Published for SISSA by Springer

Received: September 13, 2024
Accepted: November 12, 2024
Published: December 4, 2024

Dual overlaps and finite coupling 't Hooft loops

Tamas Gombor[a,b] **and Zoltán Bajnok**[a]

[a]*HUN-REN Wigner Research Centre for Physics, Konkoly-Thege Miklós u. 29-33, 1121 Budapest, Hungary*

[b]*Department of Theoretical Physics, Eötvös Loránd University, Pázmány Péter sétány 1/A, 1117 Budapest, Hungary*

E-mail: gombort@caesar.elte.hu, bajnok.zoltan@wigner.hun-ren.hu

Abstract: Integrable $su(2|2)_c$ symmetric models have integrable boundaries with $osp(2|2)$ symmetries, which can be embedded into $su(2|2)_c$ in two different ways. We dualize the previously obtained asymptotic overlap formulas for one of the embeddings to describe the other embedding and apply the results to describe the asymptotic expectation values of local operators in the presence of a 't Hooft line in $\mathcal{N}=4$ SYM. A peculiar feature of the setting is that in certain gradings only descendant states have non-vanishing overlaps with the boundary state and the overlap formula is not factorized for the Bethe roots.

Keywords: Bethe Ansatz, Integrable Field Theories, AdS-CFT Correspondence, Wilson, 't Hooft and Polyakov loops

ArXiv ePrint: 2408.14901

https://doi.org/10.1007/JHEP12(2024)034

JHEP12(2024)034

Contents

1 Introduction

Integrable boundaries have been getting considerable interest recently in statistical physics as well as in the AdS/CFT correspondence. The developments were motivated by quench problems in spin chains on the statistical physics side [1–5], while in the AdS/CFT correspondence they describe various conformal defects in the gauge theory and related probe branes in the string theory [6–13]. In most of the applications the physically relevant quantities are the overlaps of multi-particle finite volume states with integrable boundary states [14–16].

The prototypical AdS/CFT duality relates the $\mathcal{N}=4$ supersymmetric Yang-Mills (SYM) theory to strings propagating on the $AdS_5 \times S^5$ background [17]. The total superconformal symmetry of the conformal field theory $psu(2,2|4)$ is manifested as isometries of the supercoset string sigma model. By introducing a codimension 1 defect in the CFT, with prescribed boundary conditions for the fields, one can preserve half of the supersymmetry [18]. The defect breaks partially the translational symmetry, which implies that single trace operators acquire non-trivial vacuum expectation values (VEVs). The space-time dependence of the

VEV is completely determined in terms of the scaling dimension, but its proportionality factor is a non-trivial function of the 't Hooft coupling. As single trace operators correspond to multi-particle finite volume states the proportionality factor turns out to be the overlap of this state with an integrable boundary state [6, 19].

Integrable boundary states have to satisfy severe consistency requirements including the boundary Yang-Baxter equations for their K-matrix [20–22]. These equations involve the scattering matrix of the excitations, which has a factorized form with $su(2|2)_c \oplus su(2|2)_c$ symmetry, the remnant of $psu(2,2|4)$ after fixing the gauge and quantizing string theory. In our works [21, 23] we classified the solutions for integrable factorized K-matrices (one of these solutions were also found in [24] independently), which came in two versions, both having $osp(2|2)$ symmetry, such that together with $su(2|2)_c$ they form a symmetric pair. The difference lies in the embedding of this symmetry into $su(2|2)_c$, i.e. whether the bosonic $su(2)$ part of $osp(2|2)$ lies in the Lorentzian or in the R-symmetry part. The codimension 1 defect implies that it is in the Lorentzian part, thus in order to describe the expectation values of single trace operators we developed a procedure and calculated the corresponding overlaps for this type of boundary states. The result is very general and applies in all the cases with the same symmetry.

Recently a 't Hooft line was introduced in the CFT, i.e. a codimension 3 defect and its integrability properties were investigated [12]. Its presence implies specific boundary conditions for the SYM fields and the expectation values of single trace operators were connected to overlaps of multiparticle states with a boundary state, which seemed to be integrable. This was supported by directly calculating the overlaps in various subsectors at leading order. Since the geometrical setting breaks the symmetries as expected for the other $osp(2|2)$ embedding we expect that the all loop overlaps should be calculated from our integrable K-matrix. The aim of our paper is to calculate the asymptotic overlaps for the corresponding K-matrix.

Since the two K-matrices are related by a duality transformation we need to understand how duality acts on $su(2|2)_c$ overlaps. It turns out that due to the specific selection rules, highest weight Bethe states have vanishing overlaps with the boundary state and only their descendants can have non-zero overlaps. In order to investigate this phenomenon we analyze the rational version of the problem, i.e. dualities for overlaps in rational $su(2|2)$ spin chains and their action on overlaps. This is also useful to make contact with the LO 't Hooft loop calculations.

The paper is organized as follows: first, in section 2 we recall the integrable K-matrices for an $su(2|2)_c$ symmetric scattering matrix. Having introduced the Bethe ansatz equations and integrable K-matrices we present the previously calculated asymptotic overlaps for one of these K-matrices. We then calculate its weak coupling, rational limit. In section 3 we analyze the two types of rational K-matrices and the overlaps in various gradings. We pay particular attention to degenerate cases. In section 4 we elevate the results for $gl(4|4)$. The various fermionic dualities enable us to make correspondences with all the existing overlaps in the 't Hooft loop setting. In section 5, using a fermionic duality, we determine the overlap formulas for the other type of K-matrix. These results are then used in section 6 to describe the asymptotic overlaps for the 't Hooft loop. We conclude in section 7. We

JHEP12(2024)034

added several appendices, in which we investigate the condition whether the Bethe state can have a non-vanishing overlap with the boundary state, and how to calculate the overlaps of descendent states in the various cases.

2 K-matrices, overlaps and their weak coupling limit

In the prototypical duality there are 8 fermionic and 8 bosonic excitations, which scatters on each other in an integrable way. Due to the factorized $su(2|2)_c \oplus su(2|2)_c$ symmetry the scattering matrix has also a factorized form

$$\mathbb{S}(p_1, p_2) = S_0(p_1, p_2) S(p_1, p_2) \otimes S(p_1, p_2) \tag{2.1}$$

The centrally extended $su(2|2)_c$ symmetry completely fixes the matrix structure of the scattering matrix $S(p_1, p_2)$ [25] of the particles labeled by $(1, 2|3, 4)$, where $1, 2$ are considered to be bosonic, while $3, 4$ as fermionic. Their dispersion relation is also fixed by the symmetry

$$\epsilon(p_j) = ig\left(x_j^- - \frac{1}{x_j^-} - x_j^+ + \frac{1}{x_j^+}\right); \qquad x_j^\pm = e^{\pm i\frac{p_j}{2}} \frac{1 + \sqrt{1 + 16g^2 \sin^2 \frac{p_j}{2}}}{4g \sin \frac{p_j}{2}} \tag{2.2}$$

where the 't Hooft coupling is $\lambda = 16\pi^2 g$ and the momentum of the particles is simply $e^{ip_j} = \frac{x_j^+}{x_j^-}$.

2.1 Periodic $su(2|2)_c$ spin chains

In calculating the asymptotic, large volume (R-charge) spectrum one has to diagonalize an inhomogeneous $su(2|2)_c$ transfer matrix

$$t(p_0) = \mathrm{Tr}_0(S(p_0, p_1) \dots S(p_0, p_{2L})) \tag{2.3}$$

This can be done by the Bethe ansatz method (which could be coordinate or algebraic). As a first step one finds a simple eigenstate of the transfer matrix: all state being the same, which can be either bosonic or fermionic. Typically we choose the 1s, and denote this bosonic pseudovacuum as $|0^b\rangle = \otimes^{2L}|1\rangle$. One can then introduce excitations, by turning 1s into other labels and build a plane wave out of them. The y roots are the momenta of the plane waves of excitations of 3s. Finally, w labels the momenta when 4s are created by flipping 3s. The eigenvalue is then labeled by the set of magnons: y-roots $\{y_k\}_{k=1\dots N}$ and w-roots $\{w\}_{l=1\dots M}$ as

$$t(u)|\mathbf{y}, \mathbf{w}\rangle = \Lambda(u, \mathbf{y}, \mathbf{w})|\mathbf{y}, \mathbf{w}\rangle \tag{2.4}$$

where we reparametrized p_0 with u as $x_0^\pm + \frac{1}{x_0^\pm} = u \pm \frac{i}{2g}$. The eigenvalue $\Lambda(u, \mathbf{y}, \mathbf{w})$ can be written as

$$\Lambda(u, \mathbf{y}, \mathbf{w}) = e^{i\frac{p_0(u)}{2}(N-2L)} \frac{\mathcal{R}^{(+)+}}{\mathcal{R}^{(+)-}} \left\{ \frac{\mathcal{R}^{(-)+}\mathcal{R}_y^-}{\mathcal{R}^{(+)+}\mathcal{R}_y^+} - \frac{\mathcal{R}_y^- Q_w^{++}}{\mathcal{R}_y^+ Q_w} - \frac{\mathcal{B}_y^+ Q_w^{--}}{\mathcal{B}_y^- Q_w} + \frac{\mathcal{B}^{(+)-}\mathcal{B}_y^+}{\mathcal{B}^{(-)-}\mathcal{B}_y^-} \right\} \tag{2.5}$$

where

$$\mathcal{R}_y(u) = \prod_{j=1}^{N} (x(u) - y_j); \qquad Q_w(u) = \prod_{l=1}^{M} (u - w_l); \qquad \mathcal{R}^{(\pm)}(u) = \prod_{j=1}^{2L} (x(u) - x_j^\pm). \tag{2.6}$$

JHEP12(2024)034

and the $\mathcal{B}$ quantities can be obtained from the $\mathcal{R}$-s by replacing $x(u)$ with $1/x(u)$:

$$\mathcal{B}_y(u) = \prod_{j=1}^{N}(1/x(u) - y_j)\,; \qquad \mathcal{B}^{(\pm)}(u) = \prod_{j=1}^{2L}(1/x(u) - x_j^{\pm}) \tag{2.7}$$

Shifts are understood as $f^{\pm}(u) = f(u \pm \frac{i}{2g})$ and we assumed that the total momentum vanishes: $\sum_j p_j = 0$, which is enforced by the level matching condition. Finally, $x(u)$ is defined by $x(u) + 1/x(u) = u$.

The actual values for the roots can be obtained from the Bethe ansatz equations, which originate from the regularity of the transfer matrix. Indeed, the transfer matrix has to be regular at $x^+(u) = y_j$, which implies the quantization conditions

$$\frac{\mathcal{R}_y^-}{\mathcal{R}_y^+}\left\{\frac{\mathcal{R}^{(-)+}}{\mathcal{R}^{(+)+}} - \frac{Q_w^{++}}{Q_w}\right\}_{x^+(u)=y_j} = \text{regular} \quad\longrightarrow\quad \left\{\frac{\mathcal{R}^{(-)}}{\mathcal{R}^{(+)}} - \frac{Q_w^{+}}{Q_w^{-}}\right\}_{x(u)=y_j} = 0 \tag{2.8}$$

The corresponding Bethe ansatz equation reads explicitly as

$$e^{i\phi_{v_j}} := \prod_{k=1}^{2L} e^{-ip_k/2}\frac{y_j - x_k^+}{y_j - x_k^-}\prod_{k=1}^{M}\frac{v_j - w_k + \frac{i}{2g}}{v_j - w_k - \frac{i}{2g}} = 1 \tag{2.9}$$

where $v_j = y_j + y_j^{-1}$. Similarly, the w-roots are quantized via the regularity at $u = w_l$

$$\left\{\frac{\mathcal{R}_y^- Q_w^{++}}{\mathcal{R}_y^+} + \frac{\mathcal{B}_y^+ Q_w^{--}}{\mathcal{B}_y^-}\right\}_{u=w_l} = 0 \tag{2.10}$$

Using that

$$\mathcal{Q}_v := \frac{\mathcal{R}_y\mathcal{B}_y}{\prod_j(-y_j)} = \prod_{j=1}^{N}(u - v_j) \tag{2.11}$$

we can write the corresponding BA equation as

$$e^{i\phi_{w_j}} := \prod_{k=1}^{N}\frac{w_j - v_k + \frac{i}{2g}}{w_j - v_k - \frac{i}{2g}}\prod_{\substack{l=1\\ l\neq j}}^{M}\frac{w_j - w_l - \frac{i}{g}}{w_j - w_l + \frac{i}{g}} = 1 \tag{2.12}$$

We note that the Bethe ansatz method provides eigenstates, which are highest weight states for the symmetry algebra. Descendent states can be obtained by applying symmetry transformations. They show up in the Bethe ansatz equations as roots at infinities. The Bethe ansatz equations assumed a specific order of creating excitations ($1 \to 3, 3 \to 4$), i.e. they are valid in a specific *grading*. Later we explain how to switch between different gradings.

2.2 Integrable boundary states and overlaps

Integrable boundaries can be represented as boundary states in spin chains [20]. They can be parametrized by K-matrices, which satisfy the boundary K-Yang Baxter equations involving the scattering matrix S. These equations are very restrictive and in the $su(2|2)_c$ case we found two types of solutions [21]:

$$K^{(1)} = \begin{pmatrix} k_1 & k_2 + e^{(1)} & 0 & 0 \\ k_2 - e^{(1)} & k_4 & 0 & 0 \\ 0 & 0 & 0 & f^{(1)} \\ 0 & 0 & -f^{(1)} & 0 \end{pmatrix}, \quad K^{(2)} = \begin{pmatrix} 0 & f^{(2)} & 0 & 0 \\ -f^{(2)} & 0 & 0 & 0 \\ 0 & 0 & k_1 & k_2 + e^{(2)} \\ 0 & 0 & k_2 - e^{(2)} & k_4 \end{pmatrix}. \tag{2.13}$$

where

$$e^{(1)}(p) = \frac{-i}{x_s}\frac{x^+ + x_s^2 x^-}{1 + x^+ x^-}, \quad f^{(1)}(p) = \frac{i}{x_s}\sqrt{\frac{x^-}{x^+}}\frac{(x^+)^2 - x_s^2}{1 + x^+ x^-}, \tag{2.14}$$

with $k_1 k_4 - k_2^2 = 1$, and

$$e^{(2)}(p) = \frac{i}{x_s}\frac{x^- + x_s^2 x^+}{1 + x^+ x^-}, \quad f^{(2)}(p) = \frac{i}{x_s}\sqrt{\frac{x^+}{x^-}}\frac{(x^-)^2 - x_s^2}{1 + x^+ x^-}. \tag{2.15}$$

and some parameters k_i and x_s. Both matrices have $osp(2|2)$ symmetry, but they differ in which way this symmetry is embedded into $su(2|2)_c$.

Every solution of the K-Yang Baxter equation leads to an integrable boundary state. We first create a two-site state and then distribute it homogeneously aloung the chain

$$\langle \Psi_K | = \langle \psi_1 | \otimes \langle \psi_2 | \otimes \cdots \otimes \langle \psi_L | ; \qquad \langle \psi | = \sum_{a,b} \langle a | \otimes \langle b | K_{ab} \tag{2.16}$$

We investigated the overlap corresponding to the boundary state built from $K^{(1)}$ in detail and found that the non-vanishing overlap requires the following selection rules for the roots. The momenta of the particles, i.e. the inhomogeneities of the spin chain, should come in pairs $p_i = -p_{2L+1-i}$ as well as the v-s, $v_i = -v_{N+1-i}$ and w-s, $w_i = -w_{M+1-i}$, and their numbers should be related as $N = 2M$. In this case the overlap takes the form [21]

$$\frac{|\langle \Psi_{K^{(1)}} | \mathbf{y}, \mathbf{w} \rangle|^2}{\langle \mathbf{y}, \mathbf{w} | \mathbf{y}, \mathbf{w} \rangle} = k_1^{2L-N} x_s^{-N} \frac{\mathcal{R}_y(x_s)^2}{\mathcal{R}_y(0)} \frac{1}{Q_w(0) Q_w(\frac{i}{2g})} \frac{\det G^+}{\det G^-}, \tag{2.17}$$

where $G^{\pm}$ are related to the factorization of the Gaudin determinant for the paired state

$$G = \begin{vmatrix} \left(\partial_{v_i} \phi_{v_j}\right)_{N \times N} & \left(\partial_{v_i} \phi_{w_j}\right)_{N \times M} \\ \left(\partial_{w_i} \phi_{v_j}\right)_{M \times N} & \left(\partial_{w_i} \phi_{w_j}\right)_{M \times M} \end{vmatrix} = G_+ G_- \tag{2.18}$$

These formulae are valid for even M. For odd M the function $Q_w(u)$ has a zero root $w_{\frac{M+1}{2}} = 0$, $Q_w(u) = u \prod_{j=1}^{\frac{M-1}{2}} (u^2 - w_j^2) = u \bar{Q}_w(u)$. In this case $\bar{Q}_w$ has to be used, instead of Q_w. Also the factorization of the Gaudin determinant involves this special root in a particular way, see [21, 26, 27] for details.

Our result is very general and describes overlaps in various physical situations. The different cases are distinguished by how x_s depends on the coupling constant.

2.3 Weak coupling limit

At weak coupling the $su(2|2)_c$ symmetry reduces to $su(2|2)$ and the inhomogeneous spin chain becomes rational. In this spin chain we have three different type of roots $u^{(i)}$, which describe how we flip the labels starting from all the 1s, and three different type of Q_i functions, $Q_i = \prod_{j=1}^{N_i}(u - u_j^{(i)})$. The y type roots can scale at weak coupling either as $y \sim \frac{u^{(1)}}{g}$ or as $y \sim \frac{g}{u^{(3)}}$ distinguishing between type 1 and type 3 roots. The w roots scale as $w \sim \frac{u^{(2)}}{g}$ and they become the type 2 roots. A type 1 root $u^{(1)}$ describes how to create 3s from 1s, $u^{(2)}$ creates 4s from 3s, while $u^{(3)}$ creates 2s from 4s. They correspond to a specific grading.

The limit of the K-matrix depends on the behavior of x_s at small coupling. If x_s does not depend on g, the K-matrix in the weak coupling limit reads as

$$K^{(1)}(p) = \begin{pmatrix} k_1 & k_2 & 0 & 0 \\ k_2 & k_4 & 0 & 0 \\ 0 & 0 & 0 & \frac{i}{x_s} \\ 0 & 0 & -\frac{i}{x_s} & 0 \end{pmatrix}, \tag{2.19}$$

and the limit of the overlap formula is

$$\frac{|\langle \Psi_{K^{(1)}} | \mathbf{u} \rangle|^2}{\langle \mathbf{u} | \mathbf{u} \rangle} = k_1^{2L-N_1-N_3} x_s^{N_3-N_1} \frac{Q_1(0)Q_3(0)}{Q_2(0)Q_2(i/2)} \frac{\det G^+}{\det G^-}. \tag{2.20}$$

If, however, x_s is defined in a g-dependent way, say as $x_s + x_s^{-1} = \frac{is}{g}$, then the K-matrix in the weak coupling limit is degenerate

$$K^{(1)} = \begin{pmatrix} k_1 & k_2 + \frac{s}{u+i/2} & 0 & 0 \\ k_2 - \frac{s}{u+i/2} & k_4 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}. \tag{2.21}$$

and the limit of the overlap formula is

$$\frac{|\langle \Psi_{K^{(1)}} | \mathbf{u} \rangle|^2}{\langle \mathbf{u} | \mathbf{u} \rangle} = k_1^{2L-2N_1} \frac{Q_1(is)^2 Q_3(0)}{Q_1(0)Q_2(0)Q_2(i/2)} \frac{\det G^+}{\det G^-}, \tag{2.22}$$

with the selection rules $N_1 = N_2 = N_3$. These results are valid in a specific grading.

We can easily calculate the weak coupling limit of the second type of boundary K-matrix, $K^{(2)}$ and realize that $K^{(1)}$ and $K^{(2)}$ are related by the $1 \leftrightarrow 3, 2 \leftrightarrow 4$ flips. This, however, is nothing but choosing a different grading. In the following we investigate how changing the grading will transform the overlap formulas in order to describe the overlaps with the K-matrix $K^{(2)}$.

3 Overlaps and dualities for rational $su(2|2)$ spin chains

First we recall the overlaps in the $su(2)$ spin chains as we will encounter the same quantities later when we analyze the $su(2|2)$ spin chain.

The generic $\mathfrak{su}(2)$ K-matrix is nothing but the one, which appeared in the weak coupling limit in a 2 by 2 box and has the form

$$K(u) = \begin{pmatrix} k_1 & k_2 + \frac{s}{u+i/2} \\ k_2 - \frac{s}{u+i/2} & k_4 \end{pmatrix} \tag{3.1}$$

The corresponding boundary state has the following overlap with the Bethe states [28, 29]

$$\frac{|\langle \Psi_K | \mathbf{u} \rangle|^2}{\langle \mathbf{u} | \mathbf{u} \rangle} \sim \frac{Q_1(is)^2}{Q_1(0)Q_1(\frac{i}{2})} \frac{\det G^+}{\det G^-} \tag{3.2}$$

In this section we do not pay attention on scalar prefactors. The overlap has two interesting limits. For $s \to 0$ the K-matrix becomes symmetric and the overlap takes the form

$$K(u) = \begin{pmatrix} k_1 & k_2 \\ k_2 & k_4 \end{pmatrix}; \qquad \frac{|\langle \Psi_K | \mathbf{u} \rangle|^2}{\langle \mathbf{u} | \mathbf{u} \rangle} \sim \frac{Q_1(0)}{Q_1(\frac{i}{2})} \frac{\det G^+}{\det G^-} \tag{3.3}$$

while in the opposite $s \to \infty$ limit the K-matrix is anti-symmetric with the overlap

$$K(u) = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}; \qquad \frac{|\langle \Psi_K | \mathbf{u} \rangle|^2}{\langle \mathbf{u} | \mathbf{u} \rangle} \sim \frac{1}{Q_1(0)Q_1(\frac{i}{2})} \frac{\det G^+}{\det G^-} \tag{3.4}$$

Let us turn now to the $su(2|2)$ case.

3.1 Gradings, nesting and overlaps in the regular case

For the rational $su(2|2)$ spin chains there exist two types of integrable K-matrices corresponding to the two different realizations of the unbroken $osp(2|2)$ symmetry and they come with chiral pair structures:

$$K^{(1)} = \begin{pmatrix} k_1 & k_2 & 0 & 0 \\ k_2 & k_4 & 0 & 0 \\ 0 & 0 & 0 & -s \\ 0 & 0 & s & 0 \end{pmatrix}, \qquad K^{(2)} = \begin{pmatrix} 0 & -s & 0 & 0 \\ s & 0 & 0 & 0 \\ 0 & 0 & k_1 & k_2 \\ 0 & 0 & k_2 & k_4 \end{pmatrix}. \tag{3.5}$$

They are related to the $g \to 0$ limits of the two K-matrices for the $su(2|2)_c$ case. We fix the normalization of the K-matrices by demanding that $k_1 k_4 - k_2^2 = 1$. These two K-matrices are related to each other by the $1 \leftrightarrow 3, 2 \leftrightarrow 4$ changes. As these changes are also related to nestings and gradings we recall them now.

In the diagonalization of the super spin chain transfer matrices we can use different paths for the nesting. Let us choose the pseudo vacuum as A_1 and the magnons $u^{(1)}, u^{(2)}, u^{(3)}$ such a way, that they change the flavors as $A_1 \to A_2, A_2 \to A_3, A_3 \to A_4$, respectively, where $A_k \in \{1,2,3,4\}$ all distinct. We denote this nesting as (A_1, A_2, A_3, A_4). In particular, for the nesting $(1,2,3,4)$ the Bethe state with magnon numbers N_1, N_2, N_3 will have the following number of 1, 2, 3 and 4 labels: $\#1 = L - N_1$, $\#2 = N_1 - N_2$, $\#3 = N_3 - N_4$ and $\#4 = N_3$. This nesting corresponds to the grading $(++--)$ and the Dynkin-diagram $\bigcirc - \otimes - \bigcirc$, where

$\pm$ corresponds to bosonic/ferminic indices and the bosonic $\bigcirc$ versus fermionic $\otimes$ nodes encode if there is a change in the nature of the labels. Another example is the nesting $(1, 3, 4, 2)$ where $\#1 = L - N_1$, $\#2 = N_3$, $\#3 = N_1 - N_2$ and $\#4 = N_2 - N_3$. The grading is $(+ - -+)$, while the corresponding Dynkin-diagram is $\otimes - \bigcirc - \otimes$. Bethe states depend on the nesting.

The overlap of a Bethe state with a boundary state depends both on the nesting and the type of the K-matrix. For the nesting (A_1, A_2, A_3, A_4) and K-matrix $K^{(i)}$ we denote it by

$$S_i^{(A_1,A_2,A_3,A_4)}(\mathbf{u}) = \frac{|\langle \Psi_i | \mathbf{u} \rangle|^2}{\langle \mathbf{u} | \mathbf{u} \rangle} \tag{3.6}$$

Since the two K-matrices are connected by the changes $1 \leftrightarrow 3$; $2 \leftrightarrow 4$, the overlaps are also related

$$S_2^{(A_1,A_2,A_3,A_4)}(\mathbf{u}) = S_1^{(\bar{A}_1,\bar{A}_2,\bar{A}_3,\bar{A}_4)}(\mathbf{u}), \tag{3.7}$$

where $\bar{1} = 3; \bar{2} = 4; \bar{3} = 1; \bar{4} = 2$. The above changes leave invariant the S-matrix implying that the Bethe equations and the Bethe roots are the same on the two sides. We have already obtained the overlap $S_1^{(1,3,4,2)}$ from the $g \to 0$ limit as

$$S_1^{(1,3,4,2)} \sim \frac{Q_1(0)Q_3(0)}{Q_2(0)Q_2(i/2)} \frac{\det G^+}{\det G^-}, \tag{3.8}$$

with the selection rule $N_1 + N_3 = 2N_2$. By using the relation (3.7) we can easily write the overlap for the other type of K-matrix in the opposite grading

$$S_2^{(3,1,2,4)} \sim \frac{Q_1(0)Q_3(0)}{Q_2(0)Q_2(i/2)} \frac{\det G^+}{\det G^-}. \tag{3.9}$$

The natural question is, how we could describe this second overlap in the original grading or in any other gradings. In order to get this result, we have to apply fermionic dualities [30]. The fermionic duality expresses the Bethe roots $\mathbf{u}$ in a given grading with the dual Bethe roots $\tilde{\mathbf{u}}$ in the dual grading. The original overlap written in terms of the dual roots provides the overlap in the dual grading $S_2^{\mathbf{A}}(\mathbf{u}) = S_2^{\tilde{\mathbf{A}}}(\tilde{\mathbf{u}})$. There is, however an important difference. Although the original overlap corresponds to a (highest weight) Bethe state $|\mathbf{u}\rangle$ the dual overlap corresponds only to a descendent state, which can obtained by acting with some fermionic generator $|\mathbf{u}\rangle = \mathbb{Q}|\tilde{\mathbf{u}}\rangle$ on the h.w. Bethe state $|\tilde{\mathbf{u}}\rangle$ in the dual grading. This implies that

$$\frac{|\langle \Psi_{K^{(2)}} | \mathbf{u} \rangle|^2}{\langle \mathbf{u} | \mathbf{u} \rangle} = S_2^{\mathbf{A}}(\mathbf{u}) = S_2^{\tilde{\mathbf{A}}}(\tilde{\mathbf{u}}) = \frac{|\langle \Psi_{K^{(2)}} | \mathbb{Q} | \tilde{\mathbf{u}} \rangle|^2}{|\mathbb{Q}|\tilde{\mathbf{u}}\rangle|^2} \tag{3.10}$$

In order to calculate this (descendant) overlap we use the fermionic dualities. The duality transformation involves two steps: the changing of the Gaudin determinant and using the QQ-relations. The Gaudin determinant transforms as

$$\frac{\det G^+}{\det G^-} \sim \begin{cases} \frac{Q_{a-1}(i/2)Q_{a+1}(i/2)}{\bar{Q}_a(0)\bar{\tilde{Q}}_a(0)} \frac{\det \tilde{G}^+}{\det \tilde{G}^-}, & \text{if } N_a = \text{even} \\ \frac{\bar{Q}_a(0)\bar{\tilde{Q}}_a(0)}{Q_{a-1}(i/2)Q_{a+1}(i/2)} \frac{\det \tilde{G}^+}{\det \tilde{G}^-}, & \text{if } N_a = \text{odd} \end{cases} \tag{3.11}$$

where $\tilde{Q}_a$ denotes the Q-function after the duality transformation, for the new type of a root. The QQ-relation reads as

$$Q_a(u)\tilde{Q}_a(u) \sim Q_{a+1}(u+i/2)Q_{a-1}(u-i/2) - Q_{a+1}(u-i/2)Q_{a-1}(u+i/2) \tag{3.12}$$

In showing these relations, one has to be careful and investigate the role of zero roots. Their presence depends on the parity of $N_{a-1}+N_{a+1}$ and they imply non-trivial relations between Q-functions. For instance, if $N_{a-1}+N_{a+1}$ is odd then $\bar{Q}_a(0)\bar{\tilde{Q}}_a(0) \sim Q_{a+1}(i/2)Q_{a-1}(i/2)$ and the Gaudin determinants are proportional to each other. A careful analysis shows that both N_1 and N_3 should be even and $N_1+N_3=2N_2$. In this case the following transformation rule can be applied:

$$Q_a(0) \to \tilde{Q}_a(0)^{-1}Q_{a-1}(i/2)Q_{a+1}(i/2) \tag{3.13}$$

Let us start from the overlap (3.9) and perform a duality on Q_3. Using (3.13) we obtain

$$S_2^{(3,1,4,2)} \sim \frac{Q_1(0)}{Q_2(0)\tilde{Q}_3(0)}\frac{\det\tilde{G}^+}{\det\tilde{G}^-} \longrightarrow \frac{Q_1(0)}{Q_2(0)Q_3(0)}\frac{\det G^+}{\det G^-}. \tag{3.14}$$

Since we keep track of the grading in the notation it would be confusing to carry the tilde over the duality (and further dualities) thus we suppress it in the notation. Each quantity, Q , $\det G^{\pm}$ is understood in the respective grading.

A further duality at Q_1 provides the sought for overlap in the grading $(1,3,4,2)$

$$S_2^{(1,3,4,2)} \sim \frac{Q_2(\frac{i}{2})}{Q_1(0)Q_2(0)Q_3(0)}\frac{\det G^+}{\det G^-}. \tag{3.15}$$

We can also reach by dualities the overlap in the grading $(1,2,3,4)$

$$S_2^{(1,2,3,4)} \sim \frac{1}{Q_1(0)Q_1(i/2)}\frac{Q_2(0)}{1}\frac{Q_3(0)}{Q_3(i/2)}\frac{\det G^+}{\det G^-} \tag{3.16}$$

which agrees with the overlap formulas of the $su(2)$ subsectors. All of these overlaps have factorized forms but they might not correspond to Bethe states in the new grading. In the appendix we investigate carefully in which gradings the boundary state can have overlaps with Bethe states and calculate carefully the scalar coefficients.

3.2 Non-invertible K-matrices and overlaps of descendant states

Let us now investigate the non-invertible K-matrices, since they appear in the relevant applications. They again come in two types

$$K^{(1)} = \begin{pmatrix} k_1 & k_2+\frac{s}{u+i/2} & 0 & 0 \\ k_2-\frac{s}{u+i/2} & k_4 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \quad K^{(2)} = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & k_1 & k_2+\frac{s}{u+i/2} \\ 0 & 0 & k_2-\frac{s}{u+i/2} & k_4 \end{pmatrix}, \tag{3.17}$$

which are the weak coupling versions of the K-matrices when x_s is g-dependent. Since in the $g\to 0$ limit we obtained the overlap for the first type of K-matrix

$$S_1^{(1,3,4,2)} \sim \frac{Q_1(is)^2 Q_3(0)}{Q_1(0)Q_2(0)Q_2(\frac{i}{2})}\frac{\det G^+}{\det G^-} \tag{3.18}$$

we can use the transformation $1\leftrightarrow 3$; $2\leftrightarrow 4$ to obtain the overlap of the second type of K-matrix in the opposite grading

$$S_2^{(3,1,2,4)} \sim \frac{Q_1(is)^2 Q_3(0)}{Q_1(0)Q_2(0)Q_2(\frac{i}{2})}\frac{\det G^+}{\det G^-}. \tag{3.19}$$

which has a factorized form. We now would like to calculate this overlap in the original grading $(1,3,4,2)$. This amounts to use the duality transformation for Q_1:

$$S_2^{(1,3,2,4)} \sim \frac{Q_1(0)\tilde{Q}_1(is)^2 Q_3(0)}{Q_2(0)Q_2(\frac{i}{2})^2}\frac{\det G^+}{\det G^-}. \tag{3.20}$$

written in terms of the new $Q_1(0)$. But then the previous $Q_1(is)$ term, denoted by $\tilde{Q}_1(is)$ in this grading, is problematic as it should be expanded by the QQ-relations $\tilde{Q}_1 \sim Q_1^{-1}(Q_\theta^+ Q_2^- - Q_\theta^- Q_2^+)$ used at $s\neq 0$ implying that the overlap $S_2^{(1,3,2,4)}$ does not have a factorized form

$$S_2^{(1,3,2,4)} \sim \frac{Q_1(0)Q_1^{-1}((Q_\theta^+(is)Q_2^-(is)-Q_\theta^-(is)Q_2^+(is))^2 Q_3(0)}{Q_1(is)^2 Q_2(0)Q_2(\frac{i}{2})^2}\frac{\det G^+}{\det G^-}. \tag{3.21}$$

This seems to contradict with the assumptions in [30], that overlaps of Bethe states have factorized forms. The way out is what we explained, that the Bethe state under duality transforms to a descendant state. If both the Bethe state and its descendants have non-zero overlaps then they differ only by some constant factors, what we typically omit. If, however, the highest weight Bethe state has a zero overlap, and only the descendants overlap, then this overlap is not necessarily factorizing.

For $K^{(2)}$ the non-vanishing overlaps require the selection rule $\#1=\#2=0$. From considerations in appendix A it follows that there are no Bethe states with non-vanishing overlaps for the gradings $(1,2,3,4)$, $(1,3,4,2)$, $(1,3,2,4)$, $(3,1,2,4)$ and $(3,1,4,2)$ therefore only descendant states can have non-vanishing overlaps there. For the grading $(3,4,1,2)$ Bethe states can have non-trivial overlaps. This overlap is non-vanishing only in the $\mathfrak{su}(2)$ sector when $N_2=N_3=0$. Since we know the $\mathfrak{su}(2)$ overlaps, we can write that

$$S_2^{(3,4,1,2)} \sim \frac{Q_1(is)^2}{Q_1(0)Q_1(\frac{i}{2})}\frac{\det G^+}{\det G^-}. \tag{3.22}$$

This grading is very special for the boundary state corresponding to $K^{(2)}$ since this is the only one, when there are Bethe-states with non-vanishing overlaps. The corresponding overlap has actually a factorized form. For gradings where only descendant states can have non-vanishing overlaps, the overlap does not necessarily factorize. Indeed, for the grading $(3,1,2,4)$ it has but for $(1,2,3,4)$ or $(1,3,4,2)$ it does not.

In appendix A we investigate explicitly the relations between Bethe states in one grading and descendant states in other gradings. Besides determining the requirement for non-vanishing overlaps we also demonstrate how the precise overlaps for descendant states can be calculated. This amounts to use heavily the $su(2|2)$ symmetry algebra (A.2) and the $osp(2|2)$ symmetry of the boundary state (A.5).

We close the section by summarizing what we have learned so far.

- The overlaps corresponding to the two types of K-matrices (3.5) can be described by the same formulae, but they belong to opposite gradings (3.7).
- In order to obtain the overlaps of the other type of K-matrix in the original grading, or other gradings, we can use the fermionic duality formulae. These formulae, however, in special gradings can lead to non-factorizing overlaps, signaling that the overlap of the Bethe state vanishes. There are special gradings where all Bethe states have vanishing overlaps and only descendants state can overlap with the boundary state.
- For descendant states the overlaps can be expressed in terms of the overlaps with Bethe states in another grading. In doing so one has to use the duality formulas. together with the connection between Bethe states (A.1), the $osp(2|2)$ symmetry of the boundary state (A.5) and the $su(2|2)$ symmetry algebra (A.2).

We are going to face similar situation for the centrally extended $su(2|2)_c$ algebra and the corresponding K-matrices. Before turning to this situation we extend the previous analysis for $gl(4|4)$ overlaps, which appear at the weak coupling limit of the AdS/dCFT correspondence.

4 Overlaps and dualities for rational $gl(4|4)$ spin chains, the leading order 't Hooft line

The weak coupling limit of the integrable description governing the AdS_5/CFT_4 duality can be described by the $gl(4|4)$ spin chain [31]. In the following we lift the previous result for this case. The bosonic part of the $gl(4|4)$ K-matrices are diagonal sums of the form

$$\mathbb{K} = K^{(+)} \oplus K^{(-)}. \tag{4.1}$$

As for the $su(2|2)$ case, there are two solutions of the bYBE, which can be distinguished how they transform for transposition in the respective sub-spaces. For the first solution $\mathbb{K}^{(1)}$: $\left(K^{(1),(\pm)}\right)^t = \pm K^{(1),(\pm)}$, while for the second $\mathbb{K}^{(2)}$: $\left(K^{(2),(\pm)}\right)^t = \mp K^{(2),(\pm)}$. In the AdS_5/CFT_4 context the $(+)$ bosonic subspace corresponds to the isometries of S^5 while the $(-)$ bosonic subspace to the isometries of AdS_5. The boundary related to $\mathbb{K}^{(1)}$ breaks the isometries of S^5 and AdS_5 to $\mathrm{SO}(3)\times\mathrm{SO}(3)$ and $\mathrm{SO}(2,3)$, respectively, which is the bosonic symmetry of the D3-D5 domain wall defect. Contrary, the boundary related to $\mathbb{K}^{(2)}$ breaks the isometries of S^5 and AdS_5 to $\mathrm{SO}(5)$ and $\mathrm{SO}(2,1)\times\mathrm{SO}(3)$, respectively, which is the symmetry of the 't Hooft line.

Since we have already obtained the weak coupling asymptotic overlaps for $\mathbb{K}^{(1)}$, our goal is to derive the overlaps corresponding to the other K-matrix $\mathbb{K}^{(2)}$ and demonstrate that these overlaps indeed describe the 't Hooft line at weak coupling. We are going to achieve this goal by exploiting the connection between the K-matrices $K^{(1),(\pm)} = K^{(2),(\mp)}$ and using various duality transformations. Let us denote the overlap $S_k^{(\theta_1\theta_2\theta_3\theta_4\theta_5\theta_6\theta_7\theta_8)}$ for the K-matrix $\mathbb{K}^{(k)}$ in the grading θ_j ($\theta_j = \pm$). By using the connection between the K-matrices we have the relation

$$S_2^{(\theta_1\theta_2\theta_3\theta_4\theta_5\theta_6\theta_7\theta_8)} = S_1^{(-\theta_1,-\theta_2,-\theta_3,-\theta_4,-\theta_5,-\theta_6,-\theta_7,-\theta_8)}. \tag{4.2}$$

We start with duality transformations for the first type of overlaps S_1. For the alternating Dynkin diagram $\otimes - \bigcirc - \otimes - \bigcirc - \otimes - \bigcirc - \otimes$ there are two gradings $(+--++--+)$ and $(-++--++-)$, in which the massive particles (related to Q_4) correspond to the SU(2) or SL(2) sectors, respectively. For the grading $(+--++--+)$ and the K-matrix $\mathbb{K}^{(1)}$, the weak coupling overlap is

$$S_1^{(+--++--+)} = \frac{Q_1 Q_3 Q_4 Q_5 Q_7}{Q_2 Q_2^+ Q_4^+ Q_6 Q_6^+} \frac{\det G^+}{\det G^-}. \tag{4.3}$$

where $Q_j = Q_j(0)$, $Q_j^+ = Q_j(\frac{i}{2})$ denote the Q-functions in the given grading. By dualizing Q_3 we can reach the grading with the overlap

$$S_1^{(+-+-+--+)} = \frac{Q_1 Q_4 Q_5 Q_7}{Q_2 Q_3 Q_6 Q_6^+} \frac{\det G^+}{\det G^-}. \tag{4.4}$$

As we explained before the meaning of Q_3 and the ratio of determinants is different from the previous formula and they are all understood in their respective gradings. Performing a duality on Q_5 leads to

$$S_1^{(+-+--+-+)} = \frac{Q_1 Q_4 Q_4^+ Q_7}{Q_2 Q_3 Q_5 Q_6} \frac{\det G^+}{\det G^-}. \tag{4.5}$$

Finally, dualizing Q_1 and Q_7 we obtain

$$S_1^{(-++--++-)} = \frac{Q_2^+ Q_4 Q_4^+ Q_6^+}{Q_1 Q_2 Q_3 Q_5 Q_6 Q_7} \frac{\det G^+}{\det G^-}, \tag{4.6}$$

which is the overlap for the same alternating diagram, we started with, but with the massive particle in the SL(2) sector. By using the identity, which relates the two K-matrices (4.2), we can obtain the overlap of the other K-matrix for the alternating diagram with massive particles in the SU(2) sector as

$$S_2^{(+--++--+)} = \frac{Q_2^+ Q_4 Q_4^+ Q_6^+}{Q_1 Q_2 Q_3 Q_5 Q_6 Q_7} \frac{\det G^+}{\det G^-}. \tag{4.7}$$

In order to compare this overlap with results for the 't Hooft line we perform various duality transformations. Let us first calculate the overlap in the grading $\bigcirc - \otimes - \bigcirc - \bigcirc - \bigcirc - \otimes - \bigcirc$ with $(--++++--)$. Dualizing Q_1 and Q_7 leads to

$$S_2^{(-+-++-+-)} = \frac{Q_1 Q_4 Q_4^+ Q_7}{Q_2 Q_3 Q_5 Q_6} \frac{\det G^+}{\det G^-}. \tag{4.8}$$

By dualizing further Q_2 and Q_6 we obtain

$$S_2^{(--++++--)} = \frac{Q_1 Q_2 Q_4 Q_4^+ Q_6 Q_7}{Q_1^+ Q_3 Q_3^+ Q_5 Q_5^+ Q_7^+} \frac{\det G^+}{\det G^-}. \tag{4.9}$$

We can project this result into the SO(6) subsector by switching off other excitations. The overlap is simply

$$S_2^{(--++++--)} = \frac{Q_4 Q_4^+}{Q_3 Q_3^+ Q_5 Q_5^+} \frac{\det G^+}{\det G^-}. \tag{4.10}$$

JHEP12(2024)034

which completely agrees with the SO(5) overlap formula in the SO(6) sector of the 't Hooft line [12].

By dualizing further we obtain that

$$S_2^{(++-----++)} = \frac{Q_2Q_3Q_4Q_5Q_6}{Q_1Q_1^+Q_3^+Q_4^+Q_5^+Q_7Q_7^+}\frac{\det G^+}{\det G^-}, \tag{4.11}$$

which in the SL(2) subsector leads to

$$S_2^{(++-----++)} = \frac{Q_4}{Q_4^+}\frac{\det G^+}{\det G^-}, \tag{4.12}$$

in complete agreement with [12].

Performing one last duality leads to

$$S_2^{(----++++)} = \frac{Q_1Q_2Q_3Q_4Q_6Q_6^+}{Q_1^+Q_2^+Q_3^+Q_5Q_5^+Q_7Q_7^+}\frac{\det G^+}{\det G^-} \tag{4.13}$$

which provides the overlap in the gluon subsector

$$S_2^{(----++++)} = \frac{Q_3}{Q_3^+}\frac{\det G^+}{\det G^-} \tag{4.14}$$

agreeing with [12].

In conclusion, we have determined the overlaps of the second type of K-matrix, $\mathbb{K}^{(2)}$ in the various gradings and demonstrated that they reproduce all the overlaps which were available for the weak coupling limits of the 't Hooft line. Motivated by this matching we calculate the overlaps of the second type of K-matrix for the $su(2|2)_c$ spin chains in the next section.

5 Overlaps and dualities for the $su(2|2)_c$ spin chain

In the $su(2|2)_c$ spin chain there are only two non-equivalent gradings. In the first the pseudovacuum is a bosonic tensor-product state $|0^b\rangle = |1\rangle^{\otimes 2L}$ from which the y roots create a fermionic label, say 3 and then the w root another fermionic one with 4. Due to the $34 \to 12$ scattering process, states with label 2 are automatically created. We can introduce the bosonic and fermionic quantum numbers $n_b = \#1 - \#2$, $n_f = \#3 - \#4$, such that the corresponding Bethe states $|\mathbf{y}, \mathbf{w}\rangle^b$ have the quantum numbers $n_b = 2L - N$ and $n_f = N - 2M$. We have obtained the normalized overlap in this grading for the first type of K-matrix as

$$S_1^b(\mathbf{y}, \mathbf{w}) = k_1^{2L-N} x_s^{-N} \frac{\mathcal{R}_y(x_s)^2}{\mathcal{R}_y(0)} \frac{1}{Q_w(0)Q_w(\frac{i}{2g})} \frac{\det G^+}{\det G^-}, \tag{5.1}$$

where $N = 2M$.

Alternatively, we could use the fermionic pseudo vacuum $|0^f\rangle = |3\rangle^{\otimes 2L}$ to create Bethe state, by flipping fermions to bosons first, and then bosons to bosons. Actually the transformation $1 \leftrightarrow 3, 2 \leftrightarrow 4$ not only changes the gradings but also exchanges the two K-matrices. This implies that the overlap corresponding to the other K-matrix $K^{(2)}$ can be described by the same overlap formula in the other grading, when the Bethe state is created from the fermionic vacuum

$$S_2^f(\mathbf{y},\mathbf{w}) = S_1^b(\mathbf{y},\mathbf{w}), \tag{5.2}$$

thus

$$S_2^f(\mathbf{y},\mathbf{w}) = k_1^{2L-N} x_s^{-N} \frac{\mathcal{R}_y(x_s)^2}{\mathcal{R}_y(0)} \frac{1}{Q_w(0)Q_w(\frac{i}{2g})} \frac{\det G^+}{\det G^-}, \tag{5.3}$$

where again, the various Q-functions and determinants are understood in their own grading. This could be the result for the overlaps in the case of the 't Hooft line for any coupling, when wrapping effects are neglected. We, however, would like to calculate the same overlap in the bosonic grading. In doing so we need to implement the duality transformations [32].

5.1 Duality relations

Let us dualize the y-roots in order to describe the transfer matrix eigenvalue in the other grading. We start from the bosonic grading and the Bethe state $|\mathbf{y},\mathbf{w}\rangle^b$. The idea is to investigate the quantity, which is related to the y-type Bethe ansatz equation $(\mathcal{R}^{(-)}Q_w^- - \mathcal{R}^{(+)}Q_w^+)$. By definition it vanishes at $x = y_j$. Let us calculate the leading orders of this expression

$$\mathcal{R}^{(-)}Q_w^- - \mathcal{R}^{(+)}Q_w^+ = 0 \times x^{2L+M} + \sum_{c=-M+1}^{2L+M-1} c_k x^k + \left(\mathcal{R}^{(-)}(0) - \mathcal{R}^{(+)}(0)\right) \times x^{-M+1} \tag{5.4}$$

Since $\mathcal{R}^{(-)}(0) = \mathcal{R}^{(+)}(0)$ the expression $x^{M-1}(\mathcal{R}^{(-)}Q_w^- - \mathcal{R}^{(+)}Q_w^+)$ is a polynomial of order $2L + 2M - 2$ and it has zeros at $x = y_j$. Let us denote the remaining zeros by $\tilde{y}_j$ and their generating function by $\tilde{\mathcal{R}}_y$:

$$\tilde{\mathcal{R}}_y = \prod_{j=1}^{\tilde{N}} (x(u) - \tilde{y}_j) \tag{5.5}$$

They are defined by

$$x^{M-1}(\mathcal{R}^{(-)}Q_w^- - \mathcal{R}^{(+)}Q_w^+) = A\mathcal{R}_y\tilde{\mathcal{R}}_y \tag{5.6}$$

where A is a constant and by comparing the degrees we can see that $\tilde{N} = 2L + 2M - N - 2$. We have also a similar equation for the $\mathcal{B}$ quantities:

$$\frac{1}{x^{M-1}}(\mathcal{B}^{(-)}Q_w^- - \mathcal{B}^{(+)}Q_w^+) = A\mathcal{B}_y\tilde{\mathcal{B}}_y. \tag{5.7}$$

The dual Bethe roots $\tilde{y}$ satisfies the dual Bethe equations. They can be obtained by evaluating (5.6) at $x = \tilde{y}_j$ leading to

$$\left\{ \frac{\mathcal{R}^{(-)}}{\mathcal{R}^{(+)}} - \frac{Q_w^+}{Q_w^-} \right\}_{x(u)=\tilde{y}_j} = 0 \tag{5.8}$$

By evaluating (5.6) at $u \to w_l \pm \frac{i}{2g}$ and taking the ratio we have the equations

$$\left\{ \frac{\mathcal{R}_y^+}{\mathcal{R}_y^-} = -\frac{\tilde{\mathcal{R}}_y^-}{\tilde{\mathcal{R}}_y^+} \left(\frac{x^+}{x^-} \right)^{M-1} \frac{\mathcal{R}^{(+)+} Q_w^{++}}{\mathcal{R}^{(-)-} Q_w^{--}} \right\}_{u=w_l} \tag{5.9}$$

and

$$\left\{ \frac{\mathcal{B}_y^+}{\mathcal{B}_y^-} = -\frac{\tilde{\mathcal{B}}_y^-}{\tilde{\mathcal{B}}_y^+} \left(\frac{x^-}{x^+} \right)^{M-1} \frac{\mathcal{B}^{(+)+} Q_w^{++}}{\mathcal{B}^{(-)-} Q_w^{--}} \right\}_{u=w_l} \tag{5.10}$$

This implies that

$$\left\{ \frac{Q_w^{--}}{Q_w^{++}} \frac{\mathcal{R}_y^+ \mathcal{B}_y^+}{\mathcal{R}_y^- \mathcal{B}_y^-} = \frac{\tilde{\mathcal{R}}_y^-}{\tilde{\mathcal{R}}_y^+} \frac{\tilde{\mathcal{B}}_y^-}{\tilde{\mathcal{B}}_y^+} \frac{Q_w^{++}}{Q_w^{--}} = -1 \right\}_{u=w_l}$$

meaning, that the dual roots satisfy the same Bethe equations as the original ones but they correspond to different solutions. The corresponding Bethe states $|\tilde{\mathbf{y}}, \mathbf{w}\rangle^f$ have the quantum numbers $n_b = \tilde{N} - 2M = 2L - N - 2$ and $n_f = 2L - \tilde{N} = N - 2M + 2$, therefore the Bethe states $|\tilde{\mathbf{y}}, \mathbf{w}\rangle^f$ and $|\mathbf{y}, \mathbf{w}\rangle^b$ are not equal but they are in the same $\mathfrak{su}(2|2)_c$ multiplet and they are connected as

$$|\tilde{\mathbf{y}}, \mathbf{w}\rangle^f = \mathbb{Q}_3^1 \mathbb{Q}_2^{\dagger 4} |\mathbf{y}, \mathbf{w}\rangle^b \tag{5.11}$$

where we used the standard notation for the $su(2|2)_c$ generators, see appendix B for details.

5.2 Dual overlap formulae

We now derive the dual overlap formulae. At first let us check the selection rules. From the form of the K-matrix $K^{(2)}$ we can see that the non-vanishing overlap requires $n_b = 0$, i.e. $2L = N$. However, Bethe states could contain maximum $2L - 2$ y-roots, therefore they must have vanishing overlaps and only descendant overlaps can be non-zero. Let us denote a descendant of the Bethe state $|\mathbf{y}, \mathbf{w}\rangle$ by $|\mathbf{y}, \mathbf{w}, d\rangle$ and the corresponding overlap by $S_2^b(\mathbf{y}, \mathbf{w}, d)$. All these descendants are in the same $su(2|2)_c$ multiplet as $|\tilde{\mathbf{y}}, \mathbf{w}\rangle^f$. It implies that their overlaps are proportional to $S_2^f(\tilde{\mathbf{y}}, \mathbf{w})$:

$$S_2^b(\mathbf{y}, \mathbf{w}; d) = C_d S_2^f(\tilde{\mathbf{y}}, \mathbf{w}) = C_d k_1^{2L-N} x_s^{-N} \frac{\tilde{\mathcal{R}}_y(x_s)^2}{\tilde{\mathcal{R}}_y(0)} \frac{1}{Q_w(0) Q_w(\frac{i}{2g})} \frac{\det \tilde{G}^+}{\det \tilde{G}^-}, \tag{5.12}$$

d refers to the specific descendant state and C_d is its corresponding proportionality factor to be fixed.

In expressing the overlap in terms of bosonic quantities we start with the ratio of determinants. We investigated this quantity in many cases numerically and observed that

$$\frac{\det \tilde{G}^+}{\det \tilde{G}^-} = \pm \frac{\det G^+}{\det G^-} \tag{5.13}$$

was always satisfied. The sign difference is irrelevant when we compare squares of overlaps, which is natural as the phases of Bethe states are conventional. The remaining part of the dual overlap formula (5.12) is still not satisfactory as it contains the dual Bethe roots,

$\tilde{\mathcal{R}}_y$. We would like to express the overlap with the original Bethe roots only. In doing so we use the identity

$$\mathcal{R}_y(x_s)\tilde{\mathcal{R}}_y(x_s) = \frac{x_s^{M-1}}{A}(\mathcal{R}^{(-)}(x_s)Q_w^-(s) - \mathcal{R}^{(+)}(x_s)Q_w^+(s)) \tag{5.14}$$

where the prefactor A can be obtained by evaluating the identity at zero:

$$\mathcal{R}_y(0)\tilde{\mathcal{R}}_y(0) = \mathcal{R}^{(+)}(0)\frac{\frac{i}{g}(2L-2M) - A}{A}, \quad A = \left[2\sum_{j=1}^{L}\left(x_j^+ - x_j^-\right) - M\frac{i}{g}\right] \tag{5.15}$$

The overlap then can be written in terms of purely bosonic quantities as

$$\begin{aligned} S_2^b(\mathbf{y}, \mathbf{w}; d) = C_d k_1^{2L-N} x_s^{2M-N-2} &\left(\frac{\mathcal{R}^{(-)}(x_s)Q_w^-(s) - \mathcal{R}^{(+)}(x_s)Q_w^+(s)}{\mathcal{R}_y(x_s)}\right)^2 \\ &\times \frac{\mathcal{R}_y(0)}{\mathcal{R}^{(+)}(0)A(\frac{i}{g}(2L-2M)-A)}\frac{1}{Q_w(0)Q_w(\frac{i}{2g})}\frac{\det G^+}{\det G^-}. \end{aligned} \tag{5.16}$$

Clearly, this is not a factorized overlap which is related to the fact that only the descendants can overlap with the boundary state in this grading. There is only one example when the overlap is factorized which is the $s = 0$ case when

$$S_2^b(\mathbf{y}, \mathbf{w}; d) = C_d k_1^{2L-N} x_s^{2M-N-2}\frac{(\mathcal{R}^{(+)}(i) - (-1)^M\mathcal{R}^{(-)}(i))^2}{\mathcal{R}^{(+)}(0)A(\frac{i}{g}(2L-2M)-A)}\frac{\mathcal{R}_y(0)}{\mathcal{R}_y(i)^2}\frac{Q_w(\frac{i}{2g})}{Q_w(0)}\frac{\det G^+}{\det G^-}. \tag{5.17}$$

We can also take the $g \to 0$ limit. Assuming that the unfixed parameter x_s is regular in the $g \to 0$ limit (i.e. $x_s = a + \mathcal{O}(g)$ where $a \neq 0$) we obtain that

$$\lim_{g\to 0} S_2^b(\mathbf{y}, \mathbf{w}; d) \sim \frac{1}{Q_1 Q_3}\frac{Q_2^+}{Q_2}\frac{\det G^+}{\det G^-}. \tag{5.18}$$

which agrees with our previous result (3.15).

If we would like to calculate also the normalization constants (which is necessary when we want to compare the formulas with the actual overlaps) then we need to know the relation between the descendant state with non-vanishing overlap and the dual state. The simplest case happens when $N = 2L - 2$, since then the descendent state can be

$$|\mathbf{y}, \mathbf{w}; -2, 2\rangle^b := \mathbb{Q}_3^1 \mathbb{Q}_2^{\dagger\, 4}|\mathbf{y}, \mathbf{w}\rangle^b \tag{5.19}$$

which is just $|\tilde{\mathbf{y}}, \mathbf{w}\rangle^f$ implying that $C_{-2,2} = 1$. Here the descendent is labelled by its n_b and n_f quantum numbers. Clearly both $\mathbb{Q}_3^1$ and $\mathbb{Q}_2^{\dagger 4}$ decreases n_b and increases n_f by 1.

Let us see an other example for a descendant of $N = 2L-2$ with a non-vanishing overlap

$$|\mathbf{y}, \mathbf{w}; -2, 0\rangle^b := \mathbb{Q}_3^1 \mathbb{Q}_4^1|\mathbf{y}, \mathbf{w}\rangle^b \tag{5.20}$$

In this case we also need to use the symmetry properties of the boundary state and have to calculate the norm of the descendant state. The calculations are relegated to appendix B. The final result is that $C_{-2,0} = \left(\frac{x_s}{k_1}\right)^2 \frac{(igA+2L-2M)}{(igA+N-2M+1)}$. In appendix B we provide more examples how to calculate the normalizations properly.

We have tested these formulae numerically very extensively for various spin chain sizes and randomly chosen rapidity parameters, so we are quite convinced about their correctness.

JHEP12(2024)034

6 All loop overlap for $\mathbb{K}^{(2)}$, the asymptotic 't Hooft loop

In this section we present formulae for the asymptotic overlaps with $su(2|2)_c \otimes su(2|2)_c$ symmetry. We analyze the boundary K-matrix of the form

$$\mathbb{K}^{(2)}(p) = K_0(p)K^{(2)}(p) \otimes K^{(2)}(p) \tag{6.1}$$

where $K_0(p)$ can be fixed from unitarity, boundary crossing unitarity and additional physical requirements. Different $K_0(p)$ factors correspond to different physical situations. We will not distinguish them and keep $K_0(p)$ in its abstract form.

The large volume spectrum of the AdS5/CFT4 correspondence is described by the asymptotical Bethe ansatz based on the factorizing S-matrix (2.1). We put $2L$ particles in a finite volume J (the R-charge) and demand the periodicity of the wave function. This problem reduces to the diagonalization of a tensor product of two $su(2|2)_c$ transfer matrices $t(u) \otimes t(u)$. They can be done separately for the two copies. We distinguish the left and right factors by an upper index $\alpha = 1, 2$:

$$t(u)|\mathbf{y}^{(\alpha)}, \mathbf{w}^{(\alpha)}\rangle = \Lambda(u, \mathbf{y}^{(\alpha)}, \mathbf{w}^{(\alpha)})|\mathbf{y}^{(\alpha)}, \mathbf{w}^{(\alpha)}\rangle \tag{6.2}$$

They satisfies the same Bethe equations but are typically different solutions. The quantization of the momenta requires

$$e^{\phi_{p_j}} := e^{ip_j J} \prod_{k:k\neq j} S_0(p_j, p_k)\Lambda(u_j, \mathbf{y}^{(1)}, \mathbf{w}^{(1)})\Lambda(u_j, \mathbf{y}^{(2)}, \mathbf{w}^{(2)}) = 1 \tag{6.3}$$

In order to have a non-trivial overlap, momenta should come in pairs $\mathbf{p} = \{\mathbf{p}^+, \mathbf{p}^-\}$, such that $\mathbf{p}^+ = -\mathbf{p}^-$ and $J = 2L$. The overlap of the boundary state built from the tensor product $\mathbb{K}$ matrix factorizes into two $su(2|2)_c$ copies. We should decide again in which grading we are interested in the formulas. The simplest is choosing fermionic grading in both wings, which leads to

$$\frac{\left|\langle \Psi_{\mathbb{K}^{(2)}} |\, \mathbf{p}, \mathbf{y}^{(\alpha)}, \mathbf{w}^{(\alpha)} \rangle\right|^2}{\langle \mathbf{p}, \mathbf{y}^{(\alpha)}, \mathbf{w}^{(\alpha)}\, | \mathbf{p}, \mathbf{y}^{(\alpha)}, \mathbf{w}^{(\alpha)} \rangle} = \prod_{i=1}^{L} |K_0(p_i)|^2 \bar{\mathcal{S}}_2^f(\mathbf{y}^{(1)}, \mathbf{w}^{(1)}) \bar{\mathcal{S}}_2^f(\mathbf{y}^{(2)}, \mathbf{w}^{(2)}) \frac{\det G^+}{\det G^-} \tag{6.4}$$

where

$$\bar{\mathcal{S}}_2^f(\mathbf{y}, \mathbf{w}) = k_1^{2L-N} x_s^{-N} \frac{\mathcal{R}_y(x_s)^2}{\mathcal{R}_y(0)} \frac{1}{Q_w(0) Q_w(\frac{i}{2g})}, \tag{6.5}$$

and the determinants involve also differentiation w.r.t. the momenta:

$$G_{ij}^{\pm} = \left(\partial_{\hat{U}_i^+} \phi_{U_j^+} \pm \partial_{\hat{U}_i^+} \phi_{U_j^-}\right) \tag{6.6}$$

Here $\mathbf{U}^+ = \left\{\mathbf{p}^+, \mathbf{v}^{(\alpha)+}, \mathbf{w}^{(\alpha)+}\right\}$ collects all the variables and $\hat{\mathbf{U}} = \left\{\mathbf{u}^+, \hat{\mathbf{u}}^{(\alpha)+}, \hat{\mathbf{w}}^{(\alpha)+}\right\}$ is the collection of the properly normalized rapidities [21].

If, however, we are interested in the overlaps in the bosonic gradings then we have to use the bosonic overlap formulas for descendant states (5.16)

$$\frac{\left|\langle \Psi_{\mathbb{K}^{(2)}} |\, \mathbf{p}, \mathbf{y}^{(\alpha)}, \mathbf{w}^{(\alpha)}; d^{(\alpha)} \rangle\right|^2}{\langle \mathbf{p}, \mathbf{y}^{(\alpha)}, \mathbf{w}^{(\alpha)}; d^{(\alpha)}\, | \mathbf{p}, \mathbf{y}^{(\alpha)}, \mathbf{w}^{(\alpha)}; d^{(\alpha)} \rangle} = \mathcal{A} \prod_{i=1}^{L} |K_0(p_i)|^2 \bar{\mathcal{S}}_2^b(\mathbf{y}^{(1)}, \mathbf{w}^{(1)}) \bar{\mathcal{S}}_2^b(\mathbf{y}^{(2)}, \mathbf{w}^{(2)}) \frac{\det G^+}{\det G^-} \tag{6.7}$$

which do not have a factorized form. Here $\mathcal{A}$ is the proper normalization corresponding to the given descendant state and

$$\bar{\mathcal{S}}_2^b(\mathbf{y},\mathbf{w})=\left(\frac{\mathcal{R}^{(-)}(x_s)Q_w^-(s)-\mathcal{R}^{(+)}(x_s)Q_w^+(s)}{\mathcal{R}_y(x_s)}\right)^2\frac{\mathcal{R}_y(0)}{\mathcal{R}^{(+)}(0)}\frac{1}{Q_w(0)Q_w(\frac{i}{2g})}. \tag{6.8}$$

These results are very generic and are valid for any integrable boundaries with the specifically embedded $osp(2|2)\oplus osp(2|2)$ symmetries. In order to specify to a physical situation one has to fix the scalar factor K_0 and determine how the parameter x_s (which could be even different for the two factors) depend on the parameters of the model. In the case of the 't Hooft loop we can compare the weak coupling limit of the formulas with section 4 and conclude that x_s should start at weak coupling as $x_s = a + O(g)$, i.e. it does not vanish in the limit. It is a challenging task to provide an all loop expression for x_s.

7 Conclusions

In this work we developed fermionic dualities for overlaps in $su(2|2)_c$ spin chains with K-matrices having $osp(2|2)_c$ symmetries. These symmetries come in two versions depending on how the unbroken symmetry is embedded into the full symmetry (2.13). We have previously calculated the overlaps for one type of K-matrix $K^{(1)}$ in the bosonic grading, which correspond to the D3-D5 AdS/dCFT setup. In the present work we investigated the other embedding with $K^{(2)}$, which has the symmetry of the 't Hooft loop. The two K-matrices are related by a boson-fermion duality. As a consequence, the previous overlap formula for $K^{(1)}$ in the bosonic grading describes the overlap of $K^{(2)}$ in the fermionic grading. We then developed fermionic dualities to express the overlap of the boundary state coming from $K^{(2)}$ in bosonic gradings.

Gradings encode the information how nesting happens in the Bethe ansatz, i.e. in which order from a given pseudo vacuum the excitations are created. The nature of the excitations together with their scatterings depend on the grading and the nested Bethe ansatz constructs a highest weight Bethe state, which is the eigenvector of the transfer matrix. Descendent states have the same eigenvalue and are created by symmetry transformations. The same eigenvalue can be described by different gradings. A Bethe state in one grading is a descendent state in another grading. Fermionic dualities connect the Bethe ansatz equations in different gradings. In order to transform an overlap formula we need to use the QQ-relations together with the transformation of the ratio of Gaudin type determinants. We elaborated these transformations in the $su(2|2)$ and $su(2|2)_c$ spin chains together with the corresponding selections rules. We observed that in certain gradings the overlap formula does not factorize over Bethe roots. Since after the duality the Bethe state is described by a descendent state in the dual grading we actually calculate on overlap with a descendent state. If the Bethe state in the dual grading has a non-trivial overlap with the boundary state then the overlap is factorising and it differs only by a scalar factor from the descendent overlap. If, however, the Bethe state does not overlap with the boundary state, then the descendant overlap is not factorising.

We investigated carefully the requirements that Bethe states overlap with the boundary state. On the way we determined the relations between Bethe states and descendants in the various gradings, which helped to calculate the proper prefactors in the descendant overlaps.

JHEP12(2024)034

Here the symmetry properties of the boundary state was crucial. Eventually, we could describe all overlaps in all gradings in the $su(2|2)$ and $su(2|2)_c$ spin chains. By putting together two copies of overlap formulas we made a proposal for overlaps in the AdS5/CFT4 settings with symmetries of the 't Hooft line. We confirmed our proposal against all available 1 loop results in the various subsectors [12]. In order to have a precise all loop description one needs to know the scalar factor of the reflection factor K_0, together with the explicit form of the boundary parameter x_s. The absence/presence of boundary bound-states could help in this analysis.

As future works, it would be very nice to find K_0 and x_s, which correspond to the 't Hooft line setting.

As our results relied only on the symmetry of the boundary K-matrix it is valid for other problems with the same symmetries. In particular, [13] investigates the correlation functions on the Coulomb branch of planar $\mathcal{N} = 4$ SYM, where the R-symmetry is broken to SO(5) just as in the case of the 't Hooft loop. If integrability extends to all sectors and all loops, one-point functions could be described by our formulas, with appropriately chosen K_0 and x_s. Our overlaps are expected to be applied in the ABJM theory, where it could describe asymptotic 3-point function [9], domain wall [10] and Wilson loop 1-point functions [11].

Acknowledgments

We thank Charlotte Kristjansen for the discussions and the NKFIH research Grants K134946 and PD142929 for support. ZB thanks APCTP for the invitation to the workshop "Integrability, Duality and Related Topics", Pohang, 2023, where this work started. TG was also supported by the János Bolyai Research Scholarship of the Hungarian Academy of Science.

A Bethe states and descendent states

In this appendix we investigate the relations of Bethe states in different gradings. We start by parametrizing the irreducible representations of $su(2|2)$ by the Young-tableaux on figure 1. The Bethe states are highest weight states which correspond these diagrams. They are obtained by acting with raising operators on the pseudo vacuum. In the $(1, 2, 3, 4)$ grading, for example, we start with all 1s, then use $u^{(1)}$ roots to create 2s, then $u^{(2)}$ roots to flip 2s to 3s from which we flips 4s by $u^{(3)}$. In the corresponding state the multiplicities of $|1\rangle$, $|2\rangle$, $|3\rangle$, $|4\rangle$ are $2L - N_1$, $N_1 - N_2, N_2 - N_3, N_3$, respectively. They can be put in the Young-tableaux as 1s in the first row, 2s in the second row, 3s in the remaining first column and 4s in the last column. These are kept track by the Λs, as shown on figure 1. The very existence of the diagram implies among others that $\Lambda_1 \geq \Lambda_2$ and $\Lambda_3 \geq \Lambda_4$. In table 1 we relate the multiplicities of $|1\rangle$, $|2\rangle$, $|3\rangle$, $|4\rangle$ in the various gradings to the labels of the Young-tableaux.

Each Young-tableaux corresponds to a given representation of $su(2|2)$. For a given grading the nesting procedure provides a Bethe state, which is a highest weight state. Other nesting uses different creation operators and describes the same representations by different highest weight Bethe states. A highest weight Bethe state in one grading is a descendent state in another grading. Let us denote the Bethe state in the grading (A_1, A_2, A_3, A_4) as

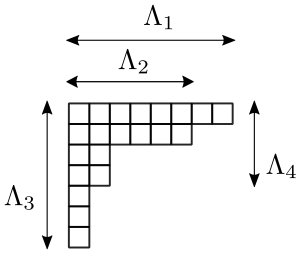

Figure 1. Young-Tableaux parametrizing the representation of $su(2|2)$.

	#1	#2	#3	#4
(1,2,3,4)	$\Lambda_1 = 2L - N_1$	$\Lambda_2 = N_1 - N_2$	$\Lambda_3 - 2 = N_2 - N_3$	$\Lambda_4 - 2 = N_3$
(1,3,4,2)	$\Lambda_1 = 2L - N_1$	$\Lambda_2 - 2 = N_3$	$\Lambda_3 - 1 = N_1 - N_2$	$\Lambda_4 - 1 = N_2 - N_3$
(1,3,2,4)	$\Lambda_1 = 2L - N_1$	$\Lambda_2 - 1 = N_2 - N_3$	$\Lambda_3 - 1 = N_1 - N_2$	$\Lambda_4 - 2 = N_3$
(3,4,1,2)	$\Lambda_1 - 2 = N_2 - N_3$	$\Lambda_2 - 2 = N_3$	$\Lambda_3 = 2L - N_1$	$\Lambda_4 = N_1 - N_2$
(3,1,2,4)	$\Lambda_1 - 1 = N_1 - N_2$	$\Lambda_2 - 1 = N_2 - N_3$	$\Lambda_3 = 2L - N_1$	$\Lambda_4 - 2 = N_3$
(3,1,4,2)	$\Lambda_1 - 1 = N_1 - N_2$	$\Lambda_2 - 2 = N_3$	$\Lambda_3 = 2L - N_1$	$\Lambda_4 - 1 = N_2 - N_3$

Table 1. Relation between the labels of the Young-tableaux and excitation numbers in the various gradings.

$|\mathbf{u}_{(A_1,A_2,A_3,A_4)}\rangle$. Bethe states in different gradings can be connected to each other as

$$|\mathbf{u}_{(\dots,A_k,A_{k+1},\dots)}\rangle = \frac{1}{\sqrt{\#A_k + \#A_{k+1}}} E_{A_k,A_{k+1}} |\mathbf{u}_{(\dots,A_{k+1},A_k,\dots)}\rangle, \tag{A.1}$$

where $E_{i,j}$ are the $su(2|2)$ generators which satisfy the relations

$$[E_{i,j}, E_{k,l}] := E_{i,j}E_{k,l} - (-1)^{([i]+[j])([k]+[l])} E_{k,l}E_{i,j} = \delta_{j,k}E_{i,l} - (-1)^{([i]+[j])([k]+[l])}\delta_{i,l}E_{k,j}. \tag{A.2}$$

The structure of $K^{(2)}$ implies that there is the selection rule $\#1 = \#2$. Looking at the table we can see that the Bethe states have non-vanishing overlaps in the gradings $(1,2,3,4)$, $(3,1,2,4)$ and $(3,4,1,2)$ for the representations $\Lambda_1 = \Lambda_2$. In the gradings $(1,3,4,2)$, $(1,3,2,4)$ and $(3,1,4,2)$ however $\Lambda_1 \geq \Lambda_2$ cannot be satisfied, thus there are no Bethe states with non-vanishing overlaps i.e. $\langle\Psi|\mathbf{u}_{(1,3,4,2)}\rangle = 0$ for all Bethe states.

In other words, using the duality we can obtain overlap expressions for every grading, however not every overlap formula corresponds to Bethe states, some of them corresponds only to descendent states. E.g., for the grading $(1,3,4,2)$, taking an irrep with $\Lambda_1 = \Lambda_2$, the overlap with the Bethe state is zero, but the following descendants have non-vanishing overlaps

$$\langle\Psi|E_{2,3}E_{2,4}|\mathbf{u}_{(1,3,4,2)}\rangle \neq 0\,; \qquad \langle\Psi|E_{3,1}E_{2,4}|\mathbf{u}_{(1,3,4,2)}\rangle \neq 0. \tag{A.3}$$

These overlaps are given by the expression (3.15) up to a combinatorial prefactor

$$\frac{|\langle\Psi|E_{2,3}E_{2,4}|\mathbf{u}_{(1,3,4,2)}\rangle|^2}{|E_{2,3}E_{2,4}|\mathbf{u}_{(1,3,4,2)}\rangle|^2} = \left(\frac{s}{k_1}\right)^2 \frac{L-N_2}{N_1-N_2+N_3+1} \frac{|\langle\Psi|\mathbf{u}_{(3,1,2,4)}\rangle|^2}{\langle\mathbf{u}_{(3,1,2,4)}|\mathbf{u}_{(3,1,2,4)}\rangle}. \tag{A.4}$$

which we determine in the next subsection.

JHEP12(2024)034

A.1 Overlap calculations in the $su(2|2)$ spin chain

Assuming that we fixed the prefactor of $S_2^{(3,1,2,4)}$ we can calculate the complete overlaps in the grading $(1,3,4,2)$ using the connection between the Bethe states of different gradings (A.1), the exchange properties of the $su(2|2)$ algebra and the transformation properties of the boundary state. We start with $|\mathbf{u}_{(1,3,4,2)}\rangle = E_{1,3}E_{4,2}|\mathbf{u}_{(3,1,2,4)}\rangle$ and write

$$\begin{aligned}\langle\Psi|E_{2,3}E_{2,4}|\mathbf{u}_{(1,3,4,2)}\rangle &= \langle\Psi|E_{2,3}E_{2,4}E_{1,3}E_{4,2}|\mathbf{u}_{(3,1,2,4)}\rangle = \langle\Psi|E_{1,3}E_{2,3}E_{2,4}E_{4,2}|\mathbf{u}_{(3,1,2,4)}\rangle \\ &= \langle\Psi|E_{1,3}E_{2,3}(E_{2,2}+E_{4,4})|\mathbf{u}_{(3,1,2,4)}\rangle = N_2\langle\Psi|E_{1,3}E_{2,3}|\mathbf{u}_{(3,1,2,4)}\rangle\end{aligned} \tag{A.5}$$

where we only used the $su(2|2)$ algebra and the fact that the Bethe state is a highest weight state, i.e. $E_{2,4}|\mathbf{u}_{(3,1,2,4)}\rangle = 0$. We now can exploit the $osp(2|2)$ symmetry of the K-matrix and the corresponding boundary state

$$\sum_k \langle\Psi|E_{i,k}K^{(2)}_{k,j} = (-1)^{[j]([j]+[k])}\sum_k K^{(2)}_{i,k}\langle\Psi|E_{j,k} \tag{A.6}$$

therefore $\langle\Psi|E_{1,3}K_{3,3} + \langle\Psi|E_{1,4}K_{3,4} = -K_{1,2}\langle\Psi|E_{3,2}$, which reads explicitly as

$$\langle\Psi|E_{1,3} = \frac{s}{k_1}\langle\Psi|E_{3,2} - \frac{k_2}{k_1}\langle\Psi|E_{1,4}. \tag{A.7}$$

Substituting this relation back into eq. (A.5) we obtain that

$$\langle\Psi|E_{2,3}E_{2,4}|\mathbf{u}_{(1,3,4,2)}\rangle = \frac{s}{k_1}N_2\langle\Psi|E_{3,2}E_{2,3}|\mathbf{u}_{(3,1,2,4)}\rangle = \frac{s}{k_1}N_2(L-N_2)\langle\Psi|\mathbf{u}_{(3,1,2,4)}\rangle. \tag{A.8}$$

which expresses the overlap of the boundary state with a descendent state in one grading to the overlap with a highest weight state in another grading. In order to get the normalized overlap we need to calculate the norm of the descendent state. This can be done by using the relations of the algebra and the relations between the different Bethe states:

$$\begin{aligned}&|E_{2,3}E_{2,4}|\mathbf{u}_{(1,3,4,2)}\rangle|^2 = \\ &= \langle\mathbf{u}_{(1,3,4,2)}|E_{4,2}E_{3,2}E_{2,3}E_{2,4}|\mathbf{u}_{(1,3,4,2)}\rangle \\ &= \langle\mathbf{u}_{(1,3,4,2)}|E_{4,2}[E_{3,2},E_{2,3}]E_{2,4}|\mathbf{u}_{(1,3,4,2)}\rangle - \langle\mathbf{u}_{(1,3,4,2)}|E_{4,2}E_{2,3}E_{3,2}E_{2,4}|\mathbf{u}_{(1,3,4,2)}\rangle \\ &= (N_1-N_2+N_3+1)\langle\mathbf{u}_{(1,3,4,2)}|E_{4,2}E_{2,4}|\mathbf{u}_{(1,3,4,2)}\rangle - \langle\mathbf{u}_{(1,3,4,2)}|E_{4,2}E_{2,3}E_{3,4}|\mathbf{u}_{(1,3,4,2)}\rangle \\ &= N_2(N_1-N_2+N_3+1)\langle\mathbf{u}_{(1,3,4,2)}|\mathbf{u}_{(1,3,4,2)}\rangle.\end{aligned} \tag{A.9}$$

together with

$$\langle\mathbf{u}_{(1,3,4,2)}|\mathbf{u}_{(1,3,4,2)}\rangle = \langle\mathbf{u}_{(3,1,2,4)}|E_{2,4}E_{3,1}E_{1,3}E_{4,2}|\mathbf{u}_{(3,1,2,4)}\rangle = N_2(L-N_2)\langle\mathbf{u}_{(3,1,2,4)}|\mathbf{u}_{(3,1,2,4)}\rangle \tag{A.10}$$

Collecting all the formulas we can express the overlap in the (1,3,4,2) grading in terms of the overlap in the (3,1,2,4) grading as

$$\frac{|\langle\Psi|E_{2,3}E_{2,4}|\mathbf{u}_{(1,3,4,2)}\rangle|^2}{|E_{2,3}E_{2,4}|\mathbf{u}_{(1,3,4,2)}\rangle|^2} = \left(\frac{s}{k_1}\right)^2\frac{L-N_2}{N_1-N_2+N_3+1}\frac{|\langle\Psi|\mathbf{u}_{(3,1,2,4)}\rangle|^2}{\langle\mathbf{u}_{(3,1,2,4)}|\mathbf{u}_{(3,1,2,4)}\rangle}. \tag{A.11}$$

B Overlap calculations in the $su(2|2)_c$ spin chain

In this appendix we calculate the overlaps of descendant states in the $su(2|2)_c$ spin chain. We start by calculating the properly normalized overlap for a descendant with $N = 2L-2$ with a non-vanishing overlap

$$|\mathbf{y},\mathbf{w};-2,0\rangle^b := \mathbb{Q}_3^{\ 1}\mathbb{Q}_4^{\ 1}|\mathbf{y},\mathbf{w}\rangle^b \tag{B.1}$$

We use the exchange relations of the $su(2|2)_c$ algebra

$$\begin{aligned}
\left[\mathbb{R}_a^{\ b},\mathbb{J}_c\right] &= \delta_c^b\mathbb{J}_a - \frac{1}{2}\delta_a^b\mathbb{J}_c, & \left[\mathbb{L}_\alpha^{\ \beta},\mathbb{J}_\gamma\right] &= \delta_\gamma^\beta\mathbb{J}_\alpha - \frac{1}{2}\delta_\alpha^\beta\mathbb{J}_\gamma,\\
\left[\mathbb{R}_a^{\ b},\mathbb{J}^c\right] &= -\delta_a^c\mathbb{J}^b + \frac{1}{2}\delta_a^b\mathbb{J}^c, & \left[\mathbb{L}_\alpha^{\ \beta},\mathbb{J}^\gamma\right] &= -\delta_\alpha^\gamma\mathbb{J}^\beta + \frac{1}{2}\delta_\alpha^\beta\mathbb{J}^\gamma,\\
\left\{\mathbb{Q}_\alpha^{\ a},\mathbb{Q}_\beta^{\ b}\right\} &= \epsilon_{\alpha\beta}\epsilon^{ab}\mathbb{C}, & \left\{\mathbb{Q}_a^{\dagger\,\alpha},\mathbb{Q}_b^{\dagger\,\beta}\right\} &= \epsilon_{ab}\epsilon^{\alpha\beta}\mathbb{C}^\dagger,\\
\left\{\mathbb{Q}_\alpha^{\ b},\mathbb{Q}_a^{\dagger\,\beta}\right\} &= \delta_\alpha^\beta\mathbb{R}_a^{\ b} + \delta_a^b\mathbb{L}_\alpha^{\ \beta} + \frac{1}{2}\delta_\alpha^\beta\delta_a^b\mathbb{H}
\end{aligned} \tag{B.2}$$

where $\mathbb{C} = \mathbb{C}^\dagger = ig\left(e^{-i\mathbb{P}/2} - e^{i\mathbb{P}/2}\right)$, together with the symmetry properties of the boundary state

$$\langle\Psi^{(2)}|\mathbb{Q}_\alpha^{\ a} = ix_s\epsilon^{ab}\eta_{\alpha\beta}\langle\Psi^{(2)}|\mathbb{Q}_b^{\dagger\,\beta} \quad , \quad \eta^{\alpha\beta}\langle\Psi^{(2)}|\mathbb{Q}_\beta^{\ a} = ix_s\epsilon^{ab}\langle\Psi^{(2)}|\mathbb{Q}_b^{\dagger\,\alpha}, \tag{B.3}$$

where

$$\eta_{\alpha\beta} = \begin{pmatrix} k_1 & k_2\\ k_2 & k_4\end{pmatrix}, \quad \eta^{\alpha\beta} = \begin{pmatrix} k_4 & -k_2\\ -k_2 & k_1\end{pmatrix}. \tag{B.4}$$

Taking into account the $osp(2|2)$ invariance property of the boundary state

$$k_1\langle\Psi^{(2)}|\mathbb{Q}_4^{\ 1} = ix_s\langle\Psi^{(2)}|\mathbb{Q}_2^{\dagger\,4} + k_2\langle\Psi^{(2)}|\mathbb{Q}_3^{\ 1} \tag{B.5}$$

the overlap simplifies as

$$\langle\Psi^{(2)}|\mathbf{y},\mathbf{w};-2,0\rangle^b := -\frac{ix_s}{k_1}\langle\Psi^{(2)}|\mathbb{Q}_2^{\dagger\,4}\mathbb{Q}_3^{\ 1}|\mathbf{y},\mathbf{w}\rangle^b = \frac{ix_s}{k_1}\langle\Psi^{(2)}|\tilde{\mathbf{y}},\mathbf{w}\rangle^f \tag{B.6}$$

In order to calculate the normalized overlap we need to determine the norm, too. It can be calculated as

$$\begin{aligned}
\langle\mathbf{y},\mathbf{w};-2,0|\mathbf{y},\mathbf{w};-2,0\rangle^b &= \langle\mathbf{y},\mathbf{w}|\mathbb{Q}_1^{\dagger\,4}\mathbb{Q}_1^{\dagger\,3}\mathbb{Q}_3^{\ 1}\mathbb{Q}_4^{\ 1}|\mathbf{y},\mathbf{w}\rangle^b = \\
&= \langle\mathbf{y},\mathbf{w}|\mathbb{Q}_1^{\dagger\,4}(\mathbb{R}_3^{\ 3}+\mathbb{L}_1^{\ 1}+\frac{1}{2}\mathbb{H})\mathbb{Q}_4^{\ 1}|\mathbf{y},\mathbf{w}\rangle^b - \langle\mathbf{y},\mathbf{w}|\{\mathbb{Q}_1^{\dagger\,4},\mathbb{Q}_3^{\ 1}\}\mathbb{R}_4^{\ 3}|\mathbf{y},\mathbf{w}\rangle^b\\
&= -(igA+1)\langle\mathbf{y},\mathbf{w}|\mathbb{Q}_1^{\dagger\,4}\mathbb{Q}_4^{\ 1}|\mathbf{y},\mathbf{w}\rangle^b - \langle\mathbf{y},\mathbf{w}|\mathbb{R}_3^{\ 4}\mathbb{R}_4^{\ 3}|\mathbf{y},\mathbf{w}\rangle^b\\
&= -(igA+1)\langle\mathbf{y},\mathbf{w}|(\mathbb{R}_4^{\ 4}+\mathbb{L}_1^{\ 1}+\frac{1}{2}\mathbb{H})|\mathbf{y},\mathbf{w}\rangle^b - 2\langle\mathbf{y},\mathbf{w}|\mathbb{R}_3^{\ 3}|\mathbf{y},\mathbf{w}\rangle^b\\
&= igA(igA+N-2M+1)\langle\mathbf{y},\mathbf{w}|\mathbf{y},\mathbf{w}\rangle^b
\end{aligned} \tag{B.7}$$

where we used that

$$\frac{1}{2}\mathbb{H}|\mathbf{y},\mathbf{w}\rangle^b = \left[2ig\sum_{k=1}^{L}(x_k^- - x_k^+) - L\right]|\mathbf{y},\mathbf{w}\rangle^b = \left[-igA - L + M\right]|\mathbf{y},\mathbf{w}\rangle^b$$

JHEP12(2024)034

We also have

$$
\begin{aligned}
\langle \tilde{\mathbf{y}}, \mathbf{w} | \tilde{\mathbf{y}}, \mathbf{w} \rangle^f &= \langle \mathbf{y}, \mathbf{w} | \mathbb{Q}_4^2 \mathbb{Q}_1^{\dagger 3} \mathbb{Q}_3^1 \mathbb{Q}_2^{\dagger 4} | \mathbf{y}, \mathbf{w} \rangle^b \\
&= \langle \mathbf{y}, \mathbf{w} | \mathbb{Q}_4^2 (\mathbb{R}_3^3 + \mathbb{L}_1^1 + \frac{1}{2} \mathbb{H}) \mathbb{Q}_2^{\dagger 4} | \mathbf{y}, \mathbf{w} \rangle^b - \langle \mathbf{y}, \mathbf{w} | \mathbb{C} \mathbb{C}^\dagger | \mathbf{y}, \mathbf{w} \rangle^b \\
&= igA(igA + 2L - 2M) \langle \mathbf{y}, \mathbf{w} | \mathbf{y}, \mathbf{w} \rangle^b
\end{aligned} \tag{B.8}
$$

where we used that

$$
\mathbb{C} | \mathbf{y}, \mathbf{w} \rangle^b = \mathbb{C}^\dagger | \mathbf{y}, \mathbf{w} \rangle^b = ig \sum_{k=1}^{2L} \left(\sqrt{\frac{x_k^+}{x_k^-}} - \sqrt{\frac{x_k^-}{x_k^+}} \right) | \mathbf{y}, \mathbf{w} \rangle^b \tag{B.9}
$$

$$
= ig \sum_{k=1}^{L} \left(\sqrt{\frac{x_k^+}{x_k^-}} - \sqrt{\frac{x_k^-}{x_k^+}} + \sqrt{\frac{x_k^-}{x_k^+}} - \sqrt{\frac{x_k^+}{x_k^-}} \right) | \mathbf{y}, \mathbf{w} \rangle^b = 0 \tag{B.10}
$$

In summary, the normalization constant turns out to be

$$
C_{-2,0} = \left(\frac{x_s}{k_1} \right)^2 \frac{(igA + 2L - 2M)}{(igA + N - 2M + 1)} \tag{B.11}
$$

We could repeat the calculation for the other descendant

$$
| \mathbf{y}, \mathbf{w}; -2, 0^* \rangle^b := \mathbb{Q}_2^{\dagger 3} \mathbb{Q}_2^{\dagger 4} | \mathbf{y}, \mathbf{w} \rangle^b ; \qquad C_{-2,0^*} = \frac{A}{((2L - N - 1)\frac{i}{g} - A)} \tag{B.12}
$$

Using similar calculations the normalization constants can be fixed.

Data Availability Statement. This article has no associated data or the data will not be deposited.

Code Availability Statement. This article has no associated code or the code will not be deposited.

References

[1] J.-S. Caux and F.H.L. Essler, *Time evolution of local observables after quenching to an integrable model*, *Phys. Rev. Lett.* **110** (2013) 257203 [arXiv:1301.3806] [INSPIRE].

[2] J. De Nardis, B. Wouters, M. Brockmann and J.-S. Caux, *Solution for an interaction quench in the Lieb-Liniger Bose gas*, *Phys. Rev. A* **89** (2014) 033601.

[3] L. Piroli, E. Vernier, P. Calabrese and B. Pozsgay, *Integrable quenches in nested spin chains I: the exact steady states*, *J. Stat. Mech.* **1906** (2019) 063103 [arXiv:1811.00432] [INSPIRE].

[4] L. Piroli, E. Vernier, P. Calabrese and B. Pozsgay, *Integrable quenches in nested spin chains II: fusion of boundary transfer matrices*, *J. Stat. Mech.* **1906** (2019) 063104 [arXiv:1812.05330] [INSPIRE].

[5] C. Rylands, B. Bertini and P. Calabrese, *Integrable quenches in the Hubbard model*, *J. Stat. Mech.* **2210** (2022) 103103 [arXiv:2206.07985] [INSPIRE].

[6] I. Buhl-Mortensen, M. de Leeuw, C. Kristjansen and K. Zarembo, *One-point functions in AdS/dCFT from matrix product states*, *JHEP* **02** (2016) 052 [arXiv:1512.02532] [INSPIRE].

[7] Y. Jiang, S. Komatsu and E. Vescovi, *Structure constants in $N = 4$ SYM at finite coupling as worldsheet g-function*, *JHEP* **07** (2020) 037 [arXiv:1906.07733] [INSPIRE].

[8] Y. Jiang, S. Komatsu and E. Vescovi, *Exact three-point functions of determinant operators in planar $N = 4$ supersymmetric Yang-Mills theory*, *Phys. Rev. Lett.* **123** (2019) 191601 [arXiv:1907.11242] [INSPIRE].

[9] P. Yang, Y. Jiang, S. Komatsu and J.-B. Wu, *Three-point functions in ABJM and Bethe ansatz*, *JHEP* **01** (2022) 002 [arXiv:2103.15840] [INSPIRE].

[10] C. Kristjansen, D.-L. Vu and K. Zarembo, *Integrable domain walls in ABJM theory*, *JHEP* **02** (2022) 070 [arXiv:2112.10438] [INSPIRE].

[11] Y. Jiang, J.-B. Wu and P. Yang, *Wilson-loop one-point functions in ABJM theory*, *JHEP* **09** (2023) 047 [arXiv:2306.05773] [INSPIRE].

[12] C. Kristjansen and K. Zarembo, *'t Hooft loops and integrability*, *JHEP* **08** (2023) 184 [arXiv:2305.03649] [INSPIRE].

[13] V. Ivanovskiy et al., *Vacuum condensates on the Coulomb branch*, arXiv:2405.19043 [INSPIRE].

[14] M. De Leeuw, C. Kristjansen and G. Linardopoulos, *Scalar one-point functions and matrix product states of AdS/dCFT*, *Phys. Lett. B* **781** (2018) 238 [arXiv:1802.01598] [INSPIRE].

[15] M. De Leeuw et al., *Spin chain overlaps and the twisted Yangian*, *JHEP* **01** (2020) 176 [arXiv:1912.09338] [INSPIRE].

[16] T. Gombor and C. Kristjansen, *Overlaps for matrix product states of arbitrary bond dimension in ABJM theory*, *Phys. Lett. B* **834** (2022) 137428 [arXiv:2207.06866] [INSPIRE].

[17] N. Beisert et al., *Review of AdS/CFT integrability: an overview*, *Lett. Math. Phys.* **99** (2012) 3 [arXiv:1012.3982] [INSPIRE].

[18] A. Karch and L. Randall, *Open and closed string interpretation of SUSY CFT's on branes with boundaries*, *JHEP* **06** (2001) 063 [hep-th/0105132] [INSPIRE].

[19] M. de Leeuw, C. Kristjansen and K. Zarembo, *One-point functions in defect CFT and integrability*, *JHEP* **08** (2015) 098 [arXiv:1506.06958] [INSPIRE].

[20] L. Piroli, B. Pozsgay and E. Vernier, *What is an integrable quench?*, *Nucl. Phys. B* **925** (2017) 362 [arXiv:1709.04796] [INSPIRE].

[21] T. Gombor and Z. Bajnok, *Boundary states, overlaps, nesting and bootstrapping AdS/dCFT*, *JHEP* **10** (2020) 123 [arXiv:2004.11329] [INSPIRE].

[22] T. Gombor, *On exact overlaps for gl(N) symmetric spin chains*, *Nucl. Phys. B* **983** (2022) 115909 [arXiv:2110.07960] [INSPIRE].

[23] T. Gombor and Z. Bajnok, *Boundary state bootstrap and asymptotic overlaps in AdS/dCFT*, *JHEP* **03** (2021) 222 [arXiv:2006.16151] [INSPIRE].

[24] S. Komatsu and Y. Wang, *Non-perturbative defect one-point functions in planar $N = 4$ super-Yang-Mills*, *Nucl. Phys. B* **958** (2020) 115120 [arXiv:2004.09514] [INSPIRE].

[25] N. Beisert, *The $SU(2|2)$ dynamic S-matrix*, *Adv. Theor. Math. Phys.* **12** (2008) 945 [hep-th/0511082] [INSPIRE].

JHEP12(2024)034

[26] M. Brockmann, J. De Nardis, B. Wouters and J.-S. Caux, *Néel-XXZ state overlaps: odd particle numbers and Lieb-Liniger scaling limit*, J. Phys. A **47** (2014) 345003.

[27] T. Gombor, *Exact overlaps for all integrable two-site boundary states of $\mathfrak{gl}(N)$ symmetric spin chains*, JHEP **05** (2024) 194 [arXiv:2311.04870] [INSPIRE].

[28] T. Gombor and B. Pozsgay, *On factorized overlaps: algebraic Bethe ansatz, twists, and separation of variables*, Nucl. Phys. B **967** (2021) 115390 [arXiv:2101.10354] [INSPIRE].

[29] B. Pozsgay, *Overlaps with arbitrary two-site states in the XXZ spin chain*, J. Stat. Mech. **1805** (2018) 053103 [arXiv:1801.03838] [INSPIRE].

[30] C. Kristjansen, D. Müller and K. Zarembo, *Overlaps and fermionic dualities for integrable super spin chains*, JHEP **03** (2021) 100 [arXiv:2011.12192] [INSPIRE].

[31] N. Beisert and M. Staudacher, *Long-range $psu(2,2|4)$ Bethe ansatze for gauge theory and strings*, Nucl. Phys. B **727** (2005) 1 [hep-th/0504190] [INSPIRE].

[32] N. Beisert and R. Roiban, *Beauty and the twist: the Bethe ansatz for twisted $N = 4$ SYM*, JHEP **08** (2005) 039 [hep-th/0505187] [INSPIRE].