}}^{G_{\mathcal{R}}}(-u). \tag{4.55}$$

If our dressing formulas are satisfied for a pair of MPS and twist $(\langle \mathrm{MPS}|, G)$ they are also satisfied for the rotated versions $(\langle \mathrm{MPS}^{\mathcal{R}}|, G_{\mathcal{R}})$. This means that our dressing formulas also work for non-diagonal twists. We just need to make sure that we choose the K-matrix and MPS that are compatible with the twist (the equations (3.5) and (3.6) are satisfied).

In practical terms, if we have a general twist and a compatible K-matrix (eq. (3.5)) then we can use a GL(4) transformation like in (4.53) to diagonalize the twist matrix G. In the rotated frame the twist matrix G is diagonal, therefore the dressing formulas are true in that frame. Finally, we can rotate back and get the dressing formulas in the original frame. These formulas will involve transfer matrices with modified twists. Any twist can be obtained from a diagonal one by a GL(4) transformation, therefore generalized dressing formulas exist for any twist.

5 Application to the D3-D5 domain wall

In this section we finally return to the overlap of relevance for the D3-D5 domain wall described in the introduction. Our aim is to derive the overlap $\langle \mathrm{MPS}_k|\bar{u}\rangle$ between the matrix product state given by eqs. (1.1) and (1.2) and the eigenstates of the integrable $\mathfrak{so}(6)$ spin chain. We first exploit the GL(4) symmetry of the monodromy and transfer matrices to perform a specific GL(4) rotation which transforms the dressing formulas (4.32) and (4.39) into formulas involving exactly the integrable matrix product state of interest to us. Secondly, by another GL(4) rotation we relate the left and the right version of the overlap for the integer spin case. Straightforward manipulations of the dressing formulas subsequently allow us to recover the result previously derive by numerical investigations [24].

As in the previous section we start from the symmetry relation for the monodromy matrix, eq. (4.46) with a GL(4) rotation $\mathcal{R}$ represented in the auxiliary and quantum spaces as in (4.47) and (4.48). In the untwisted case the transfer matrices also have GL(4) symmetry:

$$\mathcal{T}_\lambda(u)\Delta(\mathcal{R}) = \Delta(\mathcal{R})\mathcal{T}_\lambda(u). \tag{5.1}$$

This symmetry can be applied to (4.32) to get the generalized dressing formula for the "rotated" MPSs

$$\begin{aligned}\langle \mathrm{MPS}^{R,\mathcal{R}}_{2s+1}| + \langle \mathrm{MPS}^{L,\mathcal{R}}_{2s+1}| &= \langle \delta^{\mathcal{R}}|\mathcal{T}_{(s,s,0,0)}(-1/2) \\ &+ \frac{Q_\theta(s)}{Q_\theta(1)}\langle \delta^{\mathcal{R}}|\Big(\mathcal{T}_{(s-1,s-3,0,0)}(-s+1/2) - \mathcal{T}_{(s,s-2,0,0)}(-s+1/2) - \mathcal{T}_{(s-3,s-3,0,0)}(-s+1/2)\Big),\end{aligned} \tag{5.2}$$

where the rotated MPSs are defined as in eq. (4.53) and similarly for the rotated two-site state. The rotation for the other type of generalized dressing formula (4.39) is analogous. If the original MPS and two-site state are given by the formulas (3.30) and (4.12) then the rotated ones are

$$\begin{aligned}\langle \mathrm{MPS}^{\mathcal{R}}| &= \sum_{i_1,\dots,i_{2J}} \mathrm{Tr}\left[\psi^{\mathcal{R}}_{i_{2J-1},i_{2J}}(\theta_J)\dots\psi^{\mathcal{R}}_{i_1,i_2}(\theta_1)\right]\langle i_1,i_2,\dots,i_{2J-1},i_{2J}|, \\ \langle \delta^{\mathcal{R}}| &= \langle \varphi^{\mathcal{R}}|^{\otimes J},\end{aligned} \tag{5.3}$$

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where

$$\langle\varphi^{\mathcal{R}}| = \langle\varphi| \left[\mathcal{R}^{\mathbf{6}} \otimes \mathcal{R}^{\mathbf{6}}\right], \quad \psi^{\mathcal{R}}_{i,j}(u) = \sum_{k,l} \psi^{\mathcal{R}}_{k,l}(u) \left(\mathcal{R}^{\mathbf{6}}\right)_{k,i} \left(\mathcal{R}^{\mathbf{6}}\right)_{l,j}. \tag{5.4}$$

Let us consider the GL(4) transformation

$$\mathcal{R}^{\mathbf{6}} = \exp\left(\frac{i\pi}{4}\left(\mathcal{E}_{1,4} + \mathcal{E}_{4,1} + \mathcal{E}_{2,3} + \mathcal{E}_{3,2}\right)\right). \tag{5.5}$$

That leads to a $\psi^{\mathcal{R}}_{j,k}(u)$ which we give in explicit form in appendix A and which factorizes as follows for $u = 0$

$$\psi^{\mathcal{R}}_{j,k}(u) = \omega^{\mathcal{R}}_k \omega^{\mathcal{R}}_j, \tag{5.6}$$

where

$$\bar{\omega}^{\mathcal{R}} = \left\{\omega^{\mathcal{R}}_Z, \omega^{\mathcal{R}}_Y, \omega^{\mathcal{R}}_X, \omega^{\mathcal{R}}_{\bar{X}}, \omega^{\mathcal{R}}_{\bar{Y}}, \omega^{\mathcal{R}}_{\bar{Z}}\right\} = \left\{S^R_1 + S^L_2, S^R_2 + S^L_1, S^R_3 + S^L_3, S^R_3 - S^L_3, S^R_2 - S^L_1, S^R_1 - S^L_2\right\}. \tag{5.7}$$

With the $(0, s)$ representation the rotated matrices read

$$\bar{\omega}^{R,\mathcal{R}} = \left\{\omega^{R,\mathcal{R}}_Z, \omega^{R,\mathcal{R}}_Y, \omega^{R,\mathcal{R}}_X, \omega^{R,\mathcal{R}}_{\bar{X}}, \omega^{R,\mathcal{R}}_{\bar{Y}}, \omega^{R,\mathcal{R}}_{\bar{Z}}\right\} = \left\{S_1, S_2, S_3, S_3, S_2, S_1\right\}, \tag{5.8}$$

and give the usual convention for the MPS which describes the D3-D5 domain wall one-point functions [24], cf. eqs. (1.1)–(1.3). For the $(s, 0)$ representation we have $\langle\mathrm{MPS}^{L,\mathcal{R}}_{2s+1}|$ for which

$$\bar{\omega}^{L,\mathcal{R}} = \left\{\omega^{L,\mathcal{R}}_Z, \omega^{L,\mathcal{R}}_Y, \omega^{L,\mathcal{R}}_X, \omega^{L,\mathcal{R}}_{\bar{X}}, \omega^{L,\mathcal{R}}_{\bar{Y}}, \omega^{L,\mathcal{R}}_{\bar{Z}}\right\} = \left\{S_2, S_1, S_3, -S_3, -S_1, -S_2\right\}. \tag{5.9}$$

We have the identity

$$\sum_{a,b} \exp\left(\frac{i\pi}{2} S_3\right) \psi^{L,\mathcal{R}}_{a,b}(u) \exp\left(-\frac{i\pi}{2} S_3\right) \mathcal{D}^{\mathbf{6}}_{a,j} \mathcal{D}^{\mathbf{6}}_{b,k} = -\psi^{R,\mathcal{R}}_{j,k}(u), \tag{5.10}$$

where

$$\mathcal{D}^{\mathbf{6}} = \exp\left(\frac{i\pi}{4}\left(\mathcal{E}_{1,1} + \mathcal{E}_{2,2} - 3\mathcal{E}_{3,3} + \mathcal{E}_{4,4}\right)\right). \tag{5.11}$$

Using this identity it is easy to show that

$$\langle\mathrm{MPS}^{L,\mathcal{R}}|\Delta(\mathcal{D}) = (-1)^J \langle\mathrm{MPS}^{R,\mathcal{R}}|. \tag{5.12}$$

Let us apply the this equation on a Bethe state

$$\langle\mathrm{MPS}^{L,\mathcal{R}}|\Delta(\mathcal{D})|\bar{u}\rangle = (-1)^J \langle\mathrm{MPS}^{R,\mathcal{R}}|\bar{u}\rangle. \tag{5.13}$$

Since the operator $\mathcal{D}^{\mathbf{6}}$ is diagonal the co-product $\Delta(\mathcal{D})$ acts diagonally on the Bethe states:

$$\Delta(\mathcal{D})|\bar{u}\rangle = \exp\left(i\pi\left(J - n_2 + n_3\right)\right) = (-1)^{J+n_2+n_3}, \tag{5.14}$$

therefore the overlap of the state $\langle\mathrm{MPS}^{L,\mathcal{R}}|$ can be expressed with the overlap of the other state $\langle\mathrm{MPS}^{R,\mathcal{R}}|$ as

$$\langle\mathrm{MPS}^{L,\mathcal{R}}_{2s+1}|\bar{u}\rangle = (-1)^{n_2+n_3} \langle\mathrm{MPS}^{R,\mathcal{R}}_{2s+1}|\bar{u}\rangle. \tag{5.15}$$

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Assuming that the number of Bethe roots are even the MPS overlap for integer spin can be expressed as

$$\langle \mathrm{MPS}_{2s+1}^{R,\mathcal{R}}|\bar{u}\rangle = \langle \delta^{\mathcal{R}}|\bar{u}\rangle R_s, \tag{5.16}$$

where we defined the ratio as

$$\begin{aligned} R_s =& \frac{1}{2}\tau_{(s,s,0,0)}(-1/2) \\ &+ \frac{1}{2}\frac{Q_\theta(s)}{Q_\theta(1)}\left(\tau_{(s-1,s-3,0,0)}(-s+1/2) - \tau_{(s,s-2,0,0)}(-s+1/2) - \tau_{(s-3,s-3,0,0)}(-s+1/2)\right). \end{aligned} \tag{5.17}$$

Using the explicit expression for the eigenvalues (2.23) and the pair structures

$$Q_\theta(z) = Q_\theta(-z), \quad Q_k(z) = Q_k(-z), \quad \text{for } k = 1,2,3, \tag{5.18}$$

we can show that

$$R_s = \sum_{a=-s}^{s} Q_\theta(a)\frac{Q_1(a)}{Q_1(0)}\frac{Q_2(s+1/2)Q_2(1/2)}{Q_2(a+1/2)Q_2(a-1/2)}\frac{Q_3(a)}{Q_3(0)}. \tag{5.19}$$

where the spins are integers. This result agrees with the previously found formula [24].

In the half integer case MPS overlap can be expressed as

$$\langle \mathrm{MPS}_{2s+2}^{R,\mathcal{R}}|\bar{u}\rangle = \langle \sigma^{R,\mathcal{R}}|\bar{u}\rangle \tilde{R}_s, \tag{5.20}$$

where

$$\begin{aligned} \tilde{R}_s =& \frac{1}{Q_\theta(1/2)}\tau_{(s,s,0,0)}(0) \\ &+ \frac{Q_\theta(s+1/2)}{Q_\theta(1/2)^2}\left(\tau_{(s-1,s-2,0,0)}(-s) - \tau_{(s,s-1,0,0)}(-s) - \tau_{(s-2,s-2,0,0)}(-s)\right). \end{aligned} \tag{5.21}$$

Repeating the previous calculations we obtain that

$$\tilde{R}_s = \frac{1}{2}\sum_{a=-s-1/2}^{s+1/2} \frac{Q_\theta(a)}{Q_\theta(1/2)}\frac{Q_1(a)}{Q_1(1/2)}\frac{Q_2(s+1)Q_2(0)}{Q_2(a+1/2)Q_2(a-1/2)}\frac{Q_3(a)}{Q_3(1/2)}, \tag{5.22}$$

which also agrees with [24].

Notice that the detailed calculations of on-shell overlaps of this section cannot be applied in the twisted case since the twisted transfer matrices do not have GL(4) symmetry (rather the GL(4) rotations transform between different twists), and the selection rules do no imply pairing of Bethe roots as explained in section 3.4. Due to the lack of GL(4) symmetry, it is not possible to use "rotations" to relate the overlaps $\langle \mathrm{MPS}_{2s+1}^{L}|\bar{u}\rangle$ to the overlaps $\langle \mathrm{MPS}_{2s+1}^{R}|\bar{u}\rangle$, i.e. the dressing formula only gives the sum of the overlaps $\langle \mathrm{MPS}_{2s+1}^{L}|\bar{u}\rangle + \langle \mathrm{MPS}_{2s+1}^{R}|\bar{u}\rangle$. Furthermore, due to the lack of pair structure, the linear combination R_s does not simplify.

6 Discussion and conclusion

We have simplified, sharpened and extended our twisted Yangian approach to the computation of overlaps between integrable matrix product states and Bethe eigenstates. In particular, we have incorporated inhomogeneities and twists in our formalism. We have concentrated on particular matrix product states of the integrable $SO(6)$ spin chain which play a central role for the computation of one-point functions in the D3-D5 domain wall version of $\mathcal{N}=4$ SYM. These matrix product states originate from evaluation representations of the twisted Yangian $Y^+(4)$. The generic evaluation representations of $Y^+(4)$ are characterized by two spin quantum numbers, s_R and s_L, of a right and a left version of $\mathfrak{sl}_2$ but only those for which one of the $\mathfrak{sl}_2$ representations is trivial, denoted as $V_R(s)$ and $V_L(s)$, appear in the D3-D5 domain wall problem. Our main result consists of two dressing formulas, one for integer and one for half-integer s, which express the sum of the two matrix product states corresponding to respectively $V_R(s)$ and $V_L(s)$ as a linear combination of fused transfer matrices acting on the MPS of the trivial representation. We stress that these dressing formulas are highly universal. They are valid for any choice of twists and inhomogeneities and they also hold for any choice of quantum space for the spin chain, not just the space corresponding to the fundamental representation which is the one which appears in $\mathcal{N}=4$ SYM. By means of these dressing formulas we managed to fill the final gap in the analytical understanding of the overlaps in the $\mathfrak{so}(6)$ sector of $\mathcal{N}=4$ SYM.

Provided one can determine the overlap between the Bethe eigenstates and the MPS corresponding to the trivial representation of $Y^+(4)$, and provided that the left and the right version of the general MPS can be related, the dressing formulas give access to a closed overlap formula for any MPS of evaluation representation type. In the case where the twist vanishes the former overlap can be read off from [20, 21] where the overlap was determined for any two site product state for any GL(N) spin chain. Devising a method which allows one to find the same overlap in the presence of twists constitutes an interesting open problem which could possibly be approached by means of the functional SoV approach [51–53]. In the case where twists are present, however, there is no simple relation between the left and the right version of the MPS. Yet our dressing formulas give rise to a non-trivial summation rule for right and left MPS overlaps. Furthermore the overlaps with the trivial representation in the twisted case may simplify if one introduces the full $\mathcal{Q}$-system [47–50] and not only the limited set of Q-functions used here.

One may wonder if the general evaluation representations $V(s_R, s_L)$ and their corresponding integrable matrix product states have an interpretation within the AdS/CFT correspondence. An obvious place to search for such an interpretation would be the non-supersymmetric D3-D7 domain wall version of $\mathcal{N}=4$ SYM, which is characterized by the scalar fields attaining vevs which are given by two different representations of $\mathfrak{su}(2)$ [54]. A glance at the eq. (5.7) immediately shows that this interpretation can not be correct as the mapping of the spin chain states to a set of three complex scalar fields and their complex conjugates (or rather six real scalar fields) only works for the representations $V_R(s)$ and $V_L(s)$. This is in accordance with the fact that the former D3-D7 domain wall version of $\mathcal{N}=4$ SYM was shown to be non integrable [55].

It is noteworthy that one can encounter a situation where the homogeneous limit of the dressing formulas is subtle, cf. eq. (4.45). Understanding under which circumstances this happens is an interesting open problem.

It would likewise be interesting to extend the results of the present paper to other spin chains and in particular other reflection algebras. The extension of overlap formulas to other reflection algebras, albeit in the case of vanishing twists, is addressed in the paper [56]. Whereas each reflection algebra has to be treated independently the resulting formulas are highly universal.

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A K-matrices and boundary states for the evaluation representations

In this section we collect the K-matrices for the evaluation representations discussed in section 3.5. The K-matrix for the general representation $V(s_L, s_R)$ reads

$$\mathbf{K}(u) = \begin{pmatrix} 1+\frac{2}{2u+1}(S_3^R+S_3^L) & \frac{2\sqrt{2}}{2u+1}S_+^L & \frac{2\sqrt{2}}{2u+1}S_+^R & 0 \\ \frac{2\sqrt{2}}{2u+1}S_-^L & 1+\frac{2}{2u+1}(S_3^R-S_3^L) & 0 & -\frac{2\sqrt{2}}{2u+1}S_+^R \\ \frac{2\sqrt{2}}{2u+1}S_-^R & 0 & 1-\frac{2}{2u+1}(S_3^R-S_3^L) & -\frac{2\sqrt{2}}{2u+1}S_+^L \\ 0 & -\frac{2\sqrt{2}}{2u+1}S_-^R & -\frac{2\sqrt{2}}{2u+1}S_-^L & 1-\frac{2}{2u+1}(S_3^R+S_3^L) \end{pmatrix}, \tag{A.1}$$

and the corresponding fused two-site state operators, $\psi_{a,b}(u)$ take the form

$$\begin{aligned} \psi_{a,b}(u) &= \omega_b\omega_a - u^2\varphi_{a,b} \\ &+ u\begin{pmatrix} 0 & 0 & \sqrt{2}S_-^R & \sqrt{2}S_-^R & 0 & -2S_3^R+C(u) \\ 0 & 0 & \sqrt{2}S_-^L & -\sqrt{2}S_-^L & 2S_3^L+C(u) & 0 \\ -\sqrt{2}S_-^R & -\sqrt{2}S_-^L & 0 & C(u) & -\sqrt{2}S_+^L & \sqrt{2}S_+^R \\ -\sqrt{2}S_-^R & \sqrt{2}S_-^L & C(u) & 0 & \sqrt{2}S_+^L & \sqrt{2}S_+^R \\ 0 & -2S_3^L+C(u) & \sqrt{2}S_+^L & -\sqrt{2}S_+^L & 0 & 0 \\ 2S_3^R+C(u) & 0 & -\sqrt{2}S_+^R & -\sqrt{2}S_+^R & 0 & 0 \end{pmatrix}_{a,b}, \end{aligned} \tag{A.2}$$

where $a, b = 1, \ldots, 6$ and

$$C(u) = \frac{1}{u+1}\left(\left(S^L\right)^2 - \left(S^R\right)^2\right), \quad \left(S^{L/R}\right)^2 = \sum_{k=1}^{3}\left(S_k^{L/R}\right)^2, \tag{A.3}$$

$$\bar{\omega} = \{\omega_Z,\omega_Y,\omega_X,\omega_{\bar{X}},\omega_{\bar{Y}},\omega_{\bar{Z}}\} = \left\{\sqrt{2}S_-^R,\sqrt{2}S_-^L,S_3^R+S_3^L,S_3^R-S_3^L,-\sqrt{2}S_+^L,\sqrt{2}S_+^R\right\}. \tag{A.4}$$

Finally,

$$\varphi_{a,b} = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & -1 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & -1 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}_{a,b} . \tag{A.5}$$

Applying the GL(4) rotation given in eq. (5.5) the fused two site operator (A.2) can be brought on the form

$$\begin{aligned} \psi^{\mathcal{R}}_{j,k}(u) &= \omega^{\mathcal{R}}_k \omega^{\mathcal{R}}_j - u^2 \varphi^{\mathcal{R}}_{j,k} \\ &+ u \begin{pmatrix} 0 & -i\left(S^L_3 - S^R_3\right) & i\left(S^L_1 - S^R_2\right) & -i\left(S^L_1 + S^R_2\right) & i\left(S^L_3 + S^R_3\right) & C(u) \\ i\left(S^L_3 - S^R_3\right) & 0 & -i\left(S^L_2 - S^R_1\right) & i\left(S^L_2 + S^R_1\right) & C(u) & -i\left(S^L_3 + S^R_3\right) \\ -i\left(S^L_1 - S^R_2\right) & i\left(S^L_2 - S^R_1\right) & 0 & C(u) & -i\left(S^L_2 + S^R_1\right) & i\left(S^L_1 + S^R_2\right) \\ i\left(S^L_1 + S^R_2\right) & -i\left(S^L_2 + S^R_1\right) & C(u) & 0 & i\left(S^L_2 - S^R_1\right) & -i\left(S^L_1 - S^R_2\right) \\ -i\left(S^L_3 + S^R_3\right) & C(u) & i\left(S^L_2 + S^R_1\right) & -i\left(S^L_2 - S^R_1\right) & 0 & i\left(S^L_3 - S^R_3\right) \\ C(u) & i\left(S^L_3 + S^R_3\right) & -i\left(S^L_1 + S^R_2\right) & i\left(S^L_1 - S^R_2\right) & -i\left(S^L_3 - S^R_3\right) & 0 \end{pmatrix}_{j,k} , \end{aligned} \tag{A.6}$$

where the $\bar{\omega}^{\mathcal{R}}$ was given in eq. (5.7) and

$$\varphi^{\mathcal{R}}_{a,b} = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix}_{a,b} . \tag{A.7}$$

Specializing to the representation $V(0,s) \equiv V_R(s)$ we get the K-matrix

$$\mathbf{K}^R(u) = \begin{pmatrix} 1 + \frac{2}{2u+1} S_3 & 0 & \frac{2\sqrt{2}}{2u+1} S_+ & 0 \\ 0 & 1 + \frac{2}{2u+1} S_3 & 0 & -\frac{2\sqrt{2}}{2u+1} S_+ \\ \frac{2\sqrt{2}}{2u+1} S_- & 0 & 1 - \frac{2}{2u+1} S_3 & 0 \\ 0 & -\frac{2\sqrt{2}}{2u+1} S_- & 0 & 1 - \frac{2}{2u+1} S_3 \end{pmatrix} , \tag{A.8}$$

and similarly for $V(s,0) \equiv V_L(s)$

$$\mathbf{K}^L(u) = \begin{pmatrix} 1 + \frac{2}{2u+1} S_3 & \frac{2\sqrt{2}}{2u+1} S_+ & 0 & 0 \\ \frac{2\sqrt{2}}{2u+1} S_- & 1 - \frac{2}{2u+1} S_3 & 0 & 0 \\ 0 & 0 & 1 + \frac{2}{2u+1} S_3 & -\frac{2\sqrt{2}}{2u+1} S_+ \\ 0 & 0 & -\frac{2\sqrt{2}}{2u+1} S_- & 1 - \frac{2}{2u+1} S_3 \end{pmatrix} . \tag{A.9}$$

The corresponding $\bar{\omega}^{R,\mathcal{R}}$ and $\bar{\omega}^{L,\mathcal{R}}$ were given in eq. (5.8) and (5.9).

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$s_R \backslash s_L$	1	0	-1
1		(0,0,1,1)	
0	(0,1,0,1)	(1,0,0,1),(0,1,1,0)	(1,0,1,0)
-1		(1,1,0,0)	

$\rightarrow$

$s_R \backslash s_L$	1	0	-1
1		$\times 1$	
0	$\times 1$	$\times 2$	$\times 1$
-1		$\times 1$	

Table 1. The $\mathfrak{gl}_4$ and $\mathfrak{o}_4$ weights of the representation $L(1,1,0,0)$.

$s_R \backslash s_L$	1	0	-1
1			
0	$\times 1$	$\times 1$	$\times 1$
-1			

$s_R \backslash s_L$	1	0	-1
1		$\times 1$	
0		$\times 1$	
-1		$\times 1$	

Table 2. States of the representations $V_R(1)$ and $V_L(1)$ in the $\mathfrak{o}_4$ weight space.

B Derivation of dressing formulas for integer spin

In this section we explicitly derive the dressing formulas for $s = 1$, $s = 2$ and $s = 3$ by going through the iterative procedure described in section 4.2. The key idea is to express $L(s, s, 0, 0|1/2)$ in terms of $V_R(s) \oplus V_L(s)$.

B.1 Formulas for $s = 1$

Let us start with $s = 1$. For the $\mathfrak{gl}_4$ representation $\lambda = (1, 1, 0, 0)$ we have six states and the $\mathfrak{gl}_4$ and $\mathfrak{o}_4$ weights are shown in table 1. The state $(0, 0, 1, 1)$ is highest weight state and the highest weight is the same as $V_R(1)$. States of the representations $V_R(1)$ and $V_L(1)$ in the $\mathfrak{o}_4$ weight space are shown in table 2. From the patterns we are lead to the following conjecture

$$L((1, 1, 0, 0)|1/2) = V_R(1) \oplus V_L(1). \tag{B.1}$$

It can easily be shown that this is true. The state $(0, 1, 0, 1)$ is also a highest weight state and the highest weights are

$$\mu_4(u) = \frac{u - 1/2}{u + 1/2}, \quad \mu_3(u) = \frac{u + 3/2}{u + 1/2}. \tag{B.2}$$

Therefore,

$$P_1(u) = \frac{u - 1/2}{u + 3/2}, \quad P_2(u) = \frac{(u + 1/2)(u - 3/2)}{(u - 1/2)(u + 3/2)}. \tag{B.3}$$

This agrees with the ratio of highest weights of $V_L(1)$, therefore we just showed (B.1). It means that we obtained the generalized dressing formula for $s = 1$:

$$\langle \mathrm{MPS}_1^R| + \langle \mathrm{MPS}_1^L| = \langle \delta|\mathcal{T}_{(1,1,0,0)}(-1/2). \tag{B.4}$$

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B.2 Formulas for $s = 2$

Let us continue with $s = 2$. For the $\mathfrak{gl}_4$ representation $\lambda = (2,2,0,0)$ we have 20 states and the $\mathfrak{gl}_4$ and $\mathfrak{o}_4$ weights are shown in table 3. The state $(0,0,2,2)$ is a highest weight state and the highest weight is the same as for $V_R(2)$. The state $(0,2,0,2)$ is also a highest weight state and the highest weights are

$$\begin{aligned} \mu_4(u) &= \frac{u-3/2}{u+1/2}, \quad \mu_3(u) = \frac{u+5/2}{u+1/2}, \\ P_1(u) &= \frac{u-3/2}{u+5/2}, \quad P_2(u) = \frac{(u+1/2)(u-5/2)}{(u-1/2)(u+5/2)}. \end{aligned} \tag{B.5}$$

This agrees with the ratio of highest weights of $V_L(2)$. States of the representations $V_R(2)$ and $V_L(2)$ in the $\mathfrak{o}_4$ weight space are shown in table 5. From the pattern

$$\begin{array}{ccccc} & 1 & & & \\ & 1\ 2\ 1 & & & \\ & 1\ 2\ 4\ 2\ 1 & & & \\ & 1\ 2\ 1 & & & \\ & 1 & & & \end{array} = \begin{array}{c} 1 \\ 1 \\ 1 \\ 1 \\ 1 \end{array} \oplus \begin{array}{c} \\ \\ 1\ 1\ 1\ 1\ 1 \\ \\ \\ \end{array} \oplus \begin{array}{c} \\ 1\ 1\ 1 \\ 1\ 2\ 1 \\ 1\ 1\ 1 \\ \\ \end{array} \tag{B.6}$$

we are lead to the following conjecture

$$L((2,2,0,0)|1/2) = V_R(2) \oplus V_L(2) \oplus L((2,0,0,0)|\xi). \tag{B.7}$$

It can easily be shown that this is true. The dimension of the factor space $V^{(2)} = L((2,2,0,0)|1/2) \backslash [V_R(2) \oplus V_L(2)]$ is 10 which agrees with the dimension of $L((2,0,0,0)|\xi)$. The state $(0,1,1,2)$ is a highest weight state in the factor space $V^{(2)}$ and its highest weights are

$$\mu_4(u) = \frac{u-3/2}{u+1/2}, \quad \mu_3(u) = \frac{(u-3/2)(u+3/2)}{(u-1/2)(u+1/2)},$$

therefore

$$P_1(u) = \frac{u-1/2}{u+3/2}, \quad P_2(u) = 1. \tag{B.8}$$

From (4.16) we can see that (B.8) agrees with the ratio of highest weights of the representation $L((2,0,0,0)|3/2)$, therefore we just showed that

$$L((2,2,0,0)|1/2) = V_R(2) \oplus V_L(2) \oplus L((2,0,0,0)|3/2). \tag{B.9}$$

This means that we obtained the generalized dressing formula for $s = 2$:

$$\langle \text{MPS}_2^R| + \langle \text{MPS}_2^L| = \langle \delta|\mathcal{T}_{(2,2,0,0)}(-1/2) - \frac{Q_\theta(2)}{Q_\theta(1)}\langle \delta|\mathcal{T}_{(2,0,0,0)}(-3/2), \tag{B.10}$$

where the normalization factor was fitted numerically.

$s_R \backslash s_L$	2	1	0	-1	-2
2			(0,0,2,2)		
1		(0,1,1,2)	(1,0,1,2),(0,1,2,1)	(1,0,2,1)	
0	(0,2,0,2)	(0,2,1,1),(1,1,0,2)	2×(1,1,1,1),(0,2,2,0),(2,0,0,2)	(1,1,2,0),(2,0,1,1)	(2,0,2,0)
-1		(1,2,0,1)	(2,1,0,1),(1,2,1,0)	(2,1,1,0)	
-2			(2,2,0,0)		

$s_R \backslash s_L$	2	1	0	-1	-2
2			×1		
1		×1	×2	×1	
0	×1	×2	×4	×2	×1
-1		×1	×2	×1	
-2			×1		

Table 3. The $\mathfrak{gl}_4$ and $\mathfrak{o}_4$ weights of the representation $L(2,2,0,0)$.

$s_R \backslash s_L$	2	1	0	-1	-2
2					
1		(0,0,0,2)	(0,0,1,1)	(0,0,2,0)	
0		(0,1,0,1)	(1,0,0,1),(0,1,1,0)	(1,0,1,0)	
-1		(0,2,0,0)	(1,1,0,0)	(2,0,0,0)	
-2					

$s_R \backslash s_L$	2	1	0	-1	-2
2					
1		×1	×1	×1	
0		×1	×2	×1	
-1		×1	×1	×1	
-2					

Table 4. The $\mathfrak{gl}_4$ and $\mathfrak{o}_4$ weights of the representation $L(2,0,0,0)$.

$s_R \backslash s_L$	2	1	0	-1	-2
2					
1					
0	×1	×1	×1	×1	×1
-1					
-2					

$s_R \backslash s_L$	2	1	0	-1	-2
2			×1		
1			×1		
0			×1		
-1			×1		
-2			×1		

Table 5. States of the representations $V_R(2)$ and $V_L(2)$ in the $\mathfrak{o}_4$ weight space.

B.3 Formulas for $s = 3$

Let us continue with $s = 3$. For the $\mathfrak{gl}_4$ representation $\lambda = (3,3,0,0)$ we have 50 states and the $\mathfrak{gl}_4$ and $\mathfrak{o}_4$ weights are shown in the table 6. The states $(0,0,3,3)$ and $(0,3,0,3)$ are highest weight states which correspond to invariant subspaces $V_R(3)$ and $V_L(3)$. Let us define the factor space $V^{(2)} = L((3,3,0,0)|1/2) \backslash [V_R(3) \oplus V_L(3)]$, i.e.

$$L((3,3,0,0)|1/2) = V_R(3) \oplus V_L(3) \oplus V^{(2)}. \tag{B.11}$$

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The states of $V^{(2)}$ in the $\mathfrak{o}_4$ weight space look like

$$\begin{array}{c} 1 \\ 1\ 2\ 1 \\ 1\ 2\ 4\ 2\ 1 \\ 1\ 2\ 4\ 6\ 4\ 2\ 1 \\ 1\ 2\ 4\ 2\ 1 \\ 1\ 2\ 1 \\ 1 \end{array} - \left(\begin{array}{l} 1 \\ 1 \\ 1 \\ 1 \oplus 1\ 1\ 1\ 1\ 1\ 1\ 1 \\ 1 \\ 1 \\ 1 \end{array} \right) = \begin{array}{c} 1\ 1\ 1 \\ 1\ 2\ 3\ 2\ 1 \\ 1\ 3\ 4\ 3\ 1 \\ 1\ 2\ 3\ 2\ 1 \\ 1\ 1\ 1 \end{array}. \tag{B.12}$$

The state (0,1,2,3) is a highest weight state in the factor space $V^{(2)}$ and the corresponding highest weights are

$$\begin{aligned} \mu_4^{(2)}(u) &= \frac{u-5/2}{u+1/2}, \quad \mu_3^{(2)}(u) = \frac{(u-5/2)(u+3/2)}{(u-1/2)(u+1/2)}, \\ P_1^{(2)}(u) &= \frac{u-1/2}{u+3/2}, \quad P_2^{(2)}(u) = \frac{(u+5/2)(u-3/2)}{(u-5/2)(u+3/2)}, \end{aligned} \tag{B.13}$$

which agree with the highest weights of $L((3,1,0,0)|5/2)$ (see (4.16)). Since the dimension of $L((3,1,0,0)|5/2)$ is bigger then $V^{(2)}$ we have the decomposition

$$L((3,1,0,0)|5/2) = V^{(2)} \oplus V^{(3)}. \tag{B.14}$$

The states of $L((3,1,0,0)|5/2)$ are shown in table 7 and the states of $V^{(3)}$ in the $\mathfrak{o}_4$ weights space therefore look like

$$\begin{array}{c} 1\ 1\ 1 \\ 1\ 3\ 4\ 3\ 1 \\ 1\ 4\ 5\ 4\ 1 \\ 1\ 3\ 4\ 3\ 1 \\ 1\ 1\ 1 \end{array} - \begin{array}{c} 1\ 1\ 1 \\ 1\ 2\ 3\ 2\ 1 \\ 1\ 3\ 4\ 3\ 1 \\ 1\ 2\ 3\ 2\ 1 \\ 1\ 1\ 1 \end{array} = \begin{array}{c} 1\ 1\ 1 \\ 1\ 1\ 1 \\ 1\ 1\ 1 \end{array}. \tag{B.15}$$

There are three states $|\Lambda_1\rangle$, $|\Lambda_2\rangle$, $|\Lambda_3\rangle$ in $L((3,1,0,0)|5/2)$ which have $(1,1)$ spins and the corresponding GT-patterns are

$$\Lambda_1 = \begin{array}{cccc} 3 & 1 & 0 & 0 \\ & 2 & 0 & 0 \\ & 1 & 0 & \\ & & 0 & \end{array}, \quad \Lambda_2 = \begin{array}{cccc} 3 & 1 & 0 & 0 \\ & 1 & 1 & 0 \\ & 1 & 0 & \\ & & 0 & \end{array}, \quad \Lambda_3 = \begin{array}{cccc} 3 & 1 & 0 & 0 \\ & 1 & 0 & 0 \\ & 1 & 0 & \\ & & 1 & \end{array}. \tag{B.16}$$

$s_R \backslash s_L$	3	2	1	0	-1	-2	-3
3				$\times 1$			
2			$\times 1$	$\times 2$	$\times 1$		
1		$\times 1$	$\times 2$	$\times 4$	$\times 2$	$\times 1$	
0	$\times 1$	$\times 2$	$\times 4$	$\times 6$	$\times 4$	$\times 2$	$\times 1$
-1		$\times 1$	$\times 2$	$\times 4$	$\times 2$	$\times 1$	
-2			$\times 1$	$\times 2$	$\times 1$		
-3				$\times 1$			

Table 6. The $\mathfrak{gl}_4$ and $\mathfrak{o}_4$ weights of the representation $L(3,3,0,0)$.

The state $|\Lambda_1\rangle + \frac{1}{4}|\Lambda_3\rangle$ is a highest weight state which generates the invariant subspace $V^{(3)}$. The corresponding highest weights are

$$\begin{aligned} \mu_4^{(3)}(u) &= \frac{u+1/2}{u+5/2}, \quad \mu_3^{(3)}(u) = 1, \\ P_1^{(3)}(u) &= \frac{u+1/2}{u+5/2}, \quad P_2^{(3)}(u) = 1, \end{aligned} \tag{B.17}$$

which agree with the highest weights of $L((2,0,0,0)|5/2)$. Since the dimension of $L((2,0,0,0)|5/2)$ is 10 but the subspace $V^{(3)}$ has dimension 9, we have the following decomposition

$$L((2,0,0,0)|5/2) = V^{(3)} \oplus V^{(4)}. \tag{B.18}$$

The states of $V^{(4)}$ in the $\mathfrak{o}_4$ weights space look like

$$\begin{matrix} 1 & 1 & 1 \\ 1 & 2 & 1 \\ 1 & 1 & 1 \end{matrix} \; - \; \begin{matrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{matrix} \; = \quad 1, \tag{B.19}$$

therefore $V^{(4)}$ is the singlet representation. Summarizing the results, the generalized dressing formula for $s = 3$ can be written as

$$\begin{aligned} &\langle \mathrm{MPS}_3^R| + \langle \mathrm{MPS}_3^L| \\ &= \langle\delta|\mathcal{T}_{(3,3,0,0)}(-1/2) + \frac{Q_\theta(3)}{Q_\theta(1)}\langle\delta| \left(\mathcal{T}_{(2,0,0,0)}(-5/2) - \mathcal{T}_{(3,1,0,0)}(-5/2) - \mathcal{T}_{(0,0,0,0)}(-5/2)\right), \end{aligned} \tag{B.20}$$

where the normalization factor was fitted numerically.

$s_R \backslash s_L$	3	2	1	0	-1	-2	-3
3							
2			$\times 1$	$\times 1$	$\times 1$		
1		$\times 1$	$\times 3$	$\times 4$	$\times 3$	$\times 1$	
0		$\times 1$	$\times 4$	$\times 5$	$\times 4$	$\times 1$	
-1		$\times 1$	$\times 3$	$\times 4$	$\times 3$	$\times 1$	
-2			$\times 1$	$\times 1$	$\times 1$		
-3							

Table 7. The $\mathfrak{gl}_4$ and $\mathfrak{o}_4$ weights of the representation $L(3, 1, 0, 0)$.

B.4 Formulas for general integer s

Based on the previous analysis we have the following conjecture for the representation embeddings

$$\begin{aligned} L((s,s,0,0)|1/2) &= V_R(s) \oplus V_L(s) \oplus V^{(2)}, \\ L((s,s-2,0,0)|s-1/2) &= V^{(2)} \oplus V^{(3)}, \\ L((s-1,s-3,0,0)|s-1/2) &= V^{(3)} \oplus L((s-3,s-3,0,0)|s-1/2). \end{aligned} \tag{B.21}$$

The dimensions of the representations are

$$\begin{aligned} V_R(s) \oplus V_L(s) &\to 2(2s+1), \\ L((s,s,0,0)|1/2) &\to \frac{(s+1)(s+2)^2(s+3)}{12}, \\ L((s,s-2,0,0)|1/2) &\to \frac{(s-1)s(s+2)(s+3)}{4}. \end{aligned} \tag{B.22}$$

From the first two equations we can express the dimensions of the $V^{(2)}$ and $V^{(3)}$ spaces as

$$\begin{aligned} V^{(2)} &\to \frac{(s-1)(s^3+9s^2+32s+12)}{12}, \\ V^{(3)} &\to \frac{(s-2)(s-1)(5s^2+5s+3)}{6}, \end{aligned} \tag{B.23}$$

and the dimensions in the third equation are consistent with this. This is a strong indication that the formulas are correct. Comprehensive numerical checks confirm that this is indeed the case.

Using the branching rules (B.21) we arrive at the generalized dressing formula for integer spins given in eq. (4.32) where we have fitted the normalization factor numerically.

C Derivation of dressing formulas for half-integer spin

As before we denote the half integer spins as $s + 1/2$ where s is integer. Below we explicitly derive the dressing formulas for $s = 1$ and $s = 2$ by going through the iterative procedure described in section 4.2. We start from the observation made in section 4.3 that

$$L((s,s,0,0)|0) \otimes V_R(1/2) = V_R(s+1/2) \oplus V^{(2)}, \tag{C.1}$$

where $V^{(2)}$ has to be expressed in terms of irreps.

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C.1 Formulas for $s = 1$

Let us start with $s = 1$. The states of $V^{(2)}$ in the $\mathfrak{o}_4$ weight space are

$$\begin{array}{ccc} 1 \\ 1\,3\,1 \\ 1\,3\,1 \\ 1 \end{array} - \begin{array}{c} 1 \\ 1 \\ 1 \\ 1 \end{array} = \begin{array}{c} 1\,2\,1 \\ 1\,2\,1 \end{array}. \tag{C.2}$$

The state $(1, 1/2) \equiv (0, 1, 0, 1) \otimes |+\rangle$ is a highest weight state in the factor space $V^{(2)}$ for which

$$\begin{aligned} \mu_4^{(2)}(u) &= \frac{u-1}{u+1/2}, \quad \mu_3^{(2)}(u) = \frac{(u-1)(u+1)^2}{u^2(u+1/2)}, \\ P_1^{(2)}(u) &= \frac{u^2}{(u+1)^2}, \quad P_2^{(2)}(u) = \frac{(u-1)(u+1/2)}{(u+1)(u-1/2)}, \end{aligned} \tag{C.3}$$

which agree with the highest weights of the representation $L((1, 0, 0, 0)|1) \otimes V_L(1/2)$. Since the dimensions of $L((1, 0, 0, 0)|1) \otimes V_L(1/2)$ and $V^{(2)}$ agree

$$L((1, 1, 0, 0)|0) \otimes V_R(1/2) = V_R(3/2) \oplus [L((1, 0, 0, 0)|1) \otimes V_L(1/2)]\,. \tag{C.4}$$

The corresponding generalized dressing is

$$\langle \mathrm{MPS}_4^R| = \frac{1}{Q_\theta(1/2)} \langle \sigma^R|\mathcal{T}_{(1,1,0,0)}(0) - \frac{Q_\theta(3/2)}{Q_\theta(1/2)^2} \langle \sigma^L|\mathcal{T}_{(1,0,0,0)}(-1). \tag{C.5}$$

C.2 Formulas for $s = 2$

The states of $V^{(2)}$ in the $\mathfrak{o}_4$ weight space are

$$\begin{array}{c} 1 \\ 1\,3\,1 \\ 1\,3\,6\,3\,1 \\ 1\,3\,6\,3\,1 \\ 1\,3\,1 \\ 1 \end{array} - \begin{array}{c} 1 \\ 1 \\ 1 \\ 1 \\ 1 \\ 1 \end{array} = \begin{array}{c} 1\,2\,1 \\ 1\,3\,5\,3\,1 \\ 1\,3\,5\,3\,1 \\ 1\,2\,1 \end{array}. \tag{C.6}$$

The state $(1, 3/2) \equiv (0, 1, 1, 2) \otimes |+\rangle$ is highest weight in the factor space $V^{(2)}$ for which

$$\begin{aligned} \mu_4^{(2)}(u) &= \frac{u-2}{u+1/2}, \quad \mu_3^{(2)}(u) = \frac{(u-2)(u+1)^2}{u^2(u+1/2)}, \\ P_1^{(2)}(u) &= \frac{u^2}{(u+1)^2}, \quad P_2^{(2)}(u) = \frac{(u+2)(u-1)^2(u+1/2)}{(u-2)(u+1)^2(u-1/2)}, \end{aligned} \tag{C.7}$$

which agree with the highest weights of the representation $L((2, 1, 0, 0)|2) \otimes V_L(1/2)$, therefore

$$L((2, 1, 0, 0)|2) \otimes V_L(1/2) = V^{(2)} \oplus V^{(3)}. \tag{C.8}$$

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The states of $V^{(3)}$ in the $\mathfrak{o}_4$ weight space are

$$\begin{array}{c} 1\ 2\ 1 \\ 1\ 4\ 6\ 4\ 1 \\ 1\ 4\ 6\ 4\ 1 \\ 1\ 2\ 1 \end{array} - \begin{array}{c} 1\ 2\ 1 \\ 1\ 3\ 5\ 3\ 1 \\ 1\ 3\ 5\ 3\ 1 \\ 1\ 2\ 1 \end{array} = \begin{array}{c} 1\ 1\ 1 \\ 1\ 1\ 1 \end{array}. \tag{C.9}$$

The $\mathfrak{o}_4$ weight $(1, 1/2)$ corresponds to the 4-dimensional subspace which is spanned by the vectors $|\Lambda_1\rangle \otimes |-\rangle$, $|\Lambda_2\rangle \otimes |+\rangle$, $|\Lambda_3\rangle \otimes |+\rangle$ and $|\Lambda_4\rangle \otimes |+\rangle$ where

$$\Lambda_1 = \begin{array}{c} 2 \quad 1 \quad 0 \quad 0 \\ 1 \quad 0 \quad 0 \\ 1 \quad 0 \\ 0 \end{array}, \quad \Lambda_2 = \begin{array}{c} 2 \quad 1 \quad 0 \quad 0 \\ 1 \quad 1 \quad 0 \\ 1 \quad 0 \\ 0 \end{array}, \quad \Lambda_3 = \begin{array}{c} 2 \quad 1 \quad 0 \quad 0 \\ 2 \quad 0 \quad 0 \\ 1 \quad 0 \\ 0 \end{array}, \quad \Lambda_4 = \begin{array}{c} 2 \quad 1 \quad 0 \quad 0 \\ 1 \quad 0 \quad 0 \\ 1 \quad 0 \\ 1 \end{array}. \tag{C.10}$$

The vector

$$2|\Lambda_1\rangle \otimes |-\rangle - |\Lambda_2\rangle \otimes |+\rangle + |\Lambda_3\rangle \otimes |+\rangle + |\Lambda_4\rangle \otimes |+\rangle, \tag{C.11}$$

is a highest weight state for which

$$\begin{aligned} \mu_4^{(3)}(u) &= \frac{u(u+1)}{(u+1/2)(u+2)}, & \mu_3^{(3)}(u) &= \frac{u+1}{u+1/2}, \\ P_1^{(3)}(u) &= \frac{u}{u+2}, & P_2^{(3)}(u) &= \frac{(u-1)(u+1/2)}{(u+1)(u-1/2)}, \end{aligned} \tag{C.12}$$

which agree with the highest weights of the representation $L((1,0,0,0)|2) \otimes V_L(1/2)$, therefore

$$L((1,0,0,0)|2) \otimes V_L(1/2) = V^{(3)} \oplus V_R(1/2). \tag{C.13}$$

The corresponding generalized dressing is

$$\begin{aligned} \langle \mathrm{MPS}_6^R| = {}& \frac{1}{Q_\theta(1/2)} \langle \sigma^R | \mathcal{T}_{(2,2,0,0)}(0) \\ & + \frac{Q_\theta(5/2)}{Q_\theta(1/2)^2} \left(\langle \sigma^L | \mathcal{T}_{(1,0,0,0)}(-2) - \langle \sigma^L | \mathcal{T}_{(2,1,0,0)}(-2) - \langle \sigma^R | \mathcal{T}_{(0,0,0,0)}(-2) \right), \end{aligned} \tag{C.14}$$

where the normalization factor was determined numerically.

C.3 Formulas for general s

Based on the previous analysis we have the following conjecture for the representation embeddings

$$\begin{aligned} L((s,s,0,0)|0) \otimes V_R(1/2) &= V_R(s+1/2) \oplus V^{(2)}, \\ L((s,s-1,0,0)|s) \otimes V_L(1/2) &= V^{(2)} \oplus V^{(3)}, \\ L((s-1,s-2,0,0)|s) \otimes V_L(1/2) &= V^{(3)} \oplus [L((s-2,s-2,0,0)|s) \otimes V_R(1/2)]. \end{aligned} \tag{C.15}$$

We notice that the dimensions are consistent. We validated the formulas numerically. Using the branching rules (C.15) we arrive at the generalized dressing formula for half-integer spins given in eq. (4.39) where again the normalization factor was fitted numerically.

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