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THE THREE RADIAL MODES AND EVOLUTIONARY STATE OF AC ANDROMEDAE

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ABSTRACT

Analysis of 4662 yellow magnitude measures, obtained at the Konkoly Observatory on 78 nights in 1958–1962, shows that AC And has its fundamental and first and second overtone radial pulsation modes (periods $P_0 = 0^d71124243$, $P_1 = 0^d52512677$, and $P_2 = 0^d421069$) all excited and nonlinearly coupled. Fitting formulae, derived from published pulsation calculations for high-metal-content models ($Z = 0.044$), are given for the direct calculation of mass and radius from observed radial mode period pairs. The multimode excitation of 25 Cepheid-strip stars (comprising eight Cepheids, one RR Lyrae star, nine RRs stars, and seven δ Scuti stars) is discussed. AC And is found to have a mass $M \approx 3.1\ M_{\odot}$ and to be a high-mass analog of the δ Scuti stars, in the hydrogen-shell-burning and helium-core-contraction stage of evolution. Pulsation theory, evolution theory, and observation seem to be in excellent agreement for AC And and δ Sct.

Subject headings: stars: δ Scuti — stars: evolution — stars: individual — stars: pulsation — stars: RR Lyrae

I. INTRODUCTION

The unique RR Lyrae star AC And has extremely strong metallic lines ($\Delta S = -1$; Preston 1959) and has long been known to be doubly periodic. For a fairly complete bibliography on this star see Aznarova (1957). A few observations made in 1960–1961 have been reported by Lange (1964). Because of its peculiarities, this star was placed on the observing program of the Konkoly Observatory in 1958, and the present paper reports on an analysis of the extensive series of observations obtained there.

II. THE OBSERVATIONS

AC And was observed photoelectrically on a total of 78 nights in 1958, 1960, 1961, and 1962 with the 61 cm reflector of the Konkoly Observatory. Measures were secured in both the Johnson *V* and *B* bands, but here we restrict the analysis to the 4662 yellow magnitudes obtained with a silvered primary mirror, a Schott GG11 filter, and an RCA 1P21 photomultiplier. All the blue and yellow measures will be published by the Konkoly Observatory, and the yellow measures may also be obtained from either London or Odessa as file IAU(27).RAS-32. Except for the night of JD 2,436,540, all observations used as comparison the star designated "d" by Luri (1950, Fig. 7). On that one night, another star designated "c" was inadvertently used as comparison. The measures for that night have since been corrected by tie-in observations between stars "c" and "d" which yielded $V_c - V_d = -0.733 \pm 0.010$ mag. All the observations treated here are differential yellow magnitudes on the instrumental system.

III. ANALYSIS OF THE LIGHT VARIATION

The normal interval between successive yellow measures was about 5 minutes, which is much shorter than the shortest period expected in the light variation. Therefore, to minimize computing costs, after all the observations had been card-punched and transcription errors edited out, we compressed the data set to 1870 points by performing a luminosity and time average to one point of all measures falling within a time range of 0^d006. Next, a Fourier-transform amplitude spectrum covering the frequency range from 0.0 to 5.2 cycles/day (c/d) with phase locking over a 13^d5 interval (Fitch 1967) was calculated. This clearly showed the presence of two strong frequencies near 1.39 and 1.91 c/d. When these frequencies were improved to maximum frequency resolution by extending the phase locking first to single observing seasons and then to the whole data set, they yielded our best values of $f_0 = 1.40599$ and $f_1 = 1.90424$ c/d ($P_0 = 0^d711243$ and $P_1 = 0^d525144$) with uncertainties of about ± 0.00005 c/d. Since the periods given in the *General Catalog of Variable Stars* (Kukarkin *et al.* 1969, hereafter GCVS) agree with ours to our limits of precision but are more precise than ours because they cover a much longer time span, we adopt them as correct and hereafter use $f_0 = 1.4059904$ and $f_1 = 1.9043021$ c/d.

As these are clearly the frequencies of the fundamental and first-overtone radial pulsation modes, we expected that they would be nonlinearly coupled so that combination frequencies $if_0 + jf_1$ should also be present in the light variation (Fitch 1966). Further, several of these combination frequencies were obvious in the amplitude spectra. Accordingly, we carried out a least-squares fit to the data of the 20 frequencies involved in a complete fourth-order doubly periodic expansion (order = $|i| + |j|$; i,j integer). When the data had been whitened for these variations and a new low-resolution amplitude spectrum calculated, we were extremely surprised to find a third frequency $f_2 \approx 2.37$ c/d. The longest observing season spanned 80 days, and while the annual sidelobes produced no serious problems in our determination of f_0 and f_1, they did present some uncertainties in the annual cycle count for the much weaker frequency f_2. When proceeding to the highest frequency resolution we utilized all of the first-, second-, and third-order combination terms involving f_2 to try to discriminate between the two possible values of $f_2 \approx 2.3749$ or 2.3721 c/d (the average spectrum with phase locking over single observing seasons gave the best value of $f_2 \approx 2.373$ c/d). We first whitened the data by a complete third-order doubly periodic fit in f_0 and f_1, and then carried out a series of least-squares fits to the 19 frequencies remaining for a complete third-order triply periodic expansion, while letting f_2 vary by small amounts around each of its two possible values. An f_2 value of 2.37491 c/d yielded a slightly smaller mean error than did $f_2 = 2.37210$ c/d, so we have adopted the former as probably correct. We wish to emphasize that while the slightly smaller value for f_2 may instead be the correct one, this 0.1 percent uncertainty in f_2 does not affect our conclusions in § IV*b*.

With our adopted values of f_0, f_1, and f_2, we proceeded to a complete triply periodic ($if_0 + jf_1 + kf_2$) fifth-order fit (order = 5 = $|i| + |j| + |k|$) to the data. This should be done in a single fit, but computer size limitations required either that we carry out the fitting in two steps or else rewrite the program to utilize intermediate storage on disk or tape. We felt the former choice would be adequate, and first performed a complete fourth-order fit (64 frequencies) followed by a fit in the fifth order (51 frequencies) to the residuals from the fourth-order fit. All our fitting is done by linear least squares to specified frequencies, where the general term to be fitted is of the form $a \sin 2\pi f(t - T_0) + b \cos 2\pi f(t - T_0)$, with T_0 arbitrary. The fitting results are listed in Table 1 with the equivalent amplitudes A and phases ϕ. The Table 1 constants provide our analytic version of the differential yellow-magnitude variation as

$$m_{AC} - m_d = -0.0294 - 2.5 \times \log\left\{1 + \sum_{i,j,k} A_{ijk} \sin 2\pi[f_{ijk}(t - T_0) + \phi_{ijk}]\right\}, \quad (1)$$

where

$$f_{ijk} = if_0 + jf_1 + kf_2$$

and T_0 = Hel JD 2,436,000.0.

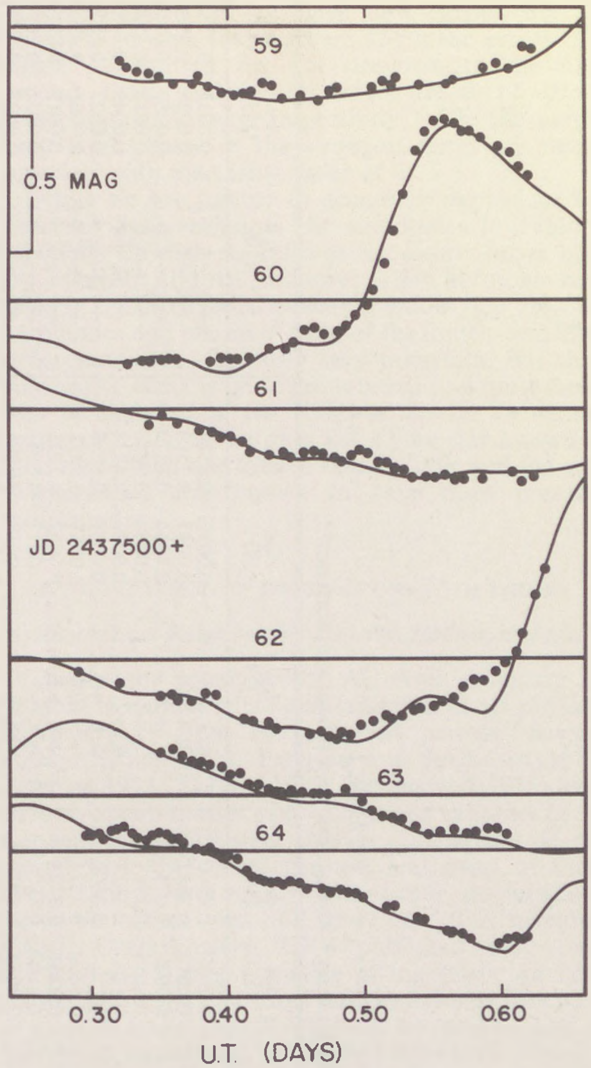

FIG. 1.—Comparison of typical observed (*solid circles*) and computed (*full line*) yellow magnitude variations of AC And. The horizontal line intersecting the light curve of each night represents the mean light level of the star.

Our adopted representation, illustrated in Figures 1 and 2 for a selection of typical observed and computed light variations, is disappointing. There are obvious systematic errors, most prominent on the ascending branches of the large-amplitude light curves, and the mean error of a single observation (in the 1870-point compressed data set) as determined by residuals from equation (1) is ±0.029 mag. We have calculated a low-resolution amplitude spectrum of the data whitened by equation (1) and covering the frequency range from 0.0 to 10.4 c/d. In this spectrum, the white-noise level is nearly constant at about ±0.007 mag, consistent with the accuracy of the observations, and the only peak which might be real, at $f \approx 4.61$ c/d, has a peak amplitude of 0.009 mag. We therefore conclude that there are probably no remaining periodicities in the

TABLE 1

FREQUENCIES,* AMPLITUDES, AND PHASES FOR EQUATION 1

i, j, k (1)	A_{ijk} (2)	ϕ_{ijk} (3)	i, j, k (4)	A_{ijk} (5)	ϕ_{ijk} (6)	i, j, k (7)	A_{ijk} (8)	ϕ_{ijk} (9)	i, j, k (10)	A_{ijk} (11)	ϕ_{ijk} (12)
+1, 0, 0	0.2038	0.1305	0, +1, −2	0.0037	0.6046	0, 3, −1	0.0035	0.8517	+2, −1, +2	0.0033	0.7753
0, +1, 0	0.1719	0.2959	0, 0, +3	0.0012	0.2882	0, +2, +2	0.0045	0.3519	+2, −1, −2	0.0026	0.3508
0, 0, +1	0.0717	0.9465	+4, 0, 0	0.008[illegible]	0.0763	0, +2, −2	0.0034	0.2370	+1, +4, 0	0.0059	0.6735
+2, 0, 0	0.0452	0.1094	+3, +1, 0	0.[illegible]4	0.2387	0, +1, +3	0.0064	0.9884	+1, −4, 0	0.0018	0.5531
+1, +1, 0	0.0774	0.2285	+3, −1, 0	0[illegible]	0.5914	0, +1, −3	0.0029	0.5924	+1, 0, +4	0.0045	0.5489
+1, −1, 0	0.0489	0.0366	+3, 0, +1	0[illegible]	0.8602	0, 0, +4	0.0051	0.5758	+1, 0, −4	0.0023	0.3556
+1, 0, +1	0.0398	0.9258	+3, 0, −1	0.[illegible]	0.3524	+5, 0, 0	0.0055	0.8952	+1, +3, +1	0.0062	0.4924
+1, 0, −1	0.0207	0.4536	+2, +2, 0	0.0241	0.3728	+4, +1, 0	0.0052	0.3203	+1, +3, −1	0.0015	0.0518
0, +2, 0	0.0307	0.4589	+2, −2, 0	0.0187	0.5874	+4, −[illegible]	[illegible].0037	0.0581	+1, −3, +1	0.0031	0.2886
0, +1, +1	0.0226	0.1086	+2, 0, +2	0.0064	0.8036	+4, 0, [illegible]	0.0029	0.9499	+1, −3, −1	0.0025	0.7350
0, +1, −1	0.0274	0.5645	+2, 0, −2	0.0053	0.5651	+4, [illegible]	0.0041	0.1994	+1, +2, +2	0.0029	0.3321
0, 0, +2	0.0062	0.8253	+2, +1, +1	0.0209	0.0693	+3, [illegible], 0	0.0052	0.3974	+1, +2, −2	0.0028	0.4944
+3, 0, 0	0.0123	0.1613	+2, +1, −1	0.0072	0.8318	+3, −2, 0	0.0068	0.5048	+1, −2, +2	0.0054	0.9767
+2, +1, 0	0.0354	0.2023	+2, −1, +1	0.0068	0.7461	+3, 0, +2	0.0012	0.5715	+1, [illegible]	0.0004	0.4763
+2, −1, 0	0.0092	0.0821	+2, −1, −1	0.0042	0.8906	+3, 0, −2	0.0044	0.1455	+1, [illegible], +3	0.0042	0.[illegible]
+2, 0, +1	0.0201	0.8655	+1, +3, 0	0.0145	0.5722	+3, +1, +1	0.0067	0.1481	+1, [illegible], −3	0.0009	0.[illegible]223
+2, 0, −1	0.0073	0.5406	+1, −3, 0	0.0113	0.4937	+3, +1, −1	0.0024	0.7070	+1, −1, +3	0.0020	0.8988
+1, +2, 0	0.0369	0.3765	+1, 0, +3	0.0059	0.7843	+3, −1, +1	0.0024	0.5732	+1, −1, −3	0.0021	0.1368
+1, −2, 0	0.0195	0.7852	+1, 0, −3	0.0064	0.2890	+3, −1, −1	0.0012	0.8866	0, +5, 0	0.0013	0.3745
+1, 0, +2	0.0106	0.8887	+1, +2, +1	0.0143	0.2825	+2, +3, 0	0.0078	0.5399	0, +4, +1	0.0033	0.9249
+1, 0, −2	0.0088	0.2309	+1, +2, −1	0.0102	0.6256	+2, −3, 0	0.0051	0.4604	0, +4, −1	0.0019	0.2269
+1, +1, +1	0.0265	0.0657	+1, −2, +1	0.0048	0.0771	+2, 0, +3	0.0012	0.7217	0, +3, +2	0.0052	0.6207
+1, +1, −1	0.0226	0.7301	+1, −2, −1	0.0061	0.8220	+2, 0, −3	0.0034	0.1232	0, +3, −2	0.0033	0.5010
+1, −1, +1	0.0228	0.8180	+1, +1, +2	0.0079	0.0836	+2, +2, +1	0.0096	0.2284	0, +2, +3	0.0053	0.2038
+1, −1, −1	0.0033	0.3934	+1, +1, −2	0.0098	0.3924	+2, +2, −1	0.0011	0.4074	0, +2, −3	0.0019	0.3474
0, +3, 0	0.0103	0.6733	+1, −1, +2	0.0045	0.7537	+2, −2, +1	0.0002	0.3556	0, +1, +4	0.0050	0.9091
0, +2, +1	0.0062	0.4149	+1, −1, −2	0.0082	0.9524	+2, −2, −1	0.0051	0.7963	0, +1, −4	0.0004	0.1516
0, +2, −1	0.0172	0.7324	0, +4, 0	0.0045	0.8939	+2, +1, +2	0.0055	0.0269	0, 0, +5	0.0031	0.5653
0, +1, +2	0.0024	0.1346	0, +3, +1	0.0067	0.6355	+2, +1, −2	0.0012	0.2616			

* $f_{ijk} = 1.4059904i + 1.9043021j + 2.37491k$ c/d.

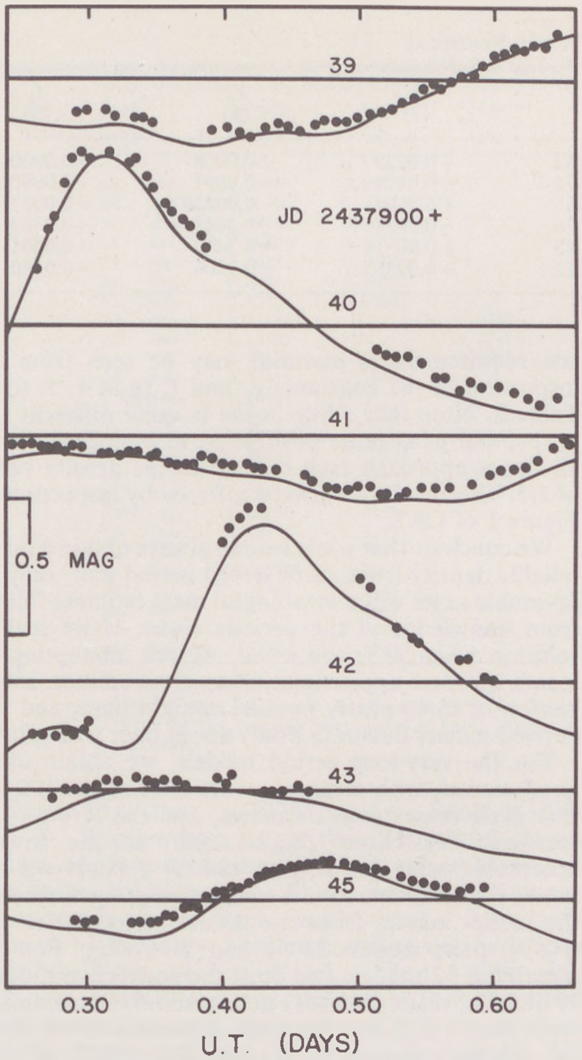

FIG. 2.—Same as Fig. 1

observed light variation. The large systematic fitting errors suggest that in AC And, as in VX Hya (Fitch 1966), there are secular changes in the relative amplitudes and phases of the various terms. To check this possibility, we made a separate 18-frequency fit, involving only the strongest terms in Table 1, to the data of each of the four observing seasons separately. If we accept the results of these last four fits as real, the total pulsational energy of AC And went through a pronounced minimum in 1960 and 1961, so that the total amplitude was about 50 percent greater in 1958 and 1962 than in 1960 and 1961. However, selection effects in the extraordinarily complex light variation are very severe here, and we cannot be certain of this conclusion. The only way we could hope to get a true picture of this star's behavior would be to monitor it continuously over several months from an orbiting observatory, but the additional information so gained would probably not justify the necessary observing time.

Prompted by the referee's concern for the credibility of our adopted fit, we have now carried out fits complete to each lower order. The mean errors of a single observation for fits complete to the first, second, third, and fourth orders are 0.112, 0.069, 0.048, and 0.035 mag, respectively, while the amplitudes and phases of the strongest terms are nearly invariant with increasing order of fit.

While we are unable to accurately reproduce the observed light variation, the amplitudes in Table 1, especially through the third order, clearly prove both the existence and the nonlinear triply harmonic coupling of the three radial pulsation modes f_0, f_1, f_2. The amplitudes and phases of most of the fourth- and fifth-order terms are obviously very uncertain, but their cumulative effect is quite pronounced and must therefore be included in the solution. So far as we are aware, AC And is the only RR Lyrae star known to have more than one excited radial mode, and the only Cepheid-strip star known to have three unstable radial modes.

IV. COMPARISON OF OBSERVATIONS WITH THEORY

a) Theoretical Relations for Density, Radius, and Mass

The unique properties of AC And prompted an attempt to extract reliable information on its physical characteristics from the observed periods. Several authors (Fitch 1970; Petersen and Jørgensen 1972; Petersen 1973; Takeuti 1973; Stellingwerf 1975) have tried to obtain masses of double-mode variables in the Cepheid instability strip from a comparison of observed with theoretical periods, and most of these investigations have relied on the linear, nonadiabatic pulsation calculations of Cox *et al.* (1972, hereafter CKS). Unfortunately, CKS published only their summarizing fitting formulae giving pulsation constants Q_k for the first three radial modes as functions of mass M and radius R. They did not publish lists of individual model parameters, and since their Figure 1 plots show small systematic fitting errors (which are discussed in Stellingwerf 1975, hereafter S75) in the region of greatest interest to us, we cannot from their published data assess the accuracy of any masses inferred from the two observed periods of a double-mode star.

After conducting a literature survey, we selected the linear, adiabatic models of Cogan (1970, hereafter Co70) as providing the best data available to us for an investigation of model fitting and mass determination. His 53 models cover the instability strip with $0.8 \leq M/M_\odot \leq 15$, $1 \leq \log L/L_\odot \leq 5$, and $0.03 \leq P_0(\text{days}) \leq 317$. The composition (used also by CKS) is for extreme Population I stars ($X = 0.602$, $Z = 0.044$), and the envelope structures (unfortunately) include the effects of convection, though most theoretical investigations suggest that Cepheid-strip variables have predominantly radiative envelopes.

Our approach to the problem was an empirical one, and after conducting a very extensive series of fitting experiments, we adopted the following procedure.

TABLE 2

CONSTANTS IN THE FITTING FORMULAE

n (1)	i (2)	j (3)	a_n (4)	b_n (5)	c_n (6)	d_n (7)	e_n (8)	f_n (9)
1.....	0	1	−1.255	+0.0356	−0.0242	+0.0229	+0.0008	−0.000062
2.....	1	2	−1.500	+0.0396	−0.0071	+0.0171	−0.0097	−0.0000066
3.....	0	2	−1.603	+0.0384	+0.686	+0.0101	−0.00259	−0.000111
4.....	0	...	+0.9490	+0.6083	−0.0276	−0.00443	+0.3645	+0.0389
5.....	1	...	+1.0376	+0.6263	−0.0083	+0.00314	+0.3452	+0.0081
6.....	2	...	+1.1136	+0.6511	+0.0071	+0.00585	+0.3306	−0.0140

With $p_k = \log_{10} P_k$(days), $q_k = \log_{10} Q_k$(days), $r = \log_{10} R/R_\odot$, and $m = \log_{10} M/M_\odot$, we find equations of the form

$$q_i = a_n + b_n p_i + c_n/(p_i - p_j) + d_n(p_i^3 - p_j^3) + e_n(p_i^4 - p_j^4) + f_n/(p_i^3 - p_j^3)\,, \quad (2, 3)$$

$$(i = 0, 1; j = n = i + 1)\,,$$

$$q_i = a_n + b_n p_i + c_n(p_i - p_j) + d_n(p_i^3 - p_j^3) + e_n(p_i^4 - p_j^4) + f_n/(p_i^3 - p_j^3)\,, \quad (4)$$

$$(i = 0; j = 2; n = 3)\,,$$

and

$$r = a_n + b_n p_i + c_n p_i^2 + d_n p_i^3 + e_n m + f_n m p_i\,, \quad (5, 6, 7)$$

$$(i = 0, 1, 2; n = i + 4)\,,$$

where the constants a_n–f_n of equations (2)–(7) were determined by least-squares fits to Co70's model data and are listed in Table 2. To minimize round-off errors in further calculations, these constants are generally given to one more significant figure than their mean errors justify. All Co70's models were used in determining the coefficients in equations (2)–(7), with the following exceptions: In equations (3), (4), and (7) we skipped model 40, which appears to have an anomalous value for Q_2, and in equation (5) we skipped his extreme convective models 13, 29, and 41. Using equations (2)–(4), the *maximum* error in q_0 or q_1 for all 159 possible model period pairs is 0.012. The mean error of q_0 (p_0, p_1) is 0.0051, of q_1 (p_1, p_2) is 0.0043, and of q_0 (p_0, p_2) is 0.0035. The mean errors of $r(p_i, m)$ are 0.0038.

Any known pair of radial mode periods yields a density via Q_0 or Q_1 from one of equations (2)–(4), and also yields two equations of the form $r = A_i + B_i m$ via equations (5)–(7). We use the density in the form $r = A_i + m/3$ and so have three equations in two unknowns which we solve by least squares. If we denote by D the coefficient of m after eliminating r from the normal equations (i.e., $D = 3\sum B_i^2 - [\sum B_i]^2$, $i = 1, 2, 3$; so that D is three times the weight of m), then the accuracy of acceptable solutions may be controlled by requiring that $D \geq D_{\min}$, where $D_{\min}$ is some suitably chosen lower limit. That this last requirement is essential may be seen from an inspection of the constants e_n and f_n $(n = 4, 5, 6)$ in Table 2. Note that while $\partial r/\partial m$ is quite different for p_0, p_1, and p_2 at large positive p_i, as p_i goes negative all slopes approach each other near the density value of 1/3. This result can also be inferred by inspection of Figure 1 of CKS.

We conclude that while we can always obtain a fairly reliable density from an observed period pair, only in favorable cases will a meaningful mass estimate follow from knowledge of the periods alone. If we seek a solution when D is too small, we are attempting to locate the best approximation to the common intersection of three nearly parallel straight lines, and the derived masses fluctuate wildly about their true values.

For the very-long-period models, we obtain quite good masses, with mean of errors in m of about 0.02. But D decreases as p_i decreases, and the errors in m increase. We chose $D_{\min} = 0.001$ as the lowest acceptable value for D, and with $D \geq 0.001$ we obtained the following results when attempting to recover the model masses from the model periods: For the (p_0, p_1) pair, models 33–51 had $D < D_{\min}$. For the remaining 32 models (we omit the convective models 13 and 29, which gave very poor results) the maximum error in m is 0.23 and the mean of errors is 0.084. With (p_1, p_2), only models 1–21 give $D \geq 0.001$, with a maximum error of 0.25 and a mean of errors of 0.10. The situation is most favorable for the (p_0, p_2) pair, where (again excluding models 13 and 29) masses were obtained for all models except 33–44, the maximum error was 0.20, and the mean of errors was 0.089. Had Co70 not included convection in his models, we could have gotten more accurate mass recovery, since the fitting errors are clearly larger for the models with intermediate convection and are very large (up to 0.5 in derived m) for extreme convective models such as model 29.

b) Multimode Cepheid-Strip Variables

Table 4 of Fitch (1970) listed 13 Cepheid-strip stars with two known radial pulsation modes. Newly published data, old data newly discovered, and new identifications of previously known results now allow us to expand the list to the 25 stars given here in Table 3. The periods and period ratios which we consider as still uncertain are marked with a colon. A few comments are necessary for some of the entries, mostly involving δ Scuti stars.

TABLE 3

MULTIMODE VARIABLES IN THE CEPHEID INSTABILITY STRIP

Star* (1)	P_o(days) (2)	P_1(days) (3)	P_2(days) (4)	P_1/P_o (5)	P_2/P_1 (6)	P_2/P_o (7)	log $\rho/\rho_\odot$ (8)	$M/M_\odot$ (9)	$R/R_\odot$ (10)
SX Phe	0.05496	0.04277		0.7782			-0.50		
CY Aqr	0.06104	0.04543:		0.7443:			-0.52:		
AE UMa	0.08602	0.06653		0.7734			-0.87		
RV Ari	0.09313	0.07195		0.7726			-0.94		
21 Mon	0.09991	0.07500		0.7507			-0.95		
BP Peg	0.10954	0.08451		0.7715			-1.07		
AI Vel	0.11157	0.08621		0.7727			-1.09		
DQ Cep	0.12420:		0.07886			0.6349:	-1.18:		
CC And	0.12491		0.07493			0.5999	-1.15		
1 Mon	0.13612		0.08261			0.6069	-1.23		
44 Tau	0.1449:	0.1046:		0.722:			-1.22:		
V703 Sco	0.14996	0.11522		0.7683			-1.33		
4 CVn	0.16553:	0.1278:		0.7721:			-1.42:		
δ Sct	0.19377		0.11636			0.6005	-1.52		
VX Hya	0.22339	0.17272		0.7732			-1.68		
VZ Cnc		0.17836	0.14280		0.8006		-1.70		
AC And	0.71124	0.52513	0.42107	0.7383	0.8018	0.5920	-2.61	3.1	10.7
V439 Oph	1.89296	1.343		0.7095			-3.37	1.6	15.5
TU Cas	2.13930	1.51823		0.7097			-3.47	1.7	17.4
U TrA	2.56844	1.8249		0.7105			-3.62	2.0	20.4
VX Pup	3.01172	2.136		0.7092			-3.75	2.3	23.4
AP Vel	3.12781	2.1993		0.7031			-3.77	2.0	22.9
BK Cen	3.1740	2.2367		0.7047			-3.79	2.1	23.4
Y Car	3.63976	2.559		0.7031			-3.90	2.3	26.3
AX Vel	3.6731	2.59285		0.7059			-3.91	2.5	27.5

*CY Aqr, Fitch 1973; AE UMa, Szeidl 1974; 21 Mon, Gupta 1973; BP Peg, Broglia 1959; DQ Cep, Fitch and Wehlau 1965; 1 Mon, Shobbrook and Stobie 1974; 44 Tau, Percy and McAlary 1974; VX Pup, Stobie 1970; Y Car, Stobie 1972; AX Vel, Stobie and Hawarden 1972. References for the other stars are given in Fitch (1970) or in the text.

From 1972 July to 1975 January, Fitch (and/or W. Z. Wisniewski), using the 1.5 m and 81 cm reflectors of the Observatorio Nacional de Mexico at San Pedro Martir and the 51 cm reflector of Steward Observatory at Kitt Peak, accumulated a large body of new, high-quality, photometric data on the δ Scuti variables δ Sct, 14 Aur, δ Del, 1 Mon, 4 CVn, and CC And. Most of this material will be fully described in future publications. Here we wish only to cite those new findings which are pertinent to the present discussion. The period P_2 for CC And was treated somewhat tentatively in Fitch (1967), but its existence is clearly proven in the new photometry obtained on 16 nights in 1974. The period P_2 for δ Sct, which was found in an analysis of Fath's (1935, 1937, 1940) data from 1935–1938 and is briefly mentioned in Fitch (1967), has been verified in new photometry obtained on 24 nights in 1972 and 1973. The period P_2 for 1 Mon was not emphasized by Shobbrook and Stobie (1974), but it was definitely present in their analysis of their extensive data. P_0 for DQ Cep seems uncertain, and therefore also the mode identification for the P_2 period, but five of the Table 3 stars have definitely established excitation of the second overtone radial mode. Further, in the three δ Scuti stars CC And, 1 Mon, and δ Sct, P_2 is weakly excited, P_1 is quiescent, and P_0 is fairly strongly excited and also either strongly modulated by tides (CC And) or else accompanied by fairly strong nonradial modes (1 Mon and δ Sct). The periods given for 4 CVn are taken from the earlier list, but we think they are probably wrong. During 1974 February–June, we obtained 27 nights of high signal/noise photometry on 4 CVn. Unfortunately, the observing times were all near full moon so the aliasing problem in the amplitude spectra is extremely severe. After making allowances for the extra sidelobes, we still find the spectra to be the most complex that we have seen. In the spectra of 4 CVn, in addition to a bewildering number of possible pulsation frequencies, there is also definitely present a low-frequency variation near either 1.39 or 2.39 c/d, and the individual light curves give evidence of slow changes in mean light level, suggesting that the primary either is an ellipsoidal variable or undergoes shallow eclipses as well as pulsating. The previously reported periods may be present in our data, but they are much weaker than other probable periods. The periods reported by Percy and McAlary (1974) for 44 Tau do not agree with either model calculations or other observed stars, and we think them of marginal observational reliability. If P_1 for CY Aqr is correct, the star must have a high Z content.

We have applied equations (2)–(7) to all period pairs in Table 3 and derived the column (8) densities, which should be accurate to 0.01 in log $\rho/\rho_\odot$ on the average, or to 0.02 at the worst. For those period pairs yielding $D \geq 0.001$, we derived masses and radii which were then used to calculate ρ. In the case of AC And, only the (P_0, P_2) pair satisfied this condition. (P_0, P_1) and (P_1, P_2) gave log $\rho/\rho_\odot = -2.60$ and -2.61, respectively. The masses of the Cepheids are larger than found by Petersen (1973) using the CKS formulae, but still much below the evolutionary masses. On the other hand, the analysis indicates that for

AC And, $m = 0.49 \pm 0.2$ at the worst, and probably m is accurate to 0.1. Even if m is as small as 0.29, $M = 1.9\ M_\odot$, and we conclude that AC And is of normal mass on its first post–main-sequence crossing of the instability strip, with luminosity derived from hydrogen-shell burning and helium-core contraction. Our finding is at sharp variance with S75, who applied a modification of the CKS formulae to the (P_0, P_1) periods in AC And, derived $M/M_\odot = 0.61 \pm 0.02$, and concluded that AC And has a normal mass for a horizontal-branch star. We suggest that since all of his RR Lyrae star models had $M/M_\odot = 0.578$, perhaps his finding agreement between initial model mass and derived stellar mass is actually independent of the particular model mass assumed. From our arguments in § IV*a*, we do not ordinarily expect to be able to obtain *reliable* masses from periods for stars with $P_0 \lesssim 1$ day. Only if both P_0 and P_2 are excited and are either short enough or long enough does there seem any possibility of getting masses in this fashion for the RR Lyrae, RRs, and δ Scuti stars.

The discussions by CKS and by Petersen (1973) indicate a very close agreement between the pulsation constants found by Co70 and by CKS for similar models. To clarify the cause of our disagreement with S75 on the mass of AC And, we applied our adopted periods to equations (2)–(4) of CKS. Since we wished to check the applicability of these formulae, which have been widely used by various investigators, we proceeded directly by successive approximations. For the (P_1, P_2) pair, the functional dependence of Q_1 and Q_2 on M and M/R is the same, and the self-consistent solution converged rapidly to yield $\log \rho/\rho_\odot = -2.601$, $\log (MR_\odot/M_\odot R) = -0.609$, $M = 2.44\ M_\odot$, and $R = 9.90\ R_\odot$. We cannot assess the accuracy of these results, but we note that they agree with our earlier findings to our limits of error. In the case of the other two period pairs, $q_0 \sim M$ while q_1, $q_2 \sim m$, so the situation is more difficult. Accordingly, we guessed a value for M, used the appropriate pair of equations with the known period ratio to find M/R by successive approximations, and then used the two equations separately to find $\rho(M, M/R, P_0)$ and $\rho(M, M/R, P_1$ or $P_2)$. We assumed that an acceptable (M, R) pair would yield the same density for both P_0 and P_1 or P_2. Our assumption was verified by calculation, but not in a usable manner. For the periods of AC And, both the (P_0, P_1) and (P_0, P_2) pairs give self-consistent densities which are quite insensitive to the value of M assumed. For example, when $M = 3.0\ M_\odot$, $m - r = -0.923$, $\log \rho/\rho_\odot = -2.5329$, and $\log \rho(P_0) - \log \rho(P_1) = -0.00003$. When $M = 1.0\ M_\odot$, $m - r = -0.699$, $\log \rho/\rho_\odot = -2.6075$, and $\log \rho(P_0) - \log \rho(P_1) = +0.00002$ (approximately, since we only carried the calculation of $m - r$ to three decimals and of $\log \rho$ to five). Formally, these results indicate that $M \approx 2\ M_\odot$, but the differences in the P_0 and P_1 densities are so small in comparison with the original fitting errors quoted for the CKS formulae that they cannot be considered to yield definitive results (i.e., self-consistent solutions cannot distinguish between correct values of M and R over a large range in these parameters). Note especially that with the CKS formulae, the derived density also varies with changing m (by $\Delta \log \rho = 0.07$ in our example). We conclude that the CKS formulae are not suitable for application to a mass determination for AC And from its periods alone.

The RR Lyrae models of S75 had a composition $X = 0.70$, $Z = 0.001$, appropriate to a normal weak-line variable, but with $\Delta S = -1$, AC And can hardly be a Population II star. The Arizona *UBV* photometry of 174 field RR Lyrae stars (Fitch *et al.* 1966; Bookmeyer *et al.* 1975), when combined with the *UBV* photometry of AC And by Notni (1963), shows that the colors of AC And are very different from those of seven typical RR Lyrae stars with $P_0 \approx 0^d.7$, and suggests that AC And is nearer the blue than the red edge of *its* instability strip. We conclude this section by noting that, while the fitting formulae adopted by S75 may be applicable to a normal weak-line RR Lyrae star, they are not adequate for AC And.

V. DISCUSSION

Fitch (1970) and others have associated mixed-mode excitation with evolutionary mode switching. There is such a wealth of excited-mode combinations displayed in Table 3, and such an extremely small scatter in period ratios (when we ignore the four uncertain stars), that we now agree with S75, who concluded that mixed-mode excitation is probably not a symptom of a star at a transition edge. S75 conducted a preliminary survey of the RR Lyrae region, using new, rapidly convergent, nonlinear pulsation calculations; and he found a few models near the red edge of the instability strip which exhibited (P_0, P_1) mixed-mode pulsation. He tentatively suggested that the cause of such excitation may lie in the star's position near the red side of the instability strip, but pointed out that many more models must be investigated in order to verify this suggestion. From our concluding remarks in § IV*b*, we think it improbable that this suggestion will explain the extremely peculiar behavior of AC And.

It is unlikely that all of the stars in Table 3 are at the same evolutionary state. Most investigators (see Petersen and Jørgensen 1972 and references therein; also Baglin *et al.* 1973) agree that the δ Scuti stars (to avoid lengthy discussion we here adopt the type definitions given in the GCVS) are Population I objects with $M/M_\odot \approx 1$–2, situated in the lower part of the instability strip between the main sequence and the horizontal branch, and have only recently evolved off the main sequence. Within the context of this description, AC And is merely a higher mass analog of the typical δ Scuti variable. The direct observational data on the masses and luminosities of the RRs stars is very sparse and generally inconclusive. However, we believe that in their theoretical survey of pulsation in this region, Petersen and Jørgensen (1972) demonstrated fairly convincingly a dichotomy of classes dependent on the composition parameter Z. For $Z \lesssim 0.005$ and $P_0 < 0^d.25$, the P_1/P_0 ratio is restricted to the range

0.760–0.776. For $0.02 \leq Z \leq 0.03$, $0.747 \leq P_1/P_0 \leq 0.757$, and, by extrapolation, if $Z > 0.03$, the lower limit on P_1/P_0 will decrease somewhat. In fact, Co70's models with $Z = 0.044$ have $0.736 \leq P_1/P_0 \leq 0.755$. It thus appears that the theoretical models give adequate range for discussing the 12 reliable mixed-mode stars with $P < 0^d.25$ (we omit from further consideration the three δ Scuti stars DQ Cep, 44 Tau, and 4 CVn, on the grounds that speculation based on insufficient data is profitless). CY Aqr is a strong-line star ($\Delta S = 2$; Preston 1959) with a definitely short beat period—only its precise value is uncertain. Theory and observation will agree if CY Aqr and 21 Mon (a δ Scuti type) have $Z \approx 0.03$–0.04. Further, the RRs stars SX Phe, AE UMa, RV Ari, BP Peg, AI Vel, V703 Sco, and VX Hya apparently must have $Z \lesssim 0.005$ if theory is reliable, and belong to either the intermediate or extreme Population II group. Again, for $P_0 < 0^d.25$, $0.611 \leq P_2/P_0 \leq 0.628$ for $Z \leq 0.005$, and $0.595 \leq P_2/P_0 \leq 0.622$ for $0.02 \leq Z \leq 0.03$. On this basis, the three δ Scuti stars CC And, 1 Mon, and δ Sct are Population I members, consistent with our previous discussion. Finally, the P_2/P_1 ratio, which for the models of Petersen and Jørgensen (1972) lies in the range 0.793–0.821, does not permit any firm conclusions on VZ Cnc, unless we restrict the periods of the models to include only, say, the range $0^d.15 \leq P_0 \leq 0^d.25$. With this restriction, we find probably $Z \geq 0.02$, in agreement with $\Delta S = 0$ for this star. Petersen's (1973) conclusion that the eight mixed-mode Cepheids are in the helium-core-burning stage seems reasonable.

Elementary considerations of continuity dictate that the mixed-mode RRs stars can neither all be at the red edge of the instability strip nor at a transition edge of the Christy (1966) type. The GCVS (1969, with supplements to 1974) lists 63 known RRs stars, and Table 3 shows that nine of these stars have mixed-mode excitation. Only if these stars evolve discontinuously across the strip could a transition-line or red-edge location explain their doubly periodic behavior. Rather, since 14 percent of the sample exhibits multiperiodicity, we conclude that this characteristic obtains in transition bands of significant width. The three modes of AC And even suggest that in some cases this band width may approach the instability strip width. Since, of the approximately 5800 RR*ab*, RR*c* stars listed in the GCVS, apparently only AC And displays mixed modes, we conclude that the concepts of transition lines and blue edges *do* apply to all *normal* RR Lyrae stars.

We have used evolutionary tracks for models with $X = 0.602$, $Z = 0.044$, and $M/M_\odot = 1.5$, 3.0, and 5.0, from the main sequence to the red-giant ascent (Robertson 1971, 1975), in order to check the consistency of our results against theory. When we interpolate linearly in m at $r = +1.035$, we find that AC And has $\log L/L_\odot = +2.0$ and $\log T_e = +3.74$, if it has our adopted composition, M, and R, and is in its first left-to-right instability-strip passage.

From an analysis based on photoelectric scans, Hγ profiles, and model-atmosphere fitting, Bessell (1969) derived $\theta_e = 0.72$, $\log g = 3.6$, and $M/M_\odot = 2.0$ for δ Sct. With a trigonometric parallax $\pi = 0''.020 \pm 0''.005$ (Jenkins 1952) and $\langle V \rangle = 4.72$ mag (Hoffleit 1964, confirmed by our own measures), we find $\langle M_V \rangle = +1.23$ mag and $\log L/L_\odot = +1.41$ ($+0.25$, -0.19). If we interpolate between the model tracks for this luminosity at $\log T_e = 3.845$, we find $M/M_\odot = 1.8$, $R/R_\odot = 3.5$, and $\log \rho/\rho_\odot = -1.36$. If instead we adopt Bessell's θ_e and our Table 3 density, interpolation between the theoretical tracks provides estimates of $M/M_\odot \approx 1.9$ and $\log L/L_\odot \approx 1.5$ for δ Sct. The grid of available models is very coarse so our roughly interpolated values are not very precise; but within our limits of accuracy we find *no* disagreement between evolution theory, pulsation theory, and observation for AC And and δ Sct (and, by inference, the rest of the δ Sct stars), provided they are now in their first post–main-sequence crossing of the instability strip. If our derived values of M and R are correct, the Table 3 Cepheids must be more evolved and already ascending the giant branch.

In conclusion, we wish to suggest that since AC And and δ Sct have apparently low values of $\rho_c/\langle \rho \rangle$ and L/M when compared with normal RR Lyrae stars, perhaps the cause of mixed-mode instability lies in abnormally massive envelopes, high envelope pulsation energies, and low pulsation amplitude growth rates, resulting in high pulsation mode "inertia." We would like to encourage the theoretical investigators to devote more time to models with unfashionable parameter values, and to make available more details of the models they calculate.

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REFERENCES

Aznarova, T. A. 1957, *Peremnye Zvesdy*, **12**, 83.
Baglin, A., Breger, M., Chevalier, C., Hauck, B., Le Contel, J. M., Sareyan, J. P., and Valtier, J. C. 1973, *Astr. and Ap.*, **23**, 221.
Bessell, M. S. 1969, *Ap. J. Suppl.*, **18**, 167.
Bookmeyer, B. B., Fitch, W. S., Lee, T. A., Wiśniewski, W. Z., and Johnson, H. L. 1975, *Comm. Lunar Planet. Lab.*, No. 202 (in press since April 1972).
Broglia, P. 1959, *Contr. Osserv. Milano-Merate, Nuovo Serie*, No. 142.
Christy, R. F. 1966, *Ann. Rev. Astr. and Ap.*, **4**, 353.
Cogan, B. C. 1970, *Ap. J.*, **162**, 139 (Co70).
Cox, J. P., King, D. S., and Stellingwerf, R. F. 1972, *Ap. J.*, **171**, 93 (CKS).
Fath, E. A. 1935, *Lick Obs. Bull.*, **17**, 175.
———. 1937, *ibid.*, **18**, 77.
———. 1940, *ibid.*, **19**, 77.
Fitch, W. S. 1966, *Ap. J.*, **143**, 852.
———. 1967, *ibid.*, **148**, 481.
———. 1970, *ibid.*, **161**, 669.
———. 1973, *Astr. and Ap.*, **27**, 161.
Fitch, W. S., and Wehlau, W. 1965, *Ap. J.*, **142**, 1616.

Fitch, W. S., Wiśniewski, W. Z., and Johnson, H. L. 1966, *Comm. Lunar Planet. Lab.*, No. 71.
Gupta, S. K. 1973, *Observatory*, **93**, 192.
Hoffleit, D. 1964, *Catalog of Bright Stars* (New Haven: Yale University Observatory).
Jenkins, L. F. 1952, *General Catalog of Trigonometric Stellar Parallaxes* (New Haven: Yale University Observatory).
Kukarkin, B. V., Kholopov, P. N., Efremov, Yu. N., Kukarkina, N. P., Kurochkin, N. E., Medvedeva, G. I., Perova, N. B., Fedorovich, V. P., and Frolov, M. S. 1969–1974, *General Catalog of Variable Stars* (Moscow: Academy of Sciences of the USSR) (GCVS).
Lange, G. A. 1964, *Peremnye Zvesdy*, **14**, 352.
Luri, T. 1950, *Peremnye Zvesdy*, **7**, 182.
Notni, P. 1963, *Astr. Nach.*, **287**, 65.
Percy, J. R., and McAlary, C. W. 1974, *Observatory*, **94**, 225.
Petersen, J. O. 1973, *Astr. and Ap.*, **27**, 89.
Petersen, J. O., and Jørgensen, H. E. 1972, *Astr. and Ap.*, **17**, 367.
Preston, G. W. 1959, *Ap. J.*, **130**, 507.
Robertson, J. W. 1971, *Ap. J.*, **170**, 353.
———. 1975, private communication.
Shobbrook, R. R., and Stobie, R. S. 1974, *M.N.R.A.S.*, **169**, 643.
Stellingwerf, R. F. 1975, *Ap. J.*, **195**, 441 (S75).
Stobie, R. S. 1970, *Observatory*, **70**, 20.
———. 1972, *M.N.R.A.S.*, **157**, 167.
Stobie, R. S., and Hawarden, T. 1972, *M.N.R.A.S.*, **157**, 157.
Szeidl, B. 1974, *IAU Commission 27, Information Bull.*, No. 903.
Takeuti, M. 1973, *Pub. Astr. Soc. Japan*, **25**, 567.

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