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Homogeneous, Monostatic Lattice Polyhedra, or the Magic Pyramid with 17 Sides

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ORIGINAL RESEARCH ARTICLE

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SUMMARY

This paper presents the explicit geometric realisation of all monostatic polyhedra, the existence of which has been previously proven. Two of them are mono-unstable, denoted as P_2 and P_3 in Domokos et al. (2020), where their existence was proven. The other two (P_C and P_B) are monostable, and their existence was proven in Goldberg–Guy (1969) and Bezdek (2011), respectively. The combinatorial structure is identical for three of them, P_2, P_3 and P_B: it is a 17-sided pyramid. For each construction, we provide integer coordinates.

KEYWORDS

monostatic polyhedron, equilibrium, convex geometry, lattice polyhedron, Conway

1. INTRODUCTION

Polyhedra and their equilibrium properties have been a fascinating subject of study in both theoretical and applied geometry (Heppes 1966; McRobie 2017). A polyhedron may be at rest on some of its faces, and we call this stable equilibrium position. It may be balanced on some of its vertices, which we call unstable equilibrium, and also may be balanced on its edges, which we call saddle-type equilibrium. We define these notions in more detail in Section 1.1. The concept of monostatic polyhedra – those with exactly one stable or one unstable equilibrium point – has received particular attention due to its mathematical uniqueness and potential applications in mechanics and design (Senechal 2022). Early works by Conway and Guy (Conway–Guy 1966; Goldberg–Guy 1969) laid the foundation for understanding homogeneous monostatic polyhedra. This theory was later expanded by Bezdek (Bezdek 2011) and Reshetov (Reshetov 2014), the latter provided the first explicit example of a homogeneous convex monostable polyhedron.

The first existence proof for homogeneous mono-unstable polyhedra appeared in Domokos et al. (2020), where the polyhedra P_2 and P_3 have been presented. The P_2, P_3 and Bezdek's polyhedron are connected in a special way: despite their different stability properties, all of them have the same combinatorial structure, a 17-sided pyramid. In this paper we pay special attention to this combinatorial structure, and its enchanting equilibrium properties. Despite these advances, except Reshetov's example, *explicit* constructions of monostatic polyhedra and especially mono-unstable polyhedra have remained undiscovered. In particular, lattice realizations of monostatic polyhedra have not been constructed.

In this study, we follow Reshetov's footsteps, and construct all the remaining known examples for monostatic polyhedra, defined on integer lattice coordinates. Lattice polyhedra represent a powerful form of constructions, as they can be precisely reconstructed.

1.1. Key definitions

Definition 1. Let P be a homogeneous convex polyhedron with f faces, v vertices, and e edges. Based on Euler's theorem

(1) $f + v = e + 2$

P may be uniquely represented by the pair (f, v). Here, we call the family of all convex polyhedra having v vertices and f faces the *primary combinatorial class* of P, and denote it by $(f, v)^C$. (Domokos et al. 2020)

Definition 2. Let P be a homogeneous convex polyhedron, with a center of mass C_P. A stable equilibrium can occur on face F_i if the projection C_{P,F_i} of C_P onto the plane of F_i lies in the interior of F_i, and we call C_{P,F_i} a stable equilibrium point of P. An unstable equilibrium can occur at a vertex V_i if the plane passing through V_i with normal vector $C_P V_i$ does not intersect P. If the condition is met, V_i is an unstable equilibrium point of P. A saddle-type equilibrium can occur on an edge E_i if the projection C_{P,E_i} of C_P onto the line carrying E_i lies in the interior of E_i and the plane passing through C_{P,E_i} with normal vector $C_P C_{P,E_i}$ does not intersect P, and we call C_{P,E_i} a saddle-type equilibrium point of P.

The numbers S, U, H denote the respective numbers of stable, unstable and saddle-type equilibria. Based on the Poincaré-Hopf theorem (Milnor 1965) we have:

(2) $S + U = H + 2.$

The triplet (S, U, H) may be uniquely represented by the pair (S, U).

Definition 3. We denote by $(S, U)^E$ the family of all convex polyhedra having S stable and U unstable equilibrium points with respect to their centres of mass, and we call this the *primary equilibrium class* of the polyhedron. (Domokos et al. 2020; Várkonyi–Domokos 2006)

Definition 4. Let P be a homogeneous convex polyhedron. If $P \in (1, U)^E$ or $P \in (S, 1)^E$ then we call P mono-stable (mono-unstable), respectively. (Bozóki et al. 2022)

Definition 5. A convex homogeneous polyhedron is called monostatic if it is either monostable or mono-unstable, and it is called mono-monostatic if it is both mono-stable and mono-unstable. (Bozóki et al. 2022)

2. GEOMETRY OF THE MONO-UNSTABLE LATTICE POLYHEDRA P_2 AND P_3

The construction of the first homogeneous mono-stable polyhedron by Conway (Goldberg–Guy 1969) was based on the unique properties of an open planar polygonal line invented by Conway, the so-called *Conway spiral*. This section outlines the use of the Conway spiral in our approach, detailing the rounding process to integer coordinates and the selection of the vertices of the mono-unstable polyhedra P_2 and P_3. The base of P_2 and P_3 is a 17-sided polygon lying in the *xy*-plane with vertices R_1,… R_{17}, the location is defined by the Conway spiral. The only vertex with nonzero z coordinate is R_{18}, the apex of the pyramid.

Definition 6. A Conway n-spiral is an open polygonal line with $n + 1$ vertices R_0, ..., R_n with $\angle OR_{i-1} R_i = \pi / 2$, i = 1 ...$n$, where the point O is the origin of the coordinate system, all points R_i with coordinates $(x_{R,i}, y_{R,i}, z_{R,i})$ lie in the *xy*-plane ($z_{R,i} = 0$), the coordinates of R_0 are $x_{R,0} = z_{R,0} = 0$, $y_{R,0} < 0$ and all remaining vertices R_i,(i = 1,2, ... n) have positive x coordinates. The central angles α_i are defined for i = 1,2, ...n as $\alpha_i = \angle R_i\ OR_{i-1}$ and the central angle α_n is given as $\alpha_n = \pi - \sum_{i=1}^{n-1} \alpha_i$. (Domokos– Kovács 2023)

Remark 1. Definition 6 is based on Domokos–Kovács (2023) with slight modifications to make it more applicable to this paper. The points of the spiral are denoted by R instead of P, and the numbering of the vertices is reversed. The y-coordinate of the starting point R_0 can be any negative number, allowing the spiral's size to remain unrestricted. The following definitions also adhere to this modified notation system.

Definition 7. We call a Conway n-spiral *classical* (Domokos–Kovács 2023) *(Fig. 1)* if all central angles $\alpha_i = \angle R_iO\ R_{i-1}$, ($i$ = 1,2,...n) are equal, i.e., we have

(3) $\alpha_1 = \alpha_2 = \cdots = \alpha_n$.

Definition 8. A *double Conway n-spiral* is a convex polygon with $2n$ – 1 vertices, which is obtained by reflecting a Conway spiral to the *yz*-plane. The coordinates for the center of mass C are denoted by (x_c, y_c, z_c). (Domokos–Kovács 2023)

Definition 9. A *lattice Conway n-spiral* is a Conway n-spiral with $\angle OR_{i-1}R_i = (\pi/2) + \delta_i$, $0 \leq \delta_i < (\pi/n)$, such that $x_{R,i}$,$y_{R,i}$ are integers. A *double lattice Conway n-spiral* is (similarly to Definition 6) a convex polygon with $2n$– 1 vertices, which is obtained by reflecting a lattice Conway spiral to the *yz*-plane.

Theorem 1. A lattice Conway n-spiral can be constructed by following these steps:

1. Construct a Conway n-spiral with vertices R_0^0,...,R_n^0 and $0 \leq \delta_0 < (\pi/n)$ such that the condition $\angle OR_{i-1}^0 R_i^0 = (\pi/2) + \delta^0$ holds.
2. Round the coordinates $x_{R,i}^0$, $y_{R,i}^0$ to integers $x_{R,i}$,$y_{R,i}$.

3. Verify that $(\pi/2) \leq \angle OR_{i-1}R_i \leq (\pi/2) + (\pi/n)$ for $i = 1,\ldots n$ remains valid with the rounded coordinates. If this condition holds then, according to Definition 9, we have a lattice Conway n-spiral with $\delta_i = \angle OR_{i-1}R_i - (\pi/2)$.
4. If condition 3 does not hold, multiply the coordinates of $R_0^0,\ldots,R_n^0$ by 10, return to Step 1 and repeat the construction.

Proof
After the a-th iteration, the coordinates of the Conway n-spiral can be expressed as

$$(4) \quad x_{R,i}^a = x_{R,i}^0 * 10^a, y_{R,i}^a = y_{R,i}^0 * 10^a,$$

where $a \in \mathbb{Z}^+$.

After the rounding, the new coordinates are

$$(5) \quad \bar{x}_{R,i}^a = x_{R,i}^a + \Delta x_i \text{ and } \bar{y}_{R,i}^a = y_{R,i}^a + \Delta y_i,$$

where the rounding errors satisfy $-0.5 < \Delta x_i, \Delta y_i \leq 0.5$.

The change in the angle $\angle OR_{i-1}^0 R_i^0$ due to rounding is

$$(6) \quad \Delta\varphi^a = \angle O\bar{R}_{i-1}^a \bar{R}_i^a - \angle OR_{i-1}^0 R_i^0.$$

To show that $\Delta\varphi^a \to 0$ as $a \to \infty$, we can express $\Delta\varphi^a$ as

$$(7) \quad \Delta\varphi^a = arctg\frac{x_{R,i}^a + \Delta x_i}{y_{R,i}^a + \Delta y_i} - arctg\frac{x_{R,i}^a}{y_{R,i}^a}, were\ y_{R,i}^a \neq 0.$$

Based on (4) and (7) we conclude

$$(8) \quad \lim_{a \to \infty} \Delta\varphi^a = 0.$$

This proves that the rounding error in the angles vanishes in the limit, ensuring that the condition $(\pi/2) \leq \angle OR_{i-1}R_i \leq (\pi/2) + (\pi/n)$ remains valid for sufficiently large a, and so the algorithm produces a lattice Conway n-spiral in a finite number of steps. □

2.1. The base of the pyramids

The classical double Conway 9-spiral (see *Figure 1b*) serves as the foundation for the construction of our polyhedra. We will refer to R_9 ($x_{R,9}=0$) as the peak vertex of the spiral. The 17-sided polygon defined by the spiral serves as the base of the 17-sided pyramids P_2 and P_3, and it is also the starting point of generating the other two polyhedra, P_B and P_C. A key property of the polygon is that if we admit inhomogeneous mass distribution (only for the sake of this explanation), we let the centre of mass C to be located below the origin ($y_C < 0$), then, apart from the peak vertex R_9, no other vertex can hold an unstable equilibrium. Additionally, except for the lower edge (R_1R_{17}), none of the edges can hold a stable equilibrium. This follows from the $\angle OR_{i-1}R_i = \pi / 2$ for $i = 1,2 \ldots n$ criterion.

This property remains valid if we replace the double Conway 9-spiral by its lattice version where the mentioned angles ($\angle OR_{i-1}R_i$) cannot be less than 90 degrees. To construct the base polygon of the pyramid, using the steps of Theorem 1 we generated a lattice Conway 9-spiral with δ = 0.5°, $y_{R,0}$ = 10^6. We rounded the coordinates of the vertices to integer values, and we verified that for all resulting triangles the condition $\pi/2 \leq \angle OR_{i-1}R_i \leq (\pi/2) + (\pi/n)$, for $i = 1,2 \ldots n$ is still true, so that the mono-monostatic condition is met. In our case, we had a small change in the angles due to the rounding, between $-8*10^{-5}<\varphi<8*10^{-5}$. This procedure resulted in a double lattice Conway 9-spiral, with 17 vertices (coordinates in *Tables 1* and *2*, rows 2–18), which established the basis for the P_2 and P_3 lattice polyhedra *(Figures 1(b)–3*, top view).

Corollary 1.
As a consequence of Theorem 1, the base polygon of the P_2 and P_3 pyramids can be constructed by generating a lattice Conway 9-spiral following the steps of the theorem.

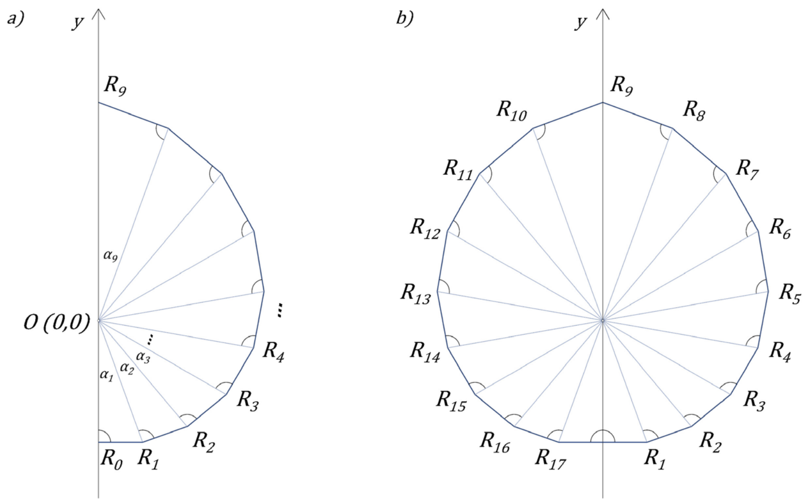

Figure 1. A Conway 9-spiral. For a classical Conway 9-spiral, $\angle R_i\, OR_{i-1} = \alpha_i = \pi / 9$, and marked angles $\angle OR_{i-1}\, R_i = \pi / 2$ (Panel (a)). For the lattice Conway 9-spiral, the coordinates of point R_i ($x_{R,i}$, $y_{R,i}$), i=1,...,9 are the integer-rounded values of the coordinates of a spiral with $\alpha_i = \pi / 9$ and $\angle OR_{i-1}\, R_i$ = 90.5°, ensuring that the inequality $\pi/2 \leq \angle OR_{i-1}\, R_i \leq (\pi/2) + (\pi/n)$ holds after the rounding. Panel (b) shows the base polygon for the P_2 and P_3 with vertices $R_1, \ldots R_{17}$, created from a double lattice Conway 9-spiral.

2.2. Selection of the apex vertex

The critical criterion for selecting the apex (R_{18}) is to avoid unstable equilibrium carried by this vertex and thus to preserve the mono-unstable property of the body. In a symmetric case ($x_{R,18}$=0) this could be guaranteed, if the angle $\angle CR_{18}\, R_0'$ is obtuse, where R_0' is the mid-point of the line segment $R_1\, R_{17}$. (We can work with C, center of mass of the base instead of C_P, the unknown center of mass of P here, because C, C_P and R_{18} are collinear). The position of the 18th vertex (R_{18}) can be determined using a Thales sphere with diameter CR_0', which defines a set of potential apex points that satisfy the required condition. In the non-symmetric cases, if $x_{R,18}>0$, then $\angle CR_{18}\ R_1$ has to be obtuse, if $x_{R,18}<0$, then $\angle CR_{18}\ R_{17}$ has to be obtuse, and the coordinates of the apex can be determined using the Thales spheres with diameter CR_1 and CR_{17}, respectively.

From these, we selected a vertex location which, after rounding, maintained the mono-unstable property (which was verified with a MATLAB code). The schematic geometry of P_3 *(Figure 3)* is symmetric, so $x_{R,18}=0$. (Note that the same base with a small amount of asymmetry at the apex still can be in class $(3,1)^E$.) *Table 2* shows integer coordinates selected for $P_3 \in (3,1)^E$, $(18,18)^C$.

During the search for the P_2 configuration we observed the non-symmetric cases, where $x_{R,18} \neq 0$. The condition defined by the Thales sphere still needed to be observed. We used a MATLAB script to examine possible vertex positions, and the stable and unstable equilibria of each configuration. Based on this analysis, we selected the final vertex coordinates for $P_2 \in (2,1)^E$, $(18,18)^C$ *(Figure 2)* and provided them in *Table 1*.

These geometric conditions ensure that the designed lattice polyhedra $P_2, P_3 \in (S, 1)^E$, thus they are both mono-unstable.

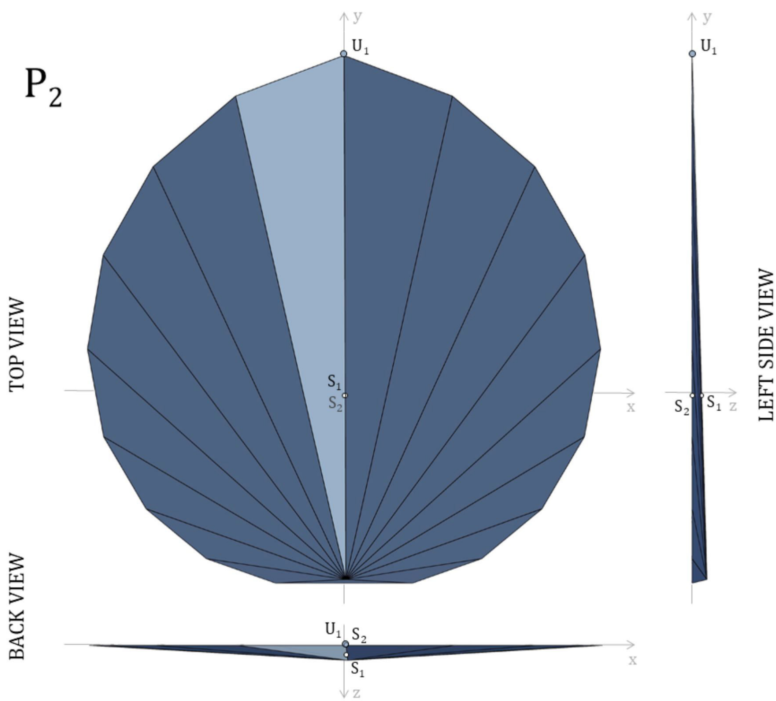

Figure 2. Polyhedron $P_2 \in (2,1)^E$, $(18,18)^C$ with coordinates from *Table 1*. The unstable vertex and the stable faces are highlighted in light blue. The 17-sided base of the pyramid, which also carries a stable equilibrium, is not visible in this view. Due to its asymmetry, the polyhedron only has two stable equilibria. The equilibrium points are marked in the figure: stable equilibria with *S* and the unstable equilibrium with *U*. The base of the pyramid is based on a double lattice Conway spiral, and the apex was chosen to eliminate any unstable equilibria on it.

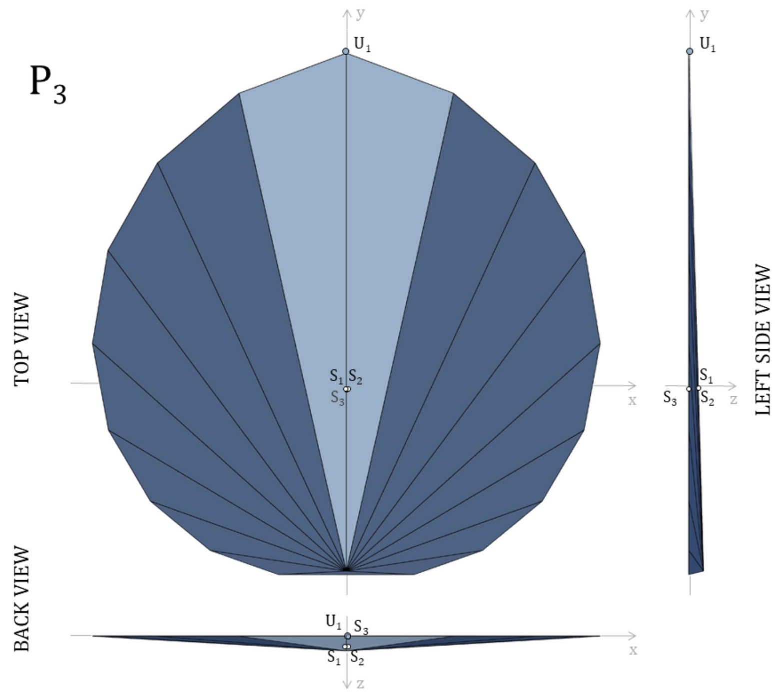

Figure 3. Polyhedron $P_3 \in (3,1)^E$, $(18,18)^C$ with coordinates from *Table 2*. The unstable vertex and the stable faces are highlighted in light blue. The 17-sided base of the pyramid, which also carries a stable equilibrium, is not visible in this view. Stable and unstable equilibrium points are marked in the figure by *S* and *U*, respectively. The polyhedron is a symmetric 17-sided pyramid. The base of the pyramid is based on a double lattice Conway spiral, and the apex was chosen to eliminate any unstable equilibrium on it.

3. GEOMETRY OF BEZDEK'S MONOSTABLE 17-SIDED PYRAMID

The polyhedron P_B, created by Bezdek (Bezdek 2011) is the first and only known homogeneous polyhedron construction in the $(1,3)^E$ class. It was also designed based on Conway's idea, as it is a pyramid version of the first homogeneous monostatic polyhedron, the Conway's polyhedron $P_C \in (1,4)^E$ (Goldberg–Guy 1969). The starting point of the construction of P_B is a pyramid built on a double Conway 9-spiral, with its apex positioned above the center of mass *C* of the base *(Figure 4)*. If $y_{C,P} < 0$, then equilibrium will only exist on the face $R_1 R_{17} R_{18}$.

To achieve monostability, the base face must be selected such that the projection of the center of mass C_P onto the plane of the base lies outside the face. In this case, the base does not hold any stable equilibrium. This can be achieved by intersecting the pyramid with an inclined plane, creating vertices $R_1,\dots R_{17}$, so that the angle $\angle R_0' R_9 C_P > 90°$, where R_0' is the mid-point of the line segment $R_1 R_{17}$.

These two criteria also ensure that the polyhedron has exactly 3 unstable equilibria. The unstable vertices are R_1, R_{17} and R_{18}. The $\angle R_0' R_9 C_P > 90°$ criterion ensures that R_9, the peak of the inclined spiral (the base of the pyramid), does not hold any unstable equilibrium, and the $y_{C,P} < 0$ criterion excludes the $R_2,\dots R_8, R_{10},\dots R_{16}$ vertices.

Considering these geometric constraints (illustrated in *Figure 4*), first a MATLAB-based search and check was conducted. However, even if we start the construction from a lattice version of the spiral, the intersection with the inclined plane does not yield integer coordinates. Direct rounding of these values would cause the 17 base vertices to no longer lie on the same plane, thus distorting the combinatorics of the polyhedron.

To resolve this, let us rotate the polyhedron around the x-axis so that the base plane aligns with the xy-plane. Once all base z-coordinates are zero, we can safely round to integer values. Rounding the apex does not alter the combinatorial structure of the pyramid.

This procedure led to the determination of the integer coordinates listed in *Table 3*. The results, verified again after the rotation and rounding with MATLAB yielded a lattice polyhedral variant of the structure made by Bezdek *(Figure 5).*

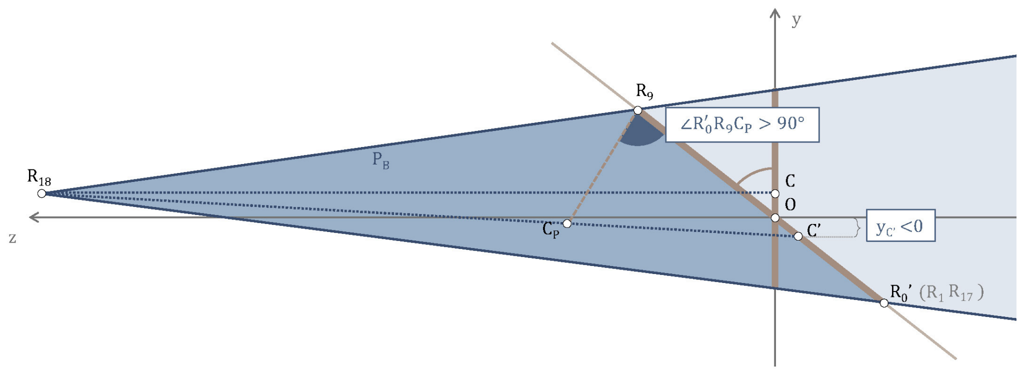

Figure 4. Not to scale section view of the P_B polyhedron along its symmetry plane (R_0' R_9 R_{18}). The initial double lattice Conway 9-spiral lies in the xy-plane with center of mass C. A regular pyramid with apex R_{18} is constructed on top of it, such that CR_{18} is perpendicular to the base. The pyramid is then intersected by an inclined plane containing the x-axis, producing vertices R_1 , R_2 ,…, R_{17}. These form a transformed spiral with center of mass C'. The angle between the planes of the base spiral and the transformed spiral (both highlighted in thick brown) is one of the two parameters adjusted during the search. The other is the pyramid height $z_{R,18}$. To satisfy the monostable property, two conditions must be met (highlighted in white rectangles): y_C'<0, and $\angle R_0'$ R_9 C_P >90°, where R_0' is the midpoint of segment R_1 R_{17}, and C_P is the centre of mass of the polyhedron, i.e., the point dividing segment $C'R_{18}$ in a 1:3 ratio closer to C'. In our construction (prior to rounding), the angle between the intersecting plane and the base spiral plane is 85.7°, and the pyramid height is $z_{R,18}$ =3×10^7.

4. FROM BEZDEK TO CONWAY: CREATING THE LATTICE CONWAY'S POLYHEDRON

The fourth known monostatic convex polyhedron is Conway's polyhedron (Goldberg–Guy 1969). It was the first of its kind, inspiring later examples. However, when thinking in reverse, one can construct its geometry from the lattice version of the P_B polyhedron, described in Section 2. If P_B is mirrored on the plane perpendicular to R_0' R_{18} in R_{18}, and subsequently, by considering the convex hull we remove R_{18}, the resulting polyhedron with 34 vertices resembles the geometry of Conway's monostable polyhedron. However, to ensure that all coordinates are inte-

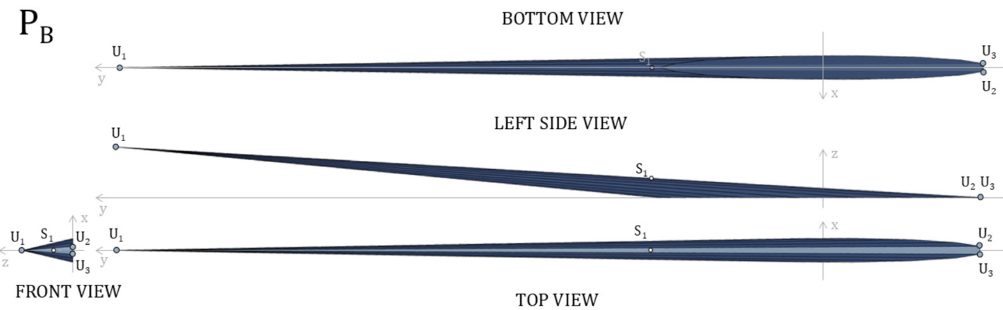

Figure 5. Polyhedron $P_B \in (1,3)^E$, $(18,18)^C$ with coordinates from *Table 3.* The stable face and the unstable vertices are highlighted in light blue. The equilibrium points are marked in the figure: the stable equilibrium with *S* and unstable equilibria with *U*. The construction is based on Bezdek's approach: a symmetric pyramid built on a double Conway 9-spiral base, intersected by an inclined plane in such a way that the base is not stable.

gers and the polyhedron maintains its monostable property, further geometric adjustments would be necessary. Therefore, we start from a double lattice Conway 9-spiral lying in the *xy*-plane, as we did in the beginning of the previous examples. In the following, we describe the construction. All main concepts are illustrated in *Figure 6.*

We start the construction of the lattice Conway polyhedron with the double lattice Conway 9-spiral, created in Subsection 2.1. lying in the *xy*-plane with a centre of mass *C*. We will denote the vertices with $R_1,\dots R_{17}$ of one heptadecagon, and with $R_{18},\dots R_{34}$ the vertices of the other one. Erect a 17-sided prism along the *z*-axis from the original double lattice Conway 9-spiral. To create the inclined faces we intersect the prism with two planes with equation

$$(9) \quad z = \pm\, m\,(y - k),$$

where $\pm\, m = \pm\,(z_0/k)$ is the slope of the inclined planes, where z_0 is the intersections of the planes with the *z*-axis, respectively, and k is the height of their intersection with axis *y*. If the slope of the inclined plane is rational ($m \in \mathbb{Q}$), the intersection points ($R_1,\dots R_{17}$) will also have rational coordinates, which can be scaled to integers. If $m, k \in \mathbb{Z}$, then the intersection points will be integers. This truncation results in the lattice Conway polyhedron. The construction is determined by two parameters: the length of the prism (l) and the slope of the inclined plane (m). Using (9), l can be expressed as

$$(10) \quad l = 2m(k - y_{R,9}).$$

These parameters must be picked in such a way that the polyhedron satisfies specific geometric and equilibrium conditions:

(I) The two, 17-sided, inclined faces must not carry a stable equilibrium. Denote the lines perpendicular to these faces and containing vertices R_9 and R_{26} by l_9, l_{26} respectively and denote the intersection point of l_9, l_{26} by L. The centre of mass of the polyhedron is denoted by C_P. Then, condition (I) can be written as

$$(11) \quad y_{C,P} \geq y_L.$$

(II) The polyhedron P_C is composed of a (middle) prismatic part P_{C0} with centre of mass P_{C0} and two wedge-shaped sections P_{C1}, P_{C2} with respective centers of masses C_{P1}, C_{P2}. In our case $C_{P0} = C$, the centre of mass of the double lattice Conway 9-spiral. The free parameters (the length of the prism (l) and the slope of the wedge (m)) must be chosen so that we have

(12) $y_{C,P}<0.$

To achieve (12), we have $y_{C,P0} = y_C$, while the centre of mass of the wedges is calculated using the formula:

(13) $y'_{C,P1} = y'_{C,P2} = (I_{x'})/(S_{x'})$

where x' is the axis parallel to the x-axis and passing through point R_9, y' is the axis parallel to the y-axis with opposite direction and also passing through point R_9. $S_{x'}$, $I_{x'}$ are the first and second-order moments of area of the Conway 9-spiral to the axis x'. $y_{C,P1}'$ and $y_{C,P2}'$ are the y' coordinates of C_{P1} and C_{P2} respectively, in the coordinate system (x' y'). (Pelikán 1972)

Following these considerations we constructed the lattice Conway polyhedron *(Figure 7, Table 4)*. Its inclined faces have slopes of $\pm m = \pm 4$, and the total length of the prismatic section P_{C0} is about half the height of the spiral, considering that k has to be an integer. The exact value of l can be computed based on (10):

(14) $k = l/8 + y_{R,9} \in \mathbb{Z}$,

so we have

(15) $l = 8 * \lfloor h/16 \rfloor$

and substituted into (14)

(16) $k = \lfloor h/16 \rfloor + y_{R,9} = \lfloor (17_{yR,9} - y_{R,1})/16 \rfloor \in \mathbb{Z}.$

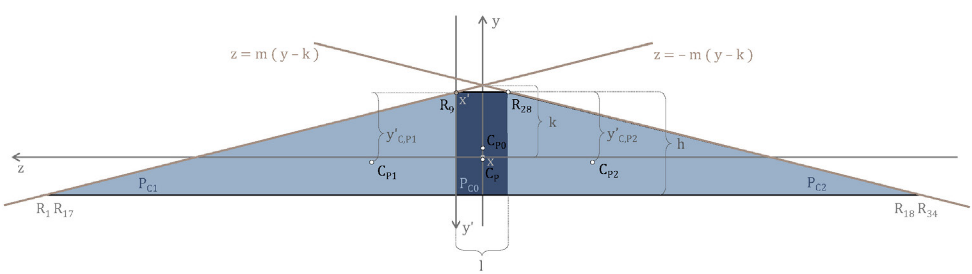

Figure 6. Cross-section of the Conway polyhedron in the *yz*-plane, illustrating the free parameters of the construction. The initial lattice double Conway spiral is in the *xy*-plane, with height *h*. The prismatic section (dark blue) is denoted as P_{C0}, with a centroid at C_{P0} and a length of *l*. The wedge-shaped sections (light blue) are denoted as P_{C1} and P_{C2}, with centroids at C_{P1} and C_{P2}, respectively. The slope of the wedges can be written as $z = \pm m (y-k)$ (brown), passing through points R_9 and R_{28}, respectively. The new coordinate system $(x' y')$, introduced for the computation of the wedge sections, has its origin at R_9. The centroid of the entire polyhedron is denoted as C_P.

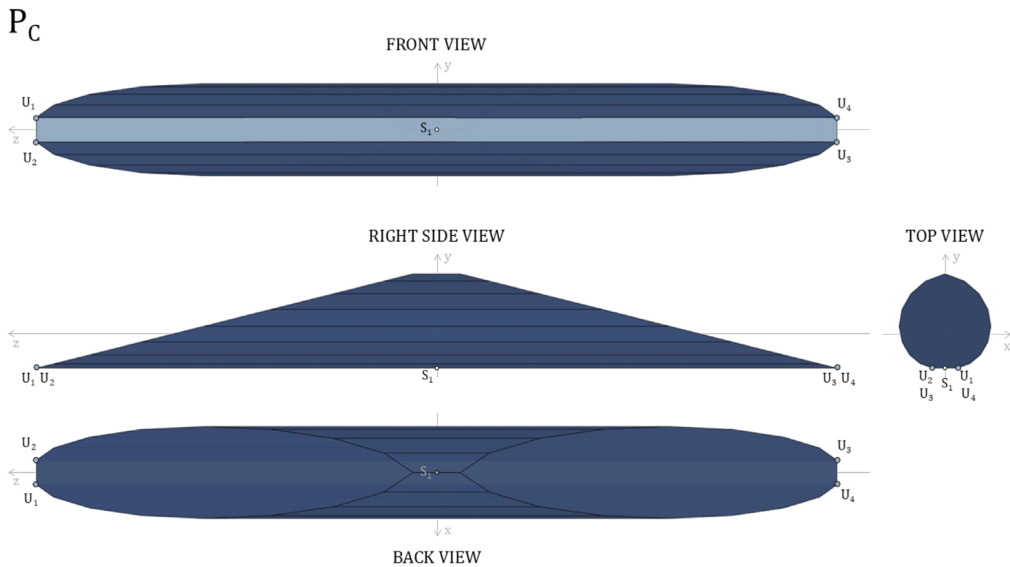

Figure 7. Polyhedron $P_C \in (1,4)^E, (19,34)^C$ with coordinates from *Table 4*. The stable face and the unstable vertices are highlighted in light blue. The equilibrium points are marked in the figure: the stable equilibrium with S and unstable equilibria with U. The construction is based on Conway's polyhedron: a prism on a lattice double Conway 9-spiral base, intersected by two inclined planes.

Table 1. Coordinates of lattice polyhedron P_2 in primary equilibrium class $(S,U)^E = (2,1)^E$

x	y	z
365130	−1003186	0
732587	−873063	0
1053703	−608356	0
1279190	−225556	0
1365623	240796	0
1282053	740193	0
1015871	1210667	0
577057	1585450	0
0	1801203	0
−577057	1585450	0
−1015871	1210667	0
−1282053	740193	0
−1365623	240796	0
−1279190	−225556	0
−1053703	−608356	0
−732587	−873063	0
−365130	−1003186	0
10000	−983976	79182

Table 2. Coordinates of lattice polyhedron P_3 in primary equilibrium class $(S,U)^E = (3,1)^E$

x	y	z
365130	–1003186	0
732587	–873063	0
1053703	–608356	0
1279190	–225556	0
1365623	240796	0
1282053	740193	0
1015871	1210667	0
577057	1585450	0
0	1801203	0
–577057	1585450	0
–1015871	1210667	0
–1282053	740193	0
–1365623	240796	0
–1279190	–225556	0
–1053703	–608356	0
–732587	–873063	0
–365130	–1003186	0
0	–983976	79182

Table 3. Coordinates of lattice polyhedron based on (Bezdek 2011) in primary equilibrium class $(S,U)^E = (1,3)^E$

x	y	z
443228	–17159512	0
865274	–14530623	0
1179765	–9597962	0
1331959	–3309437	0
1310209	3255391	0
1134551	9230112	0
837732	14068092	0
451364	17474518	0
0	19280994	0
–451364	17474518	0
–837732	14068092	0
–1134551	9230112	0
–1310209	3255391	0
–1331959	–3309437	0

Table 3. cont.

x	y	z
−1179765	−9597962	0
−865274	−14530623	0
−443228	−17159512	0
0	79798943	5677353

Table 4. Coordinates of lattice polyhedron based on (Goldberg–Guy 1969) in primary equilibrium class $(S,U)^E = (1,4)^E$

x	y	z
365130	−1003186	11918653
732587	−873063	11398161
1053703	−608356	10339333
1279190	−225556	8808133
1365623	240796	6942725
1282053	740193	4945137
1015871	1210667	3063241
577057	1585450	1564109
0	1801203	701097
−577057	1585450	1564109
−1015871	1210667	3063241
−1282053	740193	4945137
−1365623	240796	6942725
−1279190	−225556	8808133
−1053703	−608356	10339333
−732587	−873063	11398161
−365130	−1003186	11918653

x	y	z
365130	−1003186	−11918653
732587	−873063	−11398161
1053703	−608356	−10339333
1279190	−225556	−8808133
1365623	240796	−6942725
1282053	740193	−4945137
1015871	1210667	−3063241
577057	1585450	−1564109
0	1801203	−701097
−577057	1585450	−1564109
−1015871	1210667	−3063241
−1282053	740193	−4945137
−1365623	240796	−6942725
−1279190	−225556	−8808133
−1053703	−608356	−10339333
−732587	−873063	−11398161
−365130	−1003186	−11918653

5. SUMMARY

In this paper, we presented the explicit geometric realization of four monostatic polyhedra constructed on integer lattice coordinates. To achieve this, we introduced the concept of the double lattice Conway n-spiral, which provided a systematic approach for generating polyhedral structures with integer coordinates while preserving monostatic properties. Two of the polyhedra, P_2 and P_3, are mono-unstable, based on Domokos et al. (2020), while P_B and P_C is monostable, following Bezdek's and Conway's approach. Through MATLAB-based computations, we identified suitable configurations that satisfy the mono-unstable and monostable constraints and verified the stability conditions. These results contribute to the study of monostatic polyhedra by providing the first known lattice-based examples in these equilibrium classes, and have given the lattice polyhedral version of all theoretical monostatic constructions so far.

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Homogén monostatikus rácspoliéderek, avagy a mágikus 17-oldalú piramis

ÖSSZEFOGLALÓ

A cikkben bemutatom a korábban bizonyítottan létező összes monostatikus poliéder explicit geometriai megvalósítását. Kettő közülük (P_2 és P_3) mono-instabil, melyek létezését a Domokos et al. (2020) cikk bizonyítja. A másik kettő (P_C és P_B) monostabil, létezésüket Goldberg–Guy (1969) és Bezdek (2011) bizonyították. Közülük három poliéder (P_B, P_2 és P_3) kombinatorikai szerkezete azonos: egy 17-oldalú gúla. Minden konstrukció esetében a koordinátákat egész számokkal adjuk meg.

KULCSSZAVAK

monostatikus poliéder, egyensúly, konvex geometria, rácspoliéder, Conway