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Fixed points and asymptotic stability of nonlinear fractional difference equations

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Abstract In this paper, we discuss nonlinear fractional difference equations with the Caputo like difference operator. Some asymptotic stability results of equations under investigated are obtained by employing Schauder fixed point theorem and discrete Arzela-Ascoli's theorem. Three examples are also provided to illustrate our main results.

Keywords: Fractional difference equation; Caputo like difference; Asymptotic stability; Schauder fixed point theorem; Discrete Arzela-Ascoli's theorem

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1 Introduction

This paper investigates the asymptotic stability of solutions for a class of nonlinear fractional difference equations

$$\begin{cases} \Delta_*^\alpha x(t) = f(t+\alpha, x(t+\alpha)), \quad t \in N_{1-\alpha},\ 0 < \alpha \le 1, \\ x(0) = x_0, \end{cases} \tag{1}$$

where Δ_*^α is a Caputo like discrete fractional difference, $f : [0, +\infty) \times R \to R$ is continuous with respect to t and x, $N_t = \{t, t+1, t+2, \cdots\}$.

Fractional differential equations have received increasing attention during recent years since these equations have been proved to be valuable tools in the modeling of many phenomena in various fields of science and engineering, see the monographs [18, 20, 23, 25] and the papers [1, 7, 11, 19, 24, 28, 30, 31] and the references therein.

Fractional difference equations have also been studied more intensively of late [2-6, 12]. In particular, Atici and Eloe [3] investigated the commutativity properties of the fractional sum and the fractional difference operators, Atici and Sengül [6] developed Leibniz rule and summation by parts formula, Anastassiou [2] defined a Caputo like discrete fractional difference and compared it to the Riemann-Liouville fractional discrete analog, and Chen et al. [12] gave global and local existence results of solutions for nonlinear fractional difference equations with the Caputo like difference operator.

However, due to the lack of geometry interpretation of the fractional derivatives, it is difficult to find a valid tool to analyze the stability of fractional differential equations, and there are few work on the stability of solutions for either fractional differential equations or fractional difference equations. Some local asymptotical stability, Mittag-Leffler stability and linear matrix inequality (LMI) stability are discussed in [13, 15, 21, 22, 27], Chen and Zhou [13] considered the attractivity of fractional functional differential equations by Schauder fixed point theorem, Deng [15] discussed the attractivity of nonlinear fractional differential equations by means of the principle of contraction mappings, but there's no work on asymptotic stability of fractional difference equations via fixed point theorems.

To study stability properties of differential equations, Burton [10] pointed out that many difficulties of Liapunov's direct method, such as constructing Liapunov functions and functionals, ascertaining limit sets when the equation becomes unbounded or the derivative is not definite, vanish when fixed point theory is used.

Motivated by applying fixed point theory to research stability of integer-order differential equations [8-10, 16, 17, 26], in this paper, we discuss asymptotic stability of nonlinear fractional difference equations by using Schauder fixed point theorem and discrete Arzela-Ascoli's theorem.

The rest of the paper is organized as follows. In section 2, we introduce some useful preliminaries. In section 3, we prove some sufficient conditions of asymptotic stability of IVP (1). Finally, three examples are given to illustrate our main results.

2 Preliminaries

In this section, we introduce preliminary facts which are used throughout this paper.

Definition 2.1 [3, 4] Let $\nu > 0$. The $\nu-$th fractional sum x is defined by

$$\Delta^{-\nu}x(t) = \frac{1}{\Gamma(\nu)}\sum_{s=a}^{t-\nu}(t-s-1)^{(\nu-1)}x(s), \tag{2}$$

where x is defined for $s = a$ mod (1) and $\Delta^{-\nu}x$ is defined for $t = (a+\nu)$ mod (1), and $t^{(\nu)} = \frac{\Gamma(t+1)}{\Gamma(t-\nu+1)}$.

In (2), the fractional sum $\Delta^{-\nu}$ maps functions defined on N_a to functions defined on $N_{a+\nu}$. Atici and Eloe [3] pointed out that this definition is the development of the theory of the fractional calculus on time scales.

Definition 2.2 [2] Let $\mu > 0$ and $m-1 < \mu < m$, where m denotes a positive integer, $m = \lceil\mu\rceil$, $\lceil\cdot\rceil$ ceiling of number. Set $\nu = m - \mu$. The $\mu-$th fractional Caputo like difference is defined as

$$\begin{aligned}\Delta_*^{\mu}x(t) &= \Delta^{-\nu}(\Delta^m x(t)) \\ &= \frac{1}{\Gamma(\nu)}\sum_{s=a}^{t-\nu}(t-s-1)^{(\nu-1)}(\Delta^m x)(s), \quad \forall t \in N_{a+\nu},\end{aligned} \tag{3}$$

where Δ^m is the $m-$th order forward difference operator, the fractional Caputo like difference Δ_*^{μ} maps functions defined on N_a to functions defined on $N_{a-\mu}$.

Lemma 2.1 [2] For $\mu > 0$, μ non-integer, $m = \lceil\mu\rceil$, $\nu = m - \mu$, it holds:

$$x(t) = \sum_{k=0}^{m-1}\frac{(t-a)^{(k)}}{k!}\Delta^k x(a) + \frac{1}{\Gamma(\mu)}\sum_{s=a+\nu}^{t-\mu}(t-s-1)^{(\mu-1)}\Delta_*^{\mu}x(s),$$

where x is defined on N_a with $a \in Z^+$, $Z^+ = \{0, 1, 2, \cdots\}$.

In particular, when $0 < \mu < 1$ and $a = 0$, we have

$$x(t) = x(0) + \frac{1}{\Gamma(\mu)}\sum_{s=1-\mu}^{t-\mu}(t-s-1)^{(\mu-1)}\Delta_*^{\mu}x(s). \tag{4}$$

where x is defined on N_1 and $\Delta_*^{\mu}x$ is defined on $N_{1-\mu}$.

Remark 2.1 x in (4) should be defined on N_0 according to Lemma 2.1, but $t=0$ leads to $t-\mu=-\mu<1-\mu$ which makes the sum $\sum_{s=1-\mu}^{t-\mu}(t-s-1)^{(\mu-1)}\Delta_*^\mu x(s)$ no sense, then we define x on N_1.

Lemma 2.2 A solution $x: N_1 \to R$ is a solution of IVP (1) if and only if $x(t)$ is a solution of the the following fractional Taylor's difference formula

$$x(t) = x_0 + \frac{1}{\Gamma(\alpha)} \sum_{s=1-\alpha}^{t-\alpha} (t-s-1)^{(\alpha-1)} f(s+\alpha, x(s+\alpha)), \quad t \in N_1. \tag{5}$$

Proof. Suppose that x defined on N_1 is a solution of (1), i.e., $\Delta_*^\alpha x(s) = f(s+\alpha, x(s+\alpha))$ for $s \in N_{1-\alpha}$ and $x(0) = x_0$. From (4) we have

$$\begin{aligned} x(t) &= x(0) + \frac{1}{\Gamma(\alpha)} \sum_{s=1-\alpha}^{t-\alpha} (t-s-1)^{(\alpha-1)} \Delta_*^\alpha x(s) \\ &= x_0 + \frac{1}{\Gamma(\alpha)} \sum_{s=1-\alpha}^{t-\alpha} (t-s-1)^{(\alpha-1)} f(s+\alpha, x(s+\alpha)), \end{aligned}$$

which implies that (5) holds.

Conversely, if $x(t)$ is a solution of (5), comparing between with (4) and (5) we have

$$\sum_{s=1-\alpha}^{t-\alpha} (t-s-1)^{(\alpha-1)} \Delta_*^\alpha x(s) = \sum_{s=1-\alpha}^{t-\alpha} (t-s-1)^{(\alpha-1)} f(s+\alpha, x(s+\alpha)),$$

that is,

$$\sum_{s=1-\alpha}^{t-\alpha} (t-s-1)^{(\alpha-1)} [\Delta_*^\alpha x(s) - f(s+\alpha, x(s+\alpha))] \equiv 0 \tag{6}$$

for $t \in N_1$.

For $t=1$, form (6) we have

$$(\alpha-1)^{(\alpha-1)} [\Delta_*^\alpha x(1-\alpha) - f(1-\alpha+\alpha, x(1-\alpha+\alpha))] = 0,$$

which implies that

$$\Delta_*^\alpha x(1-\alpha) = f(1-\alpha+\alpha, x(1-\alpha+\alpha)). \tag{7}$$

For $t=2$, form (6) we have

$$\begin{aligned} & \alpha^{(\alpha-1)}[\Delta_*^\alpha x(1-\alpha) - f(1-\alpha+\alpha, x(1-\alpha+\alpha))] \\ + & \ (\alpha-1)^{(\alpha-1)}[\Delta_*^\alpha x(2-\alpha) - f(2-\alpha+\alpha, x(2-\alpha+\alpha))] = 0, \end{aligned}$$

which, together with (7), implies that

$$\Delta_*^\alpha x(2-\alpha) = f(2-\alpha+\alpha, x(2-\alpha+\alpha)).$$

By induction, we have that $\Delta_*^\alpha x(t) = f(t+\alpha, x(t+\alpha))$ for $t \in N_{1-\alpha}$, which implies that $x(t)$ is a solution of (1). This completes the proof.

Since $\Delta^{-\nu} t^{(\mu)} = \frac{\Gamma(\mu+1)}{\Gamma(\mu+\nu+1)} t^{(\mu+\nu)}$ ([3], Lemma 1.1), we have

$$\begin{aligned} x_0 &= x_0 t^{(0)} = \Delta^{-\alpha} \frac{\Gamma(1)x_0}{\Gamma(1-\alpha)} t^{(-\alpha)} \\ &= \frac{1}{\Gamma(\alpha)} \sum_{s=1-\alpha}^{t-\alpha} (t-s-1)^{(\alpha-1)} \frac{x_0}{\Gamma(1-\alpha)} (s+\alpha)^{(-\alpha)}. \end{aligned}$$

It follows that (5) is equivalent to the following equation

$$\begin{aligned} x(t) &= \tfrac{1}{\Gamma(\alpha)} \sum_{s=1-\alpha}^{t-\alpha} (t-s-1)^{(\alpha-1)} [\tfrac{x_0}{\Gamma(1-\alpha)} (s+\alpha)^{(-\alpha)} \\ & \quad + f(s+\alpha, x(s+\alpha))], \qquad t \in N_1. \end{aligned} \tag{8}$$

Lemma 2.3 Assume that $\beta > 1$ and $\gamma > 0$, then

$$[t^{(-\gamma)}]^\beta < \frac{\Gamma(1+\beta\gamma)}{\Gamma^\beta(1+\gamma)} t^{(-\beta\gamma)}$$

for $t \in N_1$.

Proof. Since that $1 + \frac{\beta\gamma}{t} < (1+\frac{\gamma}{t})^\beta$, for $t \in N_1$, we have

$$\begin{aligned} & \Gamma^{\beta-1}(t+1)\Gamma(t+\beta\gamma+1)\Gamma^\beta(1+\gamma) \\ = & \ [t^{\beta-1}(t-1)^{\beta-1}\cdots 1^{\beta-1}][(t+\beta\gamma)(t-1+\beta\gamma)\cdots(1+\beta\gamma) \\ & \cdot\Gamma(1+\beta\gamma)]\Gamma^\beta(1+\gamma) \\ = & \ [t^\beta(1+\frac{\beta\gamma}{t})][(t-1)^\beta(1+\frac{\beta\gamma}{t-1})]\cdots[1^\beta(1+\frac{\beta\gamma}{1})]\Gamma(1+\beta\gamma)\Gamma^\beta(1+\gamma) \\ < & \ [t^\beta(1+\frac{\gamma}{t})^\beta][(t-1)^\beta(1+\frac{\gamma}{t-1})^\beta]\cdots[1^\beta(1+\frac{\gamma}{1})^\beta]\Gamma^\beta(1+\gamma)\Gamma(1+\beta\gamma) \\ = & \ (t+\gamma)^\beta(t-1+\gamma)^\beta\cdots(1+\gamma)^\beta\Gamma^\beta(1+\gamma)\Gamma(1+\beta\gamma) \\ = & \ \Gamma^\beta(t+\gamma+1)\Gamma(1+\beta\gamma), \end{aligned}$$

that is,

$$\frac{\Gamma^{\beta-1}(t+1)}{\Gamma^{\beta}(t+\gamma+1)} < \frac{\Gamma(1+\beta\gamma)}{\Gamma^{\beta}(1+\gamma)} \cdot \frac{1}{\Gamma(t+\gamma\beta+1)}.$$

Thus,

$$[t^{(-\gamma)}]^{\beta} = \frac{\Gamma^{\beta}(t+1)}{\Gamma^{\beta}(t+\gamma+1)} < \frac{\Gamma(1+\beta\gamma)}{\Gamma^{\beta}(1+\gamma)} \cdot \frac{\Gamma(t+1)}{\Gamma(t+\gamma\beta+1)} = \frac{\Gamma(1+\beta\gamma)}{\Gamma^{\beta}(1+\gamma)} t^{(-\beta\gamma)}$$

holds for $t \in N_1$. This completes the proof.

Definition 2.3 The solution $x = \varphi(t)$ of IVP (1) is said to be

(i) stable, if for any $\varepsilon > 0$ and $t_0 \in R^+$, there exists a $\delta = \delta(t_0, \varepsilon) > 0$ such that

$$|x(t, x_0, t_0) - \varphi(t)| < \varepsilon$$

for $|x_0 - \varphi(t_0)| \leq \delta(t_0, \varepsilon)$ and all $t \geq t_0$;

(ii) attractive, if there exists $\eta(t_0) > 0$ such that $\|x_0\| \leq \eta$ implies

$$\lim_{t\to\infty} x(t, x_0, t_0) = 0;$$

(iii) asymptotically stable if it is stable and attractive.

The space $l_{n_0}^{\infty}$ is the set of real sequences defined on the set of positive integers where any individual sequence is bounded with respect to the usual supremum norm. It is well know that under the supremum norm $l_{n_0}^{\infty}$ is a Banach space [29].

Definition 2.4 [14] A set Ω of sequences in $l_{n_0}^{\infty}$ is uniformly Cauchy (or equi-Cauchy) if for every $\varepsilon > 0$, there exists an integer N such that $|x(i) - x(j)| < \varepsilon$ whenever $i, j > N$ for any $x = \{x(n)\}$ in Ω.

Theorem 2.1 [14] (Discrete Arzela-Ascoli's Theorem) A bounded, uniformly Cauchy subset Ω of $l_{n_0}^{\infty}$ is relatively compact.

The following fixed point theorem are classical, which can be seen from many books.

Theorem 2.2 (Schauder fixed point theorem) Let Ω be a closed, convex and nonempty subset of a Banach space X. Let $T : \Omega \to \Omega$ be a continuous mapping such that $T\Omega$ is a relatively compact subset of X. Then T has at least one fixed point in Ω. That is, there exists an $x \in \Omega$ such that $Tx = x$.

3 Main Results

Let l_1^∞ be the set of all real sequence $x = \{x(t)\}_{t=1}^\infty$ with norm $\|x\| = \sup_{t\in N_1} |x(t)|$. Then l_1^∞ is a Banach space.

Define the operator

$$Tx(t) = x_0 + \frac{1}{\Gamma(\alpha)} \sum_{s=1-\alpha}^{t-\alpha} (t-s-1)^{(\alpha-1)} f(s+\alpha, x(s+\alpha)), \tag{9}$$

Obviously, $x(t)$ is a solution of (1) if it is a fixed point of the operator T.

Lemma 3.1 Assume that the following condition is satisfied:

(H_1) there exist constants $\gamma_1, L_1 > 0$ such that

$$\left| x_0 + \frac{1}{\Gamma(\alpha)} \sum_{s=1-\alpha}^{t-\alpha} (t-s-1)^{(\alpha-1)} f(s+\alpha, x(s+\alpha)) \right| \le L_1 t^{(-\gamma_1)}$$

for $t \in N_1$.

Then IVP (1) exists at least one solution $x(t)$ for $t \in N_1$.

Proof. Define the set

$$S_1 = \{x(t) : |x(t)| \le L_1 t^{(-\gamma_1)} \text{ for } t \in N_1\}.$$

It is easy to know that S_1 is a closed, bounded and convex subset of R. In addition, for $t \in N_1$, we have

$$t^{(-\gamma_1)} = \frac{\Gamma(t+1)}{\Gamma(t+\gamma_1+1)} = \frac{t!}{(t+\gamma_1)\cdots(1+\gamma_1)\Gamma(1+\gamma_1)} \to 0 \quad \text{for} \quad t \to \infty.$$

To prove that T has a fixed point, we firstly show that T maps S_1 in S_1.

For $t \in N_1$, condition (H_1) implies that $|Tx(t)| \le L_1 t^{(-\gamma_1)}$, which yields that $TS_1 \subset S_1$.

Nextly, we show that T is continuous on S_1.

Let $\varepsilon > 0$ be given, there exists a $N_1 \in N_1$ such that $t > N_1$ implies that $L_1 t^{(-\gamma_1)} < \frac{\varepsilon}{2}$.

Let $\{x_n\}$ be a sequence such that $x_n \to x$. For $t \in \{1, 2, \cdots, N_1\}$, applying the continuity of f and $\sum_{s=1-\alpha}^{t-\alpha} (t-s-1)^{(\alpha-1)} = \frac{\Gamma(t+\alpha)}{\alpha\Gamma(t)}$ ([12], Lemma 2.5), we have

$$\begin{aligned}
& |Tx_n(t) - Tx(t)| \\
\leq\ & \frac{1}{\Gamma(\alpha)} \sum_{s=1-\alpha}^{t-\alpha} (t-s-1)^{(\alpha-1)} |f(s+\alpha, x_n(s+\alpha)) - f(s+\alpha, x(s+\alpha))| \\
\leq\ & \max_{s\in\{1-\alpha,\cdots,N_1-\alpha\}} |f(s+\alpha, x_n(s+\alpha)) \\
& -f(s+\alpha, x(s+\alpha))| \frac{1}{\Gamma(\alpha)} \sum_{s=1-\alpha}^{t-\alpha} (t-s-1)^{(\alpha-1)} \\
=\ & \frac{\Gamma(t+\alpha)}{\Gamma(\alpha+1)\Gamma(t)} \max_{s\in\{1-\alpha,\cdots,N_1-\alpha\}} |f(s+\alpha, x_n(s+\alpha)) - f(s+\alpha, x(s+\alpha))| \\
\leq\ & \frac{\Gamma(N_1+\alpha)}{\Gamma(\alpha+1)\Gamma(N_1)} \max_{s\in\{1-\alpha,\cdots,N_1-\alpha\}} |f(s+\alpha, x_n(s+\alpha)) \\
& -f(s+\alpha, x(s+\alpha))| \\
\to\ & 0 \quad \text{as} \quad n \to \infty.
\end{aligned}$$

For $t \in \{N_1+1, N_1+2, \cdots\}$ we have

$$\begin{aligned}
& |Tx_n(t) - Tx(t)| \\
=\ & \Big| x_0 + \frac{1}{\Gamma(\alpha)} \sum_{s=1-\alpha}^{t-\alpha} (t-s-1)^{(\alpha-1)} f(s+\alpha, x_n(s+\alpha)) \\
& -[x_0 + \frac{1}{\Gamma(\alpha)} \sum_{s=1-\alpha}^{t-\alpha} (t-s-1)^{(\alpha-1)} f(s+\alpha, x(s+\alpha))] \Big| \\
\leq\ & 2L_1 t^{(-\gamma_1)} \leq \varepsilon.
\end{aligned}$$

Thus, for all $t \in N_1$, we have

$$|Tx_n(t) - Tx(t)| \to 0 \quad \text{as} \quad n \to \infty.$$

which means that T is continuous.

Lastly, we show that TS_1 is relatively compact.

Let $t_1, t_2 \in N_1$ and $t_2 > t_1 \geq N_1$, we have

$$\begin{aligned}
& |Tx(t_2) - Tx(t_1)| \\
=\ & \frac{1}{\Gamma(\alpha)} \Big| x_0 + \sum_{s=1-\alpha}^{t_1-\alpha} (t_1-s-1)^{(\alpha-1)} f(s+\alpha, x(s+\alpha))
\end{aligned}$$

$$
\begin{aligned}
&\quad -[x_0 + \sum_{s=1-\alpha}^{t_2-\alpha} (t_2 - s - 1)^{(\alpha-1)} f(s+\alpha, x(s+\alpha))]\Big| \\
&\leq \; L_1 {t_2}^{(-\gamma_1)} + L_1 {t_1}^{(-\gamma_1)} < \varepsilon.
\end{aligned}
$$

Therefore, $\{Tx : x \in S_1\}$ is a bounded and uniformly Cauchy subset. Hence, by Theorem 2.1, TS_1 is relatively compact.

According to Theorem 2.2, we have that T has a fixed point in S_1 which is a solution of IVP (1). This completes the proof.

Theorem 3.1 Assume that condition (H_1) holds, then the solutions of (1) is attractive.

Proof. By Lemma 3.1, the solutions of (1) exist and are in S_1. All functions $x(t)$ in S_1 tend to 0 as $t \to \infty$. Then the solutions of (1) tend to zero as $t \to \infty$. This completes the proof.

Theorem 3.2 Assume that the following condition is satisfied:

(H_2) there exist constants $\gamma_2 \in (\alpha, 1)$ and $L_2 > 0$ such that

$$
|f(t+\alpha, x(t+\alpha)) - f(t+\alpha, y(t+\alpha))| \leq L_2 (t+\alpha)^{(-\gamma_2)} \|x - y\|
$$

for $t \in N_{1-\alpha}$.

Then the solutions of IVP (1) are stable provided that

$$
c = \frac{L_2 \Gamma(1-\gamma_2)}{\Gamma(1+\alpha-\gamma_2)\Gamma(2-\alpha+\gamma_2)} < 1. \tag{10}
$$

Proof Let $x(t)$ be a solution of (1), and let $\tilde{x}(t)$ be a solution of (1) satisfying the initial value condition $\tilde{x}(0) = \tilde{x}_0$. For $t \in N_1$, applying condition (H_2) we have

$$
\begin{aligned}
|x(t) - \tilde{x}(t)| &\leq |x_0 - \tilde{x}_0| + \frac{1}{\Gamma(\alpha)} \sum_{s=1-\alpha}^{t-\alpha} (t-s-1)^{(\alpha-1)} |f(s+\alpha, x(s+\alpha)) \\
&\quad - f(s+\alpha, \tilde{x}(s+\alpha))| \\
&\leq |x_0 - \tilde{x}_0| + \frac{L_2}{\Gamma(\alpha)} \sum_{s=1-\alpha}^{t-\alpha} (t-s-1)^{(\alpha-1)} (s+\alpha)^{(-\gamma_2)} \|x - \tilde{x}\| \\
&= |x_0 - \tilde{x}_0| + L_2 \Delta^{-\alpha} t^{(-\gamma_2)} \|x - \tilde{x}\| \\
&= |x_0 - \tilde{x}_0| + \frac{L_2 \Gamma(1-\gamma_2)}{\Gamma(1+\alpha-\gamma_2)} t^{(\alpha-\gamma_2)} \|x - \tilde{x}\|
\end{aligned}
$$

$$
\begin{aligned}
&\le |x_0-\tilde{x}_0| + \frac{L_2\Gamma(1-\gamma_2)}{\Gamma(1+\alpha-\gamma_2)} 1^{(\alpha-\gamma_2)}\|x-\tilde{x}\| \\
&= |x_0-\tilde{x}_0| + \frac{L_2\Gamma(1-\gamma_2)}{\Gamma(1+\alpha-\gamma_2)\Gamma(2-\alpha+\gamma_2)}\|x-\tilde{x}\| \\
&= |x_0-\tilde{x}_0| + c\|x-\tilde{x}\|,
\end{aligned}
$$

which yields that

$$
\|x-\tilde{x}\| \le \frac{1}{1-c}|x_0-\tilde{x}_0|.
$$

Then, for any $\varepsilon > 0$, let $\delta = (1-c)\varepsilon$, $|x_0-\tilde{x}_0| < \delta$ implies that $\|x-\tilde{x}\| < \varepsilon$. Therefore, the solutions of IVP (1) is stable. This completes the proof.

Combining Theorem 3.1 and Theorem 3.2, we have

Theorem 3.3 Assume that conditions (H_1) and (H_2) hold, then the solutions of IVP (1) are asymptotically stable provided that (10) holds.

Lemma 3.2 Assume that the following condition are satisfied:

(H_3) $\left|\frac{x_0}{\Gamma(1-\alpha)}(t+\alpha)^{(-\alpha)} + f(t+\alpha, x(t+\alpha))\right| \le L_3(t+\alpha)^{(-\gamma_3)}$
for $t \in N_{1-\alpha}$, where $\gamma_3 \in (\alpha, 1)$ and $L_3 > 0$.
Then IVP (1) exists at least one solution $x(t)$ on N_1.

Proof. Define the set

$$
S_2 = \{x(t) : |x(t)| \le \frac{L_3\Gamma(1-\gamma_3)}{\Gamma(1+\alpha-\gamma_3)} t^{(\alpha-\gamma_3)} \text{ for } t \in N_1\}.
$$

From the above assumption of S_2, it is easy to know that S_2 is a closed, bounded and convex subset of R.

We firstly show that T maps S_2 in S_2.

For $t \in N_1$, from condition (H_3) we have

$$
\begin{aligned}
|Tx(t)| &= \frac{1}{\Gamma(\alpha)}\sum_{s=1-\alpha}^{t-\alpha}(t-s-1)^{(\alpha-1)}\Big|\frac{x_0}{\Gamma(1-\alpha)}(s+\alpha)^{(-\alpha)} \\
&\quad + f(s+\alpha, x(s+\alpha))\Big| \\
&\le \frac{L_3}{\Gamma(\alpha)}\sum_{s=1-\alpha}^{t-\alpha}(t-s-1)^{(\alpha-1)}(s+\alpha)^{(-\gamma_3)} \\
&= L_3\Delta^{-\alpha}t^{(-\gamma_3)} \\
&= \frac{L_3\Gamma(1-\gamma_3)}{\Gamma(1+\alpha-\gamma_3)}t^{(\alpha-\gamma_3)},
\end{aligned}
$$

then $TS_2 \subset S_2$.

Nextly, we show that T is continuous on S_2.

Let $\varepsilon > 0$ be given, there exists a $N_2 \in N_1$ such that $t > N_2$ implies that

$$\frac{L_3\Gamma(1-\gamma_3)}{\Gamma(1+\alpha-\gamma_3)}t^{(\alpha-\gamma_3)} < \frac{\varepsilon}{2}.$$

Let $\{x_n\}$ be a sequence such that $x_n \to x$. For $t \in \{1, 2, \cdots, N_2\}$, similar to Lemma 3.1, we have

$$\begin{aligned}
& |Tx_n(t) - Tx(t)| \\
\leq & \frac{\Gamma(N_2+\alpha)}{\Gamma(\alpha+1)\Gamma(N_2)} \max_{s\in\{1-\alpha,\cdots,N_2-\alpha\}} |f(s+\alpha, x_n(s+\alpha)) - f(s+\alpha, x(s+\alpha))| \\
\to & 0 \quad \text{as} \quad n \to \infty.
\end{aligned}$$

For $t \in \{N_2+1, N_2+2, \cdots\}$,

$$\begin{aligned}
& |Tx_n(t) - Tx(t)| \\
= & \Big|\frac{1}{\Gamma(\alpha)}\sum_{s=1-\alpha}^{t-\alpha}(t-s-1)^{(\alpha-1)}[\frac{x_0}{\Gamma(1-\alpha)}(s+\alpha)^{(-\alpha)} + f(s+\alpha, x_n(s+\alpha))] \\
& -\frac{1}{\Gamma(\alpha)}\sum_{s=1-\alpha}^{t-\alpha}(t-s-1)^{(\alpha-1)}[\frac{x_0}{\Gamma(1-\alpha)}(s+\alpha)^{(-\alpha)} + f(s+\alpha, x(s+\alpha))]\Big| \\
\leq & \frac{2L_3}{\Gamma(\alpha)}\sum_{s=1-\alpha}^{t-\alpha}(t-s-1)^{(\alpha-1)}(t+\alpha)^{(-\gamma_3)} \\
= & 2L_3\Delta^{-\alpha}t^{(-\gamma_3)} \\
\leq & \frac{2L_3\Gamma(1-\gamma_3)}{\Gamma(1+\alpha-\gamma_3)}t^{(\alpha-\gamma_3)} \\
< & \varepsilon.
\end{aligned}$$

Thus, for all $t \in N_1$, we have

$$|Tx_n(t) - Tx(t)| \to 0 \quad \text{as} \quad n \to \infty.$$

which means that T is continuous.

The proof of TS_2 be relatively compact is similar to that of Lemma 3.1, and we omit it. By Theorem 2.2, we have that T has a fixed point in S_2 which is a solution of IVP (1). This completes the proof.

Theorem 3.4 Assume that condition (H_3) holds, then the solutions of (1) are attractive.

Theorem 3.5 Assume that conditions (H_2) and (H_3) hold, then the solutions of IVP (1) are asymptotically stable provided that (10) holds.

Lemma 3.3 Assume that the following condition are satisfied:

(H_4) There exist constants $\beta > \frac{1}{1-\alpha}$ and $L_4 > 0$ such that

$$\left|\frac{x_0}{\Gamma(1-\alpha)}(t+\alpha)^{(-\alpha)} + f(t+\alpha, x(t+\alpha))\right| \leq L_4|x(t+\alpha)|^{\beta}$$

for $t \in N_{1-\alpha}$.
Then IVP (1) exists at least one solution $x(t)$ on N_1 provided that

$$\frac{L_4\Gamma(1+\beta\gamma_4)\Gamma(1-\beta\gamma_4)}{\Gamma^{\beta}(1+\gamma_4)\Gamma(1+\alpha-\beta\gamma_4)} \leq 1, \tag{11}$$

where

$$\frac{\alpha}{\beta-1} < \gamma_4 < \frac{1}{\beta}. \tag{12}$$

Proof. From $\beta > \frac{1}{1-\alpha}$, we have that $\frac{\alpha}{\beta-1} < \frac{1}{\beta}$ which implies that γ_4 exists. In addition, $\gamma_4 < \frac{1}{\beta}$ means that $\Gamma(1-\beta\gamma_4) > 0$ and $\Gamma(1+\alpha-\beta\gamma_4) > 0$, $\frac{\alpha}{\beta-1} < \gamma_4$ implies that $\alpha - \beta\gamma_4 < -\gamma_4$.

Define the set

$$S_3 = \{x(t) : |x(t)| \leq t^{(-\gamma_4)} \text{ for } t \in N_1\}.$$

We show that T maps S_3 in S_3.

For $t \in N_1$, applying condition (H_4), Lemma 2.3 and (11), we have

$$\begin{aligned} |Tx(t)| &= \frac{1}{\Gamma(\alpha)}\sum_{s=1-\alpha}^{t-\alpha}(t-s-1)^{(\alpha-1)}\Big|\frac{x_0}{\Gamma(1-\alpha)}(s+\alpha)^{(-\alpha)} \\ &\quad + f(s+\alpha, x(s+\alpha))\Big| \\ &\leq \frac{L_4}{\Gamma(\alpha)}\sum_{s=1-\alpha}^{t-\alpha}(t-s-1)^{(\alpha-1)}|x(s+\alpha)|^{\beta} \end{aligned}$$

$$
\begin{aligned}
&\le \frac{L_4}{\Gamma(\alpha)} \sum_{s=1-\alpha}^{t-\alpha} (t-s-1)^{(\alpha-1)} \left[(s+\alpha)^{(-\gamma_4)}\right]^{\beta} \\
&\le \frac{L_4\Gamma(1+\beta\gamma_4)}{\Gamma(\alpha)\Gamma^{\beta}(1+\gamma_4)} \sum_{s=1-\alpha}^{t-\alpha} (t-s-1)^{(\alpha-1)}(s+\alpha)^{(-\beta\gamma_4)} \\
&= \frac{L_4\Gamma(1+\beta\gamma_4)\Gamma(1-\beta\gamma_4)}{\Gamma^{\beta}(1+\gamma_4)\Gamma(1+\alpha-\beta\gamma_4)} t^{(\alpha-\beta\gamma_4)} \\
&\le t^{(\alpha-\beta\gamma_4)} \\
&\le t^{(-\gamma_4)},
\end{aligned}
$$

then $TS_3 \subset S_3$.

The remaining part of the proof is similar to that of Lemma 3.2, so we omit it.

Theorem 3.6 Assume that condition (H_4) and (11) hold, then the solutions of (1) are attractive.

Theorem 3.7 Assume that conditions (H_2) and (H_4) hold, then the solutions of IVP (1) are asymptotically stable provided that (10) and (11) hold.

4 Examples

As the applications of our main results, we consider the following examples.

Example 4.1 Consider

$$
\begin{cases} \Delta_*^{0.5} x(t) = \frac{1}{\sqrt{2\pi}} (t+0.5)^{(-0.75)} \sin(x(t+0.5)), \quad t \in N_{0.5}, \\ x(0) = 0, \end{cases} \tag{13}
$$

where $f(t+0.5, x(t+0.5)) = \frac{1}{\sqrt{2\pi}}(t+0.5)^{(-0.75)} \sin(x(t+0.5))$.

The fractional Taylor's difference formula of (13) is

$$
x(t) = \frac{1}{\Gamma(0.5)} \sum_{s=0.5}^{t-0.5} (t-s-1)^{(-0.5)} \frac{1}{\sqrt{2\pi}} (s+0.5)^{(-0.75)} \sin(x(s+0.5)),\ t \in N_1.
$$

Since $1 = \Gamma(1) < \Gamma(0.75) < \Gamma(0.5) = \sqrt{\pi}$, we have

$$
\left| \frac{1}{\Gamma(0.5)} \sum_{s=0.5}^{t-0.5} (t-s-1)^{(-0.5)} \frac{1}{\sqrt{2\pi}} (s+0.5)^{(-0.75)} \sin(x(s+0.5)) \right|
$$

$$
\begin{aligned}
&\le \frac{1}{\Gamma(0.5)}\sum_{s=0.5}^{t-0.5}(t-s-1)^{(-0.5)}\frac{1}{\sqrt{2}\pi}(s+0.5)^{(-0.75)}\\
&= \frac{1}{\sqrt{2}\pi}\Delta^{-0.5}t^{(-0.75)}\\
&\le \frac{\Gamma(0.25)}{\sqrt{2}\pi\Gamma(0.75)}t^{(-0.25)}\\
&= \frac{1}{[\Gamma(0.75)]^2}t^{(-0.25)}\\
&\le t^{(-0.25)},
\end{aligned}
$$

which implies that condition (H_1) holds.

In addition,

$$
|f(t+0.5,x(t+0.5))-f(t+0.5,y(t+0.5))| \le \frac{1}{\sqrt{2}\pi}(t+0.5)^{(-0.75)}\|x-y\|,
$$

which implies that condition (H_2) holds, where $L_2=\frac{1}{\sqrt{2}\pi}$, $\gamma_2=0.75$.

Moreover,

$$
\begin{aligned}
\frac{L_2\Gamma(1-\gamma_2)}{\Gamma(1+\alpha-\gamma_2)\Gamma(2-\alpha+\gamma_2)} &= \frac{\Gamma(0.25)}{\sqrt{2}\pi\Gamma(0.75)\Gamma(2.25)}\\
&= \frac{16}{5\sqrt{2}\pi\Gamma(0.75)}\\
&\le \frac{16}{5\sqrt{2}\pi}<1,
\end{aligned}
$$

which implies that inequality (10) holds. Thus the solutions of (13) are asymptotically stable by Theorem 3.3.

Example 4.2 Consider

$$
\begin{cases}
\Delta_*^{0.5}x(t) = -\frac{x_0}{\sqrt{\pi}}(t+0.5)^{(-0.5)}\\
\qquad\qquad +\frac{1}{\sqrt{2}\pi}(t+0.5)^{(-0.75)}\sin(x(t+0.5)), \quad t\in N_{0.5},\\
x(0) = x_0,
\end{cases}
\tag{14}
$$

where x_0 is a constant and

$$
f(t+0.5,x(t+0.5)) = -\frac{x_0}{\sqrt{\pi}}(t+0.5)^{(-0.5)}+\frac{1}{\sqrt{2}\pi}(t+0.5)^{(-0.75)}\sin(x(t+0.5)).
$$

The fractional Taylor's difference formula of (14) is

$$x(t)=\frac{1}{\Gamma(0.5)}\sum_{s=0.5}^{t-0.5}(t-s-1)^{(-0.5)}\frac{1}{\sqrt{2\pi}}(s+0.5)^{(-0.75)}\sin(x(s+0.5)),\quad t\in N_1.$$

Since

$$\begin{aligned}
&\left|\frac{x_0}{\Gamma(1-\alpha)}(t+\alpha)^{(-\alpha)}+f(t+\alpha,x(t+\alpha))\right|\\
=&\left|\frac{1}{\sqrt{2\pi}}(t+0.5)^{(-0.75)}\sin(x(t+0.5))\right|\\
\le&\frac{1}{\sqrt{2\pi}}(t+0.5)^{(-0.75)}\\
<&\frac{\Gamma^2(0.75)}{\sqrt{2\pi}}(t+0.5)^{(-0.75)}\\
=&\frac{\Gamma(0.75)}{\Gamma(0.25)}(t+0.5)^{(-0.75)},
\end{aligned}$$

then condition (H_3) is satisfied.

Similar to Example 4.1, we can easily find that condition (H_2) and inequality (10) are satisfied. Thus the solutions of (14) are asymptotically stable according to Theorem 3.5.

Example 4.3 Consider

$$\begin{cases}\Delta_*^{0.1}x(t) &= 0.5x^2(t+0.1),\quad t\in N_{0.9},\\ x(0) &= 0,\end{cases}\tag{15}$$

where $\alpha=0.1$. Let $\beta=2$, $L_4=0.5$ and $\gamma_4=0.2$, then condition (H_4) and (12) hold.

Since

$$\frac{L_4\Gamma(1+\beta\gamma_4)\Gamma(1-\beta\gamma_4)}{\Gamma^\beta(1+\gamma_4)\Gamma(1+\alpha-\beta\gamma_4)}=\frac{0.5\Gamma(1.4)\Gamma(0.6)}{\Gamma^2(1.2)\Gamma(0.7)}\approx 0.6039<1,$$

then (11) is satisfied.

The solutions of (15) are attractive by Theorem 3.6.

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