Derivations for the MPS overlap formulas of rational spin chains

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Abstract

We derive a universal formula for the overlaps between integrable matrix product states (MPS) and Bethe eigenstates in $\mathfrak{gl}_N$ symmetric spin chains. This formula expresses the normalized overlap as a product of a MPS-independent Gaudin-determinant ratio and a MPS-dependent scalar factor constructed from eigenvalues of commuting operators, defined via the K-matrix associated with the MPS. Our proof is fully representation-independent and relies solely on algebraic Bethe Ansatz techniques and the KT-relation. We also propose a generalization of the overlap formula to $\mathfrak{so}_N$ and $\mathfrak{sp}_N$ spin chains, supported by algebra embeddings and low-rank isomorphisms. These results significantly broaden the class of integrable initial states for which exact overlap formulas are available, with implications for quantum quenches and defect CFTs.

1 Introduction

Overlaps between integrable boundary states and Bethe eigenstates in quantum spin chains have emerged as an important building block in the study of integrable systems. The integrable boundary states allow exact computations of overlaps with Bethe eigenstates, often yielding compact expressions involving Gaudin-like determinant ratios and scalar prefactors. The structure of these overlaps is universal across a wide class of models, including those with $\mathfrak{gl}_N$, $\mathfrak{so}_N$, and $\mathfrak{sp}_N$ symmetries, and they are relevant in both condensed matter and high-energy contexts.

In recent years, the study of non-equilibrium dynamics in integrable models has gained significant momentum, driven both by experimental advances [1, 2, 3] and theoretical developments [4, 5]. A key focus has been on quantum quenches, where a system is initialized in a specific state and its subsequent evolution is tracked. The Quench Action method has emerged as a powerful framework for analyzing such dynamics, particularly in determining the long-time steady states [6, 7, 8]. Central to this approach is the computation of overlaps between the initial state and the eigenstates of the post-quench Hamiltonian. While the method has been successfully applied to various integrable systems [9, 10, 11, 12], much of the progress has relied on initial states with simple entanglement structures, such as two-site product states, due to the tractability of their overlap calculations.

In the context of the AdS/CFT correspondence, integrability methods of the boundary states have provided powerful tools for analyzing correlation functions in the presence of defects [13, 14]. A central development has been the realization that one-point functions in defect configurations of the $\mathcal{N}=4$ super Yang-Mills (SYM) and ABJM theories can be expressed as overlaps between multi-particle Bethe states and special boundary or matrix product states (MPS) [15, 16, 17, 18]. This approach was initially applied to domain wall defects, such as the D3-D5 brane setup, where tree-level one-point functions were linked to MPS overlaps [19] and later extended to finite coupling via integrable bootstrap [20, 21, 22]. These techniques have also been generalized to other configurations, Wilson and 't Hooft lines, where similar overlap structures emerge [23, 24, 25, 26]. Moreover, overlaps have proven essential in computing certain three-point functions involving determinant and single trace operators, particularly in the AdS_5/CFT_4 and AdS_4/CFT_3 dualities [27, 28, 29]. More recent applications include the study of correlation functions on the Coulomb branch of $\mathcal{N}=4$ SYM [30], and the surface defects of Gukov-Witten type in $\mathcal{N}=4$ SYM [31, 32]. The integrable conformal defects in $\mathcal{N}=4$ SYM was also classified [33]. These developments underscore the universality of overlap structures and their growing relevance in holographic context.

We can distinguish two types of boundary states: the simpler tensor product states and the matrix product states. In the latter, an extra vector space (boundary space) is introduced, and the boundary state is constructed from the product of matrices acting on this space. If this vector space is trivial (one-dimensional), then the

MPS reduces to a simple tensor product state. Boundary states can be related to integrable boundary conditions [34, 35, 36]. It has been observed that the overlaps of tensor product states with Bethe states are proportional to the ratio of so-called Gaudin determinants [37, 38]. In early works, such overlap formulas were proven for XXX and XXZ spin chains [39, 40, 41]. These derivations heavily depended on the representation of the Hilbert space and the specific form of the tensor product state, and it was not clear how to generalize them to systems with multiple quasi-particles (nested systems). In [42], the so-called KT-relation was introduced, whose main advantage is that it enables the calculation of overlaps using purely algebraic Bethe Ansatz tools. As a result, the derivations are universal, meaning they do not depend on the specific representations of the Hilbert space. The KT-relation and the corresponding overlap proof can be generalized to $\mathfrak{gl}_N$ symmetric spin chains [43, 44]. These papers proved the overlap functions for all $\mathfrak{gl}_N$ symmetric spin chains (independent of representation), for all integrable tensor product states, which is a rather general result.

In this paper, we set an even more ambitious goal. We aim to determine a universal overlap function for all integrable MPSs of all rational spin chains. We prove these formulas for a broad class of integrable MPSs of $\mathfrak{gl}_N$ spin chains. In the remaining cases, we provide strong arguments supporting the validity of our formula. The universal overlap formula takes the following form

$$\frac{\langle \mathrm{MPS}|\bar{u}\rangle}{\sqrt{\langle \bar{u}|\bar{u}\rangle}} = \underbrace{\sum_{\ell=1}^{d_B} \beta_\ell \prod_{\nu=1}^{n_+} \prod_{k=1}^{r_\nu^+} \tilde{\mathcal{F}}_\ell^{(\nu)}(u_k^\nu)}_{\langle \mathrm{MPS}|\text{ dependent}} \times \underbrace{\sqrt{\frac{\det G^+}{\det G^-}}}_{|\bar{u}\rangle\text{ dependent}}, \tag{1.1}$$

where $|\bar{u}\rangle$ is the Bethe state and u_k^ν are the Bethe roots. The Gaudin determinants $\det G^\pm$ depend only on the Bethe Ansatz equations. The quantities β_ℓ and $\tilde{\mathcal{F}}_\ell^{(\nu)}(u)$ are eigenvalues of commuting operators $\mathbf{B}$ and $\mathbf{F}^{(\nu)}(u)$. These operators can be defined from the elements of the K-matrix associated with $\langle \mathrm{MPS}|$. Assuming that a similar universal formula exists for $\mathfrak{so}_N$ and $\mathfrak{sp}_N$ spin chains, we determine the form of the operators $\mathbf{B}$ and $\mathbf{F}^{(\nu)}(u)$. The calculation is based on the algebra embeddings $\mathfrak{sl}_{\lfloor \frac{N}{2} \rfloor} \subset \mathfrak{so}_N$, $\mathfrak{sl}_n \subset \mathfrak{sp}_{2n}$ and the isometries of low-rank cases: $\mathfrak{sp}_2 \cong \mathfrak{sl}_2$, $\mathfrak{so}_3 \cong \mathfrak{sl}_2$, $\mathfrak{so}_4 \cong \mathfrak{sl}_2 \oplus \mathfrak{sl}_2$. The universal overlap formula and the definitions of the corresponding $\mathbf{B}$ and $\mathbf{F}^{(\nu)}(u)$ operators were already published in a previous letter [45] without proof. In the current paper, we provide the proof for the $\mathfrak{gl}_N$ case and argue for the orthogonal and symplectic cases.

The paper follows the structure below. In the next section, we summarize the necessary definitions of $\mathfrak{gl}_N$ symmetric spin chains. In Section 3, we define the integrable MPSs and the corresponding K-matrices for $\mathfrak{gl}_N$ spin chains. We show how the KT-relation can be used for the systematic calculation of overlaps and introduce the nested K-matrices required for this. In Section 4, we present theorems concerning the nested K-matrices and overlaps, including the formula (1.1), which is the main result. The proofs are found in the Appendix. In Section 5, we extend the definition of the operators $\mathbf{B}$ and $\mathbf{F}^{(\nu)}(u)$ to all $\mathfrak{gl}_N$ MPSs for which the formulas in Section 4 are not applicable. In Section 6, we extend the definition of the operators $\mathbf{B}$ and $\mathbf{F}^{(\nu)}(u)$ to $\mathfrak{so}_N$ and $\mathfrak{sp}_N$ symmetric spin chains as well.

2 Definitions of the $\mathfrak{gl}_N$ symmetric spin chains

Let $T_{i,j}(z)$ be the generators of the Yangian $Y(N)$ algebra [46]. They satisfy the RTT-relation

$$R_{1,2}(u-v)T_1(u)T_2(v) = T_2(v)T_1(u)R_{1,2}(u-v). \tag{2.1}$$

The $\mathfrak{gl}_N$ symmetric R matrix is

$$R(u) = 1 + \frac{1}{u}P, \qquad P = \sum_{i,j=1}^{N} e_{i,j} \otimes e_{j,i}, \tag{2.2}$$

where $e_{i,j} \in \mathrm{End}(\mathbb{C}^N)$-s are the unit matrices with the only nonzero entry equals to 1 at the intersection of the i-th row and j-th column. The RTT-relation can be expressed with the entries

$$[T_{i,j}(u), T_{k,l}(v)] = \frac{1}{u-v}\left(T_{k,j}(v)T_{i,l}(u) - T_{k,j}(u)T_{i,l}(v)\right). \tag{2.3}$$

In this paper we will assume the following dependence of the Yangian generators on the spectral parameter

$$T_{i,j}(u) = \delta_{i,j} + \sum_{\ell=0}^{\infty} u^{-\ell-1} T_{i,j}[\ell]. \tag{2.4}$$

The generators $E_{i,j} \equiv T_{j,i}[0]$ satisfy the $\mathfrak{gl}_N$ Lie-algebra relation

$$[E_{i,j}, E_{k,l}] = \delta_{k,j} E_{i,l} - \delta_{i,l} E_{k,j}. \tag{2.5}$$

Let $\mathcal{V}_\Lambda$ and $E^\Lambda_{i,j} \in \mathrm{End}(\mathcal{V}_\Lambda)$ be vector space and a corresponding irreducible highest weight representation of $\mathfrak{gl}_N$ for which

$$\begin{aligned} E^\Lambda_{i,j}|0_\Lambda\rangle &= 0, \quad i<j, \\ E^\Lambda_{i,i}|0_\Lambda\rangle &= \Lambda_i|0_\Lambda\rangle, \end{aligned} \tag{2.6}$$

where $\Lambda = (\Lambda_1, \dots, \Lambda_N)$ is an N-tuple of scalars and $|0_\Lambda\rangle \in \mathcal{V}_\Lambda$ is the highest weight state. For finite dimensional representations $\Lambda_i - \Lambda_{i+1} \in \mathbb{N}$ for $i = 1, \dots, N-1$. The Lax operators $L^\Lambda_{i,j}(u) \in \mathrm{End}(\mathcal{V}_\Lambda)$ are the evaluation representations of $Y(N)$

$$L^\Lambda_{i,j}(u) = \delta_{i,j} + \frac{1}{u} E^\Lambda_{j,i}. \tag{2.7}$$

The Lax-operators satisfy the RTT-relation. From the co-product property of the Yangian, we can define tensor product representations on $\mathcal{H} = \mathcal{V}_{\Lambda^{(1)}} \otimes \dots \otimes \mathcal{V}_{\Lambda^{(J)}}$. The $\mathcal{H}$ is the quantum space. The monodromy matrix is the corresponding representation of the Yangian

$$T^{\bar{\Lambda},\bar{\xi}}_0(u) = L^{\Lambda^{(J)}}_{0,J}(u-\xi_J) \dots L^{\Lambda^{(1)}}_{0,1}(u-\xi_1). \tag{2.8}$$

The monodromy matrix is a *highest weight representation* of the Yangian, i.e.,

$$\begin{aligned} T^{\bar{\Lambda},\bar{\xi}}_{i,j}(u)|0\rangle &= 0, \quad i>j, \\ T^{\bar{\Lambda},\bar{\xi}}_{i,i}(u)|0\rangle &= \lambda_i(u)|0\rangle, \end{aligned} \tag{2.9}$$

where $|0\rangle = |0_{\Lambda^{(1)}}\rangle \otimes \dots \otimes |0_{\Lambda^{(J)}}\rangle$ is the pseudo-vacuum and $\lambda_i(u)$-s are the pseudo-vacuum eigenvalues which can be expressed as

$$\lambda_i(u) = \prod_{j=1}^{J} \frac{u - \xi_j + \Lambda^{(j)}_i}{u - \xi_j}. \tag{2.10}$$

For simplicity, we will omit the upper index from now on and use the notation $T_{i,j}(u)$ for arbitrary highest-weight representations of the Yangian algebra.

The crossed monodromy matrix is defined as an inverse

$$\sum_{k=1}^{N} \widehat{T}_{k,i}(z) T_{k,j}(z) = \lambda_1(z)\lambda_1(-z)\delta_{i,j}. \tag{2.11}$$

The crossed monodromy matrices satisfy the following algebra

$$\begin{aligned} R_{1,2}(u-v)\widehat{T}_1(u)\widehat{T}_2(v) &= \widehat{T}_2(v)\widehat{T}_1(u)R_{1,2}(u-v), \\ \widehat{R}_{1,2}(u-v)\widehat{T}_1(u)T_2(v) &= T_2(v)\widehat{T}_1(u)\widehat{R}_{1,2}(u-v). \end{aligned} \tag{2.12}$$

It follows from the first equation that the crossed monodromy matrix is also a representation of the Yangian algebra. In the second equation we defined the crossed R-matrix

$$\widehat{R}_{1,2}(u) = R^{t_1}_{1,2}(-u), \tag{2.13}$$

where t_1 is the partial transposition corresponding to the space 1. The crossed monodromy matrices are *lowest weight representations* of the Yangians, i.e.

$$\begin{aligned} \widehat{T}_{i,j}(u)|0\rangle &= 0, \quad i<j, \\ \widehat{T}_{i,i}(u)|0\rangle &= \hat{\lambda}_i(u)|0\rangle, \end{aligned} \tag{2.14}$$

where $|0\rangle$ is the highest weight vector (pseudo-vacuum) of the monodromy matrix $T_{i,j}$ and the crossed eigenvalues are [47][1]

$$\hat{\lambda}_i(u) = \frac{\lambda_1(u)\lambda_1(-u)}{\lambda_i(u-(i-1))} \prod_{k=1}^{i-1} \frac{\lambda_k(u-k)}{\lambda_k(u-(k-1))}. \tag{2.15}$$

[1] In this paper, we use a different convention for the crossed monodromy matrix than [47]. The connection is $\widehat{T}_{i,j} \leftrightarrow \widehat{T}_{N+1-i,N+1-j}$.

We also introduce the α-functions

$$\alpha_i(u) = \frac{\lambda_i(u)}{\lambda_{i+1}(u)}, \qquad \hat{\alpha}_i(u) = \frac{\hat{\lambda}_i(u)}{\hat{\lambda}_{i+1}(u)}. \tag{2.16}$$

The crossed α-functions can expressed from the original ones as

$$\hat{\alpha}_i(u) = 1/\alpha_i(u-i). \tag{2.17}$$

The transfer matrices can be defined in the usual way

$$\mathcal{T}(u) = \sum_{i=1}^{N} T_{i,i}(u), \qquad \widehat{\mathcal{T}}(u) = \sum_{i=1}^{N} \widehat{T}_{i,i}(u). \tag{2.18}$$

These are commutative operators

$$[\mathcal{T}(u), \mathcal{T}(v)] = \left[\mathcal{T}(u), \widehat{\mathcal{T}}(v)\right] = \left[\widehat{\mathcal{T}}(u), \widehat{\mathcal{T}}(v)\right] = 0. \tag{2.19}$$

The eigenvectors $\mathbb{B}(\bar{t})$ are called Bethe vectors. The Bethe vectors depend on the Bethe roots t_k^ν for $\nu = 1, \dots, N-1$ and $k = 1, \dots, r_\nu$. We define the sets $\bar{t}^\nu = \{t_k^\nu\}_{k=1}^{r_\nu}$ and the set of sets $\bar{t} = \{\bar{t}^\nu\}_{\nu=1}^{N-1}$. The r_ν quantum numbers are the cardinalities of the sets $\bar{t}^\nu$. The Bethe roots satisfy the Bethe equations

$$\alpha_\mu(t_k^\mu) := \frac{\lambda_\mu(t_k^\mu)}{\lambda_{\mu+1}(t_k^\mu)} = \frac{f(t_k^\mu, \bar{t}_k^\mu)}{f(\bar{t}_k^\mu, t_k^\mu)} \frac{f(\bar{t}^{\mu+1}, t_k^\mu)}{f(t_k^\mu, \bar{t}^{\mu-1})}, \tag{2.20}$$

where we introduced the following shorthand notations

$$\begin{aligned} &f(u,v) = \frac{u-v+1}{u-v}, \\ &f(u, \bar{t}^i) = \prod_{k=1}^{r_i} f(u, t_k^i), \quad f(\bar{t}^i, u) = \prod_{k=1}^{r_i} f(t_k^i, u), \quad f(\bar{t}^i, \bar{t}^j) = \prod_{k=1}^{r_i} f(t_k^i, \bar{t}^j). \end{aligned} \tag{2.21}$$

The transfer matrix eigenvalues are [47]

$$\tau(u|\bar{t}) = \sum_{i=1}^{N} \lambda_i(u) f(\bar{t}^i, u) f(u, \bar{t}^{i-1}), \tag{2.22}$$

$$\hat{\tau}(u|\bar{t}) = \sum_{i=1}^{N} \hat{\lambda}_i(u) f(\bar{t}^{i-1} + (i-1), u) f(u, \bar{t}^i + i). \tag{2.23}$$

We call the Bethe vector on-shell if the Bethe roots satisfy the Bethe equations (when the Bethe vector is an eigenvector), and off-shell for generic Bethe roots. In the literature recursive definitions are available for the off-shell Bethe vectors. For the overlap calculations we also have to know how the monodromy matrix entries act on the off-shell Bethe vectors. Fortunately it was also previously derived in [48].

One can also define the left eigenvectors $\mathbb{C}(\bar{t})$ of the transfer matrix and the square of the norm of the on-shell Bethe states satisfies the Gaudin hypothesis [49]

$$\mathbb{C}(\bar{t})\mathbb{B}(\bar{t}) = \frac{\prod_{\nu=1}^{N-1} \prod_{k \neq l} f(t_l^\nu, t_k^\nu)}{\prod_{\nu=1}^{N-2} f(\bar{t}^{\nu+1}, \bar{t}^\nu)} \det G, \tag{2.24}$$

where G is the Gaudin matrix given by

$$G_{j,k}^{(\mu,\nu)} = -\frac{\partial \log \Phi_j^{(\mu)}}{\partial t_k^\nu}. \tag{2.25}$$

3 Integrable matrix product states

In this section, we generalize the previously introduced KT-relation to MPSs. This essentially means that certain quantities which were previously scalars will now become matrices in the so-called boundary space.

3.1 KT-relations

We generalize the previously defined KT-relation [43, 44] with boundary degrees of freedom as

$$\sum_{k=1}^{N}\sum_{c=1}^{d_B} K_{i,k}^{a,c}(z)\langle\psi_{c,b}|T_{k,j}(z) = \sum_{k=1}^{N}\sum_{c=1}^{d_B}\langle\psi_{a,c}|\bar{T}_{i,k}(-z)K_{k,j}^{c,b}(z). \tag{3.1}$$

The $\langle\psi_{a,b}|$ are elements of the dual of the quantum space $\mathcal{H}$, i.e., $\langle\psi_{a,b}|\in\mathcal{H}^*$ for $a,b=1,\dots,d_B$. The KT-relations have two types: for the **uncrossed** KT-relation $\bar{T}\equiv T$ and for the **crossed** KT-relation $\bar{T}\equiv\widehat{T}$.

We can introduce a boundary vector space $\mathcal{H}_B=\mathbb{C}^{d_B}$ and we can collect the states and the coefficients to matrices in the boundary space

$$\begin{aligned}\langle\Psi| &:= \sum_{a,b=1}^{d_B}\langle\psi_{a,b}|\otimes e_{a,b}^B\in\mathcal{H}^*\otimes\mathrm{End}(\mathcal{H}_B),\\ \mathbf{K}_{i,j}(z) &:= \sum_{a,b=1}^{d_B}K_{i,j}^{a,b}(z)e_{a,b}^B\in\mathrm{End}(\mathcal{H}_B),\end{aligned} \tag{3.2}$$

where $e_{a,b}^B\in\mathrm{End}(\mathcal{H}_B)$ are the unit matrices of the boundary space for $a,b=1,\dots,d_B$. Using these notations the KT-relation simplifies as

$$\mathbf{K}_0(z)\langle\Psi|T_0(z) = \langle\Psi|\bar{T}_0(-z)\mathbf{K}_0(z), \tag{3.3}$$

where $T(z),\bar{T}(z)\in\mathrm{End}(\mathbb{C}^N)\otimes\mathrm{End}(\mathcal{H})$ are the monodromy matrices, the $\mathbf{K}(z)=\sum_{i,j=1}^N e_{i,j}\otimes\in\mathbf{K}_{i,j}(z)\in\mathrm{End}(\mathbb{C}^N)\otimes\mathrm{End}(\mathcal{H}_B)$ is the K-matrix and $\langle\Psi|\in\mathcal{H}^*\otimes\mathrm{End}(\mathcal{H}_B)$ is the boundary state.

We also define the matrix product state $\langle\mathrm{MPS}|\in\mathcal{H}^*$ as

$$\langle\mathrm{MPS}| = \sum_{\ell=1}^{d_B}\langle\psi_{\ell,\ell}| = \mathrm{Tr}_{\mathcal{H}_B}\left(\langle\Psi|\right). \tag{3.4}$$

Our goal is to calculate on-shell overlaps $\langle\mathrm{MPS}|\mathbb{B}(\bar{t})$.

3.2 Reflection algebras

The compatibility of the KT-relation with the RTT-relation (we do the calculation $\langle\Psi|T_1(u)T_2(v)\to\langle\Psi|\bar{T}_1(-u)\bar{T}_2(-v)$ in two different orders) requires the equation [43]

$$R_{1,2}(v-u)\mathbf{K}_1(u)\bar{R}_{1,2}(-u-v)\mathbf{K}_2(v) = \mathbf{K}_2(v)\bar{R}_{1,2}(-u-v)\mathbf{K}_1(u)R_{1,2}(v-u). \tag{3.5}$$

where we used the notation

$$\bar{R}(u)\equiv\begin{cases}R(u), & \text{for the uncrossed case,}\\ \widehat{R}(u), & \text{for the crossed case.}\end{cases} \tag{3.6}$$

The KT-relation and the reflection equation is invariant under the renormalization of the K-matrix $\mathbf{K}_1(u)\to f(u)\mathbf{K}_1(u)$ for any scalar function $f(u)$.

In the crossed case the reflection equation with components is

$$\begin{aligned}\left[\mathbf{K}_{i,j}(u),\mathbf{K}_{k,l}(v)\right] = &\frac{1}{u-v}\left(\mathbf{K}_{k,j}(u)\mathbf{K}_{i,l}(v)-\mathbf{K}_{k,j}(v)\mathbf{K}_{i,l}(u)\right)\\ &-\frac{1}{u+v}\left(\mathbf{K}_{i,k}(u)\mathbf{K}_{j,l}(v)-\mathbf{K}_{k,i}(v)\mathbf{K}_{l,j}(u)\right)\\ &+\frac{1}{u^2-v^2}\left(\mathbf{K}_{k,i}(u)\mathbf{K}_{j,l}(v)-\mathbf{K}_{k,i}(v)\mathbf{K}_{j,l}(u)\right).\end{aligned} \tag{3.7}$$

With a proper normalization, the K-matrix has the asymptotic limit

$$\mathbf{K}_{i,j}(u) = \mathbf{A}_{i,j}+\mathcal{O}(u^{-1}). \tag{3.8}$$

Taking the $u\to\infty$ limit of the reflection equation becomes

$$\left[\mathbf{A}_{i,j},\mathbf{K}_{k,l}(v)\right]=0. \tag{3.9}$$

We also assume that the K-matrix is irreducible, i.e. the set of matrices in the boundary space $\{\mathbf{K}_{i,j}(u)\}_{i,j=1}^N$ do not have non-trivial invariant subspace. It leads to $\mathbf{A}_{i,j} = \mathcal{U}_{i,j}\mathbf{1}$, where $\mathcal{U}_{i,j} \in \mathbb{C}$ for $i,j = 1,\dots,N$. Multiplying the reflection equation with $(u+v)$ and taking the $v=-u$ limit, we obtain

$$\begin{aligned}0 = &-(\mathbf{K}_{i,k}(u)\mathbf{K}_{j,l}(-u) - \mathbf{K}_{k,i}(-u)\mathbf{K}_{l,j}(u)) \\ &+ \frac{1}{2u}(\mathbf{K}_{k,i}(u)\mathbf{K}_{j,l}(-u) - \mathbf{K}_{k,i}(-u)\mathbf{K}_{j,l}(u)).\end{aligned} \tag{3.10}$$

Taking the $u \to \infty$ limit

$$\mathcal{U}_{i,k}\mathcal{U}_{j,l} = \mathcal{U}_{k,i}\mathcal{U}_{l,j}, \tag{3.11}$$

therefore $\mathcal{U}_{i,k} = \pm\mathcal{U}_{k,i}$. In summary, the asymptotic limit of the K-matrix is

$$\mathbf{K}_{i,j}(u) = \mathcal{U}_{i,j}\mathbf{1} + \mathcal{O}(u^{-1}). \tag{3.12}$$

The reflection equation with the series expansion (3.12) defines the twisted Yangian algebras $Y^\pm(N)$ ($Y^+(N)$ for symmetric and $Y^-(N)$ for anti-symmetric $\mathcal{U}$). [2]

For the uncrossed case the explicit form of the reflection equation is

$$\begin{aligned}[\mathbf{K}_{i,j}(u), \mathbf{K}_{k,l}(v)] = &\frac{1}{u-v}(\mathbf{K}_{k,j}(u)\mathbf{K}_{i,l}(v) - \mathbf{K}_{k,j}(v)\mathbf{K}_{i,l}(u)) \\ &+ \frac{1}{u+v}\sum_{n=1}^N(\delta_{j,k}\mathbf{K}_{i,n}(u)\mathbf{K}_{n,l}(v) - \delta_{i,l}\mathbf{K}_{k,n}(v)\mathbf{K}_{n,j}(u)) \\ &- \frac{1}{u^2-v^2}\delta_{i,j}\sum_{n=1}^N(\mathbf{K}_{k,n}(u)\mathbf{K}_{n,l}(v) - \mathbf{K}_{k,n}(v)\mathbf{K}_{n,l}(u)).\end{aligned} \tag{3.13}$$

Taking the $u \to \infty$ limit of the reflection equation we can easily show that the the asymptotic limit of the K-matrix is

$$\mathbf{K}_{i,j}(u) = \mathcal{U}_{i,j}\mathbf{1} + \mathcal{O}(u^{-1}). \tag{3.14}$$

Taking first the $v=-u$ limit and later the $u\to\infty$ limit we obtain that

$$\sum_{n=1}^N(\delta_{j,k}\mathcal{U}_{i,n}\mathcal{U}_{n,l} - \delta_{i,l}\mathcal{U}_{k,n}\mathcal{U}_{n,j}) = 0. \tag{3.15}$$

The solution of this equation is

$$\sum_{n=1}^N \mathcal{U}_{i,n}\mathcal{U}_{n,l} = a\delta_{i,l}, \tag{3.16}$$

where $a \in \mathbb{C}$. For $a=0$ we call the K-matrix singular. For non-singular K-matrices we can always choose the normalization as $a=1$. The possible eigenvalues of the matrix $\mathcal{U}$ are ±1. Let the number of -1 be M. The reflection equation and the asymptotic expansion with M number of -1 eigenvalues defines the reflection algebra $\mathcal{B}(N,M)$. [3]

3.3 Matrix product states from co-product

Let us introduce a tensor product quantum space as $\mathcal{H} = \mathcal{H}^{(1)} \otimes \mathcal{H}^{(2)}$. The corresponding monodromy matrices are $\bar{T}^{(1)}_{i,j}(u) \in \mathrm{End}(\mathcal{H}^{(1)}), \bar{T}^{(2)}_{i,j}(u) \in \mathrm{End}(\mathcal{H}^{(2)})$ and

$$\bar{T}_0(u) = \bar{T}^{(2)}_0(u)\bar{T}^{(1)}_0(u) \in \mathrm{End}(\mathbb{C}^N) \otimes \mathrm{End}(\mathcal{H}), \tag{3.17}$$

[2] In the literature (e.g., [50]), these algebras are often referred to as extended twisted Yangians, and the definitions of twisted Yangians include an additional symmetry property. This extra condition partially fixes the normalization of the K-matrix, meaning that the symmetry equation is not invariant under the rescaling $\mathbf{K}(u) \to \mu(u)\mathbf{K}(u)$ for an arbitrary function $\mu(u)$. However, our KT-relation is invariant under such rescaling, and we do not use the symmetry property in our calculations. Therefore, in this paper, we work with the extended versions, and for simplicity, we omit the term "extended."

[3] Similar to twisted Yangians, these algebras are commonly referred to as extended reflection algebras [51], and the definition of reflection algebras includes an additional unitarity condition. However, since we do not use this condition, we also omit the term "extended" here for the sake of simplicity.

Let $\langle\Psi^{(1)}\rvert \in \mathcal{H}^{(1)} \otimes \mathcal{H}_B$ and $\langle\Psi^{(2)}\rvert \in \mathcal{H}^{(2)} \otimes \mathcal{H}_B$ be boundary states with the same K-matrix i.e. they satisfy the same KT-relation

$$\mathbf{K}_0(u)\langle\Psi^{(i)}|T_0^{(i)}(u) = \langle\Psi^{(i)}|\bar{T}_0^{(i)}(-u)\mathbf{K}_0(u), \tag{3.18}$$

for $i = 1, 2$.

Proposition 1. *The co-vector*

$$\langle\Psi| = \langle\Psi^{(2)}|\langle\Psi^{(1)}| \tag{3.19}$$

is a boundary state in the tensor product quantum space $\mathcal{H}$ with the same K-matrix i.e. it satisfies the KT-relation

$$\mathbf{K}_0(u)\langle\Psi|T_0(u) = \langle\Psi|\bar{T}_0(-u)\mathbf{K}_0(u). \tag{3.20}$$

Proof. Let us start with the lhs

$$\mathbf{K}_0(u)\langle\Psi|T_0(u) = \mathbf{K}_0(u)\left(\langle\Psi^{(2)}|\langle\Psi^{(1)}|\right)\left(T_0^{(2)}(u)T_0^{(1)}(u)\right) = \mathbf{K}_0(u)\left(\langle\Psi^{(2)}|T_0^{(2)}(u)\right)\left(\langle\Psi^{(1)}|T_0^{(1)}(u)\right), \tag{3.21}$$

where we used the definition (3.19) and the co-product property of the monodromy matrix (3.17). Now we can use the KT-relation of space $\mathcal{H}^{(2)}$:

$$\mathbf{K}_0(u)\langle\Psi|T_0(u) = \left(\langle\Psi^{(2)}|\bar{T}_0^{(2)}(-u)\right)\mathbf{K}_0(u)\left(\langle\Psi^{(1)}|T_0^{(1)}(u)\right). \tag{3.22}$$

Now on $\mathcal{H}^{(1)}$:

$$\begin{aligned}\mathbf{K}_0(u)\langle\Psi|T_0(u) &= \left(\langle\Psi^{(2)}|\bar{T}_0^{(2)}(-u)\right)\left(\langle\Psi^{(1)}|\bar{T}_0^{(1)}(-u)\right)\mathbf{K}_0(u)\\ &= \left(\langle\Psi^{(2)}|\langle\Psi^{(1)}|\right)\left(\bar{T}_0^{(2)}(-u)\bar{T}_0^{(1)}(-u)\right)\mathbf{K}_0(u).\end{aligned} \tag{3.23}$$

Using (3.17) we just proved (3.20). $\square$

Using the co-product property, we can construct integrable MPSs of arbitrary length from the "elementary" solutions of the KT-equation. In the following, we present an example where only the defining representations are used. We begin with the reflection equation (3.5). After change of variables it is equivalent to

$$\mathbf{K}_0(u)\bar{R}_{0,1}(-u-\theta)\mathbf{K}_1(\theta)R_{0,1}(u-\theta) = R_{0,1}(u-\theta)\mathbf{K}_1(\theta)\bar{R}_{0,1}(-u-\theta)\mathbf{K}_0(u). \tag{3.24}$$

We can use an equivalent form

$$\mathbf{K}_0(u)\psi_{2,1}(\theta)\left[\bar{R}_{0,2}^{t_2}(-u-\theta)R_{0,1}(u-\theta)\right] = \psi_{2,1}(\theta)\left[R_{0,2}^{t_2}(u-\theta)\bar{R}_{0,1}(-u-\theta)\right]\mathbf{K}_0(u), \tag{3.25}$$

where

$$\psi_{1,2}(\theta) = \sum_{i,j=1}^{N}\langle i,j| \otimes \mathbf{K}_{i,j}(\theta). \tag{3.26}$$

The state $\psi_{2,1}(\theta)$ satisfy the KT-relation for length two spin chain with monodromy matrices

$$T_0(u) = \bar{R}_{0,2}^{t_2}(-u-\theta)R_{0,1}(u-\theta), \quad \bar{T}_0(u) = R_{0,2}^{t_2}(-u-\theta)\bar{R}_{0,1}(u-\theta). \tag{3.27}$$

Using the co-product property we can built boundary state for length $2J$ as

$$\begin{aligned}\langle\Psi| &= \psi_{2J,2J-1}(\theta_J)\dots\psi_{4,3}(\theta_2)\psi_{2,1}(\theta_1),\\ T_0(u) &= \bar{R}_{0,2J}^{t_{2J}}(-u-\theta_J)R_{0,2J-1}(u-\theta_J)\dots\bar{R}_{0,2}^{t_2}(-u-\theta_1)R_{0,1}(u-\theta_1).\end{aligned} \tag{3.28}$$

For the even sites we have

$$\bar{R}_{0,2j}^{t_{2j}}(-u-\theta_j) = \begin{cases}R_{0,2j}(u+\theta_j), & \text{for the crossed case,}\\ \widehat{R}_{0,2j}(u+\theta_j), & \text{for the uncrossed case,}\end{cases} \tag{3.29}$$

Using fusion, we can generalize the boundary states to any finite-dimensional representation of $\mathfrak{gl}_N$. The details are provided in Appendix A. For a finite-dimensional representation Λ, there is a two-site state $\psi_{2,1}^{\Lambda}(\theta)$. Using the co-product property, we can construct a boundary state

$$\langle\Psi| = \psi_{2J,2J-1}^{\Lambda^{(J)}}(\theta_J)\dots\psi_{4,3}^{\Lambda^{(2)}}(\theta_2)\psi_{2,1}^{\Lambda^{(1)}}(\theta_1). \tag{3.30}$$

This satisfies the KT-relation on a spin chain where the monodromy matrix is defined as

$$T_0(u) = \left[\bar{L}^{\Lambda^{(J)}}_{0,2J}(-u-\theta_J)\right]^{t_{2J}} L^{\Lambda^{(J)}}_{0,2J-1}(u-\theta_J)\dots\left[\bar{L}^{\Lambda^{(1)}}_{0,2}(-u-\theta_1)\right]^{t_2} L^{\Lambda^{(1)}}_{0,1}(u-\theta_1). \tag{3.31}$$

The pseudo-vacuum eigenvalue is

$$\lambda_k(u) = \begin{cases} \prod_{j=1}^{J}\lambda_k^{(j)}(u-\theta_j)\hat{\lambda}_k^{(j)}(-u-\theta_j), & \text{for the crossed case,} \\ \prod_{j=1}^{J}\lambda_k^{(j)}(u-\theta_j)\lambda_{N+1-k}^{(j)}(-u-\theta_j), & \text{for the uncrossed case,} \end{cases} \tag{3.32}$$

where

$$\lambda_k^{(j)}(u) = \frac{u+\Lambda_k^{(j)}}{u}. \tag{3.33}$$

The pseudo-vacuum eigenvalues satisfy the symmetry conditions

$$\lambda_k(u) = \begin{cases} \hat{\lambda}_k(-u), & \text{for the crossed case,} \\ \lambda_{N+1-k}(-u), & \text{for the uncrossed case.} \end{cases} \tag{3.34}$$

The symmetry properties of the α-s are

$$\alpha_k(u) = \frac{\lambda_k(u)}{\lambda_{k+1}(u)} = \begin{cases} \frac{\hat{\lambda}_k(-u)}{\hat{\lambda}_{k+1}(-u)} = \hat{\alpha}_k(-u) = \frac{1}{\alpha_k(-u-k)}, & \text{for the crossed case,} \\ \frac{\lambda_{N+1-k}(-u)}{\lambda_{N-k}(-u)} = \frac{1}{\alpha_{N-k}(-u)}, & \text{for the uncrossed case.} \end{cases} \tag{3.35}$$

3.4 Integrability and pair structure

Assuming that the K-matrices are invertible, we can product the KT-relation with the inverse:

$$\langle\Psi|T_0(z) = \mathbf{K}_0(z)^{-1}\langle\Psi|\bar{T}_0(-z)\mathbf{K}_0(z). \tag{3.36}$$

Taking the trace in the auxiliary and the boundary space we obtain that

$$\langle\text{MPS}|\mathcal{T}(z) = \langle\text{MPS}|\bar{\mathcal{T}}(-z). \tag{3.37}$$

For on-shell states we have

$$\left(\tau(u|\bar{t}) - \bar{\tau}(-u|\bar{t})\right)\langle\text{MPS}|\mathbb{B}(\bar{t}) = 0. \tag{3.38}$$

The non-vanishing overlap $\langle\text{MPS}|\mathbb{B}(\bar{t}) \neq 0$ requires that

$$\tau(u|\bar{t}) = \bar{\tau}(-u|\bar{t}). \tag{3.39}$$

For the uncrossed case, the eigenvalue $\tau(-u|\bar{t})$ can be written as

$$\begin{aligned} \tau(-u|\bar{t}) &= \sum_{i=1}^{N}\lambda_i(-u)f(\bar{t}^i,-u)f(-u,\bar{t}^{i-1}) = \\ &= \sum_{i=1}^{N}\lambda_i(u)f(-\bar{t}^{N-i},u)f(u,-\bar{t}^{N+1-i}) = \tau(u|\pi^a(\bar{t})), \end{aligned} \tag{3.40}$$

where we used the symmetry property (3.34) and introduced the following map of Bethe roots:

$$\pi^a(\bar{t}^1,\bar{t}^2,\dots,\bar{t}^{N-1}) = \left(-\bar{t}^{N-1},-\bar{t}^{N-2},\dots,-\bar{t}^1\right). \tag{3.41}$$

For the crossed case, the eigenvalue $\hat{\tau}(-u|\bar{t})$ can be written as

$$\begin{aligned} \hat{\tau}(-u|\bar{t}) &= \sum_{i=1}^{N}\hat{\lambda}_i(-u)f(\bar{t}^{i-1}+(i-1),-u)f(-u,\bar{t}^i+i) = \\ &= \sum_{i=1}^{N}\lambda_i(u)f(-\bar{t}^i-i,u)f(u,-\bar{t}^{i-1}-(i-1)) = \tau(u|\pi^c(\bar{t})), \end{aligned} \tag{3.42}$$

where we used that the symmetry property (3.34) and introduced the following map of Bethe roots:

$$\pi^c(\bar{t}^1, \bar{t}^2, \dots, \bar{t}^{N-1}) = \left(-\bar{t}^1 - 1, -\bar{t}^2 - 2, \dots, -\bar{t}^{N-1} - (N-1)\right). \tag{3.43}$$

Substituting back to (3.39) the condition for non-vanishing overlaps read as

$$\tau(u|\bar{t}) = \tau(u|\pi^{a/c}(\bar{t})). \tag{3.44}$$

Notice that the set $\pi^{a/c}(\bar{t})$ also satisfies the Bethe equations i.e. the vector $\mathbb{B}(\pi^{a/c}(\bar{t}))$ is also an on-shell Bethe vector. Since different Bethe vectors have different eigenvalues, we just obtain that the Bethe roots have to satisfy the selection rule

$$\bar{t} = \pi^{a/c}(\bar{t}), \tag{3.45}$$

for the non-vanishing overlaps. We call the Bethe roots with condition (3.45) as Bethe roots with *pair structure*.

For *achiral pair structure*, i.e. when $\bar{t} = \pi^a(\bar{t})$, every sets $\bar{t}^\nu$ satisfy the condition $\bar{t}^\nu = -\bar{t}^{N-\nu}$. For *chiral pair structure*, i.e. when $\bar{t} = \pi^c(\bar{t})$, every sets $\bar{t}^\nu$ satisfy the condition $\bar{t}^\nu = -\bar{t}^\nu - \nu$.

3.5 Recursion for off-shell overlaps

Following [43] we can obtain a method for the evaluation of off-shell overlaps. We use the recurrence relation and action formulas for the Bethe states [52, 48]. See the details in appendix B. In the following we just sketch the recursion for the overlaps and we concentrate on the number of Bethe roots and we drop the numerical coefficients which are irrelevant for this purpose. The recurrence equation of the Bethe vectors reads as

$$\mathbb{B}^{r_1,\dots,r_{N-1}} = \sum_{k=2}^{N} T_{1,k} \sum \mathbb{B}^{r_1-1,\dots,r_{k-1}-1,r_k,\dots,r_{N-1}}(\dots), \tag{3.46}$$

and the action formulas are

$$T_{i,j}\mathbb{B}^{r_1,r_2,\dots,r_{N-1}} = \begin{cases} \sum(\dots)\mathbb{B}^{r_1,\dots r_i+1,r_{i+1}+1,\dots,r_{j-1}+1,r_j,\dots r_{N-1}}, & i \leq j, \\ \sum(\dots)\mathbb{B}^{r_1,\dots r_j-1,r_{j+1}-1,\dots,r_{i-1}-1,r_i,\dots r_{N-1}}, & i > j, \end{cases} \tag{3.47}$$

and

$$\widehat{T}_{i,j}\mathbb{B}^{r_1,r_2,\dots,r_{N-1}} = \begin{cases} \sum(\dots)\mathbb{B}^{r_1,\dots r_j+1,r_{j+1}+1,\dots,r_{i-1}+1,r_i,\dots r_{N-1}}, & j \leq i, \\ \sum(\dots)\mathbb{B}^{r_1,\dots r_i-1,r_{i+1}-1,\dots,r_{j-1}-1,r_j,\dots r_{N-1}}, & j > i. \end{cases} \tag{3.48}$$

We can see that the diagonal elements $T_{i,i}$ and $\widehat{T}_{i,i}$ do not change the number of Bethe roots. For $i > j$, $T_{i,j}$ and $\widehat{T}_{j,i}$ decrease the number of certain roots by one, while $T_{j,i}$ and $\widehat{T}_{i,j}$ increase the number of those same roots by one.

3.5.1 The crossed overlaps

We use the $(1,k)$ component of the crossed KT-relation

$$\langle\Psi|\, T_{1,k}(u) = \sum_{j=1}^{N} \mathbf{K}_{1,1}^{-1}(u)\,\langle\Psi|\,\widehat{T}_{1,j}(-u)\mathbf{K}_{j,k}(u) - \sum_{i=2}^{N} \mathbf{K}_{1,1}^{-1}(u)\mathbf{K}_{1,i}(u)\,\langle\Psi|\, T_{i,k}(u), \tag{3.49}$$

which can be used to change the creation operators $T_{1,k}$ to the operators $\widehat{T}_{1,j}$ and $T_{i,k}$ where $i,k = 2,\dots,N$ and $j = 1,\dots,N$. For $j > 1$, the operators $\widehat{T}_{1,j}$ decrease the number of the first Bethe roots by one and the operators $\widehat{T}_{1,1}(-u)$ and $T_{i,k}(u)$ do not change it for $i,k \geq 2$. Using the recurrence relation (3.49) and the action formulas (3.47), (3.48) we obtain a recursion

$$\langle\Psi|\,\mathbb{B}^{r_1,\dots} = \sum(\dots)\,\langle\Psi|\,\mathbb{B}^{r_1-1,\dots} + \sum(\dots)\,\langle\Psi|\,\mathbb{B}^{r_1-2,\dots}. \tag{3.50}$$

When we changed the creation operators to the diagonal and the annihilation ones we assumed that the inverse $\mathbf{K}_{1,1}^{-1}(u)$ exists which is possible only for $Y^+(N)$ K-matrices but it is never true for $Y^-(N)$. This can be seen from the asymptotic expansion (3.12). In the symmetric case, it is possible that $\mathcal{U}_{1,1} \neq 0$, but in the anti-symmetric case

$\mathcal{U}_{1,1} = 0$. If $\mathcal{U}_{1,1} \neq 0$ then the operator $\mathbf{K}_{1,1}^{-1}(u)$ can be computed order by order in an expansion in powers of u^{-1}. In the following we concentrate on $Y^+(N)$ and deal the other case $Y^-(N)$ later.

Repeating these steps we can express the general overlap with a sum of overlaps without the first Bethe roots

$$\langle\Psi|\,\mathbb{B}^{r_1,\cdots} = \sum(\dots)\,\langle\Psi|\,\mathbb{B}^{0,\cdots}. \tag{3.51}$$

We can see that we reduced the original $\mathfrak{gl}(N)$ Bethe state to $\mathfrak{gl}(N-1)$ ones. The action of operators $\{T_{a,b}\}_{a,b=2}^N$ and $\left\{\widehat{T}_{a,b}\right\}_{a,b=2}^N$ do not lead out of the subspace generated by the Bethe vectors $\mathbb{B}^{0,\cdots}$. Naively, the crossed KT-relation is not closed for the indexes $a,b = 2,\dots,N$ since

$$\sum_{c=2}^N \mathbf{K}_{a,c}\,\langle\Psi|\,T_{c,b} + \mathbf{K}_{a,1}\,\langle\Psi|\,T_{1,b} = \sum_{c=2}^N \langle\Psi|\,\widehat{T}_{a,c}\mathbf{K}_{c,b} + \langle\Psi|\,\widehat{T}_{a,1}\mathbf{K}_{1,b}. \tag{3.52}$$

We can see that the operators $T_{1,b}$ and $\widehat{T}_{a,1}$ create the first type of Bethe roots therefore we have to eliminate them from the equation (3.52). Let us get the components $(1,b)$, $(a,1)$ and $(1,1)$ of the crossed KT-relation

$$\sum_{c=2}^N \mathbf{K}_{1,c}\,\langle\Psi|\,T_{c,b} + \mathbf{K}_{1,1}\,\langle\Psi|\,T_{1,b} \cong \langle\Psi|\,\widehat{T}_{1,1}\mathbf{K}_{1,b}, \tag{3.53}$$

$$\mathbf{K}_{a,1}\,\langle\Psi|\,T_{1,1} \cong \sum_{c=2}^N \langle\Psi|\,\widehat{T}_{a,c}\mathbf{K}_{c,1} + \langle\Psi|\,\widehat{T}_{a,1}\mathbf{K}_{1,1}, \tag{3.54}$$

$$\mathbf{K}_{1,1}\,\langle\Psi|\,T_{1,1} \cong \langle\Psi|\,\widehat{T}_{1,1}\mathbf{K}_{1,1}, \tag{3.55}$$

where $\cong$ denotes the equality in the subspace generated by the Bethe vectors $\mathbb{B}^{0,\cdots}$. Combining the equations (3.52), (3.53), (3.54) and (3.55) we obtain a new crossed KT-relation on the subspace generated by the Bethe states $\mathbb{B}^{0,\cdots}$

$$\sum_{c=2}^N \mathbf{K}^{(2)}_{a,c}\,\langle\Psi|\,T_{c,b} \cong \sum_{c=2}^N \langle\Psi|\,\widehat{T}_{a,c}\mathbf{K}^{(2)}_{c,b}, \tag{3.56}$$

where

$$\mathbf{K}^{(2)}_{a,b}(z) = \mathbf{K}_{a,b}(z) - \mathbf{K}_{a,1}(z)\mathbf{K}_{1,1}^{-1}(z)\mathbf{K}_{1,b}(z), \tag{3.57}$$

for $a,b = 2,\dots,N$. Repeating the previous method we can eliminate the sets of Bethe roots $\bar{t}^\nu$, therefore we obtain a recursion for the overlaps and in the end of the day we obtain an explicit formula for the overlap. In the $(k+1)$-th step of the nesting we have a $\mathfrak{gl}(N-k)$ K-matrix:

$$\mathbf{K}^{(k+1)}_{a,b}(z) = \mathbf{K}^{(k)}_{a,b}(z) - \mathbf{K}^{(k)}_{a,k}(z)\left[\mathbf{K}^{(k)}_{k,k}(z)\right]^{-1}\mathbf{K}^{(k)}_{k,b}(z), \tag{3.58}$$

which satisfy the KT-relations

$$\sum_{c=k}^N \mathbf{K}^{(k)}_{a,c}(u)\,\langle\Psi|\,T_{c,b}(u) \cong \sum_{c=k}^N \langle\Psi|\,\widehat{T}_{a,c}(-u)\mathbf{K}^{(k)}_{c,b}(u), \tag{3.59}$$

for $a,b = k,\dots,N$ where $\cong$ denotes the equality in the subspace generated by the Bethe vectors $\mathbb{B}^{0,\dots,0,r_k,\dots,r_{N-1}}$.

3.5.2 The uncrossed overlaps

We can use similar recursion for the uncrossed overlaps. Now we use the (N,k) component of the uncrossed KT-relation

$$\langle\Psi|\,T_{1,k}(u) = \sum_{j=1}^N \mathbf{K}_{N,1}^{-1}(u)\,\langle\Psi|\,T_{N,j}(-u)\mathbf{K}_{j,k}(u) - \sum_{i=2}^N \mathbf{K}_{N,1}^{-1}(u)\mathbf{K}_{N,i}(u)\,\langle\Psi|\,T_{i,k}(u). \tag{3.60}$$

Analogous way, we can use this equation to decrease the numbers of the first Bethe roots while the cardinality of $\bar{t}^{N-1}$ is not increased. Using the action formulas we obtain a recursion

$$\langle\Psi|\,\mathbb{B}^{r_1,\dots,r_{N-1}} = \sum(\dots)\mathbb{B}^{\tilde{r}_1,\dots,\tilde{r}_{N-1}}. \tag{3.61}$$

where $r_1 > \tilde{r}_1$ and $r_{N-1} \geq \tilde{r}_{N-1}$. We can also use the $(k,1)$ component of the uncrossed KT-relation

$$\langle\Psi| T_{k,N}(-u) = \sum_{i=1}^{N} \mathbf{K}_{k,i}(u) \langle\Psi| T_{i,1}(u)\mathbf{K}^{-1}_{N,1}(u) - \sum_{j=1}^{N-1} \langle\Psi| T_{k,j}(-u)\mathbf{K}_{j,1}(u)\mathbf{K}^{-1}_{N,1}(u). \tag{3.62}$$

Analogous way, we can use this equation to decrease the cardinality of $\bar{t}^{N-1}$ while the cardinality of $\bar{t}^1$ is not increased. Using the action formulas we obtain a recursion

$$\langle\Psi| \mathbb{B}^{r_1,\dots,r_{N-1}} = \sum(\dots)\mathbb{B}^{\tilde{r}_1,\dots,\tilde{r}_{N-1}}. \tag{3.63}$$

where $r_1 \geq \tilde{r}_1$ and $r_{N-1} > \tilde{r}_{N-1}$.

Repeating these steps we can express the general overlap with a sum of overlaps without the first and last types of Bethe roots

$$\langle\Psi| \mathbb{B}^{r_1,\dots,r_{N-1}} = \sum(\dots) \langle\Psi| \mathbb{B}^{0,\dots,0}. \tag{3.64}$$

We can see that we reduced the original $\mathfrak{gl}(N)$ Bethe state to $\mathfrak{gl}(N-2)$ ones. Naively, the uncrossed KT-relation is not closed for this $\mathfrak{gl}(N-2)$ subsector since

$$\sum_{c=2}^{N-1} \mathbf{K}_{a,c} \langle\Psi| T_{c,b} + \mathbf{K}_{a,1} \langle\Psi| T_{1,b} + \mathbf{K}_{a,N} \langle\Psi| T_{N,b} = \sum_{c=2}^{N-1} \langle\Psi| \bar{T}_{a,c}\mathbf{K}_{c,b} + \langle\Psi| \bar{T}_{a,1}\mathbf{K}_{1,b} + \langle\Psi| \bar{T}_{a,N}\mathbf{K}_{N,b}, \tag{3.65}$$

where $T_{i,j} \equiv T_{i,j}(u)$ and $\bar{T}_{i,j} \equiv T_{i,j}(-u)$. We can see that the operators $T_{1,b}$ and $T_{a,N}$ create the first and last types of Bethe roots therefore we have to eliminate them from the equation (3.65). We use the components (N,b), $(a,1)$ and $(N,1)$ of the uncrossed KT-relation

$$\sum_{c=2}^{N-1} \mathbf{K}_{N,c} \langle\Psi| T_{c,b} + \mathbf{K}_{N,1} \langle\Psi| T_{1,b} \cong \langle\Psi| \bar{T}_{N,N}\mathbf{K}_{N,b}, \tag{3.66}$$

$$\mathbf{K}_{a,1} \langle\Psi| T_{1,1} \cong \sum_{c=2}^{N-1} \langle\Psi| \bar{T}_{a,c}\mathbf{K}_{c,1} + \langle\Psi| \bar{T}_{a,N}\mathbf{K}_{N,1}, \tag{3.67}$$

$$\mathbf{K}_{N,1} \langle\Psi| T_{1,1} \cong \langle\Psi| \bar{T}_{N,N}\mathbf{K}_{N,1}, \tag{3.68}$$

where $\cong$ denotes the equality in the subspace generated by the Bethe vectors $\mathbb{B}^{0,\dots,0}$. Combining the equations (3.65), (3.66), (3.67) and (3.68) we obtain a new uncrossed KT-relation on the subspace generated by the Bethe states $\mathbb{B}^{0,\dots,0}$

$$\sum_{c=2}^{N} \mathbf{K}^{(2)}_{a,c}(u) \langle\Psi| T_{c,b}(u) \cong \sum_{c=2}^{N} \langle\Psi| T_{a,c}(-u)\mathbf{K}^{(2)}_{c,b}(u), \tag{3.69}$$

where

$$\mathbf{K}^{(2)}_{a,b}(z) = \mathbf{K}_{a,b}(z) - \mathbf{K}_{a,1}(z)\mathbf{K}^{-1}_{N,1}(z)\mathbf{K}_{N,b}(z), \tag{3.70}$$

for $a,b = 2,\dots,N-1$. Repeating the previous method we can eliminate the sets of Bethe roots $\bar{t}^{\nu}$, therefore we obtain a recursion for the overlaps and in the end of the day we obtain an explicit formula for the overlap. In the $(k+1)$-th step of the nesting we have a $\mathfrak{gl}(N-2k)$ K-matrix:

$$\mathbf{K}^{(k+1)}_{a,b}(z) = \mathbf{K}^{(k)}_{a,b}(z) - \mathbf{K}^{(k)}_{a,k}(z) \left[\mathbf{K}^{(k)}_{N+1-k,k}(z)\right]^{-1} \mathbf{K}^{(k)}_{N+1-k,b}(z), \tag{3.71}$$

which satisfy the KT-relations

$$\sum_{c=k}^{N+1-k} \mathbf{K}^{(k)}_{a,c}(u) \langle\Psi| T_{c,b}(u) \cong \sum_{c=k}^{N+1-k} \langle\Psi| T_{a,c}(-u)\mathbf{K}^{(k)}_{c,b}(u), \tag{3.72}$$

for $a,b = k,\dots,N+1-k$ where $\cong$ denotes the equality in the subspace generated by the Bethe vectors $\mathbb{B}^{\dots,0,r_k,\dots,r_{N-k},0\dots}$.

We can examine the leading order of the nested K-matrices in the asymptotic limit. It is easy to see that

$$\mathbf{K}^{(k)}_{i,j}(u) = \mathcal{U}^{(k)}_{i,j}\mathbf{1} + \mathcal{O}(u^{-1}). \tag{3.73}$$

From the recursive equations (3.71), if (3.16) holds, then the result follows

$$\begin{aligned}\sum_{n=k}^{N+1-k}\mathcal{U}_{i,n}^{(k)}\mathcal{U}_{n,l}^{(k)}&=\sum_{n=1}^{N}\mathcal{U}_{i,n}\mathcal{U}_{n,l},\\ \sum_{n=k}^{N+1-k}\mathcal{U}_{n,n}^{(k)}&=\sum_{n=1}^{N}\mathcal{U}_{n,n}.\end{aligned}\tag{3.74}$$

If the original K-matrix is a $\mathcal{B}(N,M)$ representation, then $\mathcal{U}^{(k)}$ is an $N-2k+2$ dimensional matrix with $M-k+1$ eigenvalues equal to -1 and $N-M-k+1$ eigenvalues equal to $+1$. Based on this, at step $M+1$, $\mathcal{U}^{(M+1)}$ becomes the identity matrix.

During the calculation we assumed that the inverse $\left[\mathbf{K}_{N+1-k,k}^{(k)}(z)\right]^{-1}$ exists. As long as the components $\mathcal{U}_{N+1-k,k}^{(k)}$ are non-zero, the matrices $\left[\mathbf{K}_{N+1-k,k}^{(k)}(z)\right]^{-1}$ exist. As long as the matrix $\mathcal{U}^{(k)}$ is not the identity, $\mathcal{U}_{N+1-k,k}^{(k)}\neq 0$ is guaranteed to hold after appropriate rotations. As we saw earlier, $\mathcal{U}^{(k)}$ has $M-k+1$ eigenvalues equal to -1, and the last inverse we need is at $k=\lfloor\frac{N}{2}\rfloor$, meaning the necessary inverses exist if

$$M-\left\lfloor\frac{N}{2}\right\rfloor+1\geq 1,$$

i.e., there is at least one eigenvalue equal to -1 at step $k=\lfloor\frac{N}{2}\rfloor$.

In summary, the necessary nested K-matrices exist only in the case of $\mathcal{B}(N,\lfloor\frac{N}{2}\rfloor)$. In the next section, we will precisely derive the overlaps for this case.

4 Derivations of the overlaps

For $Y^+(N)$ or $\mathcal{B}(N,\lfloor\frac{N}{2}\rfloor)$ K-matrices, with the recursions of the previous section we can eliminate all the Bethe roots and in the end the off-shell overlaps are expressed by a function of the Bethe roots t_k^ν, the α-functions α_ν, and components of the K-matrices $\mathbf{K}_{i,j}$ and the vacuum overlap $\mathbf{B}=\langle\Psi|0\rangle$, i.e. we can introduce the off-shell overlap functions as

$$\mathbf{S}_{\bar{\alpha},\mathbf{B}}(\bar{t}):=\langle\Psi|\mathbb{B}(\bar{t}).\tag{4.1}$$

In the calculations, we use only the recursion and action formulas for off-shell Bethe vectors and the KT-relation. These formulas do not depend on the specific representations used in the quantum space. In a specific case, $\alpha_k(t_j^k)$ depends on the discrete variables J, $\Lambda^{(j)}$ and the continuous variables θ_j (where J is the number of inhomogeneities and representations). For sufficiently large J, the variables t_j^k and $\alpha_k(t_j^k)$ can be considered independent. The overlap also depends on the vacuum overlap $\mathbf{B}$ which is a matrix in the boundary space. This matrix also depends on the discrete variables J, $\Lambda^{(j)}$, and the continuous variables θ_j. Let the eigenvalues of $\mathbf{B}$ be β_ℓ for $\ell=1,\dots,d_B$. For sufficiently large J, these can also be considered independent variables.

The derivations are essentially the same as those presented in the earlier paper [43]. In that earlier work, the case $d_B=1$ was studied, meaning that the additional difficulty now arises from the fact that quantities which were previously scalars are now matrices. Some parts of the derivations can be applied to matrices without difficulty. The parts that require special attention in the case $d_B>1$ are discussed in the next subsection 4.1. There, we present various theorems that describe the algebra of the nested K-matrices and define commuting subalgebras. In subsections 4.2 and 4.3, we present the theorems necessary for deriving the on-shell overlaps. These are essentially the same as those discussed in the $d_B=1$ case, with the difference that the scalar quantities depending on the K-matrix are now replaced by the commuting operators introduced in 4.1.

4.1 Theorems for the K-matrices

4.1.1 Crossed K-matrices

The nesting of the previous section defines the series of operators

$$\begin{array}{ccccccccc} Y^+(N) & \to & Y^+(N-1) & \to & \dots & \to & Y^+(2) & \to & Y^+(1)\\ \mathbf{K}^{(1)}\equiv\mathbf{K} & \to & \mathbf{K}^{(2)} & \to & \dots & \to & \mathbf{K}^{(N-1)} & \to & \mathbf{K}^{(N)}\\ \downarrow & & \downarrow & & \dots & & \downarrow & & \downarrow\\ \mathbf{G}^{(1)} & & \mathbf{G}^{(2)} & & \dots & & \mathbf{G}^{(N-1)} & & \mathbf{G}^{(N)} \end{array}$$

In the following we give the precise definitions and theorems for these nested operators.

Definition 2. The nested K-matrices are

$$\mathbf{K}^{(k+1)}_{a,b}(z) := \mathbf{K}^{(k)}_{a,b}(z) - \mathbf{K}^{(k)}_{a,k}(z) \left[\mathbf{K}^{(k)}_{k,k}(z)\right]^{-1} \mathbf{K}^{(k)}_{k,b}(z), \tag{4.2}$$

for $a, b = k+1, \ldots, N$ and

$$\mathbf{G}^{(k)}(u) := \mathbf{K}^{(k)}_{k,k}(u), \tag{4.3}$$

for $k = 1, \ldots, N$.

Theorem 3. *The crossed nested K-matrices $\mathbf{K}^{(k)}(u-(k-1)/2)$ are representations of the twisted Yangian $Y^+(N+1-k)$ and the G-operators satisfy the relations*

$$\begin{aligned} \left[\mathbf{G}^{(k)}(u_2), \mathbf{K}^{(k+1)}_{a,b}(u_1)\right] &= 0, \\ \left[\mathbf{G}^{(k)}(u_2), \mathbf{G}^{(k)}(u_1)\right] &= 0, \\ \left[\mathbf{G}^{(k)}(u_2), \mathbf{B}\right] &= 0, \end{aligned} \tag{4.4}$$

for $k = 1, \ldots, N$ and $k < a, b \leq N$.

Corollary 4. *The operators $\mathbf{G}^{(k)}(u)$ generate a Cartan subalgebra of $Y^+(N)$*

$$\left[\mathbf{G}^{(k)}(u_2), \mathbf{G}^{(l)}(u_1)\right] = 0, \tag{4.5}$$

for $k, l = 1, \ldots, N$.

4.1.2 Uncrossed K-matrices

In the uncrossed case we introduce

$$n = \left\lfloor \frac{N}{2} \right\rfloor. \tag{4.6}$$

The nesting of the uncrossed K-matrices defines the series

$$\begin{array}{ccccccccc} \mathcal{B}(2n+1,n) & \to & \mathcal{B}(2n-1,n-1) & \to & \ldots & \to & \mathcal{B}(3,1) & \to & \mathcal{B}(1,0) \\ \mathbf{K}^{(1)} \equiv \mathbf{K} & \to & \mathbf{K}^{(2)} & \to & \ldots & \to & \mathbf{K}^{(n)} & \to & \mathbf{K}^{(n+1)} \\ \downarrow & & \downarrow & & \ldots & & \downarrow & & \downarrow \\ \mathbf{G}^{(1)} & & \mathbf{G}^{(2)} & & \ldots & & \mathbf{G}^{(n)} & & \mathbf{G}^{(n+1)} \end{array}$$

for $N = 2n+1$ and

$$\begin{array}{ccccccccc} \mathcal{B}(2n,n) & \to & \mathcal{B}(2n-2,n-1) & \to & \ldots & \to & \mathcal{B}(2,1) & \to & \mathcal{B}(1,0) \\ \mathbf{K}^{(1)} \equiv \mathbf{K} & \to & \mathbf{K}^{(2)} & \to & \ldots & \to & \mathbf{K}^{(n)} & \to & \mathbf{K}^{(n+1)} \\ \downarrow & & \downarrow & & \ldots & & \downarrow & & \downarrow \\ \mathbf{G}^{(1)} & & \mathbf{G}^{(2)} & & \ldots & & \mathbf{G}^{(n)} & & \mathbf{G}^{(n+1)} \end{array}$$

for $N = 2n$.

Definition 5. The nested K-matrices are

$$\mathbf{K}^{(k+1)}_{a,b}(z) := \mathbf{K}^{(k)}_{a,b}(z) - \mathbf{K}^{(k)}_{a,k}(z) \left[\mathbf{K}^{(k)}_{N+1-k,k}(z)\right]^{-1} \mathbf{K}^{(k)}_{N+1-k,b}(z), \tag{4.7}$$

for $a, b = k+1, \ldots, N-k$ and

$$\mathbf{G}^{(k)}(u) = \mathbf{K}^{(k)}_{N+1-k,k}(u), \tag{4.8}$$

for $k = 1, \ldots, n+1$.

Theorem 6. *The uncrossed nested K-matrices* $\mathbf{K}^{(k)}(u)$ *are representations the reflection algebras* $\mathcal{B}(N+2-2k, n+1-k)$ *and the G-operators satisfy the relations*

$$\begin{aligned}\left[\mathbf{G}^{(k)}(u_2), \mathbf{K}_{a,b}^{(k+1)}(u_1)\right] &= 0,\\ \left[\mathbf{G}^{(k)}(u_2), \mathbf{G}^{(k)}(u_1)\right] &= 0,\\ \left[\mathbf{G}^{(k)}(u_2), \mathbf{B}\right] &= 0,\end{aligned} \tag{4.9}$$

for $k = 1, \ldots, n+1$ *and* $k < a, b < N+1-k$.

Corollary 7. *The operators* $\mathbf{G}^{(k)}(u)$ *generate a Cartan subalgebra of* $\mathcal{B}(N, n)$

$$\left[\mathbf{G}^{(k)}(u_2), \mathbf{G}^{(l)}(u_1)\right] = 0, \tag{4.10}$$

for $k, l = 1, \ldots, n+1$.

4.2 Off-shell overlap formulas

In this subsection we show theorems for the off-shell overlaps $\mathbf{S}_{\alpha,\mathbf{B}}(\bar{t})$. The derivations can be found in the Appendix D.

The off-shell overlap functions will be written as sums over partitions of the sets of Bethe roots. A partition $\bar{t}^s \vdash \{\bar{t}^s_{\mathrm{I}}, \bar{t}^s_{\mathrm{II}}\}$ corresponds to a decomposition into (possibly empty) disjoint subsets $\bar{t}^s_{\mathrm{I}}, \bar{t}^s_{\mathrm{II}}$ such that $\bar{t}^s = \bar{t}^s_{\mathrm{I}} \cup \bar{t}^s_{\mathrm{II}}$ and $\bar{t}^s_{\mathrm{I}} \cap \bar{t}^s_{\mathrm{II}} = \emptyset$.

Theorem 8. *The sum formula can be written as*

$$\mathbf{S}_{\bar{\alpha},\mathbf{B}}(\bar{t}) = \sum_{\mathrm{part}(\bar{t})} \frac{\prod_{\nu=1}^{N-1} f(\bar{t}^{\nu}_{II}, \bar{t}^{\nu}_{I})}{\prod_{\nu=1}^{N-2} f(\bar{t}^{\nu+1}_{II}, \bar{t}^{\nu}_{I})} \bar{\mathbf{Z}}(\bar{t}_{II}) \mathbf{B} \mathbf{Z}(\bar{t}_{I}) \prod_{\nu=1}^{N-1} \alpha_{\nu}(\bar{t}^{\nu}_{I}), \tag{4.11}$$

where the sum goes over partitions $\bar{t}^s \vdash \{\bar{t}^s_I, \bar{t}^s_{II}\}$. *The highest coefficients* $\mathbf{Z}$ *and* $\bar{\mathbf{Z}}$ satisfy

$$\mathbf{Z}^{\mathbf{K}}(\bar{t}) = (-1)^{\#\bar{t}} \left(\prod_{s=1}^{N-2} f(\bar{t}^{s+1}, \bar{t}^{s}) \right)^{-1} \left[\bar{\mathbf{Z}}^{\mathbf{K}^T}(\pi^c(\bar{t})) \right]^{t_B}, \tag{4.12}$$

in the crossed case and

$$\mathbf{Z}^{\mathbf{K}}(\bar{t}) = \left[\bar{\mathbf{Z}}^{\mathbf{K}^{\Pi}}(\pi^a(\bar{t})) \right]^{t_B}, \tag{4.13}$$

in the uncrossed case, where ^{T} *denotes the transposition on the auxiliary and boundary spaces and* $\mathbf{K}^{\Pi}_{i,j} = (\mathbf{K}_{N+1-j,N+1-i})^{t_B}$.

The HC-s have pole in the pair structure limit. In the crossed and the uncrossed cases the pair structure limits are $t^{\nu}_l \to -t^{\nu}_k - \nu$ and $t^{N-\nu}_l \to -t^{\nu}_k$, respectively. We can use a common notation $t^{\tilde{\nu}}_l \to -t^{\nu}_k - \nu\tilde{c}$ for the pair structure limits, where

$$(\tilde{\nu}, \tilde{c}) = \begin{cases} (\nu, 1), & \text{for the crossed case,} \\ (N-\nu, 0), & \text{for the uncrossed case.} \end{cases} \tag{4.14}$$

Theorem 9. *The HC-s have poles at* $t^{\tilde{\nu}}_l \to -t^{\nu}_k - \nu\tilde{c}$:

$$\mathbf{Z}(\bar{t}) \to \frac{-1}{t^{\tilde{\nu}}_l + t^{\nu}_k + \nu\tilde{c}} \frac{f(t^{\nu}_k, \bar{\tau}^{\nu}) f(t^{\tilde{\nu}}_l, \bar{\tau}^{\tilde{\nu}})}{f(t^{\nu}_k, \bar{t}^{\nu-1}) f(t^{\tilde{\nu}}_l, \bar{t}^{\tilde{\nu}-1})} \mathbf{F}^{(\nu)}(t^{\nu}_k) \mathbf{Z}(\bar{\tau}) + reg., \tag{4.15}$$

and

$$\bar{\mathbf{Z}}(\bar{t}) \to \frac{1}{t^{\tilde{\nu}}_l + t^{\nu}_k + \nu\tilde{c}} \frac{f(\bar{\tau}^{\nu}, t^{\nu}_k) f(\bar{\tau}^{\tilde{\nu}}, t^{\tilde{\nu}}_l)}{f(\bar{t}^{\nu+1}, t^{\nu}_k) f(\bar{t}^{\tilde{\nu}+1}, t^{\tilde{\nu}}_l)} \bar{\mathbf{Z}}(\bar{\tau}) \mathbf{F}^{(\nu)}(t^{\nu}_k) + reg., \tag{4.16}$$

where $\bar{\tau} = \bar{t} \backslash \{t^{\nu}_k, t^{\tilde{\nu}}_l\}$ *and*

$$\mathbf{F}^{(\nu)}(z) = (-1)^{\tilde{c}} \left[\mathbf{G}^{(\nu)}(z) \right]^{-1} \mathbf{G}^{(\nu+1)}(z). \tag{4.17}$$

We have the commuting set of operators

$$\left[\mathbf{F}^{(k)}(u),\mathbf{F}^{(l)}(v)\right]=0,\quad \left[\mathbf{F}^{(k)}(u),\mathbf{B}\right]=0, \tag{4.18}$$

therefore we can diagonalize the matrices simultaneously:

$$\left(\mathbf{F}^{(k)}(u)\right)_{n,m}=\delta_{n,m}\mathcal{F}_n^{(k)}(u),\quad (\mathbf{B})_{n,m}=\delta_{n,m}\beta_n. \tag{4.19}$$

The off-shell overlap have the sum formula

$$\begin{aligned}\langle\mathrm{MPS}|\mathbb{B}(\bar t)&=\sum_{\mathrm{part}}\frac{\prod_{\nu=1}^{N-1}f(\bar t^\nu_{\textsc{ii}},\bar t^\nu_{\textsc{i}})}{\prod_{\nu=1}^{N-2}f(\bar t^{\nu+1}_{\textsc{ii}},\bar t^\nu_{\textsc{i}})}\mathrm{Tr}\left[\bar{\mathbf{Z}}(\bar t_{\textsc{ii}})\mathbf{B}\mathbf{Z}(\bar t_{\textsc{i}})\right]\prod_\nu\alpha_\nu(\bar t^\nu_{\textsc{i}})\\&=\sum_{\ell=1}^{d_B}\beta_\ell\sum_{\mathrm{part}}\frac{\prod_{\nu=1}^{N-1}f(\bar t^\nu_{\textsc{ii}},\bar t^\nu_{\textsc{i}})}{\prod_{\nu=1}^{N-2}f(\bar t^{\nu+1}_{\textsc{ii}},\bar t^\nu_{\textsc{i}})}\left(\mathbf{Z}(\bar t_{\textsc{i}})\bar{\mathbf{Z}}(\bar t_{\textsc{ii}})\right)_{\ell,\ell}\prod_\nu\alpha_\nu(\bar t^\nu_{\textsc{i}}).\end{aligned} \tag{4.20}$$

We can define the expressions

$$S^{(\ell)}_{\bar\alpha}(\bar t):=\sum_{\mathrm{part}}\frac{\prod_{\nu=1}^{N-1}f(\bar t^\nu_{\textsc{ii}},\bar t^\nu_{\textsc{i}})}{\prod_{\nu=1}^{N-2}f(\bar t^{\nu+1}_{\textsc{ii}},\bar t^\nu_{\textsc{i}})}\left(\mathbf{Z}(\bar t_{\textsc{i}})\bar{\mathbf{Z}}(\bar t_{\textsc{ii}})\right)_{\ell,\ell}\prod_\nu\alpha_\nu(\bar t^\nu_{\textsc{i}}), \tag{4.21}$$

for which

$$\langle\mathrm{MPS}|\mathbb{B}(\bar t)=\sum_{\ell=1}^{d_B}\beta_\ell S^{(\ell)}_{\bar\alpha}(\bar t). \tag{4.22}$$

Theorem 10. *The expressions $S^{(n)}_{\bar\alpha}(\bar t)$ has the following pair structure limit*

$$\lim_{t^{\tilde\nu}_l\to -t^\nu_k-\nu\tilde c}S^{(\ell)}_{\bar\alpha}(\bar t)=\mathcal{F}^{(\nu)}_\ell(t^\nu_k)X^\nu_k\frac{f(\bar\tau^\nu,t^\nu_k)f(\bar\tau^{\tilde\nu},t^{\tilde\nu}_l)}{f(\bar t^{\nu+1},t^\nu_k)f(\bar t^{\tilde\nu+1},t^{\tilde\nu}_l)}S^{(\ell)}_{\bar\alpha^{mod}}(\bar\tau)+\tilde S, \tag{4.23}$$

where $\bar\tau=\bar t\backslash\{t^\nu_k,t^{\tilde\nu}_l\}$ and

$$X^\nu_k=-\frac{d}{dt^\nu_k}\log\alpha(t^\nu_k), \tag{4.24}$$

the $\tilde S$ does not depend on X^ν_k. The modified α-s are

$$\alpha^{mod}_\mu(u)=\alpha_\mu(u)\left(\frac{f(t^\nu_k,u)}{f(u,t^\nu_k)}\right)^{\delta_{\nu,\mu}}\left(\frac{f(-t^\nu_k-\nu\tilde c,u)}{f(u,-t^\nu_k-\nu\tilde c)}\right)^{\delta_{\tilde\nu,\mu}}\frac{f(u,t^\nu_k)^{\delta_{\nu,\mu-1}}}{f(t^\nu_k,u)^{\delta_{\nu,\mu+1}}}\frac{f(u,-t^\nu_k-\nu\tilde c)^{\delta_{\tilde\nu,\mu-1}}}{f(-t^\nu_k-\nu\tilde c,u)^{\delta_{\tilde\nu,\mu+1}}}. \tag{4.25}$$

The definition of α^{mod}_μ is compatible with the on-shell limit. In the original overlap function, the on-shell Bethe roots satisfy the Bethe equations

$$\alpha_\mu(t^\mu_k)=\frac{f(t^\mu_k,\bar t^\mu_k)}{f(\bar t^\mu_k,t^\mu_k)}\frac{f(\bar t^{\mu+1},t^\mu_k)}{f(t^\mu_k,\bar t^{\mu-1})}. \tag{4.26}$$

On the right-hand side of (4.23), we have overlaps where the Bethe roots t^ν_k and $t^{\tilde\nu}_l$ have been omitted. The remaining Bethe roots must satisfy the same equations on-shell as before. It is clear that with the original α-functions, the Bethe equations are not satisfied for the set $\bar\tau$:

$$\alpha_\mu(\tau^\mu_k)\neq\frac{f(\tau^\mu_k,\bar\tau^\mu_k)}{f(\bar\tau^\mu_k,\tau^\mu_k)}\frac{f(\bar\tau^{\mu+1},\tau^\mu_k)}{f(\tau^\mu_k,\bar\tau^{\mu-1})}. \tag{4.27}$$

However, with the modified α-functions, they are satisfied. To see this, separate the terms on the right-hand side of the Bethe equations (4.26) according to the decomposition $\bar t=\bar\tau\cup\{t^\nu_k,t^{\tilde\nu}_l\}$:

$$\frac{f(\tau^\mu_k,\bar t^\mu_k)}{f(\bar t^\mu_k,\tau^\mu_k)}\frac{f(\bar t^{\mu+1},\tau^\mu_k)}{f(\tau^\mu_k,\bar t^{\mu-1})}=\frac{f(\tau^\mu_k,\bar\tau^\mu_k)}{f(\bar\tau^\mu_k,\tau^\mu_k)}\frac{f(\bar\tau^{\mu+1},\tau^\mu_k)}{f(\tau^\mu_k,\bar\tau^{\mu-1})}\left(\frac{f(\tau^\mu_k,t^\nu_k)}{f(t^\nu_k,\tau^\mu_k)}\right)^{\delta_{\nu,\mu}}\left(\frac{f(\tau^\mu_k,t^{\tilde\nu}_l)}{f(t^{\tilde\nu}_l,\tau^\mu_k)}\right)^{\delta_{\tilde\nu,\mu}}\frac{f(t^\nu_k,\tau^\mu_k)^{\delta_{\nu,\mu+1}}}{f(\tau^\mu_k,t^\nu_k)^{\delta_{\nu,\mu-1}}}\frac{f(t^{\tilde\nu}_l,\tau^\mu_k)^{\delta_{\tilde\nu,\mu+1}}}{f(\tau^\mu_k,t^{\tilde\nu}_l)^{\delta_{\tilde\nu,\mu-1}}}. \tag{4.28}$$

Substituting back into the original Bethe equation (4.26), we obtain the result

$$\alpha^{mod}_\mu(\tau^\mu_k)=\frac{f(\tau^\mu_k,\bar\tau^\mu_k)}{f(\bar\tau^\mu_k,\tau^\mu_k)}\frac{f(\bar\tau^{\mu+1},\tau^\mu_k)}{f(\tau^\mu_k,\bar\tau^{\mu-1})}, \tag{4.29}$$

where α^{mod}_μ is defined according to (4.25), after the substitution $t^{\tilde\nu}_l=-t^\nu_k-\nu\tilde c$.

4.3 On-shell limit

We saw that the on-shell overlaps are non-vanishing only if the Bethe roots have pair structure. In the crossed case the Bethe roots have chiral pair structure. For simplicity, we assume that the numbers r_ν are even [4]. In the chiral pair structure, we can decompose the sets $\bar{t}^\nu \vdash \{\bar{t}^{+,\nu}, \bar{t}^{-,\nu}\}$, where $\#\bar{t}^{+,\nu} = \#\bar{t}^{-,\nu} = r_\nu^+ = r_\nu/2$. In the pair structure limit, $t_k^{-,\nu} = -t_k^{+,\nu} - \nu$ for $k = 1, \dots, r_\nu^+$.

In the uncrossed case the Bethe roots have achiral pair structure. For odd N, i.e., when $N = 2n+1$, we define the notations $\bar{t}^{+,\nu} = \bar{t}^\nu$ and $\bar{t}^{-,\nu} = \bar{t}^{N-\nu}$, where $\#\bar{t}^{+,\nu} = \#\bar{t}^{-,\nu} = r_\nu^+ = r_\nu$ for $\nu = 1, \dots, n$. In the pair structure limit, $t_k^{-,\nu} = -t_k^{+,\nu}$ for $k = 1, \dots, r_\nu^+$.

If $N = 2n$, then for simplicity we assume that the number r_n is even. In this case, we can decompose the set $\bar{t}^n \vdash \{\bar{t}^{+,n}, \bar{t}^{-,n}\}$ where $\#\bar{t}^{+,n} = \#\bar{t}^{-,n} = r_n^+ = r_n/2$. For the other nodes, we introduce the earlier notation $\bar{t}^{+,\nu} = \bar{t}^\nu$ and $\bar{t}^{-,\nu} = \bar{t}^{N-\nu}$ for $\nu = 1, \dots, n-1$. In the pair structure limit, $t_k^{-,\nu} = -t_k^{+,\nu}$ for $k = 1, \dots, r_\nu^+$ and $\nu = 1, \dots, n$.

In both the chiral and achiral cases, we introduce the set of sets $\bar{t}^\pm = \{\bar{t}^{\pm,\nu}\}_{\nu=1}^{n_+}$, where

$$n_+ = \begin{cases} N-1, & \text{for crossed cases,} \\ n, & \text{for uncrossed cases.} \end{cases}$$

The Theorem 10 is sufficient to prove the form of the overlaps involving Gaudin determinants.

Definition 11. The Gaudin matrices are defined by

$$G_{j,k}^{\pm,(\mu,\nu)} = -\left(\frac{\partial}{\partial t_k^{+,\nu}} \pm \frac{\partial}{\partial t_k^{-,\nu}} \right) \log \Phi_j^{+,(\mu)} \Big|_{\bar{t}^- = \pi^{a/c}(\bar{t}^+)}, \tag{4.30}$$

where $\mu, \nu = 1, \dots, n_+$, $j, k = 1, \dots, r_\nu^+$ and

$$\Phi_k^{+,(\mu)} = \alpha_\mu(t_k^{+,\mu}) \frac{f(\bar{t}_k^\mu, t_k^{+,\mu})}{f(t_k^{+,\mu}, \bar{t}_k^\mu)} \frac{f(t_k^{+,\mu}, \bar{t}^{\mu-1})}{f(\bar{t}^{\mu+1}, t_k^{+,\mu})}. \tag{4.31}$$

Corollary 12. *The Gaudin-determinant factorizes as*

$$\det G = \det G^+ \det G^-, \tag{4.32}$$

for Bethe roots with pair structure $\bar{t} = \pi^{a/c}(\bar{t})$.

After performing the pair structure limit, the elementary overlap functions $S^{(\ell)}$ depend only on the variables $t_k^{+,\mu}$, $\alpha_k^{+,\mu} \equiv \alpha_\mu(t_k^{+,\mu})$ and $X_k^{+,\mu}$. In the generalized model, these are independent variables. After the pair structure limit, we can also perform the on-shell limit via

$$\alpha_k^{+,\mu} \to \frac{f(t_k^{+,\mu}, \bar{t}_k^\mu)}{f(\bar{t}_k^\mu, t_k^{+,\mu})} \frac{f(\bar{t}^{\mu+1}, t_k^{+,\mu})}{f(t_k^{+,\mu}, \bar{t}^{\mu-1})} \tag{4.33}$$

substitution. As a result, we obtain functions that depend only on the variables $t_k^{+,\mu}$ and $X_k^{+,\mu}$:

$$S_{\bar{\alpha}}^{(n)}(\bar{t}) \to S^{(n)}(\bar{t}^+ | \bar{X}^+). \tag{4.34}$$

Theorem 13. *The crossed elementary overlaps are*

$$S^{(\ell)}(\bar{t}^+|\bar{X}^+) = \prod_{\nu=1}^{N-1} \mathcal{F}_\ell^{(\nu)}(\bar{t}^{+,\nu}) \times \prod_{\nu=1}^{N-1} \prod_{k \neq l} f(t_l^{+,\nu}, t_k^{+,\nu}) \prod_{k<l} f(t_l^{+,\nu}, -t_k^{+,\nu} - \nu c) f(-t_l^{+,\nu} - \nu c, t_k^{+,\nu}) \det G^+. \tag{4.35}$$

The uncrossed elementary overlaps are

$$S^{(\ell)}(\bar{t}^+|\bar{X}^+) = \prod_{\nu=1}^{n} \mathcal{F}_\ell^{(\nu)}(\bar{t}^{+,\nu}) \times \frac{\prod_{\nu=1}^{n} \prod_{k \neq l} f(t_l^{+,\nu}, t_k^{+,\nu})}{\prod_{\nu=1}^{n-1} f(\bar{t}^{+,\nu+1}, \bar{t}^{+,\nu}) \prod_{k \leq l} f(-t_l^{+,\frac{N-1}{2}}, t_k^{+,\frac{N-1}{2}})} \det G^+, \tag{4.36}$$

[4]The case of odd values is discussed in Section S-VIII of [45].

for $N=2n+1$ *and*

$$S^{(\ell)}(\bar{t}^{+}|\bar{X}^{+})=\prod_{\nu=1}^{n}\mathcal{F}_{\ell}^{(\nu)}(\bar{t}^{+,\nu})\times\frac{\prod_{\nu=1}^{n}\prod_{k\neq l}f(t_{l}^{+,\nu},t_{k}^{+,\nu})\prod_{k<l}f(-t_{l}^{+,\frac{N}{2}},t_{k}^{+,\frac{N}{2}})f(t_{l}^{+,\frac{N}{2}},-t_{k}^{+,\frac{N}{2}})}{\prod_{\nu=1}^{n-1}f(\bar{t}^{+,\nu+1},\bar{t}^{+,\nu})f(-\bar{t}^{+,\frac{N}{2}},\bar{t}^{+,\frac{N}{2}-1})}\det G^{+},\tag{4.37}$$

for $N=2n$.

Corollary 14. *The normalized on-shell overlaps are*

$$\frac{\langle\mathrm{MPS}|\mathbb{B}(\bar{t})}{\sqrt{\mathbb{C}(\bar{t})\mathbb{B}(\bar{t})}}=\sum_{\ell=1}^{d_B}\beta_{\ell}\prod_{\nu=1}^{n_+}\tilde{\mathcal{F}}_{\ell}^{(\nu)}(\bar{t}^{+,\nu})\sqrt{\frac{\det G^{+}}{\det G^{-}}},\tag{4.38}$$

where

$$\tilde{\mathcal{F}}_{\ell}^{(\nu)}(u)=\begin{cases}\frac{\mathcal{F}_{\ell}^{(\nu)}(u)}{\sqrt{f(-u-\nu,u)f(u,-u-\nu)}}, & \textit{for the chiral case,}\\ \mathcal{F}_{\ell}^{(\nu)}(u)\left(\frac{1}{\sqrt{f(-u,u)}}\right)^{\delta_{\nu,\frac{N-1}{2}}}\left(\frac{1}{\sqrt{f(-u,u)f(u,-u)}}\right)^{\delta_{\nu,\frac{N}{2}}}, & \textit{for the achiral case.}\end{cases}\tag{4.39}$$

4.4 Redefinition of the Bethe roots

In applications, the most common convention is the one where the Bethe equations contain product of phases. To align with this, we introduce a new convention for the Bethe roots:

$$t_k^{\nu}=\begin{cases}iu_k^{\nu}-\nu/2, & \text{for the crossed case,}\\ iu_k^{\nu}+\frac{N-2\nu}{4}, & \text{for the uncrossed case.}\end{cases}\tag{4.40}$$

The Bethe equations with these new variables are

$$\tilde{\alpha}_{\mu}(u_k^{\mu})=-\prod_{j=1}^{r_\mu}\frac{u_k^{\mu}-u_j^{\mu}-i}{u_k^{\mu}-u_j^{\mu}+i}\prod_{j=1}^{r_{\mu+1}}\frac{u_k^{\mu}-u_j^{\mu+1}+i/2}{u_k^{\mu}-u_j^{\mu+1}-i/2}\prod_{j=1}^{r_{\mu-1}}\frac{u_k^{\mu}-u_j^{\mu-1}+i/2}{u_k^{\mu}-u_j^{\mu-1}-i/2},\tag{4.41}$$

where

$$\tilde{\alpha}_{\mu}(u)=\begin{cases}\alpha_{\mu}(iu-\mu/2), & \text{for the crossed case,}\\ \alpha_{\mu}(iu+\frac{N-2\mu}{4}), & \text{for the uncrossed case.}\end{cases}\tag{4.42}$$

Let us also introduce a new notation for the Bethe vectors

$$|\bar{u}\rangle=\begin{cases}\mathbb{B}(\{i\bar{u}^{\nu}-\nu/2\}_{\nu=1}^{N-1}), & \text{for the crossed case,}\\ \mathbb{B}(\{i\bar{u}^{\nu}+\frac{N-2\nu}{4}\}_{\nu=1}^{N-1}), & \text{for the uncrossed case,}\end{cases}\tag{4.43}$$

for which the on-shell overlap formulas have the form

$$\frac{\langle\mathrm{MPS}|\bar{u}\rangle}{\sqrt{\langle\bar{u}|\bar{u}\rangle}}=\sum_{\ell=1}^{d_B}\beta_{\ell}\prod_{\nu=1}^{n_+}\tilde{\mathcal{F}}_{\ell}^{(\nu)}(\bar{u}^{+,\nu})\sqrt{\frac{\det G^{+}}{\det G^{-}}},\tag{4.44}$$

where

$$\tilde{\mathcal{F}}_{\ell}^{(\nu)}(u)=\begin{cases}\mathcal{F}_{\ell}^{(\nu)}(iu-\nu/2)\sqrt{\frac{u^2}{u^2+1/4}}, & \text{for the chiral case,}\\ \mathcal{F}_{\ell}^{(\nu)}(iu+\frac{N-2\nu}{4})\left(\sqrt{\frac{u-i/4}{u+i/4}}\right)^{\delta_{\nu,\frac{N-1}{2}}}\left(\sqrt{\frac{u^2}{u^2+1/4}}\right)^{\delta_{\nu,\frac{N}{2}}}, & \text{for the achiral case.}\end{cases}\tag{4.45}$$

5 Other reflection algebras of $\mathfrak{gl}_N$ spin chains

The formulas from the previous section can be applied to the $Y^{+}(N)$ and $\mathcal{B}(N,\lfloor\frac{N}{2}\rfloor)$ K-matrices. From now on, we assume that the universal overlap formula (1.1) also exists for the other $\mathfrak{gl}_N$ reflection algebras. In this section, we extend the definition of the $\mathbf{F}^{(\nu)}$ operators to the remaining cases as well.

5.1 Generalization for $Y^-(2n)$

First, we extend the F-operators of the $Y^+(N)$ algebra to the $Y^-(N)$ case. We begin by solving the recursive equation (4.2) for the nested K-matrix, from which we obtain the G-operators in closed form. Additionally, we formulate a new recursion for the G-operators, which we apply to the $Y^-(N)$ K-matrices.

5.1.1 Quasi-determinants

The G-operators have several equivalent definitions. To express these, it is useful to introduce new notations. Let $X = (X_{i,j})_{i,j=1}^N$ be an $N \times N$ matrix over a ring. Let $1 \leq M < N$ and introduce the following block notations,

$$X = \begin{pmatrix} A & B \\ C & D \end{pmatrix}, \tag{5.1}$$

where A is an $M \times M$, B an $M \times (N-M)$, C a $(N-M) \times M$ and D a $(N-M) \times (N-M)$ matrix over the ring. Assume that A is invertible, and then let

$$X^{(M+1)} \equiv \begin{pmatrix} A & B \\ C & \boxed{D} \end{pmatrix} := D - CA^{-1}B \tag{5.2}$$

be a $(N-M) \times (N-M)$ matrix. This matrix can also be defined in another equivalent way. Let $\mathcal{I}$ denote the inversion operation, i.e., $\mathcal{I}(X) := X^{-1}$, and let Π_M be a projection such that

$$\Pi_M(X) = \Pi_M\left(\begin{pmatrix} A & B \\ C & D \end{pmatrix}\right) := D. \tag{5.3}$$

Combining these, we introduce the following operation.

$$\omega_M := \mathcal{I} \circ \Pi_M \circ \mathcal{I}. \tag{5.4}$$

It is easy to verify that the inverse matrix can be expressed in the following way:

$$X^{-1} = \begin{pmatrix} (A - BD^{-1}C)^{-1} & -A^{-1}B(D - CA^{-1}B)^{-1} \\ -D^{-1}C(A - BD^{-1}C)^{-1} & (D - CA^{-1}B)^{-1} \end{pmatrix}, \tag{5.5}$$

therefore

$$X^{(M+1)} = \begin{pmatrix} A & B \\ C & \boxed{D} \end{pmatrix} = \omega_M(X). \tag{5.6}$$

Clearly, $\omega_{M_1+M_2} = \omega_{M_1} \circ \omega_{M_2}$, that is,

$$X^{(M_1+M_2)} = \omega_{M_1}(X^{(M_2)}). \tag{5.7}$$

These definitions can be applied to the nested K-matrices. The recursive step can be written using the new notations as follows

$$\mathbf{K}^{(s+1)} = \left(\begin{array}{c|ccc} \mathbf{K}^{(s)}_{s,s} & \mathbf{K}^{(s)}_{s,s+1} & \dots \\ \hline \mathbf{K}^{(s)}_{s+1,s} & \mathbf{K}^{(s)}_{s+1,s+1} & \dots \\ \vdots & \vdots & \ddots \end{array}\right) = \omega_1(\mathbf{K}^{(s)}).$$

Using the identity (5.7), the recursive equations can be solved

$$\mathbf{K}^{(s)} = \omega_{s-1}(\mathbf{K}) = \left(\begin{array}{cccc|ccc} \mathbf{K}_{1,1} & \dots & \mathbf{K}_{1,s-1} & & \mathbf{K}_{1,s} & \mathbf{K}_{1,s+1} & \dots \\ \vdots & \ddots & \vdots & & & \vdots & \\ \mathbf{K}_{s-1,1} & \dots & \mathbf{K}_{s-1,s-1} & & \mathbf{K}_{s-1,s} & \mathbf{K}_{s-1,s+1} & \dots \\ \hline \mathbf{K}_{s,1} & & \mathbf{K}_{s,s-1} & & \mathbf{K}_{s,s} & \mathbf{K}_{s,s+1} & \dots \\ \mathbf{K}_{s+1,1} & \dots & \mathbf{K}_{s+1,s-1} & & \mathbf{K}_{s+1,s} & \mathbf{K}_{s+1,s+1} & \dots \\ \vdots & & \vdots & & \vdots & \vdots & \ddots \end{array}\right),$$

or simply by using components

$$\mathbf{K}^{(s)}_{i,j} = \begin{pmatrix} \mathbf{K}_{1,1} & \dots & \mathbf{K}_{1,s-1} & \mathbf{K}_{1,j} \\ \vdots & \ddots & \vdots & \vdots \\ \mathbf{K}_{s-1,1} & \dots & \mathbf{K}_{s-1,s-1} & \mathbf{K}_{s-1,j} \\ \mathbf{K}_{i,1} & \dots & \mathbf{K}_{i,s-1} & \boxed{\mathbf{K}_{i,j}} \end{pmatrix}, \tag{5.8}$$

where $i, j = s, \dots, N$. Based on this, the G-operators can also be expressed in closed form.

$$\mathbf{G}^{(s)} = \begin{pmatrix} \mathbf{K}_{1,1} & \dots & \mathbf{K}_{1,s-1} & \mathbf{K}_{1,s} \\ \vdots & \ddots & \vdots & \vdots \\ \mathbf{K}_{s-1,1} & \dots & \mathbf{K}_{s-1,s-1} & \mathbf{K}_{s-1,s} \\ \mathbf{K}_{s,1} & \dots & \mathbf{K}_{s,s-1} & \boxed{\mathbf{K}_{s,s}} \end{pmatrix}. \tag{5.9}$$

The definitions above assume that the matrices

$$\begin{pmatrix} \mathbf{K}_{1,1} & \dots & \mathbf{K}_{1,s} \\ \vdots & \ddots & \vdots \\ \mathbf{K}_{s,1} & \dots & \mathbf{K}_{s,s} \end{pmatrix} \tag{5.10}$$

are invertible. This holds for the $Y^+(N)$ algebra, but for the $Y^-(2n)$ algebra, the inverse exists only for even s.

For odd K-matrices, there exists another recursion. We can introduce another series of subalgebras

$$\begin{array}{ccccccccc} Y^+(2n) & \to & Y^+(2n-2) & \to & \dots & \to & Y^+(4) & \to & Y^+(2) \\ \mathbf{K}^{(1)} \equiv \mathbf{K} & \to & \mathbf{K}^{(3)} & \to & \dots & \to & \mathbf{K}^{(2n-3)} & \to & \mathbf{K}^{(2n-1)} \\ \downarrow & & \downarrow & & \dots & & \downarrow & & \downarrow \\ \mathbf{k}^{(1)} & & \mathbf{k}^{(2)} & & \dots & & \mathbf{k}^{(n-1)} & & \mathbf{k}^{(n)} \end{array}$$

In this case, the recurrence equation is

$$\mathbf{K}^{(2k+1)} = \omega_2 \left(\mathbf{K}^{(2k-1)} \right), \tag{5.11}$$

which explicitly gives

$$\mathbf{K}^{(2k+1)}_{a,b}(z) := \mathbf{K}^{(2k-1)}_{a,b}(z) - \sum_{\alpha,\beta=2k-1}^{2k} \mathbf{K}^{(2k-1)}_{a,\alpha}(z) \widehat{\mathbf{K}}^{(2k-1)}_{\alpha,\beta}(z) \mathbf{K}^{(2k-1)}_{\beta,b}(z), \tag{5.12}$$

for $a, b = 2k+1, \dots, 2n$, where $\widehat{\mathbf{K}}^{(2k-1)}_{\alpha,\beta}(z)$ is the inverse of a 2×2 block

$$\sum_{\gamma=2k-1}^{2k} \mathbf{K}^{(2k-1)}_{\alpha,\gamma}(z) \widehat{\mathbf{K}}^{(2k-1)}_{\gamma,\beta}(z) = \delta_{\alpha,\beta} \mathbf{1}. \tag{5.13}$$

We also define the 2×2 K-matrices as

$$\mathbf{k}^{(k)}_{\alpha,\beta}(u) := \mathbf{K}^{(2k-1)}_{\alpha+2k-2,\beta+2k-2}(u), \tag{5.14}$$

for $\alpha, \beta = 1, 2$.

The nested K-matrices $\mathbf{K}^{(2k-1)}$ are the same as before, i.e., they satisfy the reflection equation (3.5). The 2×2 matrices $\mathbf{k}^{(k)}_{\alpha,\beta}(u - (k-1))$ for $\alpha, \beta = 1, 2$ generate $Y^+(2)$ subalgebras. The element $\mathbf{k}^{(k)}_{1,1} = \mathbf{K}^{(2k-1)}_{2k-1,2k-1}$, which by definition is the $\mathbf{G}^{(2k-1)}$ operator. To compute $\mathbf{G}^{(2k)}(u)$ we can use the formula (5.9) and the identity $\omega_{2k-1} = \omega_1 \circ \omega_{2k-2}$, that is,

$$\mathbf{G}^{(2k)} = \omega_{2k-1} \circ \begin{pmatrix} \mathbf{K}_{1,1} & \dots & \mathbf{K}_{1,2k-1} & \mathbf{K}_{1,2k} \\ \vdots & \ddots & \vdots & \vdots \\ \mathbf{K}_{2k-1,1} & \dots & \mathbf{K}_{2k-1,2k-1} & \mathbf{K}_{2k-1,2k} \\ \mathbf{K}_{2k,1} & \dots & \mathbf{K}_{2k,2k-1} & \mathbf{K}_{2k,2k} \end{pmatrix} = \omega_1 \circ \omega_{2k-2} \circ \begin{pmatrix} \mathbf{K}_{1,1} & \dots & \mathbf{K}_{1,2k-1} & \mathbf{K}_{1,2k} \\ \vdots & \ddots & \vdots & \vdots \\ \mathbf{K}_{2k-1,1} & \dots & \mathbf{K}_{2k-1,2k-1} & \mathbf{K}_{2k-1,2k} \\ \mathbf{K}_{2k,1} & \dots & \mathbf{K}_{2k,2k-1} & \mathbf{K}_{2k,2k} \end{pmatrix}$$

$$= \omega_1 \circ \begin{pmatrix} \mathbf{K}^{(2k-1)}_{2k-1,2k-1} & \mathbf{K}^{(2k-1)}_{2k-1,2k} \\ \mathbf{K}^{(2k-1)}_{2k,2k-1} & \mathbf{K}^{(2k-1)}_{2k,2k} \end{pmatrix} = \begin{pmatrix} \mathbf{k}^{(k)}_{1,1} & \mathbf{k}^{(k)}_{1,2} \\ \mathbf{k}^{(k)}_{2,1} & \boxed{\mathbf{k}^{(k)}_{2,2}} \end{pmatrix}. \tag{5.15}$$

In summary, the G-operators have an alternative expression

$$\begin{aligned}\mathbf{G}^{(2k-1)}(z) =&\mathbf{k}^{(k)}_{1,1}(z),\\ \mathbf{G}^{(2k)}(z) =&\mathbf{k}^{(k)}_{2,2}(z) - \mathbf{k}^{(k)}_{2,1}(z)\left[\mathbf{k}^{(k)}_{1,1}(z)\right]^{-1}\mathbf{k}^{(k)}_{1,2}(z).\end{aligned} \tag{5.16}$$

5.1.2 The $Y^-(2)$ overlaps

In the case of $Y^-(2)$ K-matrices, the asymptotic expansion begins with

$$\mathbf{K}_{i,j}(u) = \epsilon_{i,j}\mathbf{1} + \mathcal{O}(u^{-1}), \tag{5.17}$$

where $i,j = 1,2$ and $\epsilon_{i,j} = -\epsilon_{j,i}$, $\epsilon_{1,2} = 1$. It is clear that the $\mathbf{K}_{1,1}(u)$ component is not invertible, so the previous overlap formula cannot be applied. However, for finite-dimensional irreducible K-matrices, there exists a continuous deformation $\mathbf{K} \to \tilde{\mathbf{K}}$, that still solves the reflection equation and for which the $\tilde{\mathbf{K}}_{1,1}$ component is invertible. For this deformed K-matrix, the overlap formula derived in the previous section can be applied. Taking the zero limit of the deformation parameter yields the overlap formula for the original K-matrix.

This deformation is based on the theorem: every finite-dimensional irreducible representation of $Y^-(2)$ arises from a representation of $Y(2)$ via the embedding $Y^-(2) \hookrightarrow Y(2)$ [53]. That is, for every finite-dimensional irreducible $Y^-(2)$ K-matrix, there exists a Lax operator (a $Y(2)$ representation) such that

$$\mathbf{K}_{i,j}(u) = \sum_{k,l}\mathbf{L}_{k,i}(u)\epsilon_{k,l}\mathbf{L}_{l,j}(-u). \tag{5.18}$$

The $\mathbf{K}_{1,1}(u)$ component is

$$\mathbf{K}_{1,1}(u) = \mathbf{L}_{1,1}(u)\mathbf{L}_{2,1}(-u) - \mathbf{L}_{2,1}(u)\mathbf{L}_{1,1}(-u). \tag{5.19}$$

Since $Y(2)$ contains an $\mathfrak{sl}_2$ subalgebra, the boundary space is also an $\mathfrak{sl}_2$ representation. The operator $\mathbf{L}_{1,1}$ preserves the $\mathfrak{sl}_2$ quantum number, while $\mathbf{L}_{2,1}$ lowers it by one, meaning that for finite-dimensional representations, $\mathbf{K}_{1,1}(u)$ is a nilpotent operator.

There is a special scalar solution of the reflection equation ($d_B = 1$)

$$K(u|\mathfrak{b}) = \begin{pmatrix} \mathfrak{b}(u+1/2) & 1 \\ -1 & 0 \end{pmatrix}, \tag{5.20}$$

where $\mathfrak{b} \in \mathbb{C}$ is a scalar parameter. From this scalar representation and the Lax operator $\mathbf{L}$, we can construct a deformed K-matrix:

$$\tilde{\mathbf{K}}_0(u|\mathfrak{b}) = \mathbf{L}_0^{t_0}(u)K_0(u|\mathfrak{b})\mathbf{L}_0(-u), \tag{5.21}$$

that is

$$\begin{aligned}\tilde{\mathbf{K}}_{i,j}(u|\mathfrak{b}) &= \sum_{k,l}\mathbf{L}_{k,i}(u)\epsilon_{k,l}\mathbf{L}_{l,j}(-u) + \mathfrak{b}(u+1/2)\mathbf{L}_{1,i}(u)\mathbf{L}_{1,j}(-u)\\ &= \mathbf{K}_{i,j}(u) + \mathfrak{b}(u+1/2)\mathbf{L}_{1,i}(u)\mathbf{L}_{1,j}(-u).\end{aligned} \tag{5.22}$$

This matrix satisfies the reflection equation, the component $\tilde{\mathbf{K}}_{1,1}(u|\mathfrak{b})$ is invertible, and $\tilde{\mathbf{K}}_{i,j}(u|0) = \mathbf{K}_{i,j}(u)$.

Let us examine the component $\tilde{\mathbf{K}}_{1,1}$, that is, the first G-operator:

$$\tilde{\mathbf{G}}^{(1)}(u|\mathfrak{b}) = \tilde{\mathbf{K}}_{1,1}(u|\mathfrak{b}) = \mathbf{K}_{1,1}(u) + \mathfrak{b}(u+1/2)\mathbf{L}_{1,1}(u)\mathbf{L}_{1,1}(-u).$$

Since the operator $\mathbf{K}_{1,1}$ lowers the $\mathfrak{sl}_2$ spin and $\mathbf{L}_{1,1}(u)\mathbf{L}_{1,1}(-u)$ does not change it, in a specific basis, the former is an upper triangular matrix and the latter is a diagonal matrix, therefore the eigenvalues of $\tilde{\mathbf{G}}^{(1)}$ coincide with those of $\mathfrak{b}(u+1/2)\mathbf{L}_{1,1}(u)\mathbf{L}_{1,1}(-u)$.

Let us continue with the second G-operator.

$$\tilde{\mathbf{G}}^{(2)}(u|\mathfrak{b}) = \tilde{\mathbf{K}}_{2,2}(u|\mathfrak{b}) - \tilde{\mathbf{K}}_{2,1}(u|\mathfrak{b})\left[\tilde{\mathbf{K}}_{1,1}(u|\mathfrak{b})\right]^{-1}\tilde{\mathbf{K}}_{1,2}(u|\mathfrak{b}). \tag{5.23}$$

It can be shown that this operator can be expressed using the Sklyanin determinant (see (A.29) for the definition of $\tilde{\mathbf{k}}^{1,2}_{1,2}$)

$$\tilde{\mathbf{k}}^{1,2}_{1,2}(u|\mathfrak{b}) = \tilde{\mathbf{G}}^{(1)}(u+1|\mathfrak{b})\tilde{\mathbf{G}}^{(2)}(u|\mathfrak{b}). \tag{5.24}$$

The Sklyanin determinant, in turn, factorizes

$$\tilde{\mathbf{k}}^{1,2}_{1,2}(u|\mathfrak{b}) = \mathbf{l}^{1,2}_{1,2}(-u)\mathbf{l}^{1,2}_{1,2}(u+1) = \mathbf{k}^{1,2}_{1,2}(u), \tag{5.25}$$

where the quantum determinant is defined as

$$R_{1,2}(-1)\mathbf{L}_1(u-1)\mathbf{L}_2(u) = \sum_{a_i,b_i=1}^{2} e_{a_1,b_1}\otimes e_{a_2,b_2}\otimes \mathbf{l}^{b_1,b_2}_{a_1,a_2}(u). \tag{5.26}$$

(The proof matches Theorem 2.5.3 in [50]), meaning that the Sklyanin determinant does not depend on the deformation parameter $\mathfrak{b}$. Based on this, the dependence of the eigenvalues of the F-operator

$$\mathbf{F}^{(1)}(u|\mathfrak{b}) = \left[\tilde{\mathbf{G}}^{(1)}(u|\mathfrak{b})\right]^{-1}\tilde{\mathbf{G}}^{(2)}(u|\mathfrak{b}) \tag{5.27}$$

on $\mathfrak{b}$ is extremely simple

$$\mathcal{F}_\ell(u|\mathfrak{b}) = \mathfrak{b}^{-2}\mathcal{F}_\ell(u), \tag{5.28}$$

where the functions $\mathcal{F}_\ell(u)$ are the eigenvalues of the F-operator:

$$\mathbf{F}(u) := \left[\tilde{\mathbf{G}}^{(1)}(u|1)\right]^{-1}\tilde{\mathbf{G}}^{(2)}(u|1). \tag{5.29}$$

The boundary state must also be deformed. The boundary state corresponding to the deformed K-matrix, $\langle\tilde{\Psi}(\mathfrak{b})|$ can be obtained based on the description in Appendix A. The vacuum overlap in the deformed case is

$$\tilde{\mathbf{B}}(\mathfrak{b}) = \langle\tilde{\Psi}(\mathfrak{b})|0\rangle = \prod_{j=1}^{J}\prod_{l=1}^{\Lambda_1^{(j)}-\Lambda_2^{(j)}}\tilde{\mathbf{K}}_{1,1}(\theta_j-l|\mathfrak{b}) = \prod_{j=1}^{J}\prod_{l=1}^{k_j}\tilde{\mathbf{G}}^{(1)}(\theta_j-l|\mathfrak{b}). \tag{5.30}$$

Since the eigenvalues of $\tilde{\mathbf{G}}^{(1)}$ are linear in $\mathfrak{b}$, the dependence of the eigenvalues of the operator $\tilde{\mathbf{B}}$ on $\mathfrak{b}$ is

$$\tilde{\beta}_\ell(\mathfrak{b}) = \mathfrak{b}^{\frac{\Lambda_1-\Lambda_2}{2}}\beta_\ell, \tag{5.31}$$

where the β_ℓ are the eigenvalues of the operator

$$\mathbf{B} := \langle\tilde{\Psi}(1)|0\rangle. \tag{5.32}$$

Applying the overlap formula:

$$\frac{\langle\mathrm{MPS}(\mathfrak{b})|\mathbb{B}(\bar{t})}{\sqrt{\mathbb{C}(\bar{t})\mathbb{B}(\bar{t})}} = \mathfrak{b}^{\frac{\Lambda_1-\Lambda_2}{2}-r_1}\sum_{\ell=1}^{d_B}\beta_\ell\tilde{\mathcal{F}}_\ell(\bar{t}^+)\sqrt{\frac{\det G^+}{\det G^-}}. \tag{5.33}$$

Now we can take the $\mathfrak{b}\to 0$ limit. The Bethe states exists only when $2r_1 \leq \Lambda_1-\Lambda_2$ therefore the limit is non-vanishing only when

$$r_1 = \frac{\Lambda_1-\Lambda_2}{2}, \tag{5.34}$$

and the $Y^-(2)$ overlap is

$$\frac{\langle\mathrm{MPS}|\mathbb{B}(\bar{t})}{\sqrt{\mathbb{C}(\bar{t})\mathbb{B}(\bar{t})}} = \delta_{2r_1,\Lambda_1-\Lambda_2}\sum_{\ell=1}^{d_B}\beta_\ell\tilde{\mathcal{F}}_\ell(\bar{t}^+)\sqrt{\frac{\det G^+}{\det G^-}}, \tag{5.35}$$

where the β_ℓ and $\mathcal{F}_\ell$ are the eigenvalues of the operators (5.32) and (5.29).

The selection rule (5.34) means that the $Y^-(2)$ MPSs are $\mathfrak{gl}_2$ singlet states.

5.1.3 Proposal for $Y^-(2n)$ overlaps

In this section, we combine the results of 5.1.1 and 5.1.2. The definitions discussed in Section 4 cannot be used in the $Y^-(2n)$ case because those definitions involve the inverse of the $\mathbf{K}_{1,1}$ matrix, which is not invertible in this case. However, in 5.1.1 we saw that there exists another equivalent definition for which the nested K-matrices can be defined without requiring $\mathbf{K}_{1,1}$ to be invertible.

Based on Subsection 5.1.1, we have the series of subalgebras

$$\begin{array}{ccccccccc} Y^-(2n) & \to & Y^-(2n-2) & \to & \dots & \to & Y^-(4) & \to & Y^-(2) \\ \mathbf{K}^{(1)} \equiv \mathbf{K} & \to & \mathbf{K}^{(3)} & \to & \dots & \to & \mathbf{K}^{(2n-3)} & \to & \mathbf{K}^{(2n-1)} \\ \downarrow & & \downarrow & & \dots & & \downarrow & & \downarrow \\ \mathbf{k}^{(1)} & & \mathbf{k}^{(2)} & & \dots & & \mathbf{k}^{(n-1)} & & \mathbf{k}^{(n)} \end{array}$$

Based on the definition in (5.14) the K-matrices $\mathbf{k}^{(s)}(u-(s-1))$ are representations of the $Y^-(2)$. For these, the G-operators defined by (5.16) do not exist. However, we can apply the method described in 5.1.2 i.e., we first deform the $\mathbf{k}^{(s)}$ matrices so that the G-operators become definable. The deformation is carried out as follows

$$\begin{aligned} \mathbf{k}^{(s)}_{i,j}(u) &= \sum_{k,l} \mathbf{L}^{(s)}_{k,i}(u+(s-1))\epsilon_{k,l}\mathbf{L}^{(s)}_{l,j}(-u-(s-1)), \\ \tilde{\mathbf{k}}^{(s)}_{i,j}(u|\mathfrak{b}_s) &= \mathbf{k}^{(s)}_{i,j}(u) + \mathfrak{b}_s(u+(s-1/2))\mathbf{L}_{1,i}(u+(s-1))\mathbf{L}_{1,j}(-u-(s-1)). \end{aligned} \tag{5.36}$$

For these deformed matrices, we can define the G-operators as

$$\begin{aligned} \tilde{\mathbf{G}}^{(2s-1)}(u|\mathfrak{b}_s) &= \tilde{\mathbf{k}}^{(s)}_{1,1}(u|\mathfrak{b}_s), \\ \tilde{\mathbf{G}}^{(2s)}(u|\mathfrak{b}_s) &= \tilde{\mathbf{k}}^{(s)}_{2,2}(u|\mathfrak{b}_s) - \tilde{\mathbf{k}}^{(s)}_{2,1}(u|\mathfrak{b}_s)\left[\tilde{\mathbf{k}}^{(s)}_{1,1}(u|\mathfrak{b}_s)\right]^{-1}\tilde{\mathbf{k}}^{(s)}_{1,2}(u|\mathfrak{b}_s). \end{aligned} \tag{5.37}$$

Based on the previous reasoning, the eigenvalues of the $\tilde{\mathbf{G}}^{(2s-1)}$ operators are proportional, while the eigenvalues of the $\tilde{\mathbf{G}}^{(2s)}$ operators are inversely proportional to $\mathfrak{b}_s$. Thus, the dependence of the eigenvalues of the F-operators

$$\begin{aligned} \mathbf{F}^{(2s-1)}(u|\mathfrak{b}_s) &= \left[\tilde{\mathbf{G}}^{(2s-1)}(u|\mathfrak{b}_s)\right]^{-1}\tilde{\mathbf{G}}^{(2s)}(u|\mathfrak{b}_s), \\ \mathbf{F}^{(2s)}(u|\mathfrak{b}_s,\mathfrak{b}_{s+1}) &= \left[\tilde{\mathbf{G}}^{(2s)}(u|\mathfrak{b}_s)\right]^{-1}\tilde{\mathbf{G}}^{(2s+1)}(u|\mathfrak{b}_{s+1}), \end{aligned} \tag{5.38}$$

on $\mathfrak{b}_s$ is as follows

$$\begin{aligned} \mathcal{F}^{(2s-1)}_\ell(u|\mathfrak{b}_s) =& \mathfrak{b}_s^{-2}\mathcal{F}^{(2s-1)}_\ell(u), \\ \mathcal{F}^{(2s)}_\ell(u|\mathfrak{b}_s,\mathfrak{b}_{s+1}) =& \mathfrak{b}_s\mathfrak{b}_{s+1}\mathcal{F}^{(2s)}_\ell(u), \end{aligned} \tag{5.39}$$

where $\mathcal{F}^{(k)}_\ell$ are the eigenvalues of the $\mathbf{F}^{(k)}$ operators in the limit $\mathfrak{b}_s \to 1$.

The deformed vacuum overlap is given by the formula in (A.36) meaning that the dependence of the vacuum eigenvalues on $\mathfrak{b}_s$ is

$$\tilde{\beta}_\ell(\mathfrak{b}_1,\dots,\mathfrak{b}_n) = \prod_{s=1}^n \mathfrak{b}_s^{\frac{\Lambda_{2s-1}-\Lambda_{2s}}{2}}\beta_\ell, \tag{5.40}$$

where β_ℓ is the eigenvalue of the vacuum overlap in the limit $\mathfrak{b}_s \to 1$. Applying the overlap formula:

$$\frac{\langle \mathrm{MPS}|\mathbb{B}(\bar{t})}{\sqrt{\mathbb{C}(\bar{t})\mathbb{B}(\bar{t})}} = \lim_{\mathfrak{b}_s\to 0}\prod_{s=1}^n \mathfrak{b}_s^{\frac{\Lambda_{2s-1}-\Lambda_{2s}}{2}-r_{2s-1}+\frac{r_{2s-2}+r_{2s}}{2}}\sum_{\ell=1}^{d_B}\beta_\ell\tilde{\mathcal{F}}_\ell(\bar{t}^+)\sqrt{\frac{\det G^+}{\det G^-}}.$$

Now we can take the limit $\mathfrak{b}_s \to 0$. The limit is non-vanishing only when

$$r_{2s-1} = \frac{\Lambda_{2s-1}-\Lambda_{2s}+r_{2s-2}+r_{2s}}{2}, \tag{5.41}$$

for $s=1,\dots,n$ and the $Y^-(2n)$ overlap is

$$\frac{\langle \mathrm{MPS}|\mathbb{B}(\bar{t})}{\sqrt{\mathbb{C}(\bar{t})\mathbb{B}(\bar{t})}} = \sum_{\ell=1}^{d_B}\beta_\ell\tilde{\mathcal{F}}_\ell(\bar{t}^+)\sqrt{\frac{\det G^+}{\det G^-}}. \tag{5.42}$$

5.2 Generalization for $\mathcal{B}(N,M)$

In the case of uncrossed $\mathcal{B}(N,M)$ K-matrices, the nesting procedure described in 3.5.2 cannot be continued beyond step M. The nesting in this case proceeds as follows

$$\begin{array}{ccccccccc}
\mathcal{B}(N,M) & \to & \mathcal{B}(2N-2,M-1) & \to & \dots & \to & \mathcal{B}(2N-2M+2,1) & \to & \mathcal{B}(2N-2M,0) \\
\mathbf{K}^{(1)}\equiv\mathbf{K} & \to & \mathbf{K}^{(2)} & \to & \dots & \to & \mathbf{K}^{(M)} & \to & \mathbf{K}^{(M+1)} \\
\downarrow & & \downarrow & & \dots & & \downarrow & & \\
\mathbf{G}^{(1)} & & \mathbf{G}^{(2)} & & \dots & & \mathbf{G}^{(M)} & &
\end{array}$$

It is useful to first examine the case $M=0$ separately and then generalize to arbitrary M.

5.2.1 The $\mathcal{B}(N,0)$ overlaps

For the $\mathcal{B}(N,0)$ K-matrices, the asymptotic expansion begins with

$$\mathbf{K}_{i,j}(u)=\delta_{i,j}\mathbf{1}+\mathcal{O}(u^{-1}), \tag{5.43}$$

where $i,j=1,\dots,N$. It is clear that the component $\mathbf{K}_{N,1}(u)$ is not invertible, so the previously used overlap formula cannot be applied. For $N=2$, the uncrossed $\mathcal{B}(2,0)$ algebra is isomorphic to the crossed $Y^-(2)$ algebra. In this case, for finite-dimensional irreducible K-matrices, there exists a continuous deformation $\mathbf{K}\to\tilde{\mathbf{K}}$, that still solves the reflection equation and makes the matrix elements in the overlap formula invertible. The following generalizes this procedure to arbitrary N.

The deformation is based on the assumption that for every finite-dimensional $\mathcal{B}(N,0)$ K-matrix, there exists a Lax operator (a representation of $Y(N)$) such that

$$\mathbf{K}_{i,j}(u)=\sum_{k=1}^{N}\mathbf{L}'_{i,k}(u)\mathbf{L}_{k,j}(-u), \tag{5.44}$$

where

$$\sum_{k=1}^{N}\mathbf{L}'_{i,k}(u)\mathbf{L}_{k,j}(u)=\delta_{i,j}\mathbf{1}. \tag{5.45}$$

This is certainly true for $N=2$, but for $N>2$, it remains a conjecture. The component

$$\mathbf{K}_{N,1}(u)=\sum_{k=1}^{N}\mathbf{L}'_{N,k}(u)\mathbf{L}_{k,1}(-u), \tag{5.46}$$

is a nilpotent operator.

The reflection equation has a special scalar solution (with $d_B=1$)

$$K_{i,j}(u|\bar{\mathfrak{b}})=\delta_{i,j}+u\sum_{i=1}^{N/2}\mathfrak{b}_i e_{N+1-i,i}. \tag{5.47}$$

From this scalar representation and the Lax operator $\mathbf{L}$, we can construct a deformed K-matrix:

$$\tilde{\mathbf{K}}_0(u|\bar{\mathfrak{b}})=\mathbf{L}'_0(u)K_0(u|\bar{\mathfrak{b}})\mathbf{L}_0(-u), \tag{5.48}$$

that is

$$\tilde{\mathbf{K}}_{i,j}(u|\bar{\mathfrak{b}})=\mathbf{K}_{i,j}(u)+u\sum_{k=1}^{N/2}\mathfrak{b}_k\mathbf{L}'_{i,N+1-k}(u)\mathbf{L}_{k,j}(-u). \tag{5.49}$$

This matrix satisfies the reflection equation, the component $\tilde{\mathbf{K}}_{N,1}(u|\bar{\mathfrak{b}})$ is invertible, and $\tilde{\mathbf{K}}_{i,j}(u|0)=\mathbf{K}_{i,j}(u)$.

Let us now examine the component $\tilde{\mathbf{K}}_{N,1}$, i.e., the first G-operator:

$$\begin{aligned}
\tilde{\mathbf{G}}^{(1)}(u|\bar{\mathfrak{b}})=\tilde{\mathbf{K}}_{N,1}(u|\bar{\mathfrak{b}})&=\mathfrak{b}_1 u\mathbf{L}'_{N,N}(u)\mathbf{L}_{1,1}(-u)+\mathbf{K}_{N,1}(u)+u\sum_{k=2}^{N/2}\mathfrak{b}_k\mathbf{L}'_{N,N+1-k}(u)\mathbf{L}_{k,1}(-u)\\
&=\mathfrak{b}_1 u\mathbf{L}'_{N,N}(u)\mathbf{L}_{1,1}(-u)+\check{\mathbf{K}}_{N,1}(u).
\end{aligned} \tag{5.50}$$

The eigenvalues of $\tilde{\mathbf{G}}^{(1)}$ coincide with those of $\mathfrak{b}_1 u \mathbf{L}'_{N,N}(u)\mathbf{L}_{1,1}(-u)$. Let us continue with the second G-operator

$$\tilde{\mathbf{G}}^{(2)}(u|\bar{\mathfrak{b}}) = \tilde{\mathbf{K}}_{2,2}(u|\bar{\mathfrak{b}}) - \tilde{\mathbf{K}}_{2,1}(u|\bar{\mathfrak{b}}) \left[\tilde{\mathbf{K}}_{1,1}(u|\bar{\mathfrak{b}})\right]^{-1} \tilde{\mathbf{K}}_{1,2}(u|\bar{\mathfrak{b}}). \tag{5.51}$$

It can be shown that this operator can be expressed using the quantum minor (see (C.39))

$$\tilde{\mathbf{k}}^{1,2}_{\bar{1},\bar{2}}(u|\bar{\mathfrak{b}}) = \tilde{\mathbf{G}}^{(1)}(u+1|\bar{\mathfrak{b}})\tilde{\mathbf{G}}^{(2)}(u|\bar{\mathfrak{b}}), \tag{5.52}$$

where $\bar{i} = N+1-i$. The quantum minor, however, factorizes

$$\tilde{\mathbf{k}}^{1,2}_{\bar{1},\bar{2}}(u|\bar{\mathfrak{b}}) = \mathfrak{b}_1\mathfrak{b}_2 \mathbf{l}^{1,2}_{1,2}(-u)\hat{\mathbf{l}}^{\bar{1},\bar{2}}_{\bar{1},\bar{2}}(u), \tag{5.53}$$

meaning that the eigenvalues of $\tilde{\mathbf{G}}^{(2)}$ are proportional to $\mathfrak{b}_2$. This can be generalized: the eigenvalues of $\tilde{\mathbf{G}}^{(k)}$ are proportional to $\mathfrak{b}_k$ for $k \leq N/2$. Based on this, the dependence of the eigenvalues of the operators

$$\mathbf{F}^{(k)}(u|\bar{\mathfrak{b}}) = \left[\tilde{\mathbf{G}}^{(k)}(u|\bar{\mathfrak{b}})\right]^{-1} \tilde{\mathbf{G}}^{(k+1)}(u|\bar{\mathfrak{b}}) \tag{5.54}$$

on $\mathfrak{b}$ is extremely simple for $k < N/2$:

$$\mathcal{F}^{(k)}_{\ell}(u|\bar{\mathfrak{b}}) = \frac{\mathfrak{b}_{k+1}}{\mathfrak{b}_k}\mathcal{F}^{(k)}_{\ell}(u) \tag{5.55}$$

where the functions $\mathcal{F}_{\ell}(u)$ are the eigenvalues of the operator

$$\mathbf{F}^{(k)}(u) := \mathbf{F}^{(k)}(u|\bar{\mathfrak{b}})\Big|_{\mathfrak{b}_k=1}. \tag{5.56}$$

If $N = 2n+1$, then the eigenvalues of $\tilde{\mathbf{G}}^{(n+1)}(u|\bar{\mathfrak{b}})$ do not depend on the $\mathfrak{b}_k$ parameters. If $N = 2n$, then the eigenvalues of $\tilde{\mathbf{G}}^{(n+1)}(u|\bar{\mathfrak{b}})$ are inversely proportional to $\mathfrak{b}_n$, i.e.,

$$\mathcal{F}^{(n)}_{\ell}(u|\bar{\mathfrak{b}}) = \begin{cases} \frac{1}{\mathfrak{b}_n}\mathcal{F}^{(n)}_{\ell}(u), & \text{if } N = 2n+1, \\ \frac{1}{\mathfrak{b}_n^2}\mathcal{F}^{(n)}_{\ell}(u), & \text{if } N = 2n. \end{cases} \tag{5.57}$$

The boundary state must also be deformed. The boundary state $\langle\tilde{\Psi}(\mathfrak{b})|$ corresponding to the deformed K-matrix can be obtained based on the description in Appendix A. Since the eigenvalues of $\tilde{\mathbf{G}}^{(k)}$ are linear in $\mathfrak{b}_k$ the dependence of the eigenvalues of the $\tilde{\mathbf{B}}$ operator on $\mathfrak{b}$ is as follows:

$$\tilde{\beta}_{\ell}(\bar{\mathfrak{b}}) = \prod_{k=1}^{n} \mathfrak{b}_k^{\Lambda_k}\beta_{\ell}, \tag{5.58}$$

where the β_{ℓ} are the eigenvalues of the operator

$$\mathbf{B} := \langle\tilde{\Psi}(\bar{\mathfrak{b}})|0\rangle\Big|_{\mathfrak{b}_k=1}. \tag{5.59}$$

Applying the overlap formula:

$$\frac{\langle \mathrm{MPS}(\bar{\mathfrak{b}})|\mathbb{B}(\bar{t})}{\sqrt{\mathbb{C}(\bar{t})\mathbb{B}(\bar{t})}} = \prod_{k=1}^{n} \mathfrak{b}_k^{\Lambda_k - r_k + r_{k-1}} \sum_{\ell=1}^{d_B} \beta_{\ell}\tilde{\mathcal{F}}_{\ell}(\bar{t}^{+})\sqrt{\frac{\det G^{+}}{\det G^{-}}}. \tag{5.60}$$

Now we can take the $\mathfrak{b} \to 0$ limit. The limit is non-vanishing only when

$$\Lambda_k = r_k - r_{k-1}, \tag{5.61}$$

or equivalently

$$r_k = \sum_{l=1}^{k} \Lambda_l. \tag{5.62}$$

In summary, the $\mathcal{B}(N,0)$ overlap is

$$\frac{\langle \mathrm{MPS}|\mathbb{B}(\bar{t})}{\sqrt{\mathbb{C}(\bar{t})\mathbb{B}(\bar{t})}} = \sum_{\ell=1}^{d_B} \beta_{\ell} \prod_{\nu=1}^{n} \tilde{\mathcal{F}}^{(\nu)}_{\ell}(\bar{t}^{+,\nu})\sqrt{\frac{\det G^{+}}{\det G^{-}}}, \tag{5.63}$$

with the selection rule (5.62).

5.2.2 General M

For general M, we define the nested K-matrices and G-operators according to the procedure described in 4, continuing until we reach the matrix $\mathbf{K}^{(M+1)}$, which is a representation of $\tilde{\mathcal{B}}(N-2M,0)$. This matrix is then deformed as described above, and the G-operators are computed from the deformed matrices, following the same procedure:

$$
\begin{array}{ccccccccccccc}
\mathcal{B}(N,M) & \to & \dots & \to & \mathcal{B}(2N-2M+2,1) & \to & \mathcal{B}(2N-2M,0) & & & & & & \\
\mathbf{K}^{(1)}\equiv\mathbf{K} & \to & \dots & \to & \mathbf{K}^{(M)} & \to & \mathbf{K}^{(M+1)} & \to & \tilde{\mathbf{K}}^{(M+1)} & \to & \tilde{\mathbf{K}}^{(M+2)} & \to & \dots \\
\downarrow & & & & \downarrow & & & & \downarrow & & \downarrow & & \\
\mathbf{G}^{(1)} & & \dots & & \mathbf{G}^{(M)} & & & & \mathbf{G}^{(M+1)} & & \mathbf{G}^{(M+2)} & & \dots
\end{array}
$$

Specifically, the deformed K-matrix is given by

$$
\begin{aligned}
\mathbf{K}^{(M+1)}_{i,j}(u) &= \sum_{l=M+1}^{N-M} \mathbf{L}'_{i,l}(u)\mathbf{L}_{l,j}(-u), \\
\tilde{\mathbf{K}}^{(M+1)}_{i,j}(u) &= \mathbf{K}^{(M+1)}_{i,j}(u) + u\sum_{l=M+1}^{n} \mathbf{L}'_{i,N+1-l}(u)\mathbf{L}_{l,j}(-u),
\end{aligned}
\tag{5.64}
$$

where $i,j=M+1,\dots,N-M$. The G-operators $\mathbf{G}^{(s)}$ are derived from the matrix $\tilde{\mathbf{K}}^{(M+1)}$ in the usual way for $s\geq M+1$-re. Non-zero overlaps require the following selection rule:

$$
r_s = r_M + \sum_{l=M+1}^{s} \Lambda_l, \tag{5.65}
$$

for $s=M+1,\dots,n$.

6 Generalization for the orthogonal and symplectic spin chains

In this section, we generalize the overlap formulas to orthogonal and symplectic spin chains. Here, n denotes the rank of the symmetry algebra, i.e., $\mathfrak{sp}_N=\mathfrak{sp}_{2n}$ and

$$
\mathfrak{so}_N = \begin{cases} \mathfrak{so}_{2n}, & \text{if } N \text{ is even,} \\ \mathfrak{so}_{2n+1}, & \text{if } N \text{ is odd.} \end{cases}
$$

We introduce the following index sets

$$
I = \begin{cases} \{-n,\dots,,-2,-1,1,2,\dots,n\}, & \text{for } N=2n, \\ \{-n,\dots,,-2,-1,0,1,2,\dots,n\}, & \text{for } N=2n+1, \end{cases} \tag{6.1}
$$

and in summations, our convention is

$$
\sum_{j=-n}^{n} a_j \equiv \sum_{j\in I} a_j. \tag{6.2}
$$

Assuming that the universal form of the overlaps exists (1.1) for both the orthogonal and symplectic cases, we determine the unknown one-particle overlap functions $\tilde{\mathcal{F}}^{(\nu)}_{\ell}(u)$ for each node in the above subsections.

6.1 RTT- and KT-relations

The R-matrix for orthogonal and symplectic spin chains is given by:

$$
R(u) = \mathbf{1} + \frac{1}{u}\mathbf{P} - \frac{1}{u+\kappa_N}\mathbf{Q} \in \mathrm{End}(\mathbb{C}^N\otimes\mathbb{C}^N), \tag{6.3}
$$

where

$$
\mathbf{P} = \sum_{i,j=-n}^{n} e_{i,j}\otimes e_{j,i}, \quad \mathbf{Q} = \sum_{i,j=-n}^{n} \theta_i\theta_j e_{i,j}\otimes e_{-i,-j}, \tag{6.4}
$$

and $\kappa_N = \frac{N \mp 2}{2}$, where the minus and plus signs correspond to the orthogonal and symplectic cases, respectively. In the orthogonal case, $\theta_i = +1$ for all i, and in the symplectic case, $\theta_i = \mathrm{sgn}(i)$. The RTT-relation defines the Yangian algebras $Y(\mathfrak{g}_N)$

$$R_{1,2}(u-v)T_1(u)T_2(v) = T_2(v)T_1(u)R_{1,2}(u-v). \tag{6.5}$$

From this relation, it also follows that

$$\sum_{k=-n}^{n} \theta_j \theta_k T_{i,k}(u) T_{-j,-k}(u-\kappa_N) = \delta_{i,j}\gamma(u), \tag{6.6}$$

where $\gamma(u)$ is a central element of the $Y(\mathfrak{g}_N)$ algebra. This also implies that for the $\mathfrak{g}_N$ symmetric spin chains, the crossed KT-relation is equivalent to the uncrossed one, since $\widehat{T}(u) \sim T(u-\kappa_N)$.

The highest weight representations can be defined analogously as before, meaning there exists a highest weight vector for which

$$\begin{aligned} T_{i,j}(u)|0\rangle &= 0, \quad i > j, \\ T_{i,i}(u)|0\rangle &= \lambda_i(u)|0\rangle. \end{aligned} \tag{6.7}$$

Just like in the $\mathfrak{gl}_N$ case, the algebra $Y(\mathfrak{g}_N)$ also contains a $\mathfrak{g}_N$ subalgebra, and the pseudo-vacuum is the highest weight vector of this $\mathfrak{g}_N$ subalgebra as well. These $\mathfrak{gl}_N$ weights can be obtained from the asymptotic limit of the pseudo-vacuum eigenvalues

$$\lambda_{-i}(u) = 1 + \frac{1}{u}\Lambda_{-i+n+1} + \mathcal{O}(u^{-2}), \tag{6.8}$$

for $i = 1, \ldots, n$.

For these representations, we can apply the inversion relation (6.6) on the pseudo-vacuum

$$\sum_{k=-n}^{n} \theta_{-n}\theta_{-k} T_{-n,-k}(u) T_{n,k}(u-\kappa_N)|0\rangle = T_{-n,-n}(u)T_{n,n}(u-\kappa_N)|0\rangle = \gamma(u)|0\rangle, \tag{6.9}$$

i.e.,

$$\gamma(u) = \lambda_{-n}(u)\lambda_n(u-\kappa_N). \tag{6.10}$$

The Bethe Ansatz equations are the following

$$\tilde{\alpha}_a(u_j^a) = -\prod_{b=1}^{n}\prod_{k=1}^{r_b} \frac{u_j^a - u_k^b - \frac{i}{2}C_{a,b}}{u_j^a - u_k^b + \frac{i}{2}C_{a,b}}, \tag{6.11}$$

where $C_{a,b}$ is the symmetric Cartan matrix, $C_{a,b} = \vec{\rho}_a \cdot \vec{\rho}_b$, with $\vec{\rho}_a \in \mathbb{R}^n$ being the simple roots for $a = 1, \ldots, n$. In both the orthogonal and symplectic cases,

$$\rho_a = (\underbrace{0, \ldots, 0}_{a-1}, 1, -1, \underbrace{0, \ldots, 0}_{n-a-1}), \tag{6.12}$$

for $a = 1, \ldots, n-1$, and the remaining root is

$$\rho_n = \begin{cases} (0, \ldots, 0, 0, 1), & \text{for } \mathfrak{so}_{2n+1}, \\ (0, \ldots, 0, 0, 2), & \text{for } \mathfrak{sp}_{2n}, \\ (0, \ldots, 0, 1, 1), & \text{for } \mathfrak{so}_{2n}. \end{cases} \tag{6.13}$$

As previously established, for orthogonal and symplectic spin chains, there is only one type of KT-relation. For simplicity, we use the uncrossed convention:

$$\mathbf{K}_0(u)\langle\Psi|T_0(u) = \langle\Psi|T_0(-u)\mathbf{K}_0(u). \tag{6.14}$$

From the KT-relation, it also follows that

$$\lambda_k(u) = \lambda_{-k}(-u).$$

As before, the KT-relation is compatible with associativity and the RTT-relation if the K-matrix satisfies the reflection equation

$$R(v-u)\mathbf{K}_1(u)R(-u-v)\mathbf{K}_2(v) = \mathbf{K}_2(v)R(-u-v)\mathbf{K}_1(u)R(v-u). \tag{6.15}$$

The K-matrices can be classified from the asymptotic limit of the reflection equation. Using previous calculations, the asymptotic limit of the K-matrix is

$$\mathbf{K}_{i,j}(u) = \mathcal{U}_{i,j} \otimes \mathbf{1} + \mathcal{O}(u^{-1}), \tag{6.16}$$

where

$$\mathcal{U}_{i,j} = \pm\theta_i\theta_j\mathcal{U}_{-j,-i}, \qquad \sum_{j=-n}^{n} \mathcal{U}_{i,j}\mathcal{U}_{j,k} = a\delta_{i,k},$$

where $a \in \mathbb{C}$. If $a = 0$, then the K-matrix is called singular. If $a \neq 0$, then the K-matrix can be normalized such that $a = 1$. From now on, we deal with non-singular K-matrices, i.e., where $a = 1$. From the second condition, it follows that the eigenvalues of the $\mathcal{U}$ matrix are ± 1. If $\mathcal{U}_{i,j} = +\theta_i\theta_j\mathcal{U}_{-j,-i}$ then the symmetry algebras of the possible $\mathcal{U}$ matrices are $\mathfrak{g}_M \oplus \mathfrak{g}_{N-M}$. The corresponding K-matrices generate the reflection algebra $Y(\mathfrak{g}_N, \mathfrak{g}_M \oplus \mathfrak{g}_{N-M})$ [54]. If $\mathcal{U}_{i,j} = -\theta_i\theta_j\mathcal{U}_{-j,-i}$ then N is always even, and the symmetry algebra of the possible $\mathcal{U}$ matrices is $\mathfrak{gl}_n$, with the corresponding reflection algebra $Y(\mathfrak{g}_{2n}, \mathfrak{gl}_n)$ [54].

In the following, we present examples of these algebras. In the case of $Y(\mathfrak{so}_N, \mathfrak{so}_M \oplus \mathfrak{so}_{N-M})$,

$$\mathcal{U} = \sum_{j=n-M+1}^{n} (e_{j,-j} + e_{-j,j}) + \sum_{j=-n+M}^{n-M} e_{j,j}. \tag{6.17}$$

In the case of $Y(\mathfrak{so}_{2n}, \mathfrak{gl}_n)$,

$$\mathcal{U} = \sum_{j=1}^{n/2} x_j(e_{n-2j+1,2j-2-n} - e_{n-2j+2,2j-1-n}) + \sum_{j=1}^{n}(e_{j,j} - e_{-j,-j}). \tag{6.18}$$

In the case of $Y(\mathfrak{sp}_{2n}, \mathfrak{sp}_{2m} \oplus \mathfrak{sp}_{2n-2m})$,

$$\mathcal{U} = \sum_{j=1}^{m} x_j(e_{n-2j+1,2j-2-n} - e_{n-2j+2,2j-1-n}) + \sum_{j=-n}^{n} \mathfrak{s}_j e_{j,j}, \tag{6.19}$$

$$\mathfrak{s}_j = \begin{cases} +1, & |j| \leq n-2m, \\ (-1)^{n-j}, & |j| > n-2m. \end{cases}$$

In the case of $Y(\mathfrak{sp}_{2n}, \mathfrak{gl}_n)$,

$$\mathcal{U} = \sum_{j=1}^{n} x_j e_{j,-j} + \sum_{j=1}^{n}(e_{j,j} - e_{-j,-j}). \tag{6.20}$$

For better clarity, we provide a few explicit examples. In the case of $Y(\mathfrak{so}_4, \mathfrak{gl}_2)$,

$$\mathcal{U} = \left(\begin{array}{cc|cc} -1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ \hline x_1 & 0 & +1 & 0 \\ 0 & -x_1 & 0 & +1 \end{array}\right), \tag{6.21}$$

and in the case of $Y(\mathfrak{so}_6, \mathfrak{gl}_3)$,

$$\mathcal{U} = \left(\begin{array}{ccc|ccc} -1 & 0 & 0 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 & 0 & 0 \\ \hline 0 & 0 & 0 & +1 & 0 & 0 \\ x_1 & 0 & 0 & 0 & +1 & 0 \\ 0 & -x_1 & 0 & 0 & 0 & +1 \end{array}\right), \tag{6.22}$$

and in the case of $Y(\mathfrak{so}_8, \mathfrak{gl}_4)$,

$$\mathcal{U} = \left(\begin{array}{cccc|cccc} -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 & 0 & 0 & 0 & 0 \\ \hline 0 & 0 & x_2 & 0 & +1 & 0 & 0 & 0 \\ 0 & 0 & 0 & -x_2 & 0 & +1 & 0 & 0 \\ x_1 & 0 & 0 & 0 & 0 & 0 & +1 & 0 \\ 0 & -x_1 & 0 & 0 & 0 & 0 & 0 & +1 \end{array}\right). \tag{6.23}$$

In the case of $Y(\mathfrak{sp}_8, \mathfrak{sp}_2 \oplus \mathfrak{sp}_6)$,

$$\mathcal{U} = \left(\begin{array}{cccc|cccc} -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & +1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & +1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & +1 & 0 & 0 & 0 & 0 \\ \hline 0 & 0 & 0 & 0 & +1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & +1 & 0 & 0 \\ x_1 & 0 & 0 & 0 & 0 & 0 & +1 & 0 \\ 0 & -x_1 & 0 & 0 & 0 & 0 & 0 & -1 \end{array}\right), \tag{6.24}$$

and in the case of $Y(\mathfrak{sp}_8, \mathfrak{sp}_4 \oplus \mathfrak{sp}_4)$,

$$\mathcal{U} = \left(\begin{array}{cccc|cccc} -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & +1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & +1 & 0 & 0 & 0 & 0 \\ \hline 0 & 0 & x_2 & 0 & +1 & 0 & 0 & 0 \\ 0 & 0 & 0 & -x_2 & 0 & -1 & 0 & 0 \\ x_1 & 0 & 0 & 0 & 0 & 0 & +1 & 0 \\ 0 & -x_1 & 0 & 0 & 0 & 0 & 0 & -1 \end{array}\right). \tag{6.25}$$

6.2 The $Y(n)$ subalgebra

The algebra $Y(\mathfrak{g}_N)$ contains a subalgebra $Y(n)$, and the corresponding generators are denoted

$$T^{\mathfrak{gl}_n}_{n+1-i,n+1-j}(u) = T_{-i,-j}(u + \frac{\kappa_N}{2}), \tag{6.26}$$

where $i, j = 1, \ldots, n$. From this, it also follows that

$$\lambda_1^{\mathfrak{gl}_n}(u) = \lambda_{-n}(u + \frac{\kappa_N}{2}) = \lambda_n(-u - \frac{\kappa_N}{2}). \tag{6.27}$$

The Bethe vectors corresponding to this subalgebra are those for which $r_n = 0$, meaning that in this sector, the Bethe vectors are generated by the generators $T_{-i,-j}$ for $1 \leq j < i \leq n$, i.e,

$$\mathbb{B}(\{\bar{t}^\nu\}_{\nu=1}^{n-1}, \emptyset) = \mathcal{P}(T_{-i<-j})|0\rangle, \tag{6.28}$$

where $\mathcal{P}(T_{-i<-j})$ is a polynomial in the generators $T_{-i,-j}$ for $1 \leq j < i \leq n$. It follows that the Bethe vectors in the $\mathfrak{gl}_n$ sector are annihilated by the generators $T_{i,-j}$ if $0 \leq i, j \leq n$ and $i \neq -j$. Based on this, we introduce an equivalence relation in the $\mathfrak{gl}_n$ sector:

$$A \cong B, \quad \Longleftrightarrow \quad T_{i,-j}(A - B) = 0, \tag{6.29}$$

for some $0 \leq i, j \leq n$ if $i \neq -j$. The inversion relation (6.6) simplifies in the $\mathfrak{gl}_n$ sector

$$\sum_{k=1}^{n} T_{-i,-k}(u + \frac{\kappa_N}{2}) T_{j,k}(u - \frac{\kappa_N}{2}) \cong \delta_{i,j} \lambda_{-n}(u + \frac{\kappa_N}{2}) \lambda_n(u - \frac{\kappa_N}{2}), \tag{6.30}$$

when $i,j=1,\dots,n$. Here, we used that $\theta_{-j}\theta_{-k}=1$ for $i,j=1,\dots,n$. Using the correspondences (6.6) and (6.27) we get

$$\sum_{k=1}^{n} T^{\mathfrak{gl}_n}_{i,k}(u) T_{n+1-j,n+1-k}(u-\frac{\kappa_N}{2}) \cong \delta_{i,j}\lambda^{\mathfrak{gl}_n}_1(u)\lambda^{\mathfrak{gl}_n}_1(-u). \tag{6.31}$$

We can see that this matches the inversion relation (2.11) used for the $Y(n)$ algebras , i.e.,

$$\widehat{T}^{\mathfrak{gl}_n}_{j,k}(u) = T_{n+1-j,n+1-k}(u-\frac{\kappa_N}{2}). \tag{6.32}$$

The KT-relation in the $\mathfrak{gl}_n$ sector also simplifies accordingly

$$\sum_{k=1}^{n} \mathbf{K}_{n+1-i,k-n-1}(u+\frac{\kappa_N}{2})\langle\Psi|T^{\mathfrak{gl}_n}_{k,j}(u) \cong \sum_{k=1}^{n}\langle\Psi|\widehat{T}^{\mathfrak{gl}_n}_{i,k}(-u)\mathbf{K}_{n+1-k,j-n-1}(u+\frac{\kappa_N}{2}). \tag{6.33}$$

This is the KT-relation for the $\mathfrak{gl}_n$ spin chain, and the K-matrix in the $\mathfrak{gl}_n$ sector under the $\mathfrak{gl}_n$ convention is

$$\mathbf{K}^{\mathfrak{gl}_n}_{i,j}(u) = \mathbf{K}_{n+1-i,j-n-1}(u+\frac{\kappa_N}{2}), \tag{6.34}$$

where $i,j=1,\dots,n$.

Based on this, we can determine the relationship between the one-particle overlap functions $\tilde{\mathcal{F}}^{\mathfrak{g}_N,(s)}_k$ and $\tilde{\mathcal{F}}^{\mathfrak{gl}_n,(s)}_k$ for $s=1,\dots,n-1$. For this, we use the formula of the $\mathfrak{gl}_n$ G-operators (5.9). Based on this, in the $\mathfrak{g}_N$ convention, the G-operators are

$$\mathbf{G}^{(s)} = \begin{pmatrix} \mathbf{K}_{n,-n} & \dots & \mathbf{K}_{n,-n+s-2} & \mathbf{K}_{n,-n+s-1} \\ \vdots & \ddots & \vdots & \vdots \\ \mathbf{K}_{n-s+2,-n} & \dots & \mathbf{K}_{n-s+2,-n+s-2} & \mathbf{K}_{n-s+2,-n+s-1} \\ \mathbf{K}_{n-s+1,-n} & \dots & \mathbf{K}_{n-s+1,-n+s-2} & \boxed{\mathbf{K}_{n-s+1,-n+s-1}} \end{pmatrix}. \tag{6.35}$$

Using the correspondence (6.34), we can easily see that

$$\mathbf{G}^{\mathfrak{gl}_n,(s)}(u) = \mathbf{G}^{(s)}(u+\frac{\kappa_N}{2}), \tag{6.36}$$

for $s=1,\dots,n$. Based on this, we can define the $\mathfrak{g}_N$ F-operators for $s=1,\dots,n-1$ in the usual way

$$\mathbf{F}^{(s)}(u) := \left[\mathbf{G}^{(s)}(u)\right]^{-1}\mathbf{G}^{(s+1)}(u) = \mathbf{F}^{\mathfrak{gl}_n,(s)}(u-\frac{\kappa_N}{2}). \tag{6.37}$$

The one-particle overlap functions in the $\mathfrak{gl}_\mathfrak{n}$ sector are given by (4.45)

$$\tilde{\mathcal{F}}^{(s)}_\ell(u) = \mathcal{F}^{\mathfrak{gl}_n,(s)}_\ell(iu-s/2)\sqrt{\frac{u^2}{u^2+1/4}} = \mathcal{F}^{(s)}_\ell(iu+\frac{\kappa_N}{2}-s/2)\sqrt{\frac{u^2}{u^2+1/4}}, \tag{6.38}$$

for $s=1,\dots,n-1$.

6.3 Nested $\mathfrak{g}_N$ K-matrices

Using equation (5.8) we can define nested K-matrices in the $\mathfrak{gl}_\mathfrak{n}$ sector. Rewriting the quasi-determinant in the $\mathfrak{g}_N$ convention, we obtain

$$\mathbf{K}^{(s)}_{i,j} = \begin{pmatrix} \mathbf{K}_{n,-n} & \dots & \mathbf{K}_{n,-n+s-2} & \mathbf{K}_{n,j} \\ \vdots & \ddots & \vdots & \vdots \\ \mathbf{K}_{n-s+2,-n} & \dots & \mathbf{K}_{n-s+2,-n+s-2} & \mathbf{K}_{n-s+2,j} \\ \mathbf{K}_{i,-n} & \dots & \mathbf{K}_{i,-n+s-2} & \boxed{\mathbf{K}_{i,j}} \end{pmatrix}. \tag{6.39}$$

This defines a $\mathfrak{gl}_{n-s+1}$ K-matrix for $i,-j=1,\dots,n-s+1$, according to the correspondence (6.34). This algebra embedding can be extended to the full $\mathfrak{g}_N$ algebra, meaning that (6.39) defines a $\mathfrak{g}_{N-2s+2}$ K-matrix for $i,j=-n+s-1,\dots,n-s+1$. The proof can be carried out based on [55] or the proof of Theorem 6.

There are recursive definitions equivalent to quasi-determinants for nested K-matrices. One such option is the following:

$$\mathbf{K}^{(s+1)}_{a,b}(u) = \mathbf{K}^{(s)}_{a,b}(u) - \mathbf{K}^{(s)}_{a,-n+s-1}(u)\left[\mathbf{K}^{(s)}_{n+1-s,-n+s-1}(u)\right]^{-1}\mathbf{K}^{(s)}_{n+1-s,b}(u), \tag{6.40}$$

where $a, b = -n+s, \ldots, n-s$. The G-operators can be expressed in the following way.

$$\mathbf{G}^{(s)}(u) = \mathbf{K}^{(s)}_{n+1-s,-n-1+s}(u), \tag{6.41}$$

for $s = 1, \ldots, n-1$.

According to section 5.1.1, another recursive definition for odd-indexed K-matrices is:

$$\mathbf{K}^{(2s+1)}_{a,b}(u) = \mathbf{K}^{(2s-1)}_{a,b}(u) - \sum_{\alpha,\beta=n-2s+1}^{n-2s+2}\mathbf{K}^{(2s-1)}_{a,-\alpha}(u)\widehat{\mathbf{K}}^{(2s-1)}_{-\alpha,\beta}(u)\mathbf{K}^{(2s-1)}_{a,\beta}(u), \tag{6.42}$$

where $\widehat{\mathbf{K}}$ is a 2×2 inverse matrix defined as follows

$$\sum_{\alpha,\beta=n-2s+1}^{n-2s+2}\widehat{\mathbf{K}}^{(2s-1)}_{-\alpha,\beta}(u)\mathbf{K}^{(2s-1)}_{\beta,-\gamma}(u) = \delta_{\alpha,\gamma}.$$

These matrices $\mathbf{K}^{(2s-1)}$ exist for $s = 1, \ldots, \lfloor\frac{n}{2}\rfloor$. We can select $k = \lfloor\frac{n}{2}\rfloor$ number of $\mathfrak{gl}_2$ K-matrices

$$\mathbf{k}^{(s)}_{\alpha,\beta}(u) := \mathbf{K}^{(2s-1)}_{n-2s+3-\alpha,-n+2s-3+\beta}(u), \tag{6.43}$$

for $\alpha, \beta = 1, 2$. Using these, the G-operators can be defined as

$$\begin{aligned}\mathbf{G}^{(2s-1)}(u) &= \mathbf{k}^{(s)}_{1,1}(u),\\ \mathbf{G}^{(2s)}(u) &= \mathbf{k}^{(s)}_{2,2}(u) - \mathbf{k}^{(s)}_{2,1}(u)\left[\mathbf{k}^{(s)}_{1,1}(u)\right]^{-1}\mathbf{k}^{(s)}_{1,2}(u),\end{aligned} \tag{6.44}$$

for $s = 1, \ldots, k$.

We can examine the leading order of nested K-matrices in the asymptotic limit

$$\mathbf{K}^{(s)}_{i,j}(u) = \mathcal{U}^{(s)}_{i,j}\mathbf{1} + \mathcal{O}(u^{-1}). \tag{6.45}$$

From the recursive equations (6.40), it is easily derived that

$$\begin{aligned}\sum_{k=-n-1+s}^{n+1-s}\mathcal{U}^{(s)}_{i,k}\mathcal{U}^{(s)}_{k,l} &= \sum_{k=-n}^{n}\mathcal{U}_{i,k}\mathcal{U}_{k,l},\\ \sum_{k=-n-1+s}^{n+1-s}\mathcal{U}^{(s)}_{k,k} &= \sum_{k=-n}^{n}\mathcal{U}_{k,k}.\end{aligned} \tag{6.46}$$

We can also see that for the algebras $Y(\mathfrak{so}_{2n}, \mathfrak{so}_M \oplus \mathfrak{so}_{2n-M})$ and $Y(\mathfrak{sp}_{2n}, \mathfrak{gl}_n)$, the operator $\mathbf{K}^{-1}_{n,-n}$ exists, meaning that the first type of recursion (6.40) can be applied. This follows from the asymptotic limit of the K-matrices (6.17), (6.20). Since in these cases $\mathcal{U}_{n,-n} \neq 0$, the inverse of the operator $\mathbf{K}_{n,-n}(u)$ can be defined order by order in the expansion in u^{-1}. Based on this, we obtain the following recursions for these reflection algebras.

$$\begin{array}{ccccccc} Y(\mathfrak{so}_{2n}, \mathfrak{so}_M \oplus \mathfrak{so}_{2n-M}) & \to & Y(\mathfrak{so}_{2n-2}, \mathfrak{so}_{M-1} \oplus \mathfrak{so}_{2n-M-1}) & \to & \ldots & \to & Y(\mathfrak{so}_{2n-2M}, \mathfrak{so}_{2n-2M})\\ Y(\mathfrak{sp}_{2n}, \mathfrak{gl}_n) & \to & Y(\mathfrak{sp}_{2n-2}, \mathfrak{gl}_{n-1}) & \to & \ldots & \to & Y(\mathfrak{sp}_2, \mathfrak{gl}_1)\end{array}$$

In the case of $\mathfrak{so}_{2n}$, the recursion can be terminated when $M = n$ or $M = n-1$. In these cases, we have

$$\begin{array}{ccccccc} Y(\mathfrak{so}_{2n}, \mathfrak{so}_n \oplus \mathfrak{so}_n) & \to & Y(\mathfrak{so}_{2n-2}, \mathfrak{so}_{n-1} \oplus \mathfrak{so}_{n-1}) & \to & \ldots & \to & Y(\mathfrak{so}_4, \mathfrak{so}_2 \oplus \mathfrak{so}_2)\\ Y(\mathfrak{so}_{2n}, \mathfrak{so}_{n-1} \oplus \mathfrak{so}_{n+1}) & \to & Y(\mathfrak{so}_{2n-2}, \mathfrak{so}_{n-2} \oplus \mathfrak{so}_n) & \to & \ldots & \to & Y(\mathfrak{so}_4, \mathfrak{so}_3)\end{array}$$

We observe that for the algebras $Y(\mathfrak{so}_{2n}, \mathfrak{gl}_n)$ and $Y(\mathfrak{sp}_{2n}, \mathfrak{sp}_{2m} \oplus \mathfrak{sp}_{2n-2m})$ the inverse $\mathbf{K}^{-1}_{n,-n}$ does not exist, so the first recursion is not applicable. However, the second recursion (6.42) is applicable. Based on this, we obtain the following recursions for these reflection algebras.

$$\begin{array}{ccccccc} Y(\mathfrak{so}_{4k}, \mathfrak{gl}_{2k}) & \to & Y(\mathfrak{so}_{4k-4}, \mathfrak{gl}_{2k-2}) & \to & \ldots & \to & Y(\mathfrak{so}_4, \mathfrak{gl}_2)\\ Y(\mathfrak{so}_{4k+2}, \mathfrak{gl}_{2k+1}) & \to & Y(\mathfrak{so}_{4k-2}, \mathfrak{gl}_{2k-1}) & \to & \ldots & \to & Y(\mathfrak{so}_6, \mathfrak{gl}_3)\\ Y(\mathfrak{sp}_{2n}, \mathfrak{sp}_{2m} \oplus \mathfrak{sp}_{2n-2m}) & \to & Y(\mathfrak{sp}_{2n-4}, \mathfrak{sp}_{2m-2} \oplus \mathfrak{sp}_{2n-2m-2}) & \to & \ldots & \to & Y(\mathfrak{sp}_{2n-4m}, \mathfrak{sp}_{2n-4m})\end{array}$$

6.4 Small rank algebras

In this subsection, we determine the one-particle overlaps in low-rank cases based on the correspondence $\mathfrak{sp}_2 \cong \mathfrak{sl}_2$, $\mathfrak{so}_3 \cong \mathfrak{sl}_2$, $\mathfrak{so}_4 \cong \mathfrak{sl}_2 \oplus \mathfrak{sl}_2$.

6.4.1 The $\mathfrak{sp}_2$ case

The algebras $Y(\mathfrak{sp}_2) \cong Y(2)$ are equivalent, since the R-matrices satisfy the relation

$$\frac{2u+2}{2u+1} R^{\mathfrak{sp}_2}(2u) = R^{\mathfrak{gl}_2}(u), \tag{6.47}$$

meaning that the $\mathfrak{sp}_2$ K-matrices can be obtained from $\mathfrak{gl}_2$ K-matrices in the following way

$$\mathbf{K}^{\mathfrak{sp}_2}(u) = \mathbf{K}^{\mathfrak{gl}_2}(u/2), \tag{6.48}$$

Based on this, the relation between the F-operators is:

$$\mathbf{F}^{\mathfrak{sp}_2}(u) = \mathbf{F}^{\mathfrak{gl}_2}(u/2). \tag{6.49}$$

The Bethe Ansatz equations in the $\mathfrak{sp}_2$ case are

$$\tilde{\alpha}(u_j) = \prod_{k \neq j} \frac{u_j - u_k - 2i}{u_j - u_k + 2i}, \tag{6.50}$$

meaning that the normalization of the Bethe roots differs between the two conventions: $u_j^{\mathfrak{sp}_2} = 2u_j^{\mathfrak{gl}_2}$. The overlap function in the $\mathfrak{gl}_2$ convention is

$$\begin{aligned}
\tilde{\mathcal{F}}_\ell^{\mathfrak{gl}_2}(u_j^{\mathfrak{gl}_2}) = \mathcal{F}_\ell^{\mathfrak{gl}_2}(iu_j^{\mathfrak{gl}_2}) \sqrt{\frac{(u_j^{\mathfrak{gl}_2})^2}{(u_j^{\mathfrak{gl}_2})^2 + 1/4}} = \mathcal{F}_\ell^{\mathfrak{gl}_2}(iu_j^{\mathfrak{sp}_2}/2) \sqrt{\frac{(u_j^{\mathfrak{sp}_2})^2}{(u_j^{\mathfrak{sp}_2})^2 + 1}} \\
= \mathcal{F}_\ell^{\mathfrak{sp}_2}(iu_j^{\mathfrak{sp}_2}) \sqrt{\frac{(u_j^{\mathfrak{sp}_2})^2}{(u_j^{\mathfrak{sp}_2})^2 + 1}}.
\end{aligned} \tag{6.51}$$

We can see that in the $\mathfrak{sp}_2$ convention, the overlap function takes the form

$$\tilde{\mathcal{F}}_\ell^{\mathfrak{sp}_2}(u) = \mathcal{F}_\ell^{\mathfrak{sp}_2}(iu) \sqrt{\frac{u^2}{u^2 + 1/4}}. \tag{6.52}$$

The definitions of the $\mathfrak{sp}_2$ G-operators are

$$\begin{aligned}
\mathbf{G}^{(1)} &= \mathbf{K}_{1,-1}. \\
\mathbf{G}^{(2)} &= \begin{pmatrix} \mathbf{K}_{1,-1} & \mathbf{K}_{1,1} \\ \mathbf{K}_{-1,-1} & \boxed{\mathbf{K}_{-1,1}} \end{pmatrix}.
\end{aligned} \tag{6.53}$$

6.4.2 The $\mathfrak{so}_3$ case

The $\mathfrak{so}_3$ K-matrices can be obtained from $\mathfrak{gl}_2$ K-matrices via fusion. Let the $\mathfrak{gl}_2$ K-matrix be $\mathbf{k}_{i,j}$. The $\mathfrak{so}_3$ K-matrix can be obtained as

$$\begin{aligned}
R_{1,2}(1)\mathbf{k}_1(u - \tfrac{1}{2})R_{1,2}(-2u)\mathbf{k}_2(u + \tfrac{1}{2}) = \mathbf{k}_2(u + \tfrac{1}{2})R_{1,2}(-2u)\mathbf{k}_2(u - \tfrac{1}{2})R_{1,2}(1) = \\
= \sum_{a_i, b_i = 1}^{2} e_{a_1,b_1} \otimes e_{a_2,b_2} \otimes \mathbf{k}_{a_1,a_2}^{b_1,b_2}(u),
\end{aligned} \tag{6.54}$$

where $\mathbf{k}_{a_1,a_2}^{b_1,b_2}$ is symmetric in both lower and upper indices. The $\mathfrak{so}_3$ K-matrix $\mathbf{K}_{i,j}$ can be obtained using the index correspondence $(1,1) \equiv -1$, $(1,2) \to 0$, $(2,2) \to +1$, i.e.,

$$\sqrt{2}^{|i|+|j|}\mathbf{K}_{i,j}(u) = \mathbf{k}_{a_1,a_2}^{b_1,b_2}(2u), \tag{6.55}$$

where $(a_1, a_2) \equiv i$ and $(b_1, b_2) \equiv j$. The components needed for the F- and G-operators are

$$\begin{aligned}
\mathbf{K}_{1,-1}(u) &= \mathbf{k}^{+}_{2,1}\mathbf{k}^{-}_{2,1}, \quad \mathbf{K}_{1,0}(u) = \sqrt{2}\mathbf{k}^{-}_{2,1}\mathbf{k}^{+}_{2,2}, \\
\mathbf{K}_{0,-1}(u) &= \frac{1}{\sqrt{2}}\left(\frac{4u-1}{4u}\mathbf{k}^{-}_{2,1}\mathbf{k}^{+}_{1,1} + \mathbf{k}^{-}_{1,1}\mathbf{k}^{+}_{2,1} - \frac{1}{4u}\mathbf{k}^{-}_{2,2}\mathbf{k}^{+}_{2,1}\right), \\
\mathbf{K}_{0,0}(u) &= \frac{4u-1}{4u}\mathbf{k}^{-}_{2,1}\mathbf{k}^{+}_{1,2} + \mathbf{k}^{-}_{1,1}\mathbf{k}^{+}_{2,2} - \frac{1}{4u}\mathbf{k}^{-}_{2,2}\mathbf{k}^{+}_{2,2},
\end{aligned} \tag{6.56}$$

where $\mathbf{k}^{\pm}_{i,j} \equiv \mathbf{k}_{i,j}(2u \pm \frac{1}{2})$. Let us compute the following expression

$$\mathbf{K}_{0,-1}(u)\left[\mathbf{K}_{1,-1}(u)\right]^{-1}\mathbf{K}_{1,0}(u) = \frac{4u-1}{4u}\mathbf{k}^{-}_{2,1}\mathbf{k}^{+}_{1,1}\left(\mathbf{k}^{+}_{2,1}\right)^{-1}\mathbf{k}^{+}_{2,2} + \mathbf{k}^{-}_{1,1}\mathbf{k}^{+}_{2,2} - \frac{1}{4u}\mathbf{k}^{-}_{2,2}\mathbf{k}^{+}_{2,2}. \tag{6.57}$$

We combine this formula with $\mathbf{K}_{0,0}$:

$$\mathbf{K}_{0,0}(u) - \mathbf{K}_{0,-1}(u)\left[\mathbf{K}_{1,-1}(u)\right]^{-1}\mathbf{K}_{1,0}(u) = \frac{4u-1}{4u}\mathbf{k}^{-}_{2,1}\left(\mathbf{k}^{+}_{1,2} - \mathbf{k}^{+}_{1,1}\left(\mathbf{k}^{+}_{2,1}\right)^{-1}\mathbf{k}^{+}_{2,2}\right). \tag{6.58}$$

We can see that the $\mathfrak{gl}_2$ G-operator appears on the right-hand side. Based on this, we define the $\mathfrak{so}_3$ F-operator as follows

$$\mathbf{F}^{\mathfrak{so}_3}(u) := \left[\mathbf{K}_{1,-1}(u)\right]^{-1}\Big(\mathbf{K}_{0,0}(u) - \mathbf{K}_{0,-1}(u)\left[\mathbf{K}_{1,-1}(u)\right]^{-1}\mathbf{K}_{1,0}(u)\Big) = \frac{4u-1}{4u}\mathbf{F}^{\mathfrak{gl}_2}(2u+1/2).$$

The Bethe Ansatz equations in the $\mathfrak{so}_3$ case are

$$\tilde{\alpha}(u_j) = \prod_{k\neq j}\frac{u_j - u_k - i/2}{u_j - u_k + i/2},$$

meaning $u_j^{\mathfrak{so}_3} = u_j^{\mathfrak{gl}_2}/2$. The overlap function in the $\mathfrak{gl}_2$ convention is

$$\begin{aligned}
\tilde{\mathcal{F}}^{\mathfrak{gl}_2}_{\ell}(u_j^{\mathfrak{gl}_2}) &= \mathcal{F}^{\mathfrak{gl}_2}_{\ell}(iu_j^{\mathfrak{gl}_2})\sqrt{\frac{(u_j^{\mathfrak{gl}_2})^2}{(u_j^{\mathfrak{gl}_2})^2 + 1/4}} = \mathcal{F}^{\mathfrak{gl}_2}_{\ell}(2iu_j^{\mathfrak{so}_3})\sqrt{\frac{(u_j^{\mathfrak{so}_3})^2}{(u_j^{\mathfrak{so}_3})^2 + 1/16}} \\
&= \mathcal{F}^{\mathfrak{so}_3}_{\ell}(iu_j^{\mathfrak{so}_3} - 1/4)\frac{u_j^{\mathfrak{so}_3} + i/4}{u_j^{\mathfrak{so}_3} + i/2}\sqrt{\frac{(u_j^{\mathfrak{so}_3})^2}{(u_j^{\mathfrak{so}_3})^2 + 1/16}}.
\end{aligned} \tag{6.59}$$

We can see that in the $\mathfrak{so}_3$ convention, the overlap function takes the following form

$$\tilde{\mathcal{F}}^{\mathfrak{so}_3}_{\ell}(u) = \mathcal{F}^{\mathfrak{so}_3}_{\ell}(iu - 1/4)\frac{u + i/4}{u + i/2}\sqrt{\frac{u^2}{u^2 + 1/16}}. \tag{6.60}$$

6.4.3 The $\mathfrak{so}_4$ case

The algebras $Y(\mathfrak{so}_4) \cong Y(2) \oplus Y(2)$ are equivalent, meaning the $\mathfrak{so}_4$ R-matrix has a tensor product form:

$$R^{\mathfrak{so}_4}(u) = \frac{u}{u+1}R^{\mathfrak{gl}_2}(u) \otimes R^{\mathfrak{gl}_2}(u). \tag{6.61}$$

The $\mathfrak{so}_4$ reflection equation has factorizable solutions

$$\mathbf{K}(u) = \mathbf{k}^L(u) \otimes \mathbf{k}^R(u), \tag{6.62}$$

where $\mathbf{k}^{L/R}$ are solutions to the uncrossed $\mathfrak{gl}_2$ reflection equation. In this case, we have two non-interacting $\mathfrak{gl}_2$ spin chains, meaning that the overlap is the product of the overlaps of the two $\mathfrak{gl}_2$ subsystems

$$\frac{\langle \mathrm{MPS}|\bar{u}^L, \bar{u}^R\rangle}{\sqrt{\langle \bar{u}^L, \bar{u}^R|\bar{u}^L, \bar{u}^R\rangle}} = \frac{\langle \mathrm{MPS}_L|\bar{u}^L\rangle}{\sqrt{\langle \bar{u}^L|\bar{u}^L\rangle}}\frac{\langle \mathrm{MPS}_R|\bar{u}^R\rangle}{\sqrt{\langle \bar{u}^R|\bar{u}^R\rangle}}, \tag{6.63}$$

where $\langle \mathrm{MPS}_{L/R}|$ are the MPSs corresponding to the $\mathbf{k}^{L/R}(u)$ K-matrices. The K-matrix written explicitly is

$$\mathbf{K} = \begin{pmatrix} \mathbf{k}^L_{1,1}\mathbf{k}^R_{1,1} & \mathbf{k}^L_{1,1}\mathbf{k}^R_{1,2} & \mathbf{k}^L_{1,2}\mathbf{k}^R_{1,1} & \mathbf{k}^L_{1,2}\mathbf{k}^R_{1,2} \\ \mathbf{k}^L_{1,1}\mathbf{k}^R_{2,1} & \mathbf{k}^L_{1,1}\mathbf{k}^R_{2,2} & \mathbf{k}^L_{1,2}\mathbf{k}^R_{2,1} & \mathbf{k}^L_{1,2}\mathbf{k}^R_{2,2} \\ \mathbf{k}^L_{2,1}\mathbf{k}^R_{1,1} & \mathbf{k}^L_{2,1}\mathbf{k}^R_{1,2} & \mathbf{k}^L_{2,2}\mathbf{k}^R_{1,1} & \mathbf{k}^L_{2,2}\mathbf{k}^R_{1,2} \\ \mathbf{k}^L_{2,1}\mathbf{k}^R_{2,1} & \mathbf{k}^L_{2,1}\mathbf{k}^R_{2,2} & \mathbf{k}^L_{2,2}\mathbf{k}^R_{2,1} & \mathbf{k}^L_{2,2}\mathbf{k}^R_{2,2} \end{pmatrix}, \tag{6.64}$$

We define the $\mathfrak{so}_4$ G-operators as follows:

$$\begin{aligned} \mathbf{G}^{(R)}(u) &:= \mathbf{K}_{1,-1}(u) - \mathbf{K}_{1,-2}(u)\mathbf{K}^{-1}_{2,-2}(u)\mathbf{K}_{2,-1}(u) = \mathbf{k}^L_{2,1}(\mathbf{k}^R_{1,2} - \mathbf{k}^R_{1,1}\left[\mathbf{k}^R_{2,1}\right]^{-1}\mathbf{k}^R_{2,2}), \\ \mathbf{G}^{(L)}(u) &:= \mathbf{K}_{-1,1}(u) - \mathbf{K}_{-1,-2}(u)\mathbf{K}^{-1}_{2,-2}(u)\mathbf{K}_{2,1}(u) = \mathbf{k}^R_{2,1}(\mathbf{k}^L_{1,2} - \mathbf{k}^L_{1,1}\left[\mathbf{k}^L_{2,1}\right]^{-1}\mathbf{k}^L_{2,2}). \end{aligned} \tag{6.65}$$

It is evident that the $\mathfrak{gl}_2$ G-operators appear on the right-hand side. We define the $\mathfrak{so}_4$ F-operators as follows

$$\mathbf{F}^{L/R}(u) := \left[\mathbf{K}_{2,-2}(u)\right]^{-1}\mathbf{G}^{(L/R)}(u) = \mathbf{F}^{\mathfrak{gl}_2,L/R}(u). \tag{6.66}$$

We can see that the $\mathfrak{so}_4$ F-operators are equal to the $\mathfrak{gl}_2$ F-operators. In these calculations, we assumed that $\mathbf{K}_{2,-2}(u)$ is invertible, which implies that $\mathbf{k}^L_{2,1}$ and $\mathbf{k}^R_{2,1}$ are also invertible, which holds for $\mathcal{B}(2,1)$ K-matrices. Therefore, the above formulas are valid for the $Y(\mathfrak{so}_4, \mathfrak{so}_2 \oplus \mathfrak{so}_2) \cong \mathcal{B}(2,1) \oplus \mathcal{B}(2,1)$ K-matrices.

We can also say something about the case $Y(\mathfrak{so}_4, \mathfrak{gl}_2) \cong \mathcal{B}(2,1) \oplus \mathcal{B}(2,0)$. In this case, $\mathbf{K}_{2,-2}(u)$ is not invertible (see also (6.21)), but $\mathbf{K}_{2,-1}(u) = \mathbf{k}^L_{2,1}\mathbf{k}^R_{2,2}$ is! Using this, $\mathbf{F}^{\mathfrak{gl}_2,L}(u)$ can be expressed as

$$\mathbf{F}^L(u) := \left[\mathbf{K}_{2,-1}(u)\right]^{-1}\Big(\mathbf{K}_{-1,2}(u) - \mathbf{K}_{-1,-1}(u)\left[\mathbf{K}_{2,-1}(u)\right]^{-1}\mathbf{K}_{2,2}(u)\Big) = \mathbf{F}^{\mathfrak{gl}_2,L}(u). \tag{6.67}$$

The $\mathfrak{so}_4$ reflection algebra also has achiral solutions, meaning that the K-matrix does not factorize. The achiral solutions are representations of the $Y(\mathfrak{so}_4, \mathfrak{so}_3)$ reflection algebra. These K-matrices can be obtained from representations of the $Y(2)$ algebra as follows:

$$\mathbf{K}(u) = (\widehat{\mathbf{L}}(u) \otimes \mathbf{1})\check{R}(2u)(\mathbf{L}(u) \otimes \mathbf{1}), \tag{6.68}$$

where $\mathbf{L}(u)$ is a $Y(2)$ Lax-operator and $\widehat{\mathbf{L}}(u)$ is the usual inverse-transposed Lax-operator. In the above equation, we used the notation $\check{R}(u) = PR(u)$. In this case, the two non-interacting $\mathfrak{gl}_2$ systems are coupled through the boundary state. This results in an achiral pair structure, i.e., $\bar{u}^L = -\bar{u}^R$. In this case, the Gaudin determinants are equal:

$$\det G^+ = \det G^- = \det G^R = \det G^L, \quad \to \quad \frac{\det G^+}{\det G^-} = 1. \tag{6.69}$$

The overlap can be reduced to a matrix element between Bethe vectors of a $Y(2)$ transfer matrix, i.e., the overlap can be written as

$$\frac{\langle \mathrm{MPS}|\bar{u}^L, \bar{u}^R\rangle}{\sqrt{\langle \bar{u}^L, \bar{u}^R|\bar{u}^L, \bar{u}^R\rangle}} = \sum_{\ell=1}^{d_B}\beta_\ell \tilde{\mathcal{F}}^{\mathfrak{gl}_2}_\ell(\bar{u}^R), \tag{6.70}$$

where $\tilde{\mathcal{F}}^{\mathfrak{gl}_2}_\ell(u) = \mathcal{F}^{\mathfrak{gl}_2}_\ell(iu)$ are the eigenvalues of the F-operator

$$\mathbf{F}^{\mathfrak{gl}_2}(u) = \left(\mathbf{L}_{2,1}(u)\right)^{-1}\Big(\mathbf{L}_{1,2}(u) - \mathbf{L}_{1,1}(u)\left(\mathbf{L}_{2,1}(u)\right)^{-1}\mathbf{L}_{2,2}(u)\Big). \tag{6.71}$$

Now we express $\mathbf{F}^{\mathfrak{gl}_2}$ in terms of the matrix elements of the $\mathfrak{so}_4$ K-matrix (6.68) The components needed for the G- and F-operators are

$$\begin{aligned} \mathbf{K}_{2,-2}(u) &= \widehat{\mathbf{L}}_{2,1}(u)\mathbf{L}_{2,1}(u), \quad \mathbf{K}_{2,1}(u) = \widehat{\mathbf{L}}_{2,1}(u)\mathbf{L}_{2,2}(u), \\ \mathbf{K}_{1,-2}(u) &= \frac{2u-1}{2u}\widehat{\mathbf{L}}_{2,1}(u)\mathbf{L}_{1,1}(u) - \frac{1}{2u}\widehat{\mathbf{L}}_{2,2}(u)\mathbf{L}_{2,1}(u), \\ \mathbf{K}_{1,1}(u) &= \frac{2u-1}{2u}\widehat{\mathbf{L}}_{2,1}(u)\mathbf{L}_{1,2}(u) - \frac{1}{2u}\widehat{\mathbf{L}}_{2,2}(u)\mathbf{L}_{2,2}(u). \end{aligned} \tag{6.72}$$

The achiral $\mathfrak{so}_4$ G-operator is defined as follows:

$$\begin{aligned} \mathbf{G}^{(2)}(u) &:= \mathbf{K}_{1,1}(u) - \mathbf{K}_{1,-2}(u)\left[\mathbf{K}_{2,-2}(u)\right]^{-1}\mathbf{K}_{2,1}(u) \\ &= \frac{2u-1}{2u}\widehat{\mathbf{L}}_{2,1}(u)\Big(\mathbf{L}_{1,2}(u) - \mathbf{L}_{1,1}(u)\left(\mathbf{L}_{2,1}(u)\right)^{-1}\mathbf{L}_{2,2}(u)\Big). \end{aligned} \tag{6.73}$$

The achiral $\mathfrak{so}_4$ F-operator is defined as:

$$\mathbf{F}^{\mathfrak{so}_4}(u) := [\mathbf{K}_{2,-2}(u)]^{-1}\left(\mathbf{K}_{1,1}(u) - \mathbf{K}_{1,-2}(u)\,[\mathbf{K}_{2,-2}(u)]^{-1}\,\mathbf{K}_{2,1}(u)\right) = \frac{u-1/2}{u}\mathbf{F}^{\mathfrak{gl}_2}(u), \tag{6.74}$$

meaning that the one-particle overlap functions are

$$\tilde{\mathcal{F}}_\ell^{\mathfrak{so}_4}(u) = \mathcal{F}_\ell^{\mathfrak{so}_4}(iu)\frac{u}{u+i/2}. \tag{6.75}$$

6.5 F-operators without extra selections rules

In the previous sections, we determined the F-operators and functions for the $\mathfrak{gl}_n$ sector and for the remaining nodes separately. Based on these, we can now obtain the full overlap functions. The results are summarized below.

For the $\mathfrak{so}_{2n+1}$ models

$$\tilde{\mathcal{F}}_\ell^{(s)}(u) = \begin{cases} \mathcal{F}_\ell^{(s)}(iu - \frac{1}{4} + \frac{n-s}{2})\sqrt{\frac{u^2}{u^2+1/4}}, & \text{for } s = 1,\dots,n-1, \\ \mathcal{F}_\ell^{(n)}(iu - 1/4)\frac{u+i/4}{u+i/2}\sqrt{\frac{u^2}{u^2+1/16}}, & \text{for } s = n. \end{cases} \tag{6.76}$$

For the $\mathfrak{sp}_{2n}$ models

$$\tilde{\mathcal{F}}_\ell^{(s)}(u) = \begin{cases} \mathcal{F}_\ell^{(s)}(iu + \frac{n+1-s}{2})\sqrt{\frac{u^2}{u^2+1/4}}, & \text{for } s = 1,\dots,n-1, \\ \mathcal{F}_\ell^{(n)}(iu)\sqrt{\frac{u^2}{u^2+1}}, & \text{for } s = n. \end{cases} \tag{6.77}$$

For the chiral $\mathfrak{so}_{2n}$ models

$$\tilde{\mathcal{F}}_\ell^{(s)}(u) = \begin{cases} \mathcal{F}_\ell^{(s)}(iu + \frac{n-1-s}{2})\sqrt{\frac{u^2}{u^2+1/4}}, & \text{for } s = 1,\dots,n-2, \\ \mathcal{F}_\ell^{(s)}(iu)\sqrt{\frac{u^2}{u^2+1}}, & \text{for } s = n-1,n. \end{cases} \tag{6.78}$$

For the achiral $\mathfrak{so}_{2n}$ models

$$\tilde{\mathcal{F}}_\ell^{(s)}(u) = \begin{cases} \mathcal{F}_\ell^{(s)}(iu + \frac{n-1-s}{2})\sqrt{\frac{u^2}{u^2+1/4}}, & \text{for } s = 1,\dots,n-2, \\ \mathcal{F}_\ell^{(n-1)}(iu)\frac{u}{u+i/2}, & \text{for } s = n-1. \end{cases} \tag{6.79}$$

The F-operators are defined via the G-operators:

$$\mathbf{F}^{(s)}(u) = \left[\mathbf{G}^{(s)}(u)\right]^{-1}\mathbf{G}^{(s+1)}(u). \tag{6.80}$$

In the cases of $\mathfrak{so}_{2n+1}$ and $\mathfrak{sp}_{2n}$ this applies for $s = 1,\dots,n$ and in the case of $\mathfrak{so}_{2n}$ for $s = 1,\dots,n-1$. In the chiral $\mathfrak{so}_{2n}$ case, the remaining F-operator is:

$$\mathbf{F}^{(n)}(u) = \left[\mathbf{G}^{(n-1)}(u)\right]^{-1}\mathbf{G}^{(n+1)}(u). \tag{6.81}$$

The G-operators can be defined through nested K-matrices using equation (6.40). For $\mathfrak{so}_{2n+1}$ and $\mathfrak{sp}_{2n}$, the matrices $\mathbf{K}^{(s)}$ are defined for $s = 1,\dots,n$ and for $\mathfrak{so}_{2n}$, for $s = 1,\dots,n-1$. These define the G-operators for the corresponding values of s as

$$\mathbf{G}^{(s)}(u) = \mathbf{K}^{(s)}_{n+1-s,-n-1+s}(u). \tag{6.82}$$

The remaining G-operators are defined as follows.

For the $\mathfrak{so}_{2n+1}$ models

$$\mathbf{G}^{(n+1)}(u) := \mathbf{K}^{(n)}_{0,0}(u) - \mathbf{K}^{(n)}_{0,-1}(u)\left[\mathbf{K}^{(n)}_{1,-1}(u)\right]^{-1}\mathbf{K}^{(n)}_{1,0}(u). \tag{6.83}$$

For the $\mathfrak{sp}_{2n}$ models

$$\mathbf{G}^{(n+1)}(u) := \mathbf{K}^{(n)}_{-1,1}(u) - \mathbf{K}^{(n)}_{-1,-1}(u)\left[\mathbf{K}^{(n)}_{1,-1}(u)\right]^{-1}\mathbf{K}^{(n)}_{1,1}(u), \tag{6.84}$$

For the chiral $\mathfrak{so}_{2n}$ models

$$\begin{aligned}\mathbf{G}^{(n)}(u) &:= \mathbf{K}^{(n-1)}_{1,-1}(u) - \mathbf{K}^{(n-1)}_{1,-2}(u)\left[\mathbf{K}^{(n-1)}_{2,-2}(u)\right]^{-1}\mathbf{K}^{(n-1)}_{2,-1}(u),\\ \mathbf{G}^{(n+1)}(u) &:= \mathbf{K}^{(n-1)}_{-1,1}(u) - \mathbf{K}^{(n-1)}_{-1,-2}(u)\left[\mathbf{K}^{(n-1)}_{2,-2}(u)\right]^{-1}\mathbf{K}^{(n-1)}_{2,1}(u).\end{aligned} \tag{6.85}$$

For the achiral $\mathfrak{so}_{2n}$ models

$$\mathbf{G}^{(n)}(u) := \mathbf{K}^{(n-1)}_{1,1}(u) - \mathbf{K}^{(n-1)}_{1,-2}(u)\left[\mathbf{K}^{(n-1)}_{2,-2}(u)\right]^{-1}\mathbf{K}^{(n-1)}_{2,1}(u). \tag{6.86}$$

The G-operators can also be expressed using quasi-determinants. In both chiral and achiral cases, equation (6.35) defines the $\mathbf{G}^{(s)}$ operators for $s = 1, \ldots, n$ or $s = 1, \ldots, n-1$. The missing G-operators are as follows.

For the $\mathfrak{so}_{2n+1}$ models

$$\mathbf{G}^{(n+1)} = \begin{pmatrix} \mathbf{K}_{n,-n} & \ldots & \mathbf{K}_{n,-1} & \mathbf{K}_{n,0} \\ \vdots & \ddots & \vdots & \vdots \\ \mathbf{K}_{1,-n} & \ldots & \mathbf{K}_{1,-1} & \mathbf{K}_{1,0} \\ \mathbf{K}_{0,-n} & \ldots & \mathbf{K}_{0,-1} & \boxed{\mathbf{K}_{0,0}} \end{pmatrix}, \tag{6.87}$$

For the $\mathfrak{sp}_{2n}$ models

$$\mathbf{G}^{(n+1)} = \begin{pmatrix} \mathbf{K}_{n,-n} & \ldots & \mathbf{K}_{n,-1} & \mathbf{K}_{n,1} \\ \vdots & \ddots & \vdots & \vdots \\ \mathbf{K}_{1,-n} & \ldots & \mathbf{K}_{1,-1} & \mathbf{K}_{1,1} \\ \mathbf{K}_{-1,-n} & \ldots & \mathbf{K}_{-1,-1} & \boxed{\mathbf{K}_{-1,1}} \end{pmatrix}, \tag{6.88}$$

For the chiral $\mathfrak{so}_{2n}$ models

$$\mathbf{G}^{(n+1)} = \begin{pmatrix} \mathbf{K}_{n,-n} & \ldots & \mathbf{K}_{n,-2} & \mathbf{K}_{n,1} \\ \vdots & \ddots & \vdots & \vdots \\ \mathbf{K}_{2,-n} & \ldots & \mathbf{K}_{2,-2} & \mathbf{K}_{2,1} \\ \mathbf{K}_{-1,-n} & \ldots & \mathbf{K}_{-1,-2} & \boxed{\mathbf{K}_{-1,1}} \end{pmatrix}. \tag{6.89}$$

For the achiral $\mathfrak{so}_{2n}$ models

$$\mathbf{G}^{(n)} = \begin{pmatrix} \mathbf{K}_{n,-n} & \ldots & \mathbf{K}_{n,-2} & \mathbf{K}_{n,1} \\ \vdots & \ddots & \vdots & \vdots \\ \mathbf{K}_{2,-n} & \ldots & \mathbf{K}_{2,-2} & \mathbf{K}_{2,1} \\ \mathbf{K}_{1,-n} & \ldots & \mathbf{K}_{1,-2} & \boxed{\mathbf{K}_{1,1}} \end{pmatrix}. \tag{6.90}$$

These constructions assume the existence of inverses for certain operators. This is valid for the following reflection algebras

- $Y(\mathfrak{so}_{2n+1}, \mathfrak{so}_n \oplus \mathfrak{so}_{n+1})$
- $Y(\mathfrak{sp}_{2n}, \mathfrak{gl}_n)$
- $Y(\mathfrak{so}_{2n}, \mathfrak{so}_n \oplus \mathfrak{so}_n)$, which lead to chiral pair structures
- $Y(\mathfrak{so}_{2n}, \mathfrak{so}_{n-1} \oplus \mathfrak{so}_{n+1})$, which lead to achiral pair structures

For the remaining reflection algebras, the above recursive equations cannot be applied directly. However, the techniques described in section 5 still work, meaning that after suitable deformations, the F-operators can be defined. From here on, each case is treated separately. A common feature in these cases is the need for certain deformations of the nested K-matrices. These are continuous deformations, and in the final overlap functions, the zero limit of the deformation parameter must be taken. In this limit, the overlap vanishes for certain quantum numbers, leading to additional selection rules.

6.6 F-operators with extra selections rules

6.6.1 $Y(\mathfrak{so}_{2n}, \mathfrak{gl}_n)$

In this case, the K-matrix of the $\mathfrak{gl}_n$ sector is a representation of $Y^-(n)$ (see equations (6.34) and (6.18)), so we apply the trick described in section 5.1. The previous $\mathbf{K}^{(2s)}$ matrices do not exist, but the $\mathbf{K}^{(2s-1)}$ matrices do. For these matrices, we can use the recursion (6.42). From these, we select $k = \lfloor \frac{n}{2} \rfloor$ number of $Y^-(2)$ K-matrices (6.43). The G-operator formulas (6.44) cannot be applied directly. Therefore, we deform these $Y^-(2)$ K-matrices in the way described in section 5.1, i.e.,

$$\begin{aligned} \mathbf{k}^{(s)}_{i,j}(u) &= \sum_{k,l=1}^{2} \mathbf{L}^{(s)}_{k,i}(u - \kappa_{N-4s+4})\epsilon_{k,l}\mathbf{L}^{(s)}_{l,j}(-u), \\ \tilde{\mathbf{k}}^{(s)}_{i,j}(u) &= \mathbf{k}^{(s)}_{i,j}(u) + (u + 1/2 - \kappa_{N-4s+4}/2)\mathbf{L}^{(s)}_{1,i}(u - \kappa_{N-4s+4})\mathbf{L}^{(s)}_{1,j}(-u), \end{aligned} \tag{6.91}$$

and for these, the G-operators are well-defined

$$\begin{aligned} \mathbf{G}^{(2s-1)}(u) &= \tilde{\mathbf{k}}^{(s)}_{1,1}(u), \\ \mathbf{G}^{(2s)}(u) &= \tilde{\mathbf{k}}^{(s)}_{2,2}(u) - \tilde{\mathbf{k}}^{(s)}_{2,1}(u) \left[\tilde{\mathbf{k}}^{(s)}_{1,1}(u)\right]^{-1} \tilde{\mathbf{k}}^{(s)}_{1,2}(u), \end{aligned} \tag{6.92}$$

for $s = 1, \dots, k$. These define the F-operators $\mathbf{F}^{(s)}$ in the usual way for $s = 1, \dots, 2k-1$. From here on, it is practical to treat even and odd n separately.

$Y(\mathfrak{so}_{4k}, \mathfrak{gl}_{2k})$ In this case, the nesting proceeds as follows.

$$\begin{array}{ccccccc} Y(\mathfrak{so}_{4k}, \mathfrak{gl}_{2k}) & \rightarrow & Y(\mathfrak{so}_{4k-4}, \mathfrak{gl}_{2k-2}) & \rightarrow & \dots & \rightarrow & Y(\mathfrak{so}_{4}, \mathfrak{gl}_{2}) \\ \mathbf{K}^{(1)} \equiv \mathbf{K} & \rightarrow & \mathbf{K}^{(3)} & \rightarrow & \dots & \rightarrow & \mathbf{K}^{(2k-1)} \\ \downarrow & & \downarrow & & \dots & & \downarrow \\ \mathbf{k}^{(1)} & & \mathbf{k}^{(2)} & & \dots & & \mathbf{k}^{(k)} \end{array}$$

We see that in the final step, we obtain a $Y(\mathfrak{so}_4, \mathfrak{gl}_2)$ K-matrix, which has a chiral pair structure. In this case, we have $n = 2k$ F-operators, of which the first $2k-1$ were already given. We still need to define the missing $\mathbf{F}^{(2k)}$ operator. In the final step, the K-matrix is a representation of $Y(\mathfrak{so}_4, \mathfrak{gl}_2) \cong \mathcal{B}(2,1) \oplus \mathcal{B}(2,0)$. The F-operator corresponding to the $\mathcal{B}(2,0)$ subalgebra was already given, that was $\mathbf{F}^{(2k-1)}$. The F-operator corresponding to the $\mathcal{B}(2,1)$ subalgebra will be the missing $\mathbf{F}^{(2k)}$, and it can be determined using equation (6.67), i.e.,

$$\mathbf{F}^{(2k)}(u) = \left[\mathbf{K}^{(2k-1)}_{2,-1}(u)\right]^{-1} \left(\mathbf{K}^{(2k-1)}_{-1,2}(u) - \mathbf{K}^{(2k-1)}_{-1,-1}(u) \left[\mathbf{K}^{(2k-1)}_{2,-1}(u)\right]^{-1} \mathbf{K}^{(2k-1)}_{2,2}(u)\right). \tag{6.93}$$

Since the K-matrix of the $\mathfrak{gl}_n$ sector is a representation of $Y^-(n)$, there are extra selection rules (see (5.41)). The overlap is non-zero only if

$$\begin{aligned} r_{2s-1} &= \frac{\Lambda_{2s-1} - \Lambda_{2s} + r_{2s-2} + r_{2s}}{2}, \quad \text{for } s = 1, \dots, \frac{n}{2} - 1, \\ r_{n-1} &= \frac{\Lambda_{n-1} - \Lambda_n + r_{n-2}}{2}. \end{aligned} \tag{6.94}$$

$Y(\mathfrak{so}_{4k+2}, \mathfrak{gl}_{2k+1})$ In this case, the nesting proceeds as follows.

$$\begin{array}{ccccccc} Y(\mathfrak{so}_{4k+2}, \mathfrak{gl}_{2k+1}) & \rightarrow & Y(\mathfrak{so}_{4k-2}, \mathfrak{gl}_{2k-1}) & \rightarrow & \dots & \rightarrow & Y(\mathfrak{so}_{6}, \mathfrak{gl}_{3}) \\ \mathbf{K}^{(1)} \equiv \mathbf{K} & \rightarrow & \mathbf{K}^{(3)} & \rightarrow & \dots & \rightarrow & \mathbf{K}^{(2k-1)} \\ \downarrow & & \downarrow & & \dots & & \downarrow \\ \mathbf{k}^{(1)} & & \mathbf{k}^{(2)} & & \dots & & \mathbf{k}^{(k)} \end{array}$$

We see that in the final step, we obtain a $Y(\mathfrak{so}_6, \mathfrak{gl}_3)$ K-matrix. This algebra is equivalent to the $\mathcal{B}(4,1)$ reflection algebra, for which the pair structure is achiral. In the achiral case, we need $n = 2k+1$ G-operators, of which the first $2k$ were already given, and $\mathbf{G}^{(2k+1)}(u)$ is still missing. The definition (6.86) cannot be used, but (6.90) can. Additionally, we can use another equivalent definition:

$$\mathbf{G}^{(2k+1)}(u) = \mathbf{K}^{(2k-1)}_{1,1}(u) - \sum_{\alpha,\beta=2}^{3} \mathbf{K}^{(2k-1)}_{1,-\alpha}(u)\widehat{\mathbf{K}}^{(2k-1)}_{-\alpha,\beta}(u)\mathbf{K}^{(2k-1)}_{\beta,1}(u). \tag{6.95}$$

Since the K-matrix of the $\mathfrak{gl}_{n-1}$ sector is a representation of $Y^-(n-1)$, there are extra selection rules (see (5.41)). The overlap is non-zero only if

$$\begin{aligned} r_{2s-1} &= \frac{\Lambda_{2s-1} - \Lambda_{2s} + r_{2s-2} + r_{2s}}{2}, \quad \text{for } s = 1, \dots, \frac{n-3}{2}, \\ r_{n-2} &= \frac{\Lambda_{n-2} - \Lambda_{n-1} + r_{n-3} + r_{n-1} + r_n}{2}. \end{aligned} \tag{6.96}$$

6.6.2 $Y(\mathfrak{so}_{2n}, \mathfrak{o}_M \oplus \mathfrak{o}_{2n-M})$

In this case, the nesting proceeds as follows:

$$\begin{array}{ccccccc} Y(\mathfrak{so}_{2n}, \mathfrak{o}_M \oplus \mathfrak{o}_{2n-M}) & \to & Y(\mathfrak{so}_{2n-2}, \mathfrak{o}_{M-1} \oplus \mathfrak{o}_{2n-M-1}) & \to & \dots & \to & Y(\mathfrak{so}_{2n-2M}, \mathfrak{so}_{2n-2M}) \\ \mathbf{K}^{(1)} \equiv \mathbf{K} & \to & \mathbf{K}^{(2)} & \to & \dots & \to & \mathbf{K}^{(M+1)} \\ \downarrow & & \downarrow & & \dots & & \\ \mathbf{G}^{(1)} & & \mathbf{G}^{(2)} & & \dots & & \end{array}$$

In the $M+1$-th step, we obtain a $Y(\mathfrak{so}_{2n-2M}, \mathfrak{so}_{2n-2M})$ K-matrix. This means that the MPS is a singlet state with respect to the $\mathfrak{so}_{2n-2M}$ subalgebra, i.e., the overlap is non-zero only if

$$\begin{aligned} r_s &= r_M + \sum_{k=M+1}^{s} \Lambda_k, \quad \text{for } s = M+1, \dots, n-2, \\ r_{n-1} &= \frac{1}{2} r_M + \frac{1}{2} \sum_{k=M+1}^{n-1} \Lambda_k - \frac{1}{2}\Lambda_n, \\ r_n &= \frac{1}{2} r_M + \frac{1}{2} \sum_{k=M+1}^{n-1} \Lambda_k + \frac{1}{2}\Lambda_n. \end{aligned} \tag{6.97}$$

The corresponding $\mathbf{K}^{(M+1)}_{j,-j}$ components are not invertible, so the nesting cannot continue, and the operators $\mathbf{G}^{(M+1)}, \dots$ are not defined. However, the trick described for the $\mathcal{B}(N,M)$ reflection algebras can be applied here as well, i.e., we deform the factorized form of the K-matrix with a singular scalar K-matrix. This singular solution is

$$K(u) = 1 + u \sum_{j=1}^{\tilde{n}/2} (e_{\tilde{n}-2j+1, 2j-2-\tilde{n}} - e_{\tilde{n}-2j+2, 2j-1-\tilde{n}}), \tag{6.98}$$

where $\tilde{n} = n - M$. Assuming that the $Y(\mathfrak{so}_{2n-2M}, \mathfrak{so}_{2n-2M})$ matrix can be factorized into

$$\mathbf{K}^{(M+1)}_{a,b}(u) = \sum_{c=-\tilde{n}}^{\tilde{n}} \mathbf{L}_{-c,-a}(u - \kappa_{N-2M}) \mathbf{L}_{c,b}(-u), \tag{6.99}$$

the deformed K-matrix is

$$\begin{aligned} \tilde{\mathbf{K}}^{(1)}_{a,b}(u) = \mathbf{K}^{(M+1)}_{a,b}(u) + \\ + u \sum_{l=1}^{k} \left(\mathbf{L}_{-(\tilde{n}-2l+1),-a}(u - \kappa_{N-2M}) \mathbf{L}_{-\tilde{n}+2l-2,b}(-u) - \mathbf{L}_{-(\tilde{n}-2l+2),-a}(u - \kappa_{N-2M}) \mathbf{L}_{-\tilde{n}+2l-1,b}(-u) \right). \end{aligned} \tag{6.100}$$

It is clear that this deformed K-matrix forms a $Y^-(\tilde{n})$ subalgebra in the $\mathfrak{gl}_{\tilde{n}}$ sub-sector, i.e., the same nesting can be carried out as for the $Y(\mathfrak{so}_{2\tilde{n}}, \mathfrak{gl}_{\tilde{n}})$ algebra. We can define the nested K-matrices in the manner of (6.42). We can select $k = \lfloor \frac{\tilde{n}}{2} \rfloor$ number of $Y^-(2)$ K-matrices $\mathbf{k}^{(s)}$ for $s = 1, \dots, k$. These can be deformed in the manner of (6.91), and for these, the G-operators are well-defined

$$\begin{aligned} \mathbf{G}^{(M+2s-1)}(u) &= \tilde{\mathbf{k}}^{(s)}_{1,1}(u), \\ \mathbf{G}^{(M+2s)}(u) &= \tilde{\mathbf{k}}^{(s)}_{2,2}(u) - \tilde{\mathbf{k}}^{(s)}_{2,1}(u)\left[\tilde{\mathbf{k}}^{(s)}_{1,1}(u)\right]^{-1}\tilde{\mathbf{k}}^{(s)}_{1,2}(u), \end{aligned} \tag{6.101}$$

for $s = 1, \dots, k$.

If $n - M$ is **even**, i.e., $\tilde{n} = 2k$, then the pair structure is **chiral**. The G-operators $\mathbf{G}^{(s)}(u)$ previously defined for $s = 1, \dots, n$ immediately define the first $n-1$ F-operators. The missing F-operator is

$$\mathbf{F}^{(n)}(u) = \left[\tilde{\mathbf{K}}^{(2k-1)}_{2,-1}(u)\right]^{-1}\left(\tilde{\mathbf{K}}^{(2k-1)}_{-1,2}(u) - \tilde{\mathbf{K}}^{(2k-1)}_{-1,-1}(u)\left[\tilde{\mathbf{K}}^{(2k-1)}_{2,-1}(u)\right]^{-1}\tilde{\mathbf{K}}^{(2k-1)}_{2,2}(u)\right). \tag{6.102}$$

If $n - M$ is **odd**, i.e., $\tilde{n} = 2k+1$, then the pair structure is **achiral**. The G-operators $\mathbf{G}^{(s)}(u)$ previously defined for $s = 1, \dots, n-1$. The missing G-operator is

$$\mathbf{G}^{(n)}(u) = \tilde{\mathbf{K}}^{(2k-1)}_{1,1}(u) - \sum_{\alpha,\beta=2}^{3} \tilde{\mathbf{K}}^{(2k-1)}_{1,-\alpha}(u)\widehat{\mathbf{K}}^{(2k-1)}_{-\alpha,\beta}(u)\tilde{\mathbf{K}}^{(2k-1)}_{\beta,1}(u). \tag{6.103}$$

6.6.3 $Y(\mathfrak{so}_{2n+1}, \mathfrak{o}_M \oplus \mathfrak{o}_{2n+1-M})$

This case is very similar to the earlier $Y(\mathfrak{so}_{2n}, \mathfrak{o}_M \oplus \mathfrak{o}_{2n-M})$ case. Now, the nesting proceeds as follows.

$$\begin{array}{ccccccc} Y(\mathfrak{so}_{2n+1}, \mathfrak{o}_M \oplus \mathfrak{o}_{2n+1-M}) & \to & Y(\mathfrak{so}_{2n-1}, \mathfrak{o}_{M-1} \oplus \mathfrak{o}_{2n-M}) & \to & \dots & \to & Y(\mathfrak{so}_{2n+1-2M}, \mathfrak{so}_{2n+1-2M}) \\ \mathbf{K}^{(1)} \equiv \mathbf{K} & \to & \mathbf{K}^{(2)} & \to & \dots & \to & \mathbf{K}^{(M+1)} \\ \downarrow & & \downarrow & & \dots & & \\ \mathbf{G}^{(1)} & & \mathbf{G}^{(2)} & & \dots & & \end{array}$$

In the $M+1$-th step, we obtain a $Y(\mathfrak{so}_{2n+1-2M}, \mathfrak{so}_{2n+1-2M})$ K-matrix. This means that the MPS is a singlet state with respect to the $\mathfrak{so}_{2n+1-2M}$ subalgebra, i.e., the overlap is non-zero only if

$$r_s = r_M + \sum_{k=M+1}^{s} \Lambda_k, \tag{6.104}$$

for $s = M+1, \dots, n$.

To compute the F-operators, we again use the trick described in the previous section: we deform the factorized form of the K-matrix (6.99) with a singular scalar K-matrix (6.98) as in (6.100), where $\tilde{n} = n - M$. It is evident that the deformed K-matrix again forms a $Y^-(\tilde{n})$ subalgebra in the $\mathfrak{gl}_{\tilde{n}}$ sub-sector. Then we can define the nested K-matrices using (6.42). We select $k = \lfloor \frac{\tilde{n}}{2} \rfloor$ number of $Y^-(2)$ K-matrices $\mathbf{k}^{(s)}$ for $s = 1, \dots, k$. These can be deformed using (6.91), and the G-operators are well-defined for them (6.101).

If $n - M$ is **even**, i.e., $\tilde{n} = 2k$, then the G-operators $\mathbf{G}^{(s)}(u)$ have already been defined for $s = 1, \dots, n$. The missing G-operator is

$$\mathbf{G}^{(n+1)}(u) = \tilde{\mathbf{K}}^{(2k-1)}_{0,0}(u) - \sum_{\alpha,\beta=1}^{2} \tilde{\mathbf{K}}^{(2k-1)}_{0,-\alpha}(u)\widehat{\mathbf{K}}^{(2k-1)}_{-\alpha,\beta}(u)\tilde{\mathbf{K}}^{(2k-1)}_{\beta,0}(u). \tag{6.105}$$

If $n - M$ is **odd**, i.e., $\tilde{n} = 2k+1$, then the G-operators $\mathbf{G}^{(s)}(u)$ have already been defined for $s = 1, \dots, n-1$. Additionally, we need $\mathbf{G}^{(n)}$ and $\mathbf{G}^{(n+1)}$. The K-matrix $\tilde{\mathbf{K}}^{(2k+1)}$ is a representation of $Y(\mathfrak{so}_3, \mathfrak{so}_3)$. This must again be deformed using the following singular $\mathfrak{so}_3$ K-matrix:

$$K(u) = \begin{pmatrix} 1 & 0 & 0 \\ u & 1 & 0 \\ -\frac{1}{2}u(u+\frac{1}{4}) & -u & 1 \end{pmatrix}. \tag{6.106}$$

The deformed $\mathfrak{so}_3$ K-matrix $\check{\mathbf{K}}$ is obtained as

$$\begin{aligned} \tilde{\mathbf{K}}^{(k+1)}_{a,b}(u) &= \sum_{c=-1}^{1} \mathbf{L}_{-c,-a}(u-\kappa_3)\mathbf{L}_{c,b}(-u), \\ \check{\mathbf{K}}^{(k+1)}_{a,b}(u) &= \tilde{\mathbf{K}}^{(k+1)}_{a,b}(u) - \frac{1}{2}u(u+\frac{1}{4})\mathbf{L}_{-1,-a}(u-\kappa_3)\mathbf{L}_{-1,b}(-u) \\ &\quad + u\left(\mathbf{L}_{0,-a}(u-\kappa_3)\mathbf{L}_{-1,b}(-u) - \mathbf{L}_{-1,-a}(u-\kappa_3)\mathbf{L}_{0,b}(-u)\right), \end{aligned} \tag{6.107}$$

and the remaining G-operators $\mathbf{G}^{(n)}$, $\mathbf{G}^{(n+1)}$ are given by

$$\begin{aligned}\mathbf{G}^{(n)}(u) &= \check{\mathbf{K}}^{(k+1)}_{1,-1}(u),\\ \mathbf{G}^{(n+1)}(u) &= \check{\mathbf{K}}^{(k+1)}_{0,0}(u) - \check{\mathbf{K}}^{(k+1)}_{0,-1}(u)\left[\check{\mathbf{K}}^{(k+1)}_{1,-1}(u)\right]^{-1}\check{\mathbf{K}}^{(k+1)}_{1,0}(u).\end{aligned} \tag{6.108}$$

6.6.4 $Y(\mathfrak{sp}_{2n}, \mathfrak{sp}_{2m} \oplus \mathfrak{sp}_{2n-2m})$

This case is very similar to $Y(\mathfrak{so}_{2n}, \mathfrak{gl}_n)$ (see the asymptotic expansions). In this case, the nesting proceeds as follows.

$$\begin{array}{ccccccc} Y(\mathfrak{sp}_{2n}, \mathfrak{sp}_{2m} \oplus \mathfrak{sp}_{2n-2m}) & \to & Y(\mathfrak{sp}_{2n-4}, \mathfrak{sp}_{2m-2} \oplus \mathfrak{sp}_{2n-2m-2}) & \to & \dots & \to & Y(\mathfrak{sp}_{2n-4m}, \mathfrak{sp}_{2n-4m}) \\ \mathbf{K}^{(1)} \equiv \mathbf{K} & \to & \mathbf{K}^{(3)} & \to & \dots & \to & \mathbf{K}^{(2m+1)} \\ \downarrow & & \downarrow & & \dots & & \\ \mathbf{k}^{(1)} & & \mathbf{k}^{(2)} & & \dots & & \end{array}$$

In the $m+1$-th step, we obtain a $Y(\mathfrak{sp}_{2n-4m}, \mathfrak{sp}_{2n-4m})$ K-matrix. This means that the MPS is a singlet state with respect to the $\mathfrak{sp}_{2n-4m}$ subalgebra. The K-matrix of the $\mathfrak{gl}_{2m}$ sector is a representation of $Y^-(2m)$, so there are selection rules similar to (5.41). Based on these, the overlap is non-zero only if

$$\begin{aligned} r_{2s-1} &= \frac{\Lambda_{2s-1} - \Lambda_{2s} + r_{2s-2} + r_{2s}}{2}, \quad \text{for } s = 1, \dots, m, \\ r_s &= r_{2m} + \sum_{k=2m+1}^{s} \Lambda_k, \quad \text{for } s = 2m+1, \dots, n-1, \\ r_n &= \frac{1}{2} r_{2m} + \frac{1}{2} \sum_{k=2m+1}^{n} \Lambda_k. \end{aligned} \tag{6.109}$$

The nested K-matrices $\mathbf{K}^{(3)}, \dots, \mathbf{K}^{(2m+1)}$ can be obtained using the recursion (6.42). From these, we select m number of $Y^-(2)$ K-matrices (6.43). The G-operator formulas (6.44) cannot be applied directly. Therefore, we deform these $Y^-(2)$ K-matrices using (6.91), and the G-operators are well-defined for them using (6.92) for $s = 1, \dots, m$. These define the G-operators $\mathbf{G}^{(s)}$ for $s = 1, \dots, 2m$.

The operator $\mathbf{K}^{(2m+1)}$ can be deformed using the following scalar solution of the $\mathfrak{sp}_{2\tilde{n}}$ reflection equation:

$$K(u) = 1 + u \sum_{j=1}^{\tilde{n}} e_{j,-j}, \tag{6.110}$$

where $\tilde{n} = n - 2m$. The deformed K-matrix is

$$\begin{aligned} \mathbf{K}^{(2m+1)}_{a,b}(u) &= \sum_{c=-\tilde{n}}^{\tilde{n}} \theta_a \theta_c \mathbf{L}_{-c,-a}(u - \kappa_{2\tilde{n}}) \mathbf{L}_{c,b}(-u), \\ \tilde{\mathbf{K}}^{(1)}_{a,b}(u) &= \mathbf{K}^{(2m+1)}_{a,b}(u) + u \sum_{c=1}^{k} \theta_a \theta_c \mathbf{L}_{-c,-a}(u - \kappa_{2\tilde{n}}) \mathbf{L}_{-c,b}(-u). \end{aligned} \tag{6.111}$$

From here, the nested K-matrices $\tilde{\mathbf{K}}^{(2)}, \dots, \tilde{\mathbf{K}}^{(\tilde{n})}$ can be defined using the recursion (6.40), and the remaining G-operators are given by

$$\mathbf{G}^{(2m+s)}(u) = \tilde{\mathbf{K}}^{(s)}_{\tilde{n}-s+1,-\tilde{n}+s-1}(u), \tag{6.112}$$

for $s = 1, \dots, \tilde{n}$ and

$$\mathbf{G}^{(n+1)}(u) = \tilde{\mathbf{K}}^{(\tilde{n})}_{-1,1}(u) - \tilde{\mathbf{K}}^{(\tilde{n})}_{-1,-1}(u)\left[\tilde{\mathbf{K}}^{(\tilde{n})}_{1,-1}(u)\right]^{-1}\tilde{\mathbf{K}}^{(\tilde{n})}_{1,1}(u). \tag{6.113}$$

7 Conclusions

In this paper, we presented the universal formula (1.1) that provides the overlaps between integrable MPSs and Bethe states. The formula consists of a ratio of Gaudin determinants and a prefactor. The prefactor contains the one-particle overlap functions $\mathcal{F}_\ell^{(\nu)}(u)$, which are the eigenvalues of the commuting F-operators $\mathbf{F}^{(\nu)}(u)$. The formula is surprisingly broadly applicable, as the F-operators involved depend only on the quasi-determinants of the K-matrix associated with the MPS.

The main result of the paper is a precise proof of the universal formula for those MPSs of $\mathfrak{gl}_N$ symmetric spin chains where all quasi-determinants exist. This applies to a large family of possible integrable MPSs. However, we observed that for certain reflection algebras, these quasi-determinants are not well-defined because some operators are non-invertible. In such cases, we showed that through continuous deformations, these operators can be made invertible, allowing the quantities in the universal overlap formula to be determined. We also generalized the definitions of the F-operators to orthogonal and symplectic spin chains.

In the future, it would be worthwhile to extend the proofs to the orthogonal and symplectic cases as well. This requires knowledge of recursion, action, and coproduct formulas for the Bethe vectors. With these formulas in hand, the proofs in the current paper could likely be applied without difficulty. For $\mathfrak{so}_{2n+1}$ symmetric spin chains, these formulas are already available [56, 57]. Additionally, it may be worthwhile to extend the results to graded and trigonometric spin chains. The necessary formulas for Bethe vectors are already known in certain cases [52, 58].

It is worth noting that the formulas presented in the paper are not applicable in the case of twisted boundary conditions. In these cases, integrability does not manifest at the level of Bethe roots (pair structure), but rather in the full Q-system [42]. In such cases, the overlaps of the tensor product states cannot be expressed with Gaudin determinants but rather with a determinant involving Q-functions [59]. Currently, one systematic method is available for computing MPS overlaps, which is based on searching for generalized dressing formulas [60, 61]. The advantage of this method is that it can be applied even in the presence of twists; the disadvantage is that it must be carried out separately for each MPS, making it less general than the result of the present paper. A potential research direction could be the search for a similarly universal overlap formula under twisted boundary conditions.

Acknowledgments

The author would like to thank Vidas Regelskis for the helpful discussions about reflection algebras. This paper was supported by the NKFIH grant PD142929 and the János Bolyai Research Scholarship of the Hungarian Academy of Science.

A Boundary states for arbitrary representations

In this section, we focus on finite-dimensional representations of the Lie algebra $\mathfrak{gl}_N$, meaning that $\Lambda_k - \Lambda_{k+1} \in \mathbb{N}$. Without loss of generality, we assume $\Lambda_N = 0$, so $\Lambda_k \in \mathbb{N}$. Every finite-dimensional representation corresponds to a Young diagram $Y(\Lambda)$, where the k-th row contains Λ_k boxes. The total number of boxes in the diagram is $n = \sum_{k=1}^{N-1} \Lambda_k$. We introduce an n-fold tensor product space $(\mathbb{C}^N)^{\otimes n}$, indexed by $1, 2, \dots, n$. A box $x \in Y(\Lambda)$ in the Young diagram is identified by its row i and column j, i.e., $x = (i, j)$. Additionally, we can label the boxes as $x_j \in Y(\Lambda)$, where $j = 1, \dots, n$. The label of the box at position (a, b) is given by $\sum_{j=1}^{b-1} w_j + a$, where w_j is the height of the j-th column. For example, the Young diagram corresponding to $\Lambda = (3, 2, 2, 1, 0)$ can be labeled in the following way.

$$\begin{array}{|c|c|c|}\hline 1 & 5 & 8 \\ \hline 2 & 6 \\ \cline{1-2} 3 & 7 \\ \cline{1-2} 4 \\ \cline{1-1}\end{array}$$

Each space also has an associated inhomogeneity $s_j = k - l$, if $x_j = (k, l)$. We also introduce a shorthand notation for the R-matrices:

$$R_{j,k} \equiv R_{j,k}(u_j - u_k). \tag{A.1}$$

From these, a projection operator can be constructed

$$\Pi^\Lambda_{1,2,\dots,,n} = \frac{1}{A} \lim_{u_j \to s_j} (R_{n-1,n}) (R_{n-2,n} R_{n-2,n-1}) \dots (R_{1,n} \dots R_{1,3} R_{1,2}) , \to (\Pi^\Lambda_{1,2,\dots,,n})^2 = \Pi^\Lambda_{1,2,\dots,,n}, \tag{A.2}$$

where A is a normalization constant

$$A = \prod_{x \in Y(\Lambda)} \mathrm{hook}(x). \tag{A.3}$$

The subspace of the operator $\Pi^\Lambda_{1,2,\dots,,n}$ with eigenvalue $+1$ is isomorphic to the irreducible representation Λ.

A.1 Fusion of the R-matrix

The two types of R-matrices (2.2), (2.13) are representations of the Yangian algebra $Y(N)$, i.e.,

$$R_{1,2}(u_1-u_2)\bar{R}_{1,3}(u_1)\bar{R}_{2,3}(u_2)=\bar{R}_{2,3}(u_2)\bar{R}_{1,3}(u_1)R_{1,2}(u_1-u_2). \tag{A.4}$$

We can define fused R-matrices as follows

$$\begin{aligned}\bar{R}^{\Lambda,\square}_{I,0}(u)&=\left(\bar{R}_{n,0}(u+s_n)\dots\bar{R}_{2,0}(u+s_2)\bar{R}_{1,0}(u+s_1)\right)\Pi^{\Lambda}_{1,2,\dots,n}=\\&=\Pi^{\Lambda}_{1,2,\dots,n}\left(\bar{R}_{1,0}(u+s_1)\bar{R}_{2,0}(u+s_2)\dots\bar{R}_{n,0}(u+s_n)\right),\end{aligned} \tag{A.5}$$

and

$$\begin{aligned}\bar{R}^{\square,\Lambda}_{0,I}(u)&=\left(\bar{R}_{0,1}(u-s_1)\bar{R}_{0,2}(u-s_2)\dots\bar{R}_{0,n}(u-s_n)\right)\Pi^{\Lambda}_{1,2,\dots,n}=\\&=\Pi^{\Lambda}_{1,2,\dots,n}\left(\bar{R}_{0,n}(u-s_n)\dots\bar{R}_{0,2}(u-s_2)\bar{R}_{0,1}(u-s_1)\right),\end{aligned} \tag{A.6}$$

where I denotes the $+1$ eigenspace of the operator $\Pi^{\Lambda}_{1,2,\dots,,n}$. The matrices $R^{\Lambda,\square}$ and $R^{\square,\Lambda}$ satisfy the following relations

$$\begin{aligned}&R_{1,2}(u_1-u_2)R^{\square,\Lambda}_{1,3}(u_1-u_3)R^{\square,\Lambda}_{2,3}(u_2-u_3)=R^{\square,\Lambda}_{2,3}(u_2-u_3)R^{\square,\Lambda}_{1,3}(u_1-u_3)R_{1,2}(u_1-u_2),\\&R^{\Lambda,\square}_{1,2}(u_1-u_2)R^{\Lambda,\square}_{1,3}(u_1-u_3)R_{2,3}(u_2-u_3)=R_{2,3}(u_2-u_3)R^{\Lambda,\square}_{1,3}(u_1-u_3)R^{\Lambda,\square}_{1,2}(u_1-u_2),\\&R^{\square,\Lambda}_{1,2}(u_1-u_2)R_{1,3}(u_1-u_3)R^{\Lambda,\square}_{2,3}(u_2-u_3)=R^{\Lambda,\square}_{2,3}(u_2-u_3)R_{1,3}(u_1-u_3)R^{\square,\Lambda}_{1,2}(u_1-u_2),\end{aligned} \tag{A.7}$$

and unitarity

$$\begin{aligned}R^{\square,\Lambda}_{1,2}(u)R^{\Lambda,\square}_{2,1}(-u)&=\frac{u-(N-1)}{u}\prod_{k=1}^{N-1}\frac{u+(\Lambda_k-k+1)}{u+(\Lambda_k-k)},\\R^{\Lambda,\square}_{1,2}(u)R^{\square,\Lambda}_{2,1}(-u)&=\frac{u+(N-1)}{u}\prod_{k=1}^{N-1}\frac{u-(\Lambda_k-k+1)}{u-(\Lambda_k-k)}.\end{aligned} \tag{A.8}$$

The matrix $\widehat{R}^{\square,\Lambda}$ is also a representation of the Yangian algebra:

$$\begin{aligned}&R_{1,2}(u_1-u_2)\widehat{R}^{\square,\Lambda}_{1,3}(u_1-u_3)\widehat{R}^{\square,\Lambda}_{2,3}(u_2-u_3)=\widehat{R}^{\square,\Lambda}_{2,3}(u_2-u_3)\widehat{R}^{\square,\Lambda}_{1,3}(u_1-u_3)R_{1,2}(u_1-u_2),\\&\widehat{R}_{1,2}(u_1-u_2)\widehat{R}^{\square,\Lambda}_{1,3}(u_1-u_3)R^{\square,\Lambda}_{2,3}(u_2-u_3)=R^{\square,\Lambda}_{2,3}(u_2-u_3)\widehat{R}^{\square,\Lambda}_{1,3}(u_1-u_3)\widehat{R}_{1,2}(u_1-u_2).\end{aligned} \tag{A.9}$$

The partial transpose of $\widehat{R}^{\square,\Lambda}$ can be expressed using the matrix $R^{\Lambda,\square}_{I,0}$

$$\left(\widehat{R}^{\square,\Lambda}_{0,I}(u)\right)^{t_0}=\left(R_{0,n}(-u+s_n)\dots R_{0,2}(-u+s_2)R_{0,1}(-u+s_1)\right)\Pi^{\Lambda}_{1,2,\dots,n}=R^{\Lambda,\square}_{I,0}(-u), \tag{A.10}$$

i.e.,

$$R^{\square,\Lambda}_{1,2}(u)\left(\widehat{R}^{\square,\Lambda}_{1,2}(u)\right)^{t_1}=\frac{u-(N-1)}{u}\prod_{k=1}^{N-1}\frac{u+(\Lambda_k-k+1)}{u+(\Lambda_k-k)}. \tag{A.11}$$

The fused R-matrix $R^{\square,\Lambda}$ coincides with the previously defined Lax operator (2.7)

$$L^{\Lambda}(u)=R^{\square,\Lambda}(u). \tag{A.12}$$

This is a highest-weight representation of the Yangian, and the highest-weight vector is

$$|0_\Lambda\rangle=\Pi^{\Lambda}_{1,2,\dots,,n}|1,2,\dots,w_1,1,2,\dots,w_2,\dots,1,2,\dots,w_m\rangle. \tag{A.13}$$

The highest weights can be obtained from the action of $L^{\Lambda}_{i,i}(u)$:

$$L^{\Lambda}_{i,i}(u)|0_\Lambda\rangle=\lambda^{\Lambda}_i(u)|0_\Lambda\rangle\to\lambda^{\Lambda}_i(u)=\frac{u+\Lambda_i}{u}. \tag{A.14}$$

From the inversion relation, we get that

$$\widehat{L}^{\Lambda}(u) = \frac{u+\Lambda_1}{u}\frac{u-\Lambda_1}{u-(N-1)}\prod_{k=1}^{N-1}\frac{u+(\Lambda_k-k)}{u+(\Lambda_k-k+1)}\widehat{R}^{\square,\Lambda}(u), \tag{A.15}$$

where $\widehat{L}^{\Lambda}$ is defined as in (2.11). The vector $|0_\Lambda\rangle$ is the lowest-weight vector of the operator $\widehat{L}^{\Lambda}$, and the lowest weight is given by (2.15)

$$\begin{aligned}\hat{\lambda}_i^{\Lambda}(u) &= \frac{\lambda_1^{\Lambda}(u)\lambda_1^{\Lambda}(-u)}{\lambda_i^{\Lambda}(u-(i-1))}\prod_{k=1}^{i-1}\frac{\lambda_k^{\Lambda}(u-k)}{\lambda_k^{\Lambda}(u-(k-1))}\\ &= \frac{u+\Lambda_1}{u+(\Lambda_i-i+1)}\frac{u-\Lambda_1}{u}\prod_{k=1}^{i-1}\frac{u+(\Lambda_k-k)}{u+(\Lambda_k-k+1)}.\end{aligned} \tag{A.16}$$

For the Lax operator $\left(\widehat{L}_{0,I}^{\Lambda}\right)^{t_I}$, the vector $|0_\Lambda\rangle$ becomes the highest-weight vector, and the highest weight is given by (A.16).

A.2 Fusion of the K-matrix

Fusion can also be applied to the reflection equation

$$R_{1,2}(v-u)\mathbf{K}_1(u)\bar{R}_{1,2}(-u-v)\mathbf{K}_2(v) = \mathbf{K}_2(v)\bar{R}_{1,2}(-u-v)\mathbf{K}_1(u)R_{1,2}(v-u). \tag{A.17}$$

The fused reflection matrix is defined as

$$\begin{aligned}\mathbf{K}_I^{\Lambda}(u) &= \lim_{u_j\to u+s_j}\mathbf{K}_1\left(\bar{R}_{1,2}\mathbf{K}_2\right)\left(\bar{R}_{1,3}\bar{R}_{2,3}\mathbf{K}_3\right)\dots\left(\bar{R}_{1,n}\bar{R}_{2,n}\dots\bar{R}_{n-1,n}\mathbf{K}_n\right)\Pi_{1,2,\dots,n}^{\Lambda}\\ &= \Pi_{1,2,\dots,n}^{\Lambda}\left(\mathbf{K}_n\bar{R}_{n-1,n}\dots\bar{R}_{2,n}\bar{R}_{1,n}\right)\dots\left(\mathbf{K}_3\bar{R}_{2,3}\bar{R}_{1,3}\right)\left(\mathbf{K}_2\bar{R}_{1,2}\right)\mathbf{K}_1,\end{aligned} \tag{A.18}$$

where

$$\mathbf{K}_j \equiv \mathbf{K}_j(u_j), \quad \bar{R}_{j,k} \equiv \bar{R}_{j,k}(-u_j-u_k).$$

This K-matrix satisfies the reflection equation

$$L_{0,1}^{\Lambda}(u-\theta)\mathbf{K}_1^{\Lambda}(\theta)\bar{L}_{0,1}^{\Lambda}(-u-\theta)\mathbf{K}_0(u) = \mathbf{K}_0(u)\bar{L}_{0,1}^{\Lambda}(-u-\theta)\mathbf{K}_1^{\Lambda}(\theta)L_{0,1}^{\Lambda}(u-\theta). \tag{A.19}$$

This equation is equivalent to the relation

$$\mathbf{K}_0(u)\psi_{2,1}^{\Lambda}(\theta)\left[\bar{L}_{0,2}^{\Lambda}(-u-\theta)\right]^{t_2}L_{0,1}^{\Lambda}(u-\theta) = \psi_{2,1}^{\Lambda}(\theta)\left[L_{0,2}^{\Lambda}(u-\theta)\right]^{t_2}\bar{L}_{0,1}^{\Lambda}(-u-\theta)\mathbf{K}_0(u), \tag{A.20}$$

where we introduce a matrix-valued two-site state.

$$\psi_{1,2}^{\Lambda}(\theta) = \sum_{i,j=1}^{d_\Lambda}\langle i,j|\otimes\mathbf{K}_{i,j}^{\Lambda}(\theta). \tag{A.21}$$

A.3 Crossed KT-relation

In the crossed case, the fused K-matrix satisfies the following equation

$$\mathbf{K}_0(u)\psi_{2,1}^{\Lambda}(\theta)\left[\widehat{L}_{0,2}^{\Lambda}(-u-\theta)\right]^{t_2}L_{0,1}^{\Lambda}(u-\theta) = \psi_{2,1}^{\Lambda}(\theta)\left[L_{0,2}^{\Lambda}(u-\theta)\right]^{t_2}\widehat{L}_{0,1}^{\Lambda}(-u-\theta)\mathbf{K}_0(u). \tag{A.22}$$

This crossed KT-relation applies to a two-site spin chain, where the monodromy matrices are

$$T_0(u) = \left[\widehat{L}_{0,2}^{\Lambda}(-u-\theta)\right]^{t_2}L_{0,1}^{\Lambda}(u-\theta), \quad \widehat{T}_0(u) = \left[L_{0,2}^{\Lambda}(-u-\theta)\right]^{t_2}\widehat{L}_{0,1}^{\Lambda}(u-\theta). \tag{A.23}$$

It is easy to verify that these matrices satisfy the required relations, namely the RTT-relation (2.1) and the inversion relation (2.11). The pseudo-vacuum eigenvalues are

$$\lambda_k(u) = \lambda_k^{\Lambda}(u-\theta)\hat{\lambda}_k^{\Lambda}(-u-\theta), \quad \hat{\lambda}_k(u) = \hat{\lambda}_k^{\Lambda}(u-\theta)\lambda_k^{\Lambda}(-u-\theta). \tag{A.24}$$

In this two-site quantum space, the pseudo-vacuum is

$$|0\rangle = |0_\Lambda\rangle \otimes |0_\Lambda\rangle. \tag{A.25}$$

If Λ is a fundamental representation, i.e., $\Lambda = \mu^{(k)}$ where

$$\mu_j^{(k)} = \begin{cases} 1, & j \leq k, \\ 0, & j > k, \end{cases} \tag{A.26}$$

then the highest weight vector is

$$|0_{\mu^{(k)}}\rangle = \sum_{i_1,\dots,i_k} \epsilon_{i_1,\dots,i_k} |i_1,\dots,i_k\rangle, \tag{A.27}$$

where ϵ a Levi-Civita tensor. In these cases, the fused transfer matrices coincide with the Sklyanin minors

$$\mathbf{K}^{\mu^{(k)}} \to \mathbf{k}_{i_1,i_2,\dots,i_k}^{j_1,j_2,\dots,j_k}, \tag{A.28}$$

which are defined as

$$\Pi_{1,2,\dots,,k}^{\mu^{(k)}} \left(\mathbf{K}_k \widehat{R}_{k-1,k} \dots \widehat{R}_{2,k} \widehat{R}_{1,k}\right) \dots \left(\mathbf{K}_3 \widehat{R}_{2,3} \widehat{R}_{1,3}\right) \left(\mathbf{K}_2 \widehat{R}_{1,2}\right) \mathbf{K}_1 = \sum_{a_i,b_i} e_{a_1,b_1} \otimes \dots \otimes e_{a_k,b_k} \otimes \mathbf{k}_{a_1,\dots,a_k}^{b_1,\dots,b_k}, \tag{A.29}$$

where $\mathbf{k}$ is antisymmetric under the exchanges $i_k \leftrightarrow i_l$ and $j_k \leftrightarrow j_l$. The pseudo-vacuum overlap is

$$\psi_{2,1}^{\mu^{(k)}}(\theta)|0\rangle = \mathbf{k}_{1,2,\dots,k}^{1,2,\dots,k}(\theta), \tag{A.30}$$

which is the Sklyanin determinant in the subalgebra generated by $\{\mathbf{K}_{i,j}\}_{i,j=1}^k$. For a general representation $\Lambda = \sum_{k=1}^{N-1} d_k \mu^{(k)}$, the pseudo-vacuum overlap is

$$\mathbf{B}^\Lambda(\theta) = \psi_{2,1}^\Lambda(\theta)|0\rangle = B(\theta) \prod_{k=1}^{N-1} \prod_{l=1}^{d_k} \mathbf{k}_{1,2,\dots,k}^{1,2,\dots,k}(\theta - (D_k + l - 1)), \tag{A.31}$$

where $D_k = \sum_{l=k+1}^{N-1} d_k$, and $B(\theta)$ is a scalar depending on θ. Since the normalization of the MPS is arbitrary, the specific value of $B(\theta)$ is irrelevant.

Using the coproduct property of boundary states (Lemma 1), we can construct more general boundary states

$$\langle\Psi| = \psi_{2J,2J-1}^{\Lambda^{(J)}}(\theta_J) \dots \psi_{4,3}^{\Lambda^{(2)}}(\theta_2) \psi_{2,1}^{\Lambda^{(1)}}(\theta_1), \tag{A.32}$$

where the $\Lambda^{(j)}$ are arbitrary representations and the θ_j are arbitrary inhomogeneities. The monodromy matrices are

$$\begin{aligned} T_0(u) &= \left[\widehat{L}_{0,2J}^{\Lambda^{(J)}}(-u-\theta_J)\right]^{t_{2J}} L_{0,2J-1}^{\Lambda^{(J)}}(u-\theta_J) \dots \left[\widehat{L}_{0,2}^{\Lambda^{(1)}}(-u-\theta_1)\right]^{t_2} L_{0,1}^{\Lambda^{(1)}}(u-\theta_1), \\ \widehat{T}_0(u) &= \left[L_{0,2J}^{\Lambda^{(J)}}(-u-\theta_J)\right]^{t_{2J}} \widehat{L}_{0,2J-1}^{\Lambda^{(J)}}(u-\theta_J) \dots \left[L_{0,2}^{\Lambda^{(1)}}(-u-\theta_1)\right]^{t_2} \widehat{L}_{0,1}^{\Lambda^{(1)}}(u-\theta_1). \end{aligned} \tag{A.33}$$

The pseudo-vacuum eigenvalues are

$$\lambda_k(u) = \prod_{j=1}^J \lambda_k^{\Lambda^{(j)}}(u-\theta_j) \hat{\lambda}_k^{\Lambda^{(j)}}(-u-\theta_j), \quad \hat{\lambda}_k(u) = \prod_{j=1}^J \hat{\lambda}_k^{\Lambda^{(j)}}(u-\theta_j) \lambda_k^{\Lambda^{(j)}}(-u-\theta_j). \tag{A.34}$$

We observe a symmetry property

$$\lambda_k(u) = \hat{\lambda}_k(-u). \tag{A.35}$$

The pseudo-vacuum overlap is

$$\mathbf{B}(\theta) = \langle\Psi|0\rangle = \prod_{j=1}^J \mathbf{B}^{\Lambda^{(j)}}(\theta_j). \tag{A.36}$$

A.4 Uncrossed KT-relation

In the uncrossed case, the fused K-matrix satisfies the following equation

$$\mathbf{K}_0(u)\psi^{\Lambda}_{2,1}(\theta)\left[L^{\Lambda}_{0,2}(-u-\theta)\right]^{t_2}L^{\Lambda}_{0,1}(u-\theta). = \psi^{\Lambda}_{2,1}(\theta)\left[L^{\Lambda}_{0,2}(u-\theta)\right]^{t_2}L^{\Lambda}_{0,1}(-u-\theta)\mathbf{K}_0(u). \tag{A.37}$$

We evaluate the transposed Lax operator.

$$\begin{aligned}\left[L^{\Lambda}_{1,2}(-u)\right]^{t_2} &= e_{i,j}\otimes\left(\delta_{i,j}\mathbf{1}-\frac{1}{u}(E^{\Lambda}_{j,i})^t\right) = e_{i,j}\otimes\left(\delta_{i,j}\mathbf{1}+\frac{1}{u}E^{\Lambda^{cg}}_{j,i}\right)\\ &= L^{\Lambda^{cg}}_{1,2}(u).\end{aligned} \tag{A.38}$$

Here we introduced the contra-gradient representation

$$E^{\Lambda^{cg}}_{i,j} := -(E^{\Lambda}_{i,j})^t. \tag{A.39}$$

Let the highest weight vector of the contra-gradient representation be $|\bar{0}\rangle^{\Lambda}$, for which

$$\begin{aligned}E^{\Lambda^{cg}}_{i,j}|\bar{0}_{\Lambda}\rangle &= 0, \quad \text{for } i<j,\\ E^{\Lambda^{cg}}_{i,i}|\bar{0}_{\Lambda}\rangle &= -\Lambda_{N+1-i}|\bar{0}_{\Lambda}\rangle.\end{aligned} \tag{A.40}$$

We see that (A.37) is an uncrossed KT-relation for a two-site spin chain, where the monodromy matrix is

$$T_0(u) = L^{\Lambda^{cg}}_{0,2}(u+\theta)L^{\Lambda}_{0,1}(u-\theta). \tag{A.41}$$

In this two-site quantum space, the pseudo-vacuum is

$$|0\rangle = |0_{\Lambda}\rangle\otimes|\bar{0}_{\Lambda}\rangle. \tag{A.42}$$

The pseudo-vacuum eigenvalue is

$$\lambda_k(u) = \frac{u-\theta+\Lambda_k}{u-\theta}\,\frac{u+\theta-\Lambda_{N+1-k}}{u+\theta}. \tag{A.43}$$

For fundamental representations, we can introduce the same notation for the components of the fused K-matrix as before. The highest weight vectors are

$$\begin{aligned}|0_{\mu^{(k)}}\rangle &= \sum_{i_1,\dots,i_k=1}^{k}\epsilon_{i_1,\dots,i_k}|i_1,\dots,i_k\rangle,\\ |\bar{0}_{\mu^{(k)}}\rangle &= \sum_{i_1,\dots,i_k=1}^{k}\epsilon_{i_1,\dots,i_k}|\bar{i}_1,\dots,\bar{i}_k\rangle,\end{aligned} \tag{A.44}$$

where $\bar{i}=N+1-i$. Since the construction is symmetric under the exchange $\Lambda\leftrightarrow\Lambda^{cg}$ we can assume without loss of generality that $k\leq N/2$. The pseudo-vacuum overlap is

$$\psi^{\mu^{(k)}}_{2,1}(\theta)|0\rangle = \mathbf{k}^{1,2,\dots,k}_{\bar{1},\bar{2},\dots,\bar{k}}(\theta), \tag{A.45}$$

where the quantum minor is defined as follows

$$\Pi^{\mu^{(k)}}_{1,2,\dots,k}\left(\mathbf{K}_k R_{k-1,k}\dots R_{2,k}R_{1,k}\right)\dots\left(\mathbf{K}_3R_{2,3}R_{1,3}\right)\left(\mathbf{K}_2R_{1,2}\right)\mathbf{K}_1 = \sum_{a_i,b_i} e_{a_1,b_1}\otimes\dots\otimes e_{a_k,b_k}\otimes\mathbf{k}^{b_1,\dots,b_k}_{a_1,\dots,a_k}. \tag{A.46}$$

If $k\leq N/2$, then $\left\{\mathbf{K}_{\bar{i},j}\right\}_{i,j=1}^{k}$ forms a $Y(k)$ subalgebra, so $\mathbf{k}^{1,2,\dots,k}_{\bar{1},\bar{2},\dots,\bar{k}}(\theta)$ is a quantum determinant in the $Y(k)$ subalgebra. For a general representation $\Lambda=\sum_{k=1}^{N/2}d_k\mu^{(k)}$, the pseudo-vacuum overlap is

$$\mathbf{B}^{\Lambda}(\theta) = \psi^{\Lambda}_{2,1}(\theta)|0\rangle = B(\theta)\prod_{k=1}^{N/2}\prod_{l=1}^{d_k}\mathbf{k}^{1,2,\dots,k}_{\bar{1},\bar{2},\dots,\bar{k}}(\theta-(D_k+l-1)), \tag{A.47}$$

where $B(\theta)$ is a scalar depending on θ.

Using the coproduct property of boundary states (3.19), we can construct more general boundary states

$$\langle\Psi| = \psi^{\Lambda^{(J)}}_{2J,2J-1}(\theta_J)\dots\psi^{\Lambda^{(2)}}_{4,3}(\theta_2)\psi^{\Lambda^{(1)}}_{2,1}(\theta_1), \tag{A.48}$$

where the $\Lambda^{(j)}$ are arbitrary representations and the θ_j are arbitrary inhomogeneities. The monodromy matrix is

$$T_0(u) = L^{\Lambda^{(J)},cg}_{0,2J}(u+\theta_J)L^{\Lambda^{(J)}}_{0,2J-1}(u-\theta_J)\dots L^{\Lambda^{(1)},cg}_{0,2}(u+\theta_1)L^{\Lambda^{(1)}}_{0,1}(u-\theta_1). \tag{A.49}$$

The pseudo-vacuum eigenvalues are as follows

$$\lambda_k(u) = \prod_{j=1}^{J}\frac{u-\theta_j+\Lambda^{(j)}_k}{u-\theta_j}\frac{u+\theta_j-\Lambda^{(j)}_{N+1-k}}{u+\theta_j}. \tag{A.50}$$

We observe a symmetry property

$$\lambda_k(u) = \lambda_k(-u). \tag{A.51}$$

The pseudo-vacuum overlap is

$$\mathbf{B}(\theta) = \langle\Psi|0\rangle = \prod_{j=1}^{J}\mathbf{B}^{\Lambda^{(j)}}(\theta_j). \tag{A.52}$$

B Recursions for the off-shell Bethe states

In this section, we summarize the formulas for the off-shell Bethe vectors that are necessary for our purposes. These formulas can be found in the papers [52, 48]. The Bethe vectors satisfy the following recursive equations

$$\mathbb{B}(\{z,\bar t^1\},\{\bar t^k\}_{k=2}^{N-1}) = \sum_{j=2}^{N}\frac{T_{1,j}(z)}{\lambda_2(z)}\sum_{\mathrm{part}(\bar t)}\mathbb{B}(\bar t^1,\{\bar t^k_{\rm II}\}_{k=2}^{j-1},\{\bar t^k\}_{k=j}^{N-1})\frac{\prod_{\nu=2}^{j-1}\alpha_\nu(\bar t^\nu_{\rm I})g(\bar t^\nu_{\rm I},\bar t^{\nu-1}_{\rm I})f(\bar t^\nu_{\rm II},\bar t^\nu_{\rm I})}{\prod_{\nu=1}^{j-1}f(\bar t^{\nu+1},\bar t^\nu_{\rm I})}, \tag{B.1}$$

where the sum goes over the partitions $\bar t^s\vdash\{\bar t^s_{\rm I},\bar t^s_{\rm II}\}$ for $s=2,\dots,j-1$ such that $\#\bar t^s_{\rm I}=1$. We have another recursion

$$\mathbb{B}(\{\bar t^k\}_{k=1}^{N-2},\{z,\bar t^{N-1}\}) = \sum_{j=1}^{N-1}\frac{T_{j,N}(z)}{\lambda_N(z)}\sum_{\mathrm{part}(\bar t)}\mathbb{B}(\{\bar t^k\}_{k=1}^{j-1},\{\bar t^k_{\rm II}\}_{k=j}^{N-2},\bar t^{N-1})\frac{\prod_{\nu=j}^{N-2}g(\bar t^{\nu+1}_{\rm I},\bar t^\nu_{\rm I})f(\bar t^\nu_{\rm I},\bar t^\nu_{\rm II})}{\prod_{\nu=j}^{N-1}f(\bar t^\nu_{\rm I},\bar t^{\nu-1})}, \tag{B.2}$$

where the sum goes over the partitions $\bar t^s\vdash\{\bar t^s_{\rm I},\bar t^s_{\rm II}\}$ for $s=j,\dots,N-2$ such that $\#\bar t^s_{\rm I}=1$.

We also use the following action formula

$$\begin{aligned}T_{i,j}(z)\mathbb{B}(\bar t) = \lambda_N(z)\sum_{\mathrm{part}(\bar w)}\mathbb{B}(\bar w_{\rm II})\frac{\prod_{s=j}^{i-1}f(\bar w^s_{\rm I},\bar w^s_{\rm III})}{\prod_{s=j}^{i-2}f(\bar w^{s+1}_{\rm I},\bar w^s_{\rm III})}\times&\\ \prod_{s=1}^{i-1}\frac{f(\bar w^s_{\rm I},\bar w^s_{\rm II})}{h(\bar w^s_{\rm I},\bar w^{s-1}_{\rm I})f(\bar w^s_{\rm I},\bar w^{s-1}_{\rm II})}\prod_{s=j}^{N-1}\frac{\alpha_s(\bar w^s_{\rm III})f(\bar w^s_{\rm II},\bar w^s_{\rm III})}{h(\bar w^{s+1}_{\rm III},\bar w^s_{\rm III})f(\bar w^{s+1}_{\rm II},\bar w^s_{\rm III})},&\end{aligned} \tag{B.3}$$

where $\bar w^\nu=\{z,\bar t^\nu\}$. The sum goes over all the partitions of $\bar w^\nu\vdash\{\bar w^\nu_{\rm I},\bar w^\nu_{\rm II},\bar w^\nu_{\rm III}\}$ where $\#\bar w^\nu_{\rm I}=\Theta(i-1-\nu)$ and $\#\bar w^\nu_{\rm III}=\Theta(\nu-j)$. We also set $\bar w^0_{\rm I}=\bar w^N_{\rm III}=\{z\}$ and $\bar w^0_{\rm II}=\bar w^0_{\rm III}=\bar w^N_{\rm I}=\bar w^N_{\rm II}=\emptyset$. We also used the unit step function

$$\Theta(k) = \begin{cases}1, & k\geq 0,\\ 0, & k<0.\end{cases}$$

We also need the action formula for the crossed monodromy matrix [47, 43]

$$\begin{aligned}\widehat T_{i,j}(z)\mathbb{B}(\bar t) = (-1)^{i-j}\hat\lambda_1(z)\sum_{\mathrm{part}(\bar w)}\mathbb{B}(\bar w_{\rm II})\frac{\prod_{s=2}^{N-1}f(\bar t^{s-1}-1,\bar t^s)}{\prod_{s=2}^{N-1}f(\bar w^{s-1}_{\rm II}-1,\bar w^s_{\rm II})}\frac{\prod_{s=i}^{j-1}f(\bar w^s_{\rm I},\bar w^s_{\rm III})}{\prod_{s=i+1}^{j-1}f(\bar w^{s-1}_{\rm I}-1,\bar w^s_{\rm III})}\times&\\ \prod_{s=i}^{N-1}\frac{f(\bar w^s_{\rm I},\bar w^s_{\rm II})}{h(\bar w^s_{\rm I},\bar w^{s+1}_{\rm I}+1)f(\bar w^s_{\rm I},\bar w^{s+1}_{\rm II}+1)}\prod_{s=1}^{j-1}\frac{\alpha_s(\bar w^s_{\rm III})f(\bar w^s_{\rm II},\bar w^s_{\rm III})}{h(\bar w^{s-1}_{\rm III}-1,\bar w^s_{\rm III})f(\bar w^{s-1}_{\rm II}-1,\bar w^s_{\rm III})},&\end{aligned} \tag{B.4}$$

where $\bar{w}^\nu = \{z-\nu, \bar{t}^\nu\}$. The sum goes over all the partitions of $\bar{w}^\nu \vdash \{\bar{w}^\nu_{\text{I}}, \bar{w}^\nu_{\text{II}}, \bar{w}^\nu_{\text{III}}\}$ where $\#\bar{w}^\nu_{\text{I}} = \Theta(\nu - i)$, $\#\bar{w}^\nu_{\text{III}} = \Theta(j-1-\nu)$. We also set $\bar{w}^0_{\text{III}} = \{z\}$, $\bar{w}^N_{\text{I}} = \{z-N\}$ and $\bar{w}^0_{\text{I}} = \bar{w}^0_{\text{II}} = \bar{w}^N_{\text{II}} = \bar{w}^N_{\text{III}} = \emptyset$.

Finally, we also need the co-product formula of the Bethe states. Let $\mathcal{H}^{(1)}, \mathcal{H}^{(2)}$ be two quantum spaces for which $\mathcal{H} = \mathcal{H}^{(1)} \otimes \mathcal{H}^{(2)}$ and the corresponding off-shell states are $\mathbb{B}^{(1)}(\bar{t}), \mathbb{B}^{(2)}(\bar{t})$. The co-product formula reads as

$$\mathbb{B}(\bar{t}) = \sum_{\text{part}(\bar{t})} \frac{\prod_{\nu=1}^{N-1} \alpha^{(2)}_\nu(\bar{t}^\nu_{\text{I}}) f(\bar{t}^\nu_{\text{II}}, \bar{t}^\nu_{\text{I}})}{\prod_{\nu=1}^{N-2} f(\bar{t}^{\nu+1}_{\text{II}}, \bar{t}^\nu_{\text{I}})} \mathbb{B}^{(1)}(\bar{t}_{\text{I}}) \mathbb{B}^{(2)}(\bar{t}_{\text{II}}), \tag{B.5}$$

where the sum goes over the partitions $\bar{t}^s \vdash \{\bar{t}^s_{\text{I}}, \bar{t}^s_{\text{II}}\}$.

C Theorems for the nested K-matrices

In this section, we prove the theorems concerning nested K-matrices from subsection 4.1.

C.1 Crossed K-matrices

Lemma 15. *The operator $\mathbf{K}_{1,1}(u)$ satisfies the following commutation relations*

$$[\mathbf{K}_{1,1}(u), \mathbf{B}] = [\mathbf{K}_{1,1}(u), \mathbf{K}_{1,1}(v)] = 0. \tag{C.1}$$

Proof. Let us get the $(1,1)$ component of the KT-relation

$$\sum_{k=1}^N \mathbf{K}_{1,k}(z) \langle\Psi| T_{k,1}(z) = \sum_{k=1}^N \langle\Psi| \widehat{T}_{1,k}(-z) \mathbf{K}_{k,1}(z). \tag{C.2}$$

Let us apply it on the pseudo-vacuum:

$$\mathbf{K}_{1,1}(z) \langle\Psi| T_{1,1}(z) |0\rangle = \langle\Psi| \widehat{T}_{1,1}(-z) |0\rangle \mathbf{K}_{1,1}(z), \tag{C.3}$$

where we used that $T_{i,j}(z)|0\rangle = \widehat{T}_{j,i}(z)|0\rangle = 0$ for $i > j$. The equation above simplifies as

$$\mathbf{K}_{1,1}(z) \mathbf{B} \lambda_1(z) = \mathbf{B} \mathbf{K}_{1,1}(z) \hat{\lambda}_1(-z). \tag{C.4}$$

Using the symmetry property (3.34), we obtain that

$$[\mathbf{K}_{1,1}(z), \mathbf{B}] = 0. \tag{C.5}$$

From the nested KT-relations we can derive analogous equations as (C.5). For the nested K-matrices we have

$$[\mathbf{K}^{(k)}_{k,k}(z), \mathbf{B}] = 0. \tag{C.6}$$

For the other relation, we can use the $(1,1),(1,1)$ component of the reflection equation (3.7). After simplifications

$$[\mathbf{K}_{1,1}(u), \mathbf{K}_{1,1}(v)] = 0. \tag{C.7}$$

□

Lemma 16. *The component $\mathbf{K}_{1,1}(u)$ commutes with the entire nested K-matrix, i.e.,*

$$\left[\mathbf{K}_{1,1}(v), \mathbf{K}^{(2)}_{a,b}(u)\right] = 0, \tag{C.8}$$

for $a, b = 2, \ldots, N$.

Proof. Let us get the $((1,1),(1,b))$, $((1,1),(a,1))$ and $((1,1),(a,b))$ components of the boundary Yang-Baxter equation (3.7) $(a,b>1)$

$$
\begin{aligned}
\frac{u-v-1}{u-v}\frac{u+v+1}{u+v}\mathbf{K}_{1,1}(u)\mathbf{K}_{1,b}(v) &= \mathbf{K}_{1,b}(v)\mathbf{K}_{1,1}(u) - \frac{1}{u-v}\frac{u+v+1}{u+v}\mathbf{K}_{1,1}(v)\mathbf{K}_{1,b}(u) \\
&+ \frac{1}{u+v}\mathbf{K}_{1,1}(v)\mathbf{K}_{b,1}(u), \qquad (C.9)\\
\mathbf{K}_{1,1}(u)\mathbf{K}_{a,1}(v) &= \frac{u-v-1}{u-v}\frac{u+v+1}{u+v}\mathbf{K}_{a,1}(v)\mathbf{K}_{1,1}(u) + \frac{1}{u-v}\frac{u+v+1}{u+v}\mathbf{K}_{a,1}(u)\mathbf{K}_{1,1}(v) \\
&- \frac{1}{u+v}\mathbf{K}_{1,a}(u)\mathbf{K}_{1,1}(v), \qquad (C.10)\\
[\mathbf{K}_{1,1}(u),\mathbf{K}_{a,b}(v)] &= \frac{1}{u-v}\frac{u+v+1}{u+v}\left(\mathbf{K}_{a,1}(u)\mathbf{K}_{1,b}(v) - \mathbf{K}_{a,1}(v)\mathbf{K}_{1,b}(u)\right) \\
&- \frac{1}{u+v}\left(\mathbf{K}_{1,a}(u)\mathbf{K}_{1,b}(v) - \mathbf{K}_{a,1}(v)\mathbf{K}_{b,1}(u)\right). \qquad (C.11)
\end{aligned}
$$

Let us apply (C.10) to obtain

$$
\begin{aligned}
\mathbf{K}_{1,1}(u)\left(\mathbf{K}_{a,1}(v)\mathbf{K}^{-1}_{1,1}(v)\mathbf{K}_{1,b}(v)\right) &= \frac{u-v-1}{u-v}\frac{u+v+1}{u+v}\mathbf{K}_{a,1}(v)\mathbf{K}_{1,1}(u)\mathbf{K}^{-1}_{1,1}(v)\mathbf{K}_{1,b}(v) + \\
&+ \frac{1}{u-v}\frac{u+v+1}{u+v}\mathbf{K}_{a,1}(u)\mathbf{K}_{1,b}(v) - \frac{1}{u+v}\mathbf{K}_{1,a}(u)\mathbf{K}_{1,b}(v). \qquad (C.12)
\end{aligned}
$$

On the first term on the right-hand side, we use the commutation relations (C.1) and (C.9)

$$
\begin{aligned}
&\frac{u-v-1}{u-v}\frac{u+v+1}{u+v}\mathbf{K}_{a,1}(v)\mathbf{K}_{1,1}(u)\mathbf{K}^{-1}_{1,1}(v)\mathbf{K}_{1,b}(v) = \mathbf{K}_{a,1}(v)\mathbf{K}^{-1}_{1,1}(v)\left(\frac{u-v-1}{u-v}\frac{u+v+1}{u+v}\mathbf{K}_{1,1}(u)\mathbf{K}_{1,b}(v)\right) = \\
&\left(\mathbf{K}_{a,1}(v)\mathbf{K}^{-1}_{1,1}(v)\mathbf{K}_{1,b}(v)\right)\mathbf{K}_{1,1}(u) - \frac{1}{u-v}\frac{u+v+1}{u+v}\mathbf{K}_{a,1}(v)\mathbf{K}_{1,b}(u) + \frac{1}{u+v}\mathbf{K}_{a,1}(v)\mathbf{K}_{b,1}(u). \qquad (C.13)
\end{aligned}
$$

Combining the last two equations we have

$$
\begin{aligned}
\left[\mathbf{K}_{1,1}(u),\mathbf{K}_{a,1}(v)\mathbf{K}^{-1}_{1,1}(v)\mathbf{K}_{1,b}(v)\right] &= \frac{1}{u-v}\frac{u+v+1}{u+v}\left(\mathbf{K}_{a,1}(u)\mathbf{K}_{1,b}(v) - \mathbf{K}_{a,1}(v)\mathbf{K}_{1,b}(u)\right) \\
&- \frac{1}{u+v}\left(\mathbf{K}_{1,a}(u)\mathbf{K}_{1,b}(v) - \mathbf{K}_{a,1}(v)\mathbf{K}_{b,1}(u)\right). \qquad (C.14)
\end{aligned}
$$

We can see the r.h.s. agrees with the r.h.s. of the commutation relation (C.11) therefore we just proved

$$\left[\mathbf{K}_{1,1}(u_2),\mathbf{K}^{(2)}_{a,b}(u_1)\right] = 0, \qquad (C.15)$$

for $a,b = 2,\ldots,N$. $\square$

Now we turn to the proof of Theorem 3.

Proof. First, we prove that the nested K-matrices $\mathbf{K}^{(k)}(u-(k-1)/2)$ form a representation of the algebra $Y^+(N+1-k)$. In the section 5.1.1, we saw that the nested K-matrices can be expressed using quasi-determinants. Let us decompose the K-matrix into block form

$$\mathbf{K} = \begin{pmatrix} \mathbf{A} & \mathbf{B} \\ \mathbf{C} & \mathbf{D} \end{pmatrix}, \qquad (C.16)$$

where $\mathbf{A}$ is a $(k-1)\times(k-1)$ matrix and $\mathbf{D}$ is an $(N-k+1)\times(N-k+1)$ matrix. The nested K-matrices $\mathbf{K}^{(k)}$ can be expressed as follows

$$\mathbf{K}^{(k)} = \begin{pmatrix} \mathbf{A} & \mathbf{B} \\ \mathbf{C} & \boxed{\mathbf{D}} \end{pmatrix} = \mathbf{D} - \mathbf{C}\mathbf{A}^{-1}\mathbf{B}. \qquad (C.17)$$

First, we invert the reflection equation

$$R_{1,2}(u-v)\mathbf{K}_1^{-1}(u)\widehat{R}_{1,2}(u+v+N)\mathbf{K}_2^{-1}(v) = \mathbf{K}_2^{-1}(v)\widehat{R}_{1,2}(u+v+N)\mathbf{K}_1^{-1}(u)R_{1,2}(u-v), \qquad (C.18)$$

where we used the unitarity relations of the R-matrices

$$R_{1,2}(u)R_{1,2}(-u) = \frac{u^2-1}{u^2}, \quad \widehat{R}_{1,2}(u)\widehat{R}_{1,2}(-u+N) = 1. \tag{C.19}$$

We decompose the inverse matrix into block form

$$\mathbf{K}^{-1} = \begin{pmatrix} \tilde{\mathbf{A}} & \tilde{\mathbf{B}} \\ \tilde{\mathbf{C}} & \tilde{\mathbf{D}} \end{pmatrix}, \tag{C.20}$$

where $\tilde{\mathbf{D}}$ can be expressed in the following form

$$\tilde{\mathbf{D}} = \left(\mathbf{D} - \mathbf{C}\mathbf{A}^{-1}\mathbf{B}\right)^{-1}. \tag{C.21}$$

The matrix $\tilde{\mathbf{D}}$ satisfies the following reflection equation

$$R_{1,2}(u-v)\tilde{\mathbf{D}}_1(u)\widehat{R}_{1,2}(u+v+N)\tilde{\mathbf{D}}_2(v) = \tilde{\mathbf{D}}_2(v)\widehat{R}_{1,2}(u+v+N)\tilde{\mathbf{D}}_1(u)R_{1,2}(u-v), \tag{C.22}$$

where R and $\widehat{R}$ are $\mathfrak{gl}_{N-k+1}$ symmetric R-matrices. We can invert the equation again

$$R_{1,2}(v-u)\tilde{\mathbf{D}}_1^{-1}(u)\widehat{R}_{1,2}(-u-v-k+1)\tilde{\mathbf{D}}_2^{-1}(v) = \tilde{\mathbf{D}}_2^{-1}(v)\widehat{R}_{1,2}(-u-v-k+1)\tilde{\mathbf{D}}_1^{-1}(u)R_{1,2}(v-u). \tag{C.23}$$

We see that the operator $\tilde{\mathbf{D}}^{-1}(u-(k-1)/2)$ satisfies the $\mathfrak{gl}_{N-k+1}$ reflection equation. Combining equations (C.17), (C.21) and (C.23), we see that the operator $\mathbf{K}^{(k)}(u-(k-1)/2)$ is a representation of the reflection algebra $Y^+(N-k+1)$.

Since the operator $\mathbf{K}^{(k)}$ satisfies the $\mathfrak{gl}_{N-k+1}$ reflection equation, the Lemmas 15 and 16 are also applicable to the operator $\mathbf{K}^{(k)}$, i.e.,

$$\left[\mathbf{K}_{k,k}^{(k)}(v), \mathbf{K}_{k,k}^{(k)}(u)\right] = \left[\mathbf{K}_{k,k}^{(k)}(v), \mathbf{K}_{a,b}^{(k+1)}(u)\right] = 0, \tag{C.24}$$

where $a, b = k+1, \dots, N$. □

C.2 Uncrossed K-matrices

Lemma 17. *The operator $\mathbf{K}_{N,1}(u)$ satisfies the following commutation relations*

$$[\mathbf{K}_{N,1}(u), \mathbf{B}] = [\mathbf{K}_{N,1}(u), \mathbf{K}_{N,1}(v)] = 0. \tag{C.25}$$

Proof. Let us get the $(N,1)$ component of the KT-relation

$$\sum_{k=1}^{N} \mathbf{K}_{N,k}(z)\langle\Psi|T_{k,1}(z) = \sum_{k=1}^{N} \langle\Psi|T_{N,k}(-z)\mathbf{K}_{k,1}(z). \tag{C.26}$$

Let us apply it on the pseudo-vacuum:

$$\mathbf{K}_{N,1}(z)\langle\Psi|T_{1,1}(z)|0\rangle = \langle\Psi|T_{N,N}(-z)|0\rangle\mathbf{K}_{N,1}(z), \tag{C.27}$$

where we used that $T_{i,j}(z)|0\rangle = 0$ for $i > j$. The equation above simplifies as

$$\mathbf{K}_{N,1}(z)\mathbf{B}\lambda_1(z) = \mathbf{B}\mathbf{K}_{N,1}(z)\lambda_N(-z). \tag{C.28}$$

Using the symmetry property $\lambda_1(z) = \lambda_N(-z)$, we obtain that

$$[\mathbf{K}_{N,1}(z), \mathbf{B}] = 0. \tag{C.29}$$

From the nested KT-relations we can derive analogous equations as (C.29). For the nested K-matrices we have

$$[\mathbf{K}_{N+1-k,k}^{(k)}(z), \mathbf{B}] = 0. \tag{C.30}$$

For the other relation we can use the $(N,1),(N,1)$ component of the reflection equation (3.7). After simplifications

$$[\mathbf{K}_{N,1}(u), \mathbf{K}_{N,1}(v)] = 0.$$

□

Lemma 18. *The component* $\mathbf{K}_{N,1}(u)$ *commutes with the entire nested* K*-matrix, i.e.,*

$$\left[\mathbf{K}_{N,1}(v), \mathbf{K}^{(2)}_{a,b}(u)\right] = 0, \tag{C.31}$$

for $a, b = 2, \dots, N-1$.

Proof. Let us get the $((N,1),(N,b))$, $((N,1),(a,1))$ and $((N,1),(a,b))$ components of the boundary Yang-Baxter equation $(1 < a, b < N)$

$$\frac{u-v-1}{u-v}\mathbf{K}_{N,1}(u)\mathbf{K}_{N,b}(v) = \mathbf{K}_{N,b}(v)\mathbf{K}_{N,1}(u) - \frac{1}{u-v}\mathbf{K}_{N,1}(v)\mathbf{K}_{1,b}(u) \tag{C.32}$$

$$\mathbf{K}_{N,1}(u)\mathbf{K}_{a,1}(v) = \frac{u-v-1}{u-v}\mathbf{K}_{a,1}(v)\mathbf{K}_{N,1}(u) + \frac{1}{u-v}\mathbf{K}_{a,1}(u)\mathbf{K}_{N,1}(v) \tag{C.33}$$

$$[\mathbf{K}_{N,1}(u), \mathbf{K}_{a,b}(v)] = \frac{1}{u-v}\left(\mathbf{K}_{a,1}(u)\mathbf{K}_{N,b}(v) - \mathbf{K}_{a,1}(v)\mathbf{K}_{N,b}(u)\right) \tag{C.34}$$

Let us apply (C.33) to obtain

$$\begin{aligned}\mathbf{K}_{N,1}(u)\left(\mathbf{K}_{a,1}(v)\mathbf{K}^{-1}_{N,1}(v)\mathbf{K}_{N,b}(v)\right) = &\frac{u-v-1}{u-v}\mathbf{K}_{a,1}(v)\mathbf{K}_{N,1}(u)\mathbf{K}^{-1}_{N,1}(v)\mathbf{K}_{N,b}(v) + \\ &+ \frac{1}{u-v}\mathbf{K}_{a,1}(u)\mathbf{K}_{N,b}(v).\end{aligned} \tag{C.35}$$

On the first term on the right-hand side, we use the commutation relations (C.25) and (C.32):

$$\begin{aligned}\frac{u-v-1}{u-v}\mathbf{K}_{a,1}(v)\mathbf{K}_{N,1}(u)\mathbf{K}^{-1}_{N,1}(v)\mathbf{K}_{1,b}(v) = &\mathbf{K}_{a,1}(v)\mathbf{K}^{-1}_{N,1}(v)\left(\frac{u-v-1}{u-v}\mathbf{K}_{N,1}(u)\mathbf{K}_{N,b}(v)\right) \\ &\left(\mathbf{K}_{a,1}(v)\mathbf{K}^{-1}_{N,1}(v)\mathbf{K}_{N,b}(v)\right)\mathbf{K}_{N,1}(u) - \frac{1}{u-v}\mathbf{K}_{a,1}(v)\mathbf{K}_{N,b}(u).\end{aligned} \tag{C.36}$$

Combining the last two equations we have

$$\left[\mathbf{K}_{N,1}(u), \mathbf{K}_{a,1}(v)\mathbf{K}^{-1}_{N,1}(v)\mathbf{K}_{N,b}(v)\right] = \frac{1}{u-v}\left(\mathbf{K}_{a,1}(u)\mathbf{K}_{N,b}(v) - \mathbf{K}_{a,1}(v)\mathbf{K}_{N,b}(u)\right). \tag{C.37}$$

We can see the r.h.s. agrees with the r.h.s. of the commutation relation (C.34) therefore we just proved

$$\left[\mathbf{K}_{N,1}(u), \mathbf{K}^{(2)}_{a,b}(v)\right] = 0, \tag{C.38}$$

for $a, b = 2, \dots, N-1$. □

Now we turn to the proof of Theorem 6.

Proof. First, let us prove that the nested K-matrix $\mathbf{K}^{(2)}(u)$ satisfies the reflection equation. The proof is very similar to the proof of Theorem 3.1 in [55]. We start from the reflection equation

$$\begin{aligned}\mathbf{k}_{1,2}(u) = R_{1,2}(-1)\mathbf{K}_1(u+1)R_{1,2}(-2u-1)\mathbf{K}_2(u) &= \mathbf{K}_2(u)R_{1,2}(-2u-1)\mathbf{K}_1(u+1)R_{1,2}(-1) \\ &= \sum_{a_i,b_i} e_{a_1,b_1}\otimes e_{a_2,b_2}\otimes \mathbf{k}^{b_1,b_2}_{a_1,a_2}(u).\end{aligned} \tag{C.39}$$

Since $R_{1,2}(-1)$ is a projection operator onto the antisymmetrized subspace, the quantum minor $\mathbf{k}^{b_1,b_2}_{a_1,a_2}(u)$ is antisymmetric in both its upper and lower indices. Let us examine the specific components $\mathbf{k}^{1,b}_{N,a}(u)$:

$$\mathbf{k}^{1,b}_{N,a}(u) = \mathbf{K}_{N,1}(u+1)\mathbf{K}_{a,b}(u) - \mathbf{K}_{a,1}(u+1)\mathbf{K}_{N,b}(u). \tag{C.40}$$

Substitute $(u,v) \to (u+1,u)$ into (C.33):

$$\mathbf{K}_{N,1}(u+1)\mathbf{K}_{a,1}(u) = \mathbf{K}_{a,1}(u+1)\mathbf{K}_{N,1}(u). \tag{C.41}$$

Using this, we find that $\mathbf{k}^{1,b}_{N,a}$ can be expressed in terms of $\mathbf{K}^{(2)}_{a,b}$:

$$\mathbf{k}^{1,b}_{N,a}(u) = \mathbf{K}_{N,1}(u+1)\mathbf{K}^{(2)}_{a,b}(u). \tag{C.42}$$

We define an $(N-2)\times(N-2)$ matrix as follows:

$$\tilde{\mathbf{k}}_1(u) = \sum_{a_1,b_1=2}^{N-1} \tilde{e}_{a_1,b_1} \otimes \mathbf{k}^{1,b_1}_{N,a_1}(u), \tag{C.43}$$

where $\tilde{e}_{a,b}$ are the unit matrices of size $(N-2)\times(N-2)$. In this notation,

$$\tilde{\mathbf{k}}_1(u) = \mathbf{G}(u+1)\mathbf{K}^{(2)}_1(u). \tag{C.44}$$

Now consider four auxiliary spaces and define an R-matrix on them

$$\begin{aligned} R_{1,2,3,4}(u) &= A_{1,2}A_{3,4}R_{4,1}(u-1)R_{3,1}(u)R_{4,2}(u)R_{3,2}(u+1) \\ &= R_{3,2}(u+1)R_{3,1}(u)R_{4,2}(u)R_{4,1}(u-1)A_{1,2}A_{3,4} \\ &= \sum_{a_i,b_i=1}^{N} e_{a_1,b_1}\otimes e_{a_2,b_2}\otimes e_{a_3,b_3}\otimes e_{a_4,b_4}\otimes R^{b_1,b_2,b_3,b_4}_{a_1,a_2,a_3,a_4}(u). \end{aligned} \tag{C.45}$$

This R-matrix is antisymmetric in the first-second and third-fourth indices, i.e.,

$$P_{1,2}R_{1,2,3,4}(u) = R_{1,2,3,4}(u)P_{1,2} = P_{3,4}R_{1,2,3,4}(u) = R_{1,2,3,4}(u)P_{3,4} = -R_{1,2,3,4}(u). \tag{C.46}$$

The quantum minor satisfies the following equation:

$$R_{1,2,3,4}(v-u)\mathbf{k}_{1,2}(u)R_{1,2,3,4}(-u-v-1)\mathbf{k}_{3,4}(v) = \mathbf{k}_{3,4}(v)R_{1,2,4,3}(-u-v-1)\mathbf{k}_{1,2}(u)R_{1,2,3,4}(v-u). \tag{C.47}$$

We define operators where the components are fixed in the first and third spaces

$$\begin{aligned} R^{\circ}_{1,2}(u) &= \sum_{a_i,b_i=2}^{N-1} \tilde{e}_{a_2,b_2}\otimes\tilde{e}_{a_4,b_4}\otimes R^{1,b_2,1,b_4}_{1,a_2,1,a_4}(u), \\ R^{\bullet}_{1,2}(u) &= \sum_{a_i,b_i=2}^{N-1} \tilde{e}_{a_2,b_2}\otimes\tilde{e}_{a_4,b_4}\otimes R^{1,b_2,N,b_4}_{1,a_2,N,a_4}(u), \end{aligned} \tag{C.48}$$

again using $\tilde{e}_{a,b}$ a $(N-2)\times(N-2)$ as the unit matrices. Taking the $(N,1)$ components of equation (C.47) in the first and third spaces, we obtain a reflection equation for the $\tilde{\mathbf{k}}$ matrices

$$R^{\circ}_{1,2}(v-u)\tilde{\mathbf{k}}_1(u)R^{\bullet}_{1,2}(-u-v-1)\tilde{\mathbf{k}}_2(v) = \tilde{\mathbf{k}}_2(v)R^{\bullet}_{1,2}(-u-v-1)\tilde{\mathbf{k}}_1(u)R^{\circ}_{1,2}(v-u), \tag{C.49}$$

where we used the identities $R^{1,b_2,1,b_4}_{1,a_2,1,a_4}(u) = R^{N.b_2,N,b_4}_{N,a_2,N,a_4}(u)$ and $R^{1,b_2,N,b_4}_{1,a_2,N,a_4}(u) = R^{N.b_2,1,b_4}_{N,a_2,1,a_4}(u)$ and the fact that $R^{b_1,b_2,b_3,b_4}_{a_1,a_2,a_3,a_4}$ is antisymmetric in the first-second and third-fourth indices. The matrices R° and $R^{\bullet}$ can be expressed using the $\mathfrak{gl}_{N-2}$ symmetric R-matrix as follows:

$$\begin{aligned} R^{\circ}_{1,2}(u) &= \frac{u+2}{u+1}R^{(N-2)}_{1,2}(u), \\ R^{\bullet}_{1,2}(u) &= R^{(N-2)}_{1,2}(u+1). \end{aligned} \tag{C.50}$$

Substituting back into equation (C.49), we get:

$$R^{(N-2)}_{1,2}(v-u)\mathbf{K}^{(2)}_1(u)R^{(N-2)}_{1,2}(-u-v)\mathbf{K}^{(2)}_2(v) = \mathbf{K}^{(2)}_2(v)R^{(N-2)}_{1,2}(-u-v)\mathbf{K}^{(2)}_1(u)R^{(N-2)}_{1,2}(v-u), \tag{C.51}$$

where we used the relation (C.44) and the commutation relation (C.31). We see that the nested K-matrix $\mathbf{K}^{(2)}$ satisfies the $\mathfrak{gl}_{N-2}$ reflection equation. By repeatedly applying the above method, we see that $\mathbf{K}^{(k)}$ satisfies the $\mathfrak{gl}_{N-2k+2}$ reflection equation. In 3.5.2 , we saw that the asymptotic limit of the nested K-matrices satisfies equation (3.74). Combining this with the reflection equation just derived, we see that $\mathbf{K}^{(k)}$ is a representation of the reflection algebra $\mathcal{B}(N-2k+2, M-k+1)$.

Since the operator $\mathbf{K}^{(k)}$ satisfies the $\mathfrak{gl}_{N-2k+2}$ reflection equation, the Lemmas 17 and 18 are also applicable to the operator $\mathbf{K}^{(k)}$, i.e.,

$$\left[\mathbf{G}^{(k)}(v), \mathbf{G}^{(k)}(u)\right] = \left[\mathbf{G}^{(k)}(v), \mathbf{K}^{(k+1)}_{a,b}(u)\right] = 0, \tag{C.52}$$

where $a, b = k+1, \ldots, N-k$. $\square$

D Proofs for the theorems of overlaps

In this section, we prove the theorems related to the overlap formula from sections 4.2 and 4.3.

D.1 Proof for the sum formula

Lemma 19. *The off-shell overlap has the sum formula*

$$\mathbf{S}_{\bar{\alpha},\mathbf{B}}(\bar{t}) = \sum_{\text{part}(\bar{t})} \mathbf{W}_{\mathbf{B}}(\bar{t}_{I}|\bar{t}_{II}) \prod_{\nu=1}^{N-1} \alpha_{\nu}(\bar{t}^{\nu}_{I}), \tag{D.1}$$

where the sum goes through the partitions $\bar{t}^{\nu} = \bar{t}^{\nu}_{I} \cup \bar{t}^{\nu}_{II}$*, and* $\mathbf{W}_{\mathbf{B}}$ is a matrix valued function which depends only on *the variables* $\bar{t}_{I}, \bar{t}_{II}$*, components of the* K*-matrices* $\mathbf{K}_{i,j}$ *and the vacuum overlap* $\mathbf{B}$.

Proof. The proof is the same as Appendix C in [43]. In that earlier case, the quantities $\mathbf{K}_{i,j}$, $\mathbf{W}$ and $\mathbf{S}$ were still scalars, but in the current paper, they are matrices. However, this does not require any changes in the derivation of the sum formula. Appendix C of [43] applies directly even in the presence of nontrivial boundary spaces. □

Lemma 20. *The weights are factorized as*

$$\mathbf{W}_{\mathbf{B}}(\bar{t}_{i}|\bar{t}_{ii}) = \frac{\prod_{\nu=1}^{N-1} f(\bar{t}^{\nu}_{II}, \bar{t}^{\nu}_{I})}{\prod_{\nu=1}^{N-2} f(\bar{t}^{\nu+1}_{II}, \bar{t}^{\nu}_{I})} \bar{\mathbf{Z}}(\bar{t}_{II}) \mathbf{B} \mathbf{Z}(\bar{t}_{I}), \tag{D.2}$$

where the highest coefficients (HC) $\mathbf{Z}(\bar{t})$ *and* $\bar{\mathbf{Z}}(\bar{t})$ *depend only on the variables* $\bar{t}$ *and the entries of the* K*-matrix.*

Proof. The proof is similar than Appendix D in [43]. The derivation is based on co-product formula of the Bethe states (B.5). Let us renormalize the Bethe states and overlaps as

$$\tilde{\mathbb{B}}(\bar{t}) = \prod_{\nu=1}^{N-1} \lambda_{\nu+1}(\bar{t}^{\nu}) \mathbb{B}(\bar{t}), \quad \tilde{\mathbf{S}}_{\bar{\lambda},\mathbf{B}}(\bar{t}) = \prod_{\nu=1}^{N-1} \lambda_{\nu+1}(\bar{t}^{\nu}) \mathbf{S}_{\bar{\alpha},\mathbf{B}}(\bar{t}). \tag{D.3}$$

The sum formula (Lemma 19) of the renormalized overlap is

$$\tilde{\mathbf{S}}_{\bar{\lambda},\mathbf{B}}(\bar{t}) = \sum_{\text{part}(\bar{t})} \mathbf{W}_{\mathbf{B}}(\bar{t}_{\mathrm{I}}|\bar{t}_{\mathrm{II}}) \prod_{\nu=1}^{N-1} \lambda_{\nu}(\bar{t}^{\nu}_{\mathrm{I}}) \lambda_{\nu+1}(\bar{t}^{\nu}_{\mathrm{II}}). \tag{D.4}$$

Let $\mathcal{H}^{(1)}, \mathcal{H}^{(2)}$ be two subsystems of the quantum space for which $\mathcal{H} = \mathcal{H}^{(1)} \otimes \mathcal{H}^{(2)}$ and the renormalized co-product formula is

$$\tilde{\mathbb{B}}(\bar{t}) = \sum \frac{\prod_{\nu=1}^{N-1} \lambda^{(2)}_{\nu}(\bar{t}^{\nu}_{\mathrm{I}}) \lambda^{(1)}_{\nu+1}(\bar{t}^{\nu}_{\mathrm{II}}) f(\bar{t}^{\nu}_{\mathrm{II}}, \bar{t}^{\nu}_{\mathrm{I}})}{\prod_{\nu=1}^{N-2} f(\bar{t}^{\nu+1}_{\mathrm{II}}, \bar{t}^{\nu}_{\mathrm{I}})} \tilde{\mathbb{B}}^{(1)}(\bar{t}_{\mathrm{I}}) \tilde{\mathbb{B}}^{(2)}(\bar{t}_{\mathrm{II}}), \tag{D.5}$$

where $\tilde{\mathbb{B}}^{(1/2)}$ and $\lambda^{(1/2)}_{\nu}$ are the off-shell Bethe vectors and pseudo-vacuum eigenvalues on the subsystems $\mathcal{H}^{(1/2)}$. We can also use the co-product property of the boundary state, Lemma 1. Combining the co-product formulas of the boundary state and the Bethe states, the of-shell overlaps can be written as

$$\tilde{\mathbf{S}}_{\bar{\lambda},\mathbf{B}}(\bar{t}) = \sum \frac{\prod_{\nu=1}^{N-1} \lambda^{(2)}_{\nu}(\bar{t}^{\nu}_{\mathrm{I}}) \lambda^{(1)}_{\nu+1}(\bar{t}^{\nu}_{\mathrm{II}}) f(\bar{t}^{\nu}_{\mathrm{II}}, \bar{t}^{\nu}_{\mathrm{I}})}{\prod_{\nu=1}^{N-2} f(\bar{t}^{\nu+1}_{\mathrm{II}}, \bar{t}^{\nu}_{\mathrm{I}})} \tilde{\mathbf{S}}_{\bar{\lambda}^{(2)},\mathbf{B}^{(2)}}(\bar{t}_{\mathrm{II}}) \tilde{\mathbf{S}}_{\bar{\lambda}^{(1)},\mathbf{B}^{(1)}}(\bar{t}_{\mathrm{I}}), \tag{D.6}$$

where $\mathbf{B}^{(1/2)}$ are the pseudo-vacuum overlaps.

Now let us fix a particular partition $\bar{t} = \bar{t}_{\mathrm{I}} \cup \bar{t}_{\mathrm{II}}$ and choose the highest weights as

$$\begin{aligned} \lambda^{(1)}_{\nu+1}(u) &= 0, \qquad \text{for all } u \in \bar{t}^{\nu}_{\mathrm{I}}, \\ \lambda^{(2)}_{\nu}(u) &= 0, \qquad \text{for all } u \in \bar{t}^{\nu}_{\mathrm{II}}, \end{aligned} \tag{D.7}$$

therefore in the sum rule of the overlap (D.4) and the co-product formula (D.6) there is only one non-vanishing term:

$$\begin{aligned}\tilde{\mathbf{S}}_{\bar{\lambda},\mathbf{B}}(\bar{t}) &= \mathbf{W}_{\mathbf{B}}(\bar{t}_{\text{I}}|\bar{t}_{\text{II}}) \prod_{\nu=1}^{N-1} \lambda_\nu(\bar{t}_{\text{I}}^\nu)\lambda_{\nu+1}(\bar{t}_{\text{II}}^\nu),\\ \tilde{\mathbf{S}}_{\bar{\lambda}^{(1)},\mathbf{B}^{(1)}}(\bar{t}_{\text{I}}) &= \mathbf{W}_{\mathbf{B}^{(1)}}(\bar{t}_{\text{I}}|\emptyset) \prod_{\nu=1}^{N-1} \lambda_\nu^{(1)}(\bar{t}_{\text{I}}^\nu),\\ \tilde{\mathbf{S}}_{\bar{\lambda}^{(2)},\mathbf{B}^{(2)}}(\bar{t}_{\text{II}}) &= \mathbf{W}_{\mathbf{B}^{(2)}}(\emptyset|\bar{t}_{\text{II}}) \prod_{\nu=1}^{N-1} \lambda_{\nu+1}^{(2)}(\bar{t}_{\text{II}}^\nu).\end{aligned} \tag{D.8}$$

Substituting back, we obtain that

$$\mathbf{W}_{\mathbf{B}}(\bar{t}_{\text{I}}|\bar{t}_{\text{II}}) = \frac{\prod_{\nu=1}^{N-1} f(\bar{t}_{\text{II}}^\nu,\bar{t}_{\text{I}}^\nu)}{\prod_{\nu=1}^{N-2} f(\bar{t}_{\text{II}}^{\nu+1},\bar{t}_{\text{I}}^\nu)} \mathbf{W}_{\mathbf{B}^{(2)}}(\emptyset|\bar{t}_{\text{II}})\mathbf{W}_{\mathbf{B}^{(1)}}(\bar{t}_{\text{I}}|\emptyset). \tag{D.9}$$

Applying the last equation for $\bar{t}_{\text{II}} = \emptyset$ we have

$$\mathbf{W}_{\mathbf{B}}(\bar{t}|\emptyset) = \mathbf{W}_{\mathbf{B}^{(2)}}(\emptyset|\emptyset)\mathbf{W}_{\mathbf{B}^{(1)}}(\bar{t}|\emptyset). \tag{D.10}$$

Since $\mathbf{W}_{\mathbf{B}^{(2)}}(\emptyset|\emptyset)$ is just the vacuum overlap we have

$$\mathbf{W}_{\mathbf{B}}(\bar{t}|\emptyset) = \mathbf{B}^{(2)}\mathbf{W}_{\mathbf{B}^{(1)}}(\bar{t}|\emptyset). \tag{D.11}$$

From the co-product property of the boundary state (Lemma 1) we know that $\mathbf{B} = \mathbf{B}^{(2)}\mathbf{B}^{(1)}$. Since the equation (D.11) is true for any decomposition of the quantum space, the $\mathbf{B}$-dependence of the weights $\mathbf{W}_{\mathbf{B}}(\bar{t}|\emptyset)$ should be

$$\mathbf{W}_{\mathbf{B}}(\bar{t}|\emptyset) = \mathbf{B}\mathbf{Z}(\bar{t}). \tag{D.12}$$

Analogous way, for $\bar{t}_i = \emptyset$ we have

$$\mathbf{W}_{\mathbf{B}}(\emptyset|\bar{t}) = \mathbf{W}_{\mathbf{B}^{(2)}}(\emptyset|\bar{t})\mathbf{B}^{(1)}, \tag{D.13}$$

therefore we can introduce another HC as

$$\mathbf{W}_{\mathbf{B}}(\emptyset|\bar{t}) = \bar{\mathbf{Z}}(\bar{t})\mathbf{B}. \tag{D.14}$$

Substituting back we obtain that

$$\mathbf{W}_{\mathbf{B}}(\bar{t}_{\text{I}}|\bar{t}_{\text{II}}) = \frac{\prod_{\nu=1}^{N-1} f(\bar{t}_{\text{II}}^\nu,\bar{t}_{\text{I}}^\nu)}{\prod_{\nu=1}^{N-2} f(\bar{t}_{\text{II}}^{\nu+1},\bar{t}_{\text{I}}^\nu)} \bar{\mathbf{Z}}(\bar{t}_{\text{II}})\mathbf{B}\mathbf{Z}(\bar{t}_{\text{I}}). \tag{D.15}$$

□

Lemma 21. *Let us introduce the following modified monodromy matrix*

$$\tilde{T}(u) = VT^t(-u)V, \tag{D.16}$$

which also satisfies the RTT-relation. Denote the Bethe vector defined with the components of $\tilde{T}$ as $\tilde{\mathbb{B}}(\bar{t})$. The new Bethe vectors can be expressed in terms of the original Bethe vectors.

$$\frac{1}{\prod_{\nu=1}^{N-1}\tilde{\alpha}_\nu(\bar{t}^\nu)}\tilde{\mathbb{B}}(\bar{t}) = \mathbb{B}(\pi^a(\bar{t})). \tag{D.17}$$

Proof. We can prove the statement by induction. The pseudo-vacuum is also the highest weight state for $\tilde{T}$, since

$$\tilde{T}_{i,j}(u)|0\rangle = T_{N+1-j,N+1-i}(-u)|0\rangle = 0, \tag{D.18}$$

for $i > j$ and

$$\tilde{T}_{i,i}(u)|0\rangle = T_{N+1-i,N+1-i}(-u)|0\rangle = \lambda_{N+1-i}(-u)|0\rangle, \tag{D.19}$$

therefore

$$\tilde{\lambda}_i(u) = \lambda_{N+1-i}(-u). \tag{D.20}$$

This proves the statement for $r_j = 0$ for $j = 1, \ldots, N-1$.

Now assume the statement holds when the number of Bethe roots of the first type is at most r_1. Now use the recursive equation for $r_1 + 1$:

$$\frac{1}{\tilde{\alpha}_1(z)\prod_{\nu=1}^{N-1}\tilde{\alpha}_\nu(\bar{t}^\nu)}\tilde{\mathbb{B}}(\{z,\bar{t}^1\},\{\bar{t}^k\}_{k=2}^{N-1}) = \frac{1}{\tilde{\alpha}_1(z)\prod_{\nu=1}^{N-1}\tilde{\alpha}_\nu(\bar{t}^\nu)}\sum_{j=2}^{N}\frac{\tilde{T}_{1,j}(z)}{\tilde{\lambda}_2(z)}\sum_{\mathrm{part}(\bar{t})}\tilde{\mathbb{B}}(\bar{t}^1,\{\bar{t}^k_{\rm II}\}_{k=2}^{j-1},\{\bar{t}^k\}_{k=j}^{N-1})\frac{\prod_{\nu=2}^{j-1}\tilde{\alpha}_\nu(\bar{t}^\nu_{\rm I})g(\bar{t}^\nu_{\rm I},\bar{t}^{\nu-1}_{\rm I})f(\bar{t}^\nu_{\rm II},\bar{t}^\nu_{\rm I})}{\prod_{\nu=1}^{j-1}f(\bar{t}^{\nu+1},\bar{t}^\nu_{\rm I})}. \tag{D.21}$$

Apply the induction hypothesis to the right-hand side.

$$\frac{1}{\tilde{\alpha}_1(z)\prod_{\nu=1}^{N-1}\tilde{\alpha}_\nu(\bar{t}^\nu)}\tilde{\mathbb{B}}(\{z,\bar{t}^1\},\{\bar{t}^k\}_{k=2}^{N-1}) = \sum_{j=2}^{N}\frac{\tilde{T}_{1,j}(z)}{\tilde{\lambda}_1(z)}\sum_{\mathrm{part}(\bar{t})}\mathbb{B}(\{-\bar{t}^{N-k}\}_{k=1}^{N-j},\{-\bar{t}^{N-k}_{\rm II}\}_{k=N-j+1}^{N-2},-\bar{t}^1)\frac{\prod_{\nu=2}^{j-1}g(\bar{t}^\nu_{\rm I},\bar{t}^{\nu-1}_{\rm I})f(\bar{t}^\nu_{\rm II},\bar{t}^\nu_{\rm I})}{\prod_{\nu=1}^{j-1}f(\bar{t}^{\nu+1},\bar{t}^\nu_{\rm I})}. \tag{D.22}$$

Now we can express $\tilde{T}_{1,j}$ and $\tilde{\lambda}_1$ in terms of the original quantities.

$$\frac{1}{\tilde{\alpha}_1(z)\prod_{\nu=1}^{N-1}\tilde{\alpha}_\nu(\bar{t}^\nu)}\tilde{\mathbb{B}}(\{z,\bar{t}^1\},\{\bar{t}^k\}_{k=2}^{N-1}) = \sum_{j=1}^{N-1}\frac{T_{j,N}(-z)}{\lambda_N(-z)}\sum_{\mathrm{part}(\bar{t})}\mathbb{B}(\{-\bar{t}^{N-k}\}_{k=1}^{j-1},\{-\bar{t}^{N-k}_{\rm II}\}_{k=j}^{N-2},-\bar{t}^1)\frac{\prod_{\nu=2}^{N-j}g(\bar{t}^\nu_{\rm I},\bar{t}^{\nu-1}_{\rm I})f(\bar{t}^\nu_{\rm II},\bar{t}^\nu_{\rm I})}{\prod_{\nu=1}^{N-j}f(\bar{t}^{\nu+1},\bar{t}^\nu_{\rm I})}. \tag{D.23}$$

After rearrangements, we get

$$\frac{1}{\tilde{\alpha}_1(z)\prod_{\nu=1}^{N-1}\tilde{\alpha}_\nu(\bar{t}^\nu)}\tilde{\mathbb{B}}(\{z,\bar{t}^1\},\{\bar{t}^k\}_{k=2}^{N-1}) = \sum_{j=1}^{N-1}\frac{T_{j,N}(-z)}{\lambda_N(-z)}\sum_{\mathrm{part}(\bar{t})}\mathbb{B}(\{-\bar{t}^{N-k}\}_{k=1}^{j-1},\{-\bar{t}^{N-k}_{\rm II}\}_{k=j}^{N-2},-\bar{t}^1)\frac{\prod_{\nu=j}^{N-2}g(-\bar{t}^{N-\nu-1}_{\rm I},-\bar{t}^{N-\nu}_{\rm I})f(-\bar{t}^{N-\nu}_{\rm I},-\bar{t}^{N-\nu}_{\rm II})}{\prod_{\nu=1}^{N-j}f(-\bar{t}^{N-\nu}_{\rm I},-\bar{t}^{N-\nu+1})}. \tag{D.24}$$

We see that the right-hand side matches the right-hand side of the recursive equation (B.2), i.e.,

$$\frac{1}{\tilde{\alpha}_1(z)\prod_{\nu=1}^{N-1}\tilde{\alpha}_\nu(\bar{t}^\nu)}\tilde{\mathbb{B}}(\{z,\bar{t}^1\},\{\bar{t}^k\}_{k=2}^{N-1}) = \mathbb{B}(\{-\bar{t}^{N-k}\}_{k=1}^{N-2},\{-z,-\bar{t}^1\}), \tag{D.25}$$

which proves the inductive step. □

Lemma 22. *Let us introduce the following modified monodromy matrix*

$$\tilde{T}(u) = \widehat{T}^t(-u), \tag{D.26}$$

which also satisfies the RTT-relation. Denote the Bethe vector defined with the components of $\tilde{T}$ as $\tilde{\mathbb{B}}(\bar{t})$. The new Bethe vectors can be expressed in terms of the original Bethe vectors

$$\frac{1}{\prod_{\nu=1}^{N-1}\tilde{\alpha}_\nu(\bar{t}^\nu)}\tilde{\mathbb{B}}(\bar{t}) = A(\bar{t})\mathbb{B}(\pi^c(\bar{t})), \tag{D.27}$$

where

$$A(\bar{t}) = (-1)^{\#\bar{t}}\left(\prod_{s=1}^{N-2}f(\bar{t}^{s+1},\bar{t}^s)\right)^{-1}. \tag{D.28}$$

Proof. The matrix $\tilde{T}$ can be expressed in two steps

$$\tilde{T}(u) = V\check{T}^t(-u)V. \tag{D.29}$$

This is the relation that appeared in Lemma 21, i.e., $\tilde{\mathbb{B}}$ can be expressed in terms of $\check{\mathbb{B}}$ as follows

$$\frac{1}{\prod_{\nu=1}^{N-1}\tilde{\alpha}_\nu(\bar{t}^\nu)}\tilde{\mathbb{B}}(\bar{t})=\check{\mathbb{B}}(\pi^a(\bar{t})). \tag{D.30}$$

At the same time, $\check{\mathbb{B}}$ can be expressed in terms of the original Bethe vector as shown in [47]

$$\hat{\mathbb{B}}(\bar{t})=A(\bar{t})\mathbb{B}(\mu(\bar{t})), \tag{D.31}$$

where

$$\mu(\bar{t})=\{\bar{t}^{N-1}-1,\bar{t}^{N-2}-2,\dots,\bar{t}^1-(N-1)\}. \tag{D.32}$$

Substituting back, we get

$$\frac{1}{\prod_{\nu=1}^{N-1}\tilde{\alpha}_\nu(\bar{t}^\nu)}\tilde{\mathbb{B}}(\bar{t})=A(\pi^a(\bar{t}))\mathbb{B}(\mu(\pi^a(\bar{t}))). \tag{D.33}$$

It is easy to show that $A(\pi^a(\bar{t}))=A(\bar{t})$ and $\mu(\pi^a(\bar{t}))=\pi^c(\bar{t})$, which completes the proof. □

Lemma 23. *The uncrossed HC-s have the following property*

$$\mathbf{Z}^{\mathbf{K}}(\bar{t})=\left[\bar{\mathbf{Z}}^{\mathbf{K}^\Pi}(\pi^a(\bar{t}))\right]^{t_B}, \tag{D.34}$$

where we have indicated the K-matrix dependence of the HCs and

$$\mathbf{K}^\Pi(u)=V\mathbf{K}^T(u)V, \tag{D.35}$$

where ^{T} denotes the transposition on the auxiliary and boundary spaces.

Proof. By rearranging the KT-relation, we obtain:

$$\mathbf{K}^\Pi(u)\langle\Psi^{t_B}|\left[VT^t(-u)V\right]=\langle\Psi^{t_B}|\left[VT^{t_0}(u)V\right]\mathbf{K}^\Pi(u), \tag{D.36}$$

where ^{t_B} denotes the transposition on the boundary space. Let us introduce the following modified monodromy matrix:

$$\tilde{T}(u)=VT^t(-u)V. \tag{D.37}$$

Denote the Bethe vectors defined by the components of $\tilde{T}$ as $\tilde{\mathbb{B}}(\bar{t})$. These Bethe vectors can be expressed in terms of the original Bethe vectors, according to Lemma 21:

$$\tilde{\mathbb{B}}(\bar{t})=\mathbb{B}(\pi^a(\bar{t}))\prod_{\nu=1}^{N-1}\alpha_\nu(\bar{t}^\nu). \tag{D.38}$$

Since the modified monodromy matrix satisfies the KT-relation

$$\mathbf{K}^\Pi(u)\langle\Psi^{t_B}|\tilde{T}(u)=\langle\Psi^{t_B}|\tilde{T}(-u)\mathbf{K}^\Pi(u), \tag{D.39}$$

the off-shell overlap satisfies the sum formula

$$\langle\Psi^{t_B}|\tilde{\mathbb{B}}(\bar{t})=\sum_{\mathrm{part}(\bar{t})}\mathbf{W}^{\mathbf{K}^\Pi}_{\mathbf{B}^{t_B}}(\bar{t}_{\rm I}|\bar{t}_{\rm II})\prod_{\nu=1}^{N-1}\alpha_\nu(\bar{t}^\nu_{\rm I}), \tag{D.40}$$

where we used (3.35) and the symmetry property (3.35). Now we can use the relation between Bethe overlaps, from which we obtain:

$$\langle\Psi^{t_B}|\tilde{\mathbb{B}}(\bar{t})=(\langle\Psi|\mathbb{B}(\pi^a(\bar{t})))^{t_B}\prod_{\nu=1}^{N-1}\alpha_\nu(\bar{t}^\nu)=\sum_{\mathrm{part}(\bar{t})}\left(\mathbf{W}^{\mathbf{K}}_{\mathbf{B}}(\pi^a(\bar{t}_{\rm I})|\pi^a(\bar{t}_{\rm II}))\right)^{t_B}\prod_{\nu=1}^{N-1}\alpha_\nu(\bar{t}^\nu_{\rm II}), \tag{D.41}$$

that is,

$$\left(\mathbf{W}^{\mathbf{K}}_{\mathbf{B}}(\pi^a(\bar{t}_{\rm I})|\pi^a(\bar{t}_{\rm II}))\right)^{t_B}=\mathbf{W}^{\mathbf{K}^\Pi}_{\mathbf{B}^{t_B}}(\bar{t}_{\rm II}|\bar{t}_{\rm I}). \tag{D.42}$$

Using the explicit form of the weights, we obtain the desired formula (D.34). □

Lemma 24. *The crossed HC-s have the following property*

$$\mathbf{Z}^{\mathbf{K}}(\bar{t}) = (-1)^{\#\bar{t}} \left(\prod_{s=1}^{N-2} f(\bar{t}^{s+1}, \bar{t}^s) \right)^{-1} \left[\bar{\mathbf{Z}}^{\mathbf{K}^{\Pi}}(\pi^c(\bar{t})) \right]^{t_B}, \tag{D.43}$$

where we have indicated the K-matrix dependence of the HCs and

$$\mathbf{K}^{\Pi}(u) = \mathbf{K}^T(u), \tag{D.44}$$

where ^{T} denotes the transposition on the auxiliary and boundary spaces.

Proof. By rearranging the KT-relation, we obtain:

$$\mathbf{K}^{\Pi}(u)\langle\Psi^{t_B}| \left[\widehat{T}^t(-u)\right] = \langle\Psi^{t_B}| \left[T^t(u)\right] \mathbf{K}^{\Pi}(u). \tag{D.45}$$

Let us introduce the following modified monodromy matrix:

$$\tilde{T}(u) = \widehat{T}^t(-u). \tag{D.46}$$

Denote the Bethe vectors defined by the components of $\tilde{T}$ as $\tilde{\mathbb{B}}(\bar{t})$. These Bethe vectors can be expressed in terms of the original Bethe vectors, according to Lemma 21.

$$\tilde{\mathbb{B}}(\bar{t}) = A(\bar{t})\mathbb{B}(\pi^c(\bar{t})) \prod_{\nu=1}^{N-1} \alpha_\nu(\bar{t}^\nu). \tag{D.47}$$

Since the modified monodromy matrix satisfies the KT-relation, the off-shell overlap satisfies the sum formula

$$\langle\Psi^{t_B}|\tilde{\mathbb{B}}(\bar{t}) = \sum_{\mathrm{part}(\bar{t})} \mathbf{W}^{\mathbf{K}^{\Pi}}_{\mathbf{B}^{t_B}}(\bar{t}_{\mathrm{I}}|\bar{t}_{\mathrm{II}}) \prod_{\nu=1}^{N-1} \alpha_\nu(\bar{t}^\nu_{\mathrm{I}}). \tag{D.48}$$

Now we can use the relation between Bethe overlaps, from which we obtain:

$$\langle\Psi^{t_B}|\tilde{\mathbb{B}}(\bar{t}) = A(\bar{t})\left(\langle\Psi|\mathbb{B}(\pi^c(\bar{t}))\right)^{t_B} \prod_{\nu=1}^{N-1} \alpha_\nu(\bar{t}^\nu) = A(\bar{t}) \sum_{\mathrm{part}(\bar{t})} \left(\mathbf{W}^{\mathbf{K}}_{\mathbf{B}}(\pi^c(\bar{t}_{\mathrm{I}})|\pi^c(\bar{t}_{\mathrm{II}}))\right)^{t_B} \prod_{\nu=1}^{N-1} \alpha_\nu(\bar{t}^\nu_{\mathrm{II}}), \tag{D.49}$$

that is,

$$A(\bar{t}) \left(\mathbf{W}^{\mathbf{K}}_{\mathbf{B}}(\pi^c(\bar{t}_{\mathrm{I}})|\pi^c(\bar{t}_{\mathrm{II}}))\right)^{t_B} = \mathbf{W}^{\mathbf{K}^{\Pi}}_{\mathbf{B}^{t_B}}(\bar{t}_{\mathrm{II}}|\bar{t}_{\mathrm{I}}). \tag{D.50}$$

Using the explicit form of the weights and the identity $A(\pi^c(\bar{t}))^{-1} = A(\bar{t})$, we obtain the desired formula (D.43). □

Combining the lemmas from this section, we complete the proof of Theorem 8.

D.2 Poles of the HC-s

In this subsection, we prove two lemmas that provide the HC $\bar{\mathbf{Z}}(\bar{t})$ recursively. The crossed and uncrossed cases must be treated separately. In the proof, we follow the calculations in Appendices E and F of [43]. There, the case $d_B = 1$ was derived, i.e., when the quantities $\bar{\mathbf{Z}}$ and $\mathbf{K}_{i,j}$ are scalars. Now, these calculations must be generalized to matrix-valued quantities. The computation is essentially the same; we just need to pay attention to the order of the matrices.

From the sum formula, we see that the off-shell overlap $\langle\Psi|\mathbb{B}(\bar{t})$ is simply equal to the HC $\bar{\mathbf{Z}}(\bar{t})$ when all $\alpha_\nu(t^\nu_k)$ are set to zero. That is, in computing $\bar{\mathbf{Z}}(\bar{t})$, it is sufficient to consider only the terms independent of α. We introduce the notation $\cong$ to mean that two quantities are equal in the limit $\alpha_\nu(t^\nu_k) \to 0$, i.e.,

$$\langle\Psi|\mathbb{B}(\bar{t}) \cong \bar{\mathbf{Z}}(\bar{t})\mathbf{B}. \tag{D.51}$$

Lemma 25. *The crossed HC has the following recursion*

$$\bar{\mathbf{Z}}(\{z,\bar{t}^1\},\{\bar{t}^\nu\}_{\nu=2}^{N-1}) = -\sum_{\text{part}} \mathbf{K}_{1,1}^{-1}(z)\bar{\mathbf{Z}}(\bar{t}^1_{\scriptscriptstyle II},\{\bar{t}^\nu\}_{\nu=2}^{N-1})\mathbf{G}^{(2)}(z)\frac{f(\bar{t}^1,-z-1)}{f(\bar{t}^2,-z-1)f(\bar{t}^2,z)}\frac{f(\bar{t}^1_{\scriptscriptstyle I},\bar{t}^1_{\scriptscriptstyle II})}{h(\bar{t}^1_{\scriptscriptstyle I},-z-1)} -$$
$$-\sum_{i=2}^{N}\mathbf{K}_{1,1}^{-1}(z)\mathbf{K}_{1,i}(z)\sum_{\text{part}}\bar{\mathbf{Z}}(\bar{w}^1_{\scriptscriptstyle II},\{\bar{t}^\nu_{\scriptscriptstyle II}\}_{\nu=2}^{i-1},\{\bar{t}^\nu\}_{\nu=i}^{N-1})\frac{f(\bar{w}^1_{\scriptscriptstyle I},\bar{w}^1_{\scriptscriptstyle II})}{h(\bar{w}^1_{\scriptscriptstyle I},z)}\frac{f(\bar{t}^2_{\scriptscriptstyle I},\bar{t}^2_{\scriptscriptstyle II})}{h(\bar{t}^2_{\scriptscriptstyle I},\bar{w}^1_{\scriptscriptstyle I})f(\bar{t}^2_{\scriptscriptstyle I},\bar{w}^1_{\scriptscriptstyle II})}\prod_{s=3}^{i-1}\frac{f(\bar{t}^s_{\scriptscriptstyle I},\bar{t}^s_{\scriptscriptstyle II})}{h(\bar{t}^s_{\scriptscriptstyle I},\bar{t}^{s-1}_{\scriptscriptstyle I})f(\bar{t}^s_{\scriptscriptstyle I},\bar{t}^{s-1}_{\scriptscriptstyle II})}, \tag{D.52}$$

In the first line the sum goes over all the partitions of $\bar{t}^1 \vdash \{\bar{t}^1_{\scriptscriptstyle I},\bar{t}^1_{\scriptscriptstyle II}\}$ *where* $\#\bar{t}^1_{\scriptscriptstyle I}=1$. *In the second line* $\bar{w}^1=\{z,\bar{t}^1\}$ *and the sum goes over all the partitions of* $\bar{w}^1\vdash\{\bar{w}^1_{\scriptscriptstyle I},\bar{w}^1_{\scriptscriptstyle II}\}$, $\bar{t}^\nu\vdash\{\bar{t}^\nu_{\scriptscriptstyle I},\bar{t}^\nu_{\scriptscriptstyle II}\}$ *where* $\#\bar{w}^1_{\scriptscriptstyle I}=\#\bar{t}^\nu_{\scriptscriptstyle I}=1$ *and* $\nu=2,\dots,i-1$.

Proof. We can see that this formula matches equation (4.14) from the paper [43] for scalar quantities. We will follow the proof described in section E.2 of [43].

Let us start with the recurrence equation

$$\langle\Psi|\,\mathbb{B}(\{z,\bar{t}^1\},\{\bar{t}^\nu\}_{\nu}^{N-1}) \cong \langle\Psi|\,\frac{T_{1,2}(z)}{\lambda_2(z)f(\bar{t}^2,z)}\mathbb{B}(\bar{t}), \tag{D.53}$$

and the crossed KT-relation

$$\mathbf{K}_{1,1}(z)\,\langle\Psi|\,T_{1,2}(z) = \sum_{j=1}^{N}\langle\Psi|\,\widehat{T}_{1,j}(-z)\mathbf{K}_{j,2}(z) - \sum_{i=2}^{N}\mathbf{K}_{1,i}(z)\,\langle\Psi|\,T_{i,2}(z). \tag{D.54}$$

The $T_{1,2}(z)$ term in the r.h.s. of (D.53) can be expressed with $\widehat{T}_{1,k}(-z)$ and $T_{j,2}(z)$ where $k=1,\dots,N$ and $j=2,\dots,N$ therefore

$$\langle\Psi|\,\mathbb{B}(\{z,\bar{t}^1\},\{\bar{t}^\nu\}_{\nu=2}^{N-1}) \cong$$
$$\frac{1}{f(\bar{t}^2,z)}\left[\sum_{j=1}^{N}\mathbf{K}_{1,1}^{-1}(z)\,\langle\Psi|\,\frac{\widehat{T}_{1,k}(-z)}{\lambda_2(z)}\mathbb{B}(\bar{t})\mathbf{K}_{k,2}(z) - \sum_{i=2}^{N}\mathbf{K}_{1,1}^{-1}(z)\mathbf{K}_{1,i}(z)\,\langle\Psi|\,\frac{T_{i,2}(z)}{\lambda_2(z)}\mathbb{B}(\bar{t})\right]. \tag{D.55}$$

The next step is to apply the action formulas to the expressions

$$\langle\Psi|\,\frac{\widehat{T}_{1,k}(-z)}{\lambda_2(z)}\mathbb{B}(\bar{t}) \quad\text{and}\quad \langle\Psi|\,\frac{T_{i,2}(z)}{\lambda_2(z)}\mathbb{B}(\bar{t}). \tag{D.56}$$

According to section E.2 of [43] only the terms with $k=1,2$ contain α-independent contributions, and these are the following:

$$\begin{aligned}
\langle\Psi|\,\frac{\widehat{T}_{1,1}(-z)}{\lambda_2(z)}\mathbb{B}(\bar{t}) &= \sum_{\text{part}}\bar{\mathbf{Z}}(\bar{t}^1_{\rm II},\{\bar{t}^\nu\}_{\nu=2}^{N-1})\mathbf{B}\mathbf{Z}(\{-z-1\},\emptyset^{\times N-2})\frac{f(\bar{t}^1,-z-1)}{f(\bar{t}^2,-z-1)}\frac{f(\bar{t}^1_{\rm I},\bar{t}^1_{\rm II})}{h(\bar{t}^1_{\rm I},-z-1)},\\
\langle\Psi|\,\frac{\widehat{T}_{1,2}(-z)}{\lambda_2(z)}\mathbb{B}(\bar{t}) &= (-1)\sum_{\text{part}}\bar{\mathbf{Z}}(\bar{t}^1_{\rm II},\{\bar{t}^\nu\}_{\nu=2}^{N-1})\mathbf{B}\frac{f(\bar{t}^1,-z-1)}{f(\bar{t}^2,-z-1)}\frac{f(\bar{t}^1_{\rm I},\bar{t}^1_{\rm II})}{h(\bar{t}^1_{\rm I},-z-c)},
\end{aligned} \tag{D.57}$$

see equations (E.27), (E.29), (E.33), and (E.34) in [43]. The sums goes over all the partitions of $\bar{t}^1\vdash\{\bar{t}^1_{\rm I},\bar{t}^1_{\rm II}\}$where $\#\bar{t}^1_{\rm I}=1$. The first formula contains the HC $\mathbf{Z}(\{-z-1\},\emptyset^{\times N-2})$. We can calculate it from the one-excitation overlap $\langle\Psi|\mathbb{B}(\{z\},\emptyset^{\times N-2})$. First, we use the recursive formula for the Bethe vectors, then the KT-relation (D.54)

$$\langle\Psi|\mathbb{B}(\{z\},\emptyset^{\times N-2}) = \frac{1}{\lambda_2(z)}\,\langle\Psi|\,T_{1,2}(z)|0\rangle = \frac{\mathbf{K}_{1,1}^{-1}(z)}{\lambda_2(z)}\left(\hat{\lambda}_1(-z)\mathbf{B}\mathbf{K}_{1,2}(z)-\lambda_2(z)\mathbf{K}_{1,2}(z)\mathbf{B}\right). \tag{D.58}$$

Here we used the fact that the pseudo-vacuum is a highest weight state. Exploiting the symmetry property (3.35) and the exchange relation (4.4), we obtain:

$$\langle\Psi|\mathbb{B}(\{z\},\emptyset^{\times N-2}) = \alpha_1(z)\mathbf{B}\mathbf{K}_{1,1}^{-1}(z)\mathbf{K}_{1,2}(z) - \mathbf{K}_{1,1}^{-1}(z)\mathbf{K}_{1,2}(z)\mathbf{B}. \tag{D.59}$$

From this, the highest coefficients (HCs) can be expressed:

$$\bar{\mathbf{Z}}(\{z\},\emptyset^{\times N-2}) = -\mathbf{K}_{1,1}^{-1}(z)\mathbf{K}_{1,2}(z), \quad \mathbf{Z}(\{z\},\emptyset^{\times N-2}) = \mathbf{K}_{1,1}^{-1}(z)\mathbf{K}_{1,2}(z). \tag{D.60}$$

Using the previously derived identity between the HCs (4.12), we obtain an equivalent formula for $\mathbf{Z}(\{z\},\emptyset^{\times N-2})$:

$$\mathbf{Z}(\{z\},\emptyset^{\times N-2}) = -\left[\bar{\mathbf{Z}}^{\mathbf{K}^T}(\{-z-1\},\emptyset^{\times N-2})\right]^{t_B} = \mathbf{K}_{2,1}(-z-1)\mathbf{K}_{1,1}^{-1}(-z-1). \tag{D.61}$$

Substituting back, we get:

$$\langle\Psi|\,\frac{\widehat{T}_{1,1}}{\lambda_2(z)}(-z)\mathbb{B}(\bar{t})\mathbf{K}_{1,2}(z) + \langle\Psi|\,\frac{\widehat{T}_{1,2}}{\lambda_2(z)}(-z)\mathbb{B}(\bar{t})\mathbf{K}_{2,2}(z) = \sum_{\text{part}} \bar{\mathbf{Z}}(\bar{t}^1_{\text{II}},\{\bar{t}^\nu\}_{\nu=2}^{N-1})\mathbf{B}\left(\mathbf{K}_{2,1}(z)\mathbf{K}_{1,1}^{-1}(z)\mathbf{K}_{1,2}(z)-\mathbf{K}_{2,2}(z)\right)\frac{f(\bar{t}^1,-z-1)}{f(\bar{t}^2,-z-1)}\frac{f(\bar{t}^1_{\text{I}},\bar{t}^1_{\text{II}})}{h(\bar{t}^1_{\text{I}},-z-1)}. \tag{D.62}$$

We observe that on the right-hand side, the quantity $\mathbf{K}^{(2)}_{2,2}(z)=\mathbf{G}^{(2)}(z)$ appears, which commutes with $\mathbf{B}$.

Let us continue with the action of $T_{i,2}$. According to section E.2 of [43], the α-independent terms are:

$$\langle\Psi|\,\frac{T_{i,2}(z)}{\lambda_2(z)}\mathbb{B}(\bar{t}) \cong \sum_{\text{part}} \bar{\mathbf{Z}}(\bar{w}^1_{\text{II}},\{\bar{t}^\nu_{\text{II}}\}_{\nu=2}^{i-1},\{\bar{t}^\nu\}_{\nu=i}^{N-1})\mathbf{B}\frac{f(\bar{w}^1_{\text{I}},\bar{w}^1_{\text{II}})}{h(\bar{w}^1_{\text{I}},z)}\frac{f(\bar{t}^2_{\text{I}},\bar{t}^2_{\text{II}})f(\bar{t}^2,z)}{h(\bar{t}^2_{\text{I}},\bar{w}^1_{\text{I}})f(\bar{t}^2_{\text{I}},\bar{w}^1_{\text{II}})}\prod_{s=3}^{i-1}\frac{f(\bar{t}^s_{\text{I}},\bar{t}^s_{\text{II}})}{h(\bar{t}^s_{\text{I}},\bar{t}^{s-1}_{\text{I}})f(\bar{t}^s_{\text{I}},\bar{t}^{s-1}_{\text{II}})}, \tag{D.63}$$

see equation (E.35) in [43]. The sum goes over all the partitions of $\bar{w}^1 \vdash \{\bar{w}^1_{\text{I}},\bar{w}^1_{\text{II}}\}$, $\bar{t}^\nu \vdash \{\bar{t}^\nu_{\text{I}},\bar{t}^\nu_{\text{II}}\}$ where $\bar{w}^1=\{z,\bar{t}^1\}$ and $\#\bar{w}^1_{\text{I}}=\#\bar{t}^\nu_{\text{I}}=1$ for $\nu=2,\dots,i-1$. Substituting back to (D.55) we obtain a recurrence formula (D.52) for the HC after the simplification with $\mathbf{B}$. □

Lemma 26. *The uncrossed HC has the following recursions. The recursion for the set $\bar{t}^1$ is*

$$\begin{aligned}\bar{\mathbf{Z}}(\{z,\bar{t}^1\},\{\bar{t}^\nu\}_{\nu=2}^{N-1}) = &\sum_{\text{part}}\mathbf{K}^{-1}_{N,1}(z)\bar{\mathbf{Z}}(\{\bar{\omega}^\nu_{II}\}_{\nu=1}^{N-2},\bar{t}^{N-1}_{II})\mathbf{G}^{(2)}(z)\prod_{s=1}^{N-2}\frac{f(\omega^s_I,\bar{\omega}^s_{II})}{h(\omega^s_I,\omega^{s-1}_I)f(\omega^s_I,\bar{\omega}^{s-1}_{II})}\frac{f(\bar{t}^{N-1}_I,\bar{t}^{N-1}_{II})f(\bar{t}^{N-1},-z)}{h(\bar{t}^{N-1}_I,\omega^{N-2}_I)f(\bar{t}^{N-1}_I,\bar{\omega}^{N-2}_{II})f(\bar{t}^2,z)}-\\ &-\sum_{i=2}^{N}\mathbf{K}^{-1}_{N,1}(z)\mathbf{K}_{N,i}(z)\sum_{\text{part}}\bar{\mathbf{Z}}(\bar{w}^1_{II},\{\bar{t}^\nu_{II}\}_{\nu=2}^{i-1},\{\bar{t}^\nu\}_{\nu=i}^{N-1})\frac{f(\bar{w}^1_I,\bar{w}^1_{II})}{h(\bar{w}^1_I,z)}\frac{f(\bar{t}^2_I,\bar{t}^2_{II})}{h(\bar{t}^2_I,\bar{w}^1_I)f(\bar{t}^2_I,\bar{w}^1_{II})}\prod_{s=3}^{i-1}\frac{f(\bar{t}^s_I,\bar{t}^s_{II})}{h(\bar{t}^s_I,\bar{t}^{s-1}_I)f(\bar{t}^s_I,\bar{t}^{s-1}_{II})},\end{aligned} \tag{D.64}$$

In the second line the sum goes for the partitions $\bar{t}^{N-1}\vdash\bar{t}^{N-1}_I\cup\bar{t}^{N-1}_{II}$,$\bar{\omega}^\nu\vdash\bar{\omega}^\nu_I\cup\bar{\omega}^\nu_{II}$, for $\nu=1,\dots,N-2$ where $\bar{\omega}^\nu=\{-z,\bar{t}^\nu\}$ and $\#\bar{\omega}^\nu_I=\#\bar{t}^{N-1}_I=1$. We also set $\bar{\omega}^0_I=\{-z\}$ and $\bar{\omega}^0_{II}=\emptyset$. In the second line $\bar{w}^1=\{z,\bar{t}^1\}$ and the sum goes over all the partitions of $\bar{w}^1\vdash\{\bar{w}^1_I,\bar{w}^1_{II}\}$, $\bar{t}^\nu\vdash\{\bar{t}^\nu_I,\bar{t}^\nu_{II}\}$ where $\#\bar{w}^1_I=\#\bar{t}^\nu_I=1$ and $\nu=2,\dots,i-1$.

The recursion for the set $\bar{t}^{N-1}$ is

$$\begin{aligned}\bar{\mathbf{Z}}(\{\bar{t}^s\}_{s=1}^{N-2},\{z,\bar{t}^{N-1}\}) = \sum_{i=1}^{N}\sum_{j=1}^{N-1}f(\bar{t}^1,-z)\sum_{\text{part}}\mathbf{K}_{j,i}(-z)\bar{\mathbf{Z}}(\{\bar{t}^s_{II}\}_{s=1}^{N-2},\bar{t}^{N-1})\mathbf{K}^{-1}_{N,1}(-z)\times\\ \prod_{s=1}^{i-1}\frac{f(\bar{t}^s_I,\bar{t}^s_{II})}{h(\bar{t}^s_I,\bar{t}^{s-1}_I)f(\bar{t}^s_I,\bar{t}^{s-1}_{II})}\frac{\prod_{\nu=j}^{N-2}g(\bar{t}^{\nu+1}_{III},\bar{t}^\nu_{III})f(\bar{t}^\nu_{III},\bar{t}^\nu_{II})f(\bar{t}^\nu_{III},\bar{t}^\nu_I)}{\prod_{\nu=j}^{N-1}f(\bar{t}^\nu_{III},\bar{t}^{\nu-1})},\end{aligned} \tag{D.65}$$

where we sum up to the partitions $\bar{t}^\nu\vdash\bar{t}^\nu_I\cup\bar{t}^\nu_{II}\cup\bar{t}^\nu_{III}$ for $\nu=1,\dots,N-2$ where $\#\bar{t}^\nu_I=\Theta(i-1-\nu)$, $\#\bar{t}^\nu_{III}=\Theta(\nu-j)$ and $\bar{t}^{N-1}_{III}=\{z\}$, $\bar{t}^{N-1}_{II}=\bar{t}^{N-1}$ and $\bar{t}^0_I=\{-z\}$.

Proof. We can see that this formula matches equations (4.11) and (4.12) from the paper [43] for scalar quantities. In the lemma concerning the crossed highest coefficient (HC), we saw that the proof can be generalized to matrix-valued quantities with minimal modifications. This also holds true in the uncrossed case. If we follow the derivation in section E.1 of [43] and apply the substitution

$$\mathbf{G}^{(2)}(z)=\mathbf{K}^{(2)}_{N-1,2}(z)=\mathbf{K}_{N-1,2}(z)-\mathbf{K}_{N-1,1}(z)\mathbf{K}^{-1}_{N,1}(z)\mathbf{K}_{N,2}(z) \tag{D.66}$$

using the commutation relation $[\mathbf{B},\mathbf{G}^{(2)}(z)]=0$, we obtain the proof of the lemma. □

Lemma 27. *The crossed HC has poles at $t^1_l\to -t^1_k-1$:*

$$\bar{\mathbf{Z}}(\bar{t})\to\frac{1}{t^1_l+t^1_k+1}\frac{f(\bar{\tau}^1,t^1_l)f(\bar{\tau}^1,t^1_k)}{f(\bar{t}^2,t^1_l)f(\bar{t}^2,t^1_k)}\bar{\mathbf{Z}}(\bar{\tau})\mathbf{F}^{(1)}(t^1_k)+reg. \tag{D.67}$$

where $\bar{\tau}=\bar{t}\backslash\{t^1_k,t^1_l\}$.

Proof. We follow section F.1 ($\nu = 1$) of [43], which concerns the poles of the scalar HC. The proof proceeds by induction on r_1.

Let us calculate the pole for $r_1 = 2$. Substitute into (D.52). At the pole $t^1 + z + 1 \to 0$, only the first term on the right-hand side contains a contribution.

$$\bar{\mathbf{Z}}(\{z, t^1\}, \{\bar{t}^\nu\}_{\nu=2}^{N-1}) \to -\frac{1}{t^1 + z + 1} \frac{1}{f(\bar{t}^2, -z-1) f(\bar{t}^2, z)} \mathbf{K}_{1,1}^{-1}(z) \bar{\mathbf{Z}}(\emptyset, \{\bar{t}^\nu\}_{\nu=2}^{N-1}) \mathbf{G}^{(2)}(z) + reg. \tag{D.68}$$

Since the HC $\bar{\mathbf{Z}}(\emptyset, \{\bar{t}^\nu\}_{\nu=2}^{N-1})$ is built from the components of second nested K-mátrix $\mathbf{K}^{(2)}$ which commute with $\mathbf{K}_{1,1}^{-1}$, we have

$$\left[\mathbf{K}_{1,1}^{-1}(z), \bar{\mathbf{Z}}(\emptyset, \{\bar{t}^\nu\}_{\nu=2}^{N-1})\right] = 0, \tag{D.69}$$

therefore we just proved (D.67) for $r_2 = 2$.

Now assume the statement holds for r_1 or fewer Bethe roots of type $\bar{t}^1$. Use the recursive equation (D.52) for $r_1 + 1$ first-type roots:

$$\begin{aligned} \bar{\mathbf{Z}}(\{z, \bar{t}^1\}, \{\bar{t}^\nu\}_{\nu=2}^{N-1}) = &-\sum_{\text{part}} \mathbf{K}_{1,1}^{-1}(z) \bar{\mathbf{Z}}(\bar{t}_{\text{II}}^1, \{\bar{t}^\nu\}_{\nu=2}^{N-1}) \mathbf{G}^{(2)}(z) \frac{f(\bar{t}^1, -z-1)}{f(\bar{t}^2, -z-1) f(\bar{t}^2, z)} \frac{f(\bar{t}_{\text{I}}^1, \bar{t}_{\text{II}}^1)}{h(\bar{t}_{\text{I}}^1, -z-1)} - \\ &- \sum_{i=2}^{N} \mathbf{K}_{1,1}^{-1}(z) \mathbf{K}_{1,i}(z) \sum_{\text{part}} \bar{\mathbf{Z}}(\bar{w}_{\text{II}}^1, \{\bar{t}_{\text{II}}^\nu\}_{\nu=2}^{i-1}, \{\bar{t}^\nu\}_{\nu=i}^{N-1}) \frac{f(\bar{w}_{\text{I}}^1, \bar{w}_{\text{II}}^1)}{h(\bar{w}_{\text{I}}^1, z)} \frac{f(\bar{t}_{\text{I}}^2, \bar{t}_{\text{II}}^2)}{h(\bar{t}_{\text{I}}^2, \bar{w}_{\text{I}}^1) f(\bar{t}_{\text{I}}^2, \bar{w}_{\text{II}}^1)} \prod_{s=3}^{i-1} \frac{f(\bar{t}_{\text{I}}^s, \bar{t}_{\text{II}}^s)}{h(\bar{t}_{\text{I}}^s, \bar{t}_{\text{I}}^{s-1}) f(\bar{t}_{\text{I}}^s, \bar{t}_{\text{II}}^{s-1})}. \end{aligned} \tag{D.70}$$

Now examine the pole at $t_k^1 + t_l^1 + 1 = 0$. Since the HC is a symmetric function of the first-type roots, it is sufficient to prove the statement for this pair. On the right-hand side of the equation, only the HCs contain such a pole, but in those HCs, there are r_1 or fewer first-type Bethe roots. We apply the induction hypothesis to those. The pole appears in the first line only if $t_k^1, t_l^1 \in \bar{t}_{\text{II}}^1$, and in the second line if $t_k^1, t_l^1 \in \bar{w}_{\text{II}}^1$. The pole can be written in the following form

$$\bar{\mathbf{Z}}(\{z, \bar{t}^1\}, \{\bar{t}^\nu\}_{\nu=2}^{N-1}) \to \frac{1}{t_l^1 + t_k^1 + 1} \frac{f(\bar{\mathrm{w}}^1, t_l^1) f(\bar{\mathrm{w}}^1, t_k^1)}{f(\bar{t}^2, t_l^1) f(\bar{t}^2, t_k^1)} \mathbf{Q}(\bar{\mathrm{w}}^1, \{\bar{t}^\nu\}_{\nu=2}^{N-1}) \mathbf{F}^{(1)}(t_k^1) + reg. \tag{D.71}$$

where $\bar{\mathrm{w}}^1 = \bar{w}^1 \backslash \{t_k^1, t_l^1\} = \{z, \bar{\tau}^1\}$. Additionally, we introduce the notation $\bar{\mathrm{w}}_{\text{II}}^1 = \bar{w}_{\text{II}}^1 \backslash \{t_k^1, t_l^1\}$. Using the induction hypothesis, the right-hand side of (D.70) for each HC gives a Q-operator that can be expressed as follows

$$\begin{aligned} \mathbf{Q}(\{z, \bar{\tau}^1\}, \{\bar{t}^\nu\}_{\nu=2}^{N-1}) = &-\sum_{\text{part}} \mathbf{K}_{1,1}^{-1}(z) \bar{\mathbf{Z}}(\bar{\tau}_{\text{II}}^1, \{\bar{t}^\nu\}_{\nu=2}^{N-1}) \mathbf{G}^{(2)}(z) \frac{f(\bar{\tau}^1, -z-1)}{f(\bar{t}^2, -z-1) f(\bar{t}^2, z)} \frac{f(\bar{\tau}_{\text{I}}^1, \bar{\tau}_{\text{II}}^1)}{h(\bar{\tau}_{\text{I}}^1, -z-1)} - \\ &- \sum_{i=2}^{N} \mathbf{K}_{1,1}^{-1}(z) \mathbf{K}_{1,i}(z) \sum_{\text{part}} \bar{\mathbf{Z}}(\bar{\mathrm{w}}_{\text{II}}^1, \{\bar{t}_{\text{II}}^\nu\}_{\nu=2}^{i-1}, \{\bar{t}^\nu\}_{\nu=i}^{N-1}) \frac{f(\bar{\mathrm{w}}_{\text{I}}^1, \bar{\mathrm{w}}_{\text{II}}^1)}{h(\bar{\mathrm{w}}_{\text{I}}^1, z)} \frac{f(\bar{t}_{\text{I}}^2, \bar{t}_{\text{II}}^2)}{h(\bar{t}_{\text{I}}^2, \bar{\mathrm{w}}_{\text{I}}^1) f(\bar{t}_{\text{I}}^2, \bar{\mathrm{w}}_{\text{II}}^1)} \prod_{s=3}^{i-1} \frac{f(\bar{t}_{\text{I}}^s, \bar{t}_{\text{II}}^s)}{h(\bar{t}_{\text{I}}^s, \bar{t}_{\text{I}}^{s-1}) f(\bar{t}_{\text{I}}^s, \bar{t}_{\text{II}}^{s-1})}, \end{aligned} \tag{D.72}$$

where we used the commutation relation

$$[\mathbf{F}^{(1)}(u), \mathbf{G}^{(2)}(v)] = 0. \tag{D.73}$$

We observe that the right-hand side of (D.72) is exactly the right-hand side of the recursive equation (D.52), i.e.,

$$\mathbf{Q}(\{z, \bar{\tau}^1\}, \{\bar{t}^\nu\}_{\nu=2}^{N-1}) = \bar{\mathbf{Z}}(\{z, \bar{\tau}^1\}, \{\bar{t}^\nu\}_{\nu=2}^{N-1}). \tag{D.74}$$

Substituting back into (D.71), the lemma holds for $r_1 + 1$ first-type Bethe roots. □

Lemma 28. *The uncrossed HC-s have a pole at $t_l^{N-1} \to -t_k^1$:*

$$\bar{\mathbf{Z}}(\bar{t}) \to \frac{1}{t_l^{N-1} + t_k^1} \frac{f(\bar{\tau}^1, t_k^1) f(\bar{\tau}^{N-1}, t_l^{N-1})}{f(\bar{t}^2, t_k^1)} \bar{\mathbf{Z}}(\bar{\tau}) \mathbf{F}^{(1)}(t_k^1) + reg. \tag{D.75}$$

where $\bar{\tau} = \bar{t} \backslash \{t_k^1, t_l^{N-1}\}$.

Proof. We follow section F.2 ($\nu = 1$) of [43], which concerns the poles of the scalar highest coefficient (HC). The proof proceeds by induction on both r_1 and r_{N-1}.

Let us calculate the pole for $r_1 = r_{N-1} = 1$:

$$\bar{\mathbf{Z}}(\{z\},\{\bar{t}^\nu\}_{\nu=2}^{N-2},\{t^{N-1}\}) \to \frac{1}{t^{N-1}+z}\times$$
$$\sum_{\text{part}} \mathbf{K}_{N,1}^{-1}(z)\bar{\mathbf{Z}}(\emptyset,\{\bar{\omega}_{\text{II}}^\nu\}_{\nu=2}^{N-2},\emptyset)\mathbf{G}^{(2)}(z)\prod_{s=2}^{N-2}\frac{f(\omega_{\text{I}}^s,\bar{\omega}_{\text{II}}^s)}{h(\omega_{\text{I}}^s,\omega_{\text{I}}^{s-1})f(\omega_{\text{I}}^s,\bar{\omega}_{\text{II}}^{s-1})}\frac{1}{h(-z,\omega_{\text{I}}^{N-2})f(-z,\bar{\omega}_{\text{II}}^{N-2})f(\bar{t}^2,z)}+reg. \tag{D.76}$$

In the denominator there is a factor $f(-z,\bar{\omega}_{\text{II}}^{N-2})$ therefore the residue is nonzero only when $-z\notin\bar{\omega}_{\text{II}}^{N-2}\Rightarrow\bar{\omega}_{\text{I}}^{N-2}=\{-z\}$. In an analogous way, the terms $f(\bar{\omega}_{\text{I}}^s,\bar{\omega}_{\text{II}}^{s-1})$ imply that $\bar{\omega}_{\text{I}}^s=\{-z\}$ for $s=2,\ldots,N-3$ which means there is only one non-vanishing term in the sum

$$\bar{\mathbf{Z}}(\{z\},\{\bar{t}^\nu\}_{\nu=2}^{N-2},\{t^{N-1}\}) \to \frac{1}{t^{N-1}+z}\frac{1}{f(\bar{t}^2,z)}\mathbf{K}_{N,1}^{-1}(z)\bar{\mathbf{Z}}(\emptyset,\{\bar{t}^\nu\}_{\nu=2}^{N-2},\emptyset)\mathbf{G}^{(2)}(z)+reg. \tag{D.77}$$

Since the HC $\bar{\mathbf{Z}}(\emptyset,\{\bar{t}^\nu\}_{\nu=2}^{N-2},\emptyset)$ is build from the components of second nested K-mátrix $\mathbf{K}^{(2)}$ which commute with $\mathbf{K}_{N,1}^{-1}$, we have

$$\left[\mathbf{K}_{N,1}^{-1}(z),\bar{\mathbf{Z}}(\emptyset,\{\bar{t}^\nu\}_{\nu=2}^{N-2},\emptyset)\right]=0, \tag{D.78}$$

therefore we just proved (D.75) for $r_1=r_{N-1}=1$.

The proof for $r_1+r_{N-1}>2$ can be done by induction. In the lemma concerning the crossed HCs, we saw that the proof is entirely analogous to the scalar case, provided we use the commutation relations of the F- and G-operators. The proof here is also analogous to the scalar case found in Appendix F.2 ($\nu=1$) of [43], if we use the relation

$$[\mathbf{F}^{(1)}(u),\mathbf{G}^{(1)}(v)]=[\mathbf{F}^{(1)}(u),\mathbf{G}^{(2)}(v)]=0. \tag{D.79}$$

□

We see that Lemmas 27 and 28 together prove Theorem 9 for $\nu=1$. Theorem 9 can also be proven by another induction, which involves a very similar computation to the $\nu=1$ case. The proof for the scalar HC in the case $\nu>1$ is found in Appendices F.1 and F.2 of [43], and this can be applied to the matrix-valued HCs without modification, one only needs to use the relation

$$[\mathbf{F}^{(\nu)}(u),\mathbf{G}^{(\mu)}(v)]=0. \tag{D.80}$$

D.3 Proofs for the overlap functions $S^{(\ell)}$

D.3.1 Pair structure limit of the off-shell overlaps

In this section, we prove Theorem 10.

Proof. To prove the theorem, we need to take the limit of the off-shell overlap as $t_l^{\tilde{\nu}}\to -t_k^\nu-\nu\tilde{c}$. Since the overlap depends only on the α-functions, the derivative terms X_k^ν can arise from the poles of the HCs. From the sum rule of the overlap, we see that the HC $\mathbf{Z}(\bar{t}_{\text{I}})$ has a pole if $t_k^\nu,t_l^{\tilde{\nu}}\in\bar{t}_{\text{I}}$ and $\bar{\mathbf{Z}}(\bar{t}_{\text{I}})$ has a pole if $t_k^\nu,t_l^{\tilde{\nu}}\in\bar{t}_{\text{II}}$. Introducing the usual notation $\tau=\bar{t}\backslash\{t_k^\nu,t_l^{\tilde{\nu}}\}$ the part of the overlap formula proportional to the poles comes from summing over the partitions of $\bar{\tau}$. For a given partition $\bar{\tau}\vdash\bar{\tau}_{\text{I}}\cup\bar{\tau}_{\text{II}}$ two terms contribute from the original sum: $\bar{t}_{\text{I}}=\bar{\tau}_{\text{I}}\cup\{t_k^\nu,t_l^{\tilde{\nu}}\}$, $\bar{t}_{\text{II}}=\bar{\tau}_{\text{II}}$ and $\bar{t}_{\text{I}}=\bar{\tau}_{\text{I}}$, $\bar{t}_{\text{II}}=\bar{\tau}_{\text{II}}\cup\{t_k^\nu,t_l^{\tilde{\nu}}\}$. Based on this, the limit of the overlap as $t_l^{\tilde{\nu}}\to -t_k^\nu-\nu\tilde{c}$ can be written as follows

$$\mathbf{S}_{\bar{\alpha},\mathbf{B}}(\bar{t})\to\frac{1}{t_l^{\tilde{\nu}}+t_k^\nu+\nu\tilde{c}}\left(1-\alpha_\nu(t_k^\nu)\alpha_{\tilde{\nu}}(t_l^{\tilde{\nu}})\right)\sum_{\text{part}(\bar{\tau})}\frac{\prod_{s=1}^{N-1}f(\bar{\tau}_{\text{II}}^s,\bar{\tau}_{\text{I}}^s)}{\prod_{s=1}^{N-2}f(\bar{\tau}_{\text{II}}^{s+1},\bar{\tau}_{\text{I}}^s)}\bar{\mathbf{Z}}(\bar{\tau}_{\text{II}})\mathbf{F}^{(\nu)}(t_k^\nu)\mathbf{B}\mathbf{Z}(\bar{\tau}_{\text{I}})\times$$
$$\times\frac{f(t_k^\nu,\bar{\tau}_{\text{I}}^\nu)}{f(t_k^\nu,\bar{\tau}_{\text{I}}^{\nu-1})}\frac{f(\bar{\tau}_{\text{II}}^\nu,t_k^\nu)}{f(\bar{\tau}_{\text{II}}^{\nu+1},t_k^\nu)}\frac{f(t_l^{\tilde{\nu}},\bar{\tau}_{\text{I}}^{\tilde{\nu}})}{f(t_l^{\tilde{\nu}},\bar{\tau}_{\text{I}}^{\tilde{\nu}-1})}\frac{f(\bar{\tau}_{\text{II}}^{\tilde{\nu}},t_l^{\tilde{\nu}})}{f(\bar{\tau}_{\text{II}}^{\tilde{\nu}+1},t_l^{\tilde{\nu}})}\prod_{s=1}^{N-1}\alpha_s(\bar{\tau}_{\text{I}}^s)+reg. \tag{D.81}$$

Here we used the commutation relations (4.4) and (4.9). After rearrangement, we obtain:

$$\mathbf{S}_{\bar{\alpha},\mathbf{B}}(\bar{t})\to\frac{1}{t_l^{\tilde{\nu}}+t_k^\nu+\nu\tilde{c}}\left(1-\alpha_\nu(t_k^\nu)\alpha_{\tilde{\nu}}(t_l^{\tilde{\nu}})\right)\frac{f(\bar{\tau}^\nu,t_k^\nu)}{f(\bar{t}^{\nu+1},t_k^\nu)}\frac{f(\bar{\tau}^{\tilde{\nu}},t_l^{\tilde{\nu}})}{f(\bar{t}^{\tilde{\nu}+1},t_l^{\tilde{\nu}})}\sum_{\text{part}(\bar{\tau})}\frac{\prod_{s=1}^{N-1}f(\bar{\tau}_{\text{II}}^s,\bar{\tau}_{\text{I}}^s)}{\prod_{s=1}^{N-2}f(\bar{\tau}_{\text{II}}^{s+1},\bar{\tau}_{\text{I}}^s)}\bar{\mathbf{Z}}(\bar{\tau}_{\text{II}})\mathbf{F}^{(\nu)}(t_k^\nu)\mathbf{B}\mathbf{Z}(\bar{\tau}_{\text{I}})\times$$
$$\times\frac{f(t_k^\nu,\bar{\tau}_{\text{I}}^\nu)}{f(\bar{\tau}_{\text{I}}^\nu,t_k^\nu)}\frac{f(\bar{\tau}_{\text{I}}^{\nu+1},t_k^\nu)}{f(t_k^\nu,\bar{\tau}_{\text{I}}^{\nu-1})}\frac{f(t_l^{\tilde{\nu}},\bar{\tau}_{\text{I}}^{\tilde{\nu}})}{f(\bar{\tau}_{\text{I}}^{\tilde{\nu}},t_l^{\tilde{\nu}})}\frac{f(\bar{\tau}_{\text{I}}^{\tilde{\nu}+1},t_l^{\tilde{\nu}})}{f(t_l^{\tilde{\nu}},\bar{\tau}_{\text{I}}^{\tilde{\nu}-1})}\prod_{s=1}^{N-1}\alpha_s(\bar{\tau}_{\text{I}}^s)+reg. \tag{D.82}$$

Using the definitions of X_k^ν (4.24) and α_s^{mod} (4.25), we get:

$$\mathbf{S}_{\bar{\alpha},\mathbf{B}}(\bar{t}) \to X_k^\nu \frac{f(\bar{\tau}^\nu, t_k^\nu)}{f(\bar{t}^{\nu+1}, t_k^\nu)} \frac{f(\bar{\tau}^{\tilde{\nu}}, t_l^{\tilde{\nu}})}{f(\bar{t}^{\tilde{\nu}+1}, t_l^{\tilde{\nu}})} \times \\ \sum_{\text{part}(\bar{\tau})} \frac{\prod_{s=1}^{N-1} f(\bar{\tau}_{\text{II}}^s, \bar{\tau}_{\text{I}}^s)}{\prod_{s=1}^{N-2} f(\bar{\tau}_{\text{II}}^{s+1}, \bar{\tau}_{\text{I}}^s)} \bar{\mathbf{Z}}(\bar{\tau}_{\text{II}}) \mathbf{F}^{(\nu)}(t_k^\nu) \mathbf{B} \mathbf{Z}(\bar{\tau}_{\text{I}}) \prod_{s=1}^{N-1} \alpha_s^{mod}(\bar{\tau}_{\text{I}}^s) + \tilde{\mathbf{S}}_{\alpha,\mathbf{B}}, \tag{D.83}$$

where $\tilde{\mathbf{S}}_{\alpha,\mathbf{B}}$ does not depend on X_k^ν. Taking the trace:

$$\text{tr}_{\mathcal{V}_B}\left[\mathbf{S}_{\bar{\alpha},\mathbf{B}}(\bar{t})\right] \to X_k^\nu \frac{f(\bar{\tau}^\nu, t_k^\nu)}{f(\bar{t}^{\nu+1}, t_k^\nu)} \frac{f(\bar{\tau}^{\tilde{\nu}}, t_l^{\tilde{\nu}})}{f(\bar{t}^{\tilde{\nu}+1}, t_l^{\tilde{\nu}})} \times \\ \sum_{\ell=1}^{d_B} \mathcal{F}_\ell^{(\nu)}(t_k^\nu) \beta_\ell \sum_{\text{part}(\bar{\tau})} \frac{\prod_{s=1}^{N-1} f(\bar{\tau}_{\text{II}}^s, \bar{\tau}_{\text{I}}^s)}{\prod_{s=1}^{N-2} f(\bar{\tau}_{\text{II}}^{s+1}, \bar{\tau}_{\text{I}}^s)} \left(\mathbf{Z}(\bar{\tau}_{\text{I}}) \bar{\mathbf{Z}}(\bar{\tau}_{\text{II}})\right)_{\ell,\ell} \prod_{s=1}^{N-1} \alpha_s^{mod}(\bar{\tau}_{\text{I}}^s) + \tilde{\mathbf{S}}_{\alpha,\mathbf{B}}, \tag{D.84}$$

i.e.,

$$\sum_{\ell=1}^{d_B} \beta_\ell S_{\bar{\alpha}}^{(\ell)}(\bar{t}) \to X_k^\nu \frac{f(\bar{\tau}^\nu, t_k^\nu)}{f(\bar{t}^{\nu+1}, t_k^\nu)} \frac{f(\bar{\tau}^{\tilde{\nu}}, t_l^{\tilde{\nu}})}{f(\bar{t}^{\tilde{\nu}+1}, t_l^{\tilde{\nu}})} \sum_{\ell=1}^{d_B} \mathcal{F}_\ell^{(\nu)}(t_k^\nu) \beta_\ell S_{\bar{\alpha}^{mod}}^{(\ell)}(\bar{\tau}) + \text{tr}\tilde{\mathbf{S}}_{\alpha,\mathbf{B}}. \tag{D.85}$$

Assuming that the β_ℓ variables are linearly independent, we obtain the desired formula. □

D.3.2 On-shell limit

In this section, we prove Theorem 13. To prove the theorem, we need to define the Korepin criterion and state a lemma related to it.

Definition 29. Let $\mathcal{N}^{(\bar{r})}(\bar{t}|\bar{X})$ be a function of $2\sum_{s=1}^n r_s$ variables, where $\bar{t} = \{\bar{t}^s\}_{s=1}^n$ and $\bar{X} = \{\bar{X}^s\}_{s=1}^n$ with cardinalities $\#\bar{t}^s = \#\bar{X}^s = r_s$. The Korepin criterion is defined as follows:

(i) The function $\mathcal{N}^{(\bar{r})}(\bar{t}|\bar{X})$ is symmetric over the replacement of the pairs $(X_j^\mu, t_j^\mu) \leftrightarrow (X_k^\mu, t_k^\mu)$.

(ii) It is linear function of each X_j^μ.

(iii) $\mathcal{N}^{(\bar{1}_\nu)}(\dots, \emptyset, \{t^\nu\}, \emptyset, \dots | \dots, \emptyset, \{X^\nu\}, \emptyset, \dots) = X^\nu$, where $\bar{1}_\nu \in \mathbb{N}^n$ is an n-component vector whose components are defined as $(\mathbf{1}_\nu)_k = \delta_{k,\nu}$ for $k = 1, \dots, n$.

(iv) The coefficient of X_j^μ is given by the function $\mathcal{N}^{(\bar{r}-\bar{1}_\mu)}$ with modified parameters X_k^ν

$$\frac{\partial \mathcal{N}^{(\bar{r})}(\bar{t}|\bar{X})}{\partial X_j^\mu} = \mathcal{N}^{(\bar{r}-\bar{1}_\mu)}(\bar{\tau}, \bar{\mathcal{X}}^{mod}), \tag{D.86}$$

where $\bar{\tau} = \bar{t}\backslash\{t_k^\mu\}$, $\bar{\mathcal{X}} = \bar{X}\backslash\{X_k^\mu\}$ and the original variables X_k^ν should be replaced by $X_k^{\nu,mod}$ which are defined with (4.24) and (4.25).

(v) $\mathcal{N}^{(\bar{r})}(\bar{t}|\bar{X}) = 0$, if all $X_j^\mu = 0$.

Lemma 30. *If a set of functions $\mathcal{N}^{(\bar{r}^+)}(\bar{t}^+|\bar{X}^+)$ satisfies the Korepin criterion, then*

$$\mathcal{N}^{(\bar{r}^+)}(\bar{t}^+|\bar{X}^+) = \det G_+. \tag{D.87}$$

Proof. The proof proceeds recursively in the total number of variables $\mathbf{r}^+ = \sum_{s=1}^n r_s^+$. This is the same as Proposition 4.1 in [49]. □

Using the above lemma and the earlier Theorem 10 we can now easily prove Theorem 13 for on-shell overlaps.

Proof. The derivation of Theorem 13 follows Appendix H of [43]. We begin with the crossed case and introduce the normalized overlap functions.

$$\mathcal{N}^{(\bar{r}^+)}(\bar{t}^+|\bar{X}^+) = \frac{S^{(\ell)}(\bar{t}^+|\bar{X}^+)}{\prod_{\nu=1}^{N-1}\mathcal{F}_\ell^{(\nu)}(\bar{t}^{+,\nu})\prod_{k\neq l} f(t_l^{+,\nu},t_k^{+,\nu})\prod_{k<l} f(t_l^{+,\nu},-t_k^{+,\nu}-\nu)f(-t_k^{+,\nu}-\nu,t_l^{+,\nu})}. \tag{D.88}$$

According to Lemma 30, it is sufficient to show that the functions $\mathcal{N}^{(\bar{r}^+)}$ satisfy the Korepin conditions. Property (i) follows from the definition of the overlaps. Properties (ii) and (iv) follow from Theorem 10. Property (v) follows from the fact that for a Bethe vector that does not satisfy the pair structures, the on-shell overlap is zero, i.e., the overlap

$$\langle \text{MPS}|\mathbb{B}(\bar{t}) = \sum_{\ell=1}^{d_B}\beta_\ell S_{\bar{\alpha}}^{(\ell)}(\bar{t}) \tag{D.89}$$

vanishes in the on-shell limit. In the generalized model, the variables $\alpha_k^\nu \equiv \alpha_\nu(t_k^\nu)$ are algebraically independent of the t_k^ν, so any set of Bethe roots can be on-shell with appropriately chosen α's. Thus, in the generalized model, the on-shell limit is equivalent to:

$$\alpha_k^\nu \to \mathcal{A}_k^\nu(\bar{t}) \equiv \frac{f(t_k^\mu,\bar{t}_k^\mu)}{f(\bar{t}_k^\mu,t_k^\mu)}\frac{f(\bar{t}^{\mu+1},t_k^\mu)}{f(t_k^\mu,\bar{t}^{\mu-1})}. \tag{D.90}$$

The on-shell limit of a general overlap is zero.

$$\lim_{\alpha_k^\nu\to\mathcal{A}_k^\nu(\bar{t})}\langle \text{MPS}|\mathbb{B}(\bar{t}) = \sum_{\ell=1}^{d_B}\beta_\ell\left(\lim_{\alpha_k^\nu\to\mathcal{A}_k^\nu(\bar{t})} S_{\bar{\alpha}}^{(\ell)}(\bar{t})\right) = 0. \tag{D.91}$$

Assuming the β_ℓ are also independent variables, the overlap functions vanish in the on-shell limit.

$$\lim_{\alpha_k^\nu\to\mathcal{A}_k^\nu(\bar{t})} S_{\bar{\alpha}}^{(\ell)}(\bar{t}) = 0. \tag{D.92}$$

The definition of $\mathcal{N}^{(\bar{r}^+)}$ involves two limits: the pair structure limit and the on-shell limit. These limits are not interchangeable, if we take the on-shell limit first, we get zero; however, if we start with the pair structure limit, we obtain $\mathcal{N}^{(\bar{r}^+)}$, i.e.,

$$\lim_{\alpha_k^\nu\to\mathcal{A}_k^\nu(\bar{t})}\lim_{t^-\to\pi(\bar{t}^+)} S_{\bar{\alpha}}^{(\ell)}(\bar{t}) = S^{(\ell)}(\bar{t}^+|\bar{X}^+). \tag{D.93}$$

The reason the two limits are not interchangeable is that the overlap function contains formal first-order poles, and in the pair structure limit, the first-order terms of the expressions $\alpha_\nu(t_k^{-,\nu})$ in $(t_k^{-,\nu}+t_k^{+,\nu}+\nu)$ are needed:

$$\alpha_\nu(t_k^{-,\nu}) = \alpha_\nu(t_k^{+,\nu})^{-1} + (t_k^{-,\nu}+t_k^{+,\nu}+\nu)\alpha_\nu(t_k^{+,\nu})^{-2}\alpha'_\nu(t_k^{+,\nu}) + \mathcal{O}((t_k^{-,\nu}+t_k^{+,\nu}+\nu)^2). \tag{D.94}$$

It is evident that the limits are not interchangeable because if we take the pair structure limit first, we get terms proportional to $\alpha'_\nu(t_k^{+,\nu})$ which we do not get if we start with the on-shell limit. If $X_k^\nu = 0$, then $\alpha'_\nu(t_k^{+,\nu}) = 0$, and in this case, the two limits are interchangeable, i.e.,

$$S^{(\ell)}(\bar{t}^+|\bar{0}) = 0, \tag{D.95}$$

meaning that (v) also holds. Property (iii) follows from the Theorem 10 and (v). The proof in the non-crossed case is completely analogous. □

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