



## 2-Reachable Subsets in Two-Colored Graphs

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### Abstract

A subset  $X$  of vertices in a graph  $G$  is a *diameter 2 subset* if the distance of any two vertices of  $X$  is at most two in  $G[X]$ . Relaxing this notion, a subset  $X$  of vertices in a graph  $G$  is a *2-reachable subset* if the distance of any two vertices of  $X$  is at most two in  $G$ . Related to recent attempts to strengthen a well-known conjecture of Ryser, English et al. conjectured that the vertices of a 2-edge-colored cocktail party graph (the graph obtained from a complete graph with an even number of vertices by deleting a perfect matching) can be covered by the vertices of two monochromatic diameter 2 subsets. In this note we prove the relaxed form of this conjecture, replacing diameter 2 by 2-reachable. An immediate corollary is that 2-colored cocktail party graphs on  $n$  vertices must contain a monochromatic 2-reachable subset with at least  $\frac{n}{2}$  vertices (and this is best possible).

**Keywords** Diameter in Ryser’s conjecture · 2-coloring cocktail party graphs

### 1 Introduction

In a 2-edge colored graph  $G = (V, E)$ , let the two colors be color 1 and 2, i.e.  $E = E_1 \cup E_2$ ,  $E_1 \cap E_2 = \emptyset$ , and let  $G_1 = (V, E_1)$ ,  $G_2 = (V, E_2)$ . For  $v \in V$ ,  $i \in [2]$ , let  $S_i(v)$  denote the stars with center  $v$  and with all edges of  $E_i$  incident to  $v$ . Similarly,  $N_i(v)$  denotes the set of vertices adjacent to  $v$  in color  $i$ . The color of an edge  $(x, y)$  is denoted by  $col(x, y)$ . Sometimes for convenience we think of color 1 as red and color 2 as blue.

A subset  $X$  of vertices in a graph  $G$  is a *diameter 2 subset* if the distance of any two vertices of  $X$  is at most two in  $G[X]$ . Relaxing this notion, a subset  $X$  of vertices

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in a graph  $G$  is called a *2-reachable subset* if the distance of any two vertices of  $X$  is at most two in  $G$  (thus we might use a middle vertex outside of  $X$ ). This relaxation originated from a Ramsey-type question of Erdős and Fowler: is it true that any 2-colored complete graph  $K_n$  has a monochromatic diameter 2 subgraph with at least  $\lceil \frac{3n}{4} \rceil$  vertices? The relaxed question (with a short proof) was solved in [8]. Using a more elaborate form of that proof method, Erdős and Fowler [4] answered affirmatively the original question as well.

The covering version of the Ramsey-type question above is motivated by a possible strengthening of a classical conjecture attributed to Ryser (stated in the thesis of Henderson [6]). The version below is from [7] (for details, see [3], page 3).

**Conjecture 1** *Let  $r \geq 2$  be an integer and  $G = (V, E)$  a graph with  $\alpha(G) = \alpha$ . Then in every  $r$ -edge coloring of  $G$  there is a cover of  $V(G)$  by  $(r - 1)\alpha$  monochromatic connected components.*

Milićević in [12] then DeBiasio et al. in [3] conjectured that the diameter of the monochromatic components in Conjecture 1 can be bounded (for some details and partial results see Section 3). The best bound for the diameter (namely 3) for the  $\alpha = 1, r = 2$  case comes from the folklore result that any 2-colored complete graph has a spanning monochromatic subgraph of diameter at most three. For graphs with  $\alpha = 2, r = 2$  we need a cover by two monochromatic graphs. If there is no missing edge at a vertex, the two monochromatic stars provide a cover by two diameter 2 sets. The first graph when this trivial solution does not work is the “cocktail party” graph  $G^c = (V, E)$ , obtained by deleting a perfect matching from a complete graph with an even number of vertices. For any vertex  $v \in V$  the unique vertex non-adjacent to  $v$  in  $G^c$  is denoted by  $v'$ . (Note that  $G^c$  can also be viewed as a complete partite graph, where each partite class has size 2.) Surprisingly, it does not seem easy to decide whether a cover by two monochromatic diameter 2 sets exists. English, McCourt, Mattes and Phillips conjectured a positive answer.

**Conjecture 2** ([5]). *In every 2-coloring of the edges of the cocktail party graph  $G^c = (V, E)$ , there exist  $A, B \subseteq V$  and two colors  $i, j \in [2]$  such that  $A \cup B = V$  and  $A, B$  are diameter 2 subsets of  $G_i^c, G_j^c$ , respectively.*

In this note we prove the relaxed version of Conjecture 2.

**Theorem 1** *In every 2-coloring of the edges of the cocktail party graph  $G^c = (V, E)$ , there exist  $A, B \subseteq V$  and  $i, j \in [2]$  such that  $A \cup B = V$  and  $A, B$  are 2-reachable subsets of  $G_i^c, G_j^c$ , respectively.*

It is natural to ask what happens with the Ramsey-type problem mentioned above if 2-colored complete graphs are replaced by 2-colored cocktail party graphs. Theorem 1 has the following immediate corollary.

**Corollary 2** *Any 2-colored  $G^c$  with  $n$  vertices contains a monochromatic 2-reachable subset with at least  $\frac{n}{2}$  vertices.*

The bound  $\frac{n}{2}$  of Corollary 2 seems weak compared to  $\frac{3n}{4}$  in case of complete graphs. However, it is best possible, as the following simple example shows.

**Example 3** Let  $G^c = (V, E)$ , where  $|V| = n$ . Write  $V = X \cup Y$  where  $X, Y$  are complete subgraphs in  $G^c$ . Color  $G^c[X], G^c[Y]$  red and all other edges blue.

In this example there is no monochromatic 2-reachable (let alone diameter 2)  $S \subset V$  with  $|S| > \frac{n}{2}$ . Indeed, otherwise by the pigeonhole principle there are nonadjacent vertices  $v, v' \in S$  and their distance is larger than two in both colors (infinite in red and 3 in blue), a contradiction.

## 2 Proof of Theorem 1

From now on  $G^c = (V, E)$  is a 2-colored cocktail party graph. A pair of vertices  $x, y \in A \subseteq V$  (not necessarily an edge) is *critical for color  $i \in [2]$*  if the distance of  $x, y$  in  $G_i^c$  is greater than two. Note that when  $A = V$ , a critical pair for color  $i$  witnesses that  $G_i^c$  has diameter greater than two. Thus we must have at least one critical pair in  $V$  for both colors, otherwise  $V$  itself has diameter at most two in one of the colors, satisfying the requirement of Theorem 1. Let  $e = (x, y), f = (p, q)$  be critical pairs for colors 1 and 2, respectively.

**Observation 4** *If  $e = (x, y) \notin E$  or  $f = (p, q) \notin E$ , then  $V$  can be covered by two monochromatic stars of the same color. If  $e = (x, y) \in E$ , then  $e \in E(G_2^c)$ . If  $f = (p, q) \in E$ , then  $f \in E(G_1^c)$ .*

Indeed, since we are working in the cocktail party graph,  $e = (x, y) \notin E$  implies that  $x' = y$  and  $y' = x$ , so every  $z \in V \setminus \{x, y\}$  is connected to *both*  $x$  and  $y$  by an edge. Furthermore, the criticality of  $e = (x, y)$  means that there is no color 1 path  $x, z, y$  in  $G^c$ . Thus every  $z$  is connected to either  $x$  or  $y$  (or both) by a color 2 edge. Thus  $S_2(x) \cup S_2(y) = V$ . Note that the covering stars may be degenerate, i.e.  $S_2(x) = \{x\}$  or  $S_2(y) = \{y\}$  is possible. The argument is similar for  $f = (p, q)$ . The statements for  $e, f \in E$  are immediate from the definition of critical pairs. Thus we may always assume that  $e \in E(G_2^c)$  and  $f \in E(G_1^c)$ , otherwise we are done.

Let  $\mathcal{C}_5^+$ , the blow-up of a five-cycle, be the family of graphs obtained from a five-cycle where vertices are replaced with independent sets and the edges between the parts corresponding to edges of the five-cycle are replaced by complete bipartite graphs. Note that the diameter of any graph in  $\mathcal{C}_5^+$  is two. The next lemma is the backbone of the proof; we show that we may assume that  $e$  and  $f$  are disjoint.

**Lemma 5** *If  $e \cap f \neq \emptyset$  then  $V$  can be covered by two monochromatic stars or by two monochromatic graphs from  $\mathcal{C}_5^+$ .*

Before proving Lemma 5 we show how to finish the proof of Theorem 1 from it. Consider all critical edges of  $G^c$ , let  $P \subseteq V, Q \subseteq V$  denote the vertex sets covered by the critical edges of colors 1 and 2, respectively. Then we apply Lemma 5 for all pairs of critical edges such that one is inside  $P$ , the other is inside  $Q$ . We have two possibilities. If Lemma 5 provides the required covering of  $G^c$  (by stars or  $\mathcal{C}_5^+$ -s) for *any* pair, the proof is finished. Otherwise  $P \cap Q = \emptyset$ , thus for  $A = V \setminus P, B = V \setminus Q$  we have  $A \cup B = V$ . Now we claim that  $A, B$  provide the required covering of  $V$ .

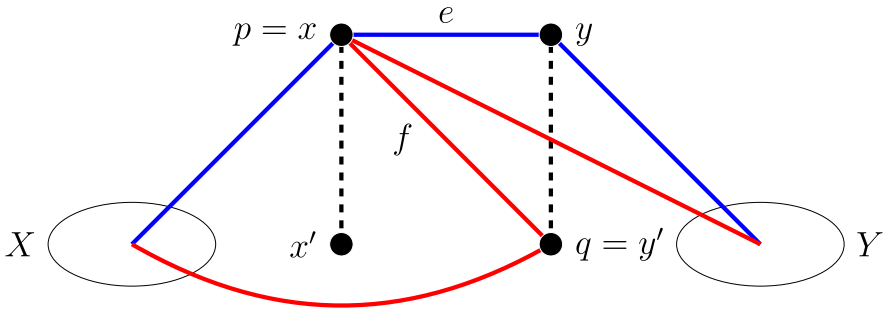


Fig. 1 The structure in the proof of Lemma 5, where color 1 is red, color 2 is blue.

Indeed, there is no critical edge of color 1 inside  $A$ , thus  $A$  is 2-reachable in color 1. Similarly  $B$  is 2-reachable in color 2, finishing the proof. Therefore it is enough to prove Lemma 5.

**Proof of Lemma 5:** Suppose w.l.o.g. that  $e \cap f = \{x\}$ , say  $x = p$ . (From Observation 4 we know that  $|e \cap f| = 1$ ). Define  $X, Y \subset V$  as (see Figure 1)

$$X = N_2(x) \setminus \{y\}, Y = N_2(y) \setminus (\{x\} \cup \{x'\} \cup N_2(x)).$$

Note that by definition  $X \cap Y = \emptyset$ , and also since  $e$  is critical in color 1, we have

$$V = \{x, x', y, y'\} \cup X \cup Y. \tag{1}$$

We start with the following claim.

**Claim 6**  $q = y'$ .

Indeed,  $(y, q) \in E(G^c)$  would contradict the definition of  $e, f$ , since if  $(y, q)$  has color 1, then  $e$  is not critical in color 1 and if  $(y, q)$  has color 2, then  $f$  is not critical in color 2. Thus  $(y, q) \notin E(G^c)$ , implying  $q = y'$ .

From the definition of  $Y, N_2(x) \cap Y = \emptyset$ , so we get

**Observation 7**  $Y \cup \{y'\} = N_1(x)$ .

Also, from the criticality of  $f$ , we have no path of length two from  $p$  to  $q = y'$  in color 2, so we get

**Observation 8**  $X \subseteq N_1(y')$ .

The structure of  $G^c$  is shown in Figure 1. Consider  $x'$  and note that  $x'$  is connected to all vertices in  $X \cup Y$ . We partition  $X, Y$  as follows. For  $i \in [2]$  set

$$X_i = X \cap N_i(x'), Y_i = Y \cap N_i(x').$$

Next we start looking for coverings by two monochromatic stars. It is easy to check the following statements.

- (i) if  $col(x', y) = 2$  then  $S_1(y') \cup S_2(y) = V$ ,
- (ii) if  $col(x', y') = 1$  then  $S_1(y') \cup S_2(y) = V$ .

From (i),(ii) we may assume that  $col(x', y) = 1, col(x', y') = 2$  (note that these must be edges). Then, using this, we look at the effect of a potential degeneracy in the partitions  $X_1, X_2$  and  $Y_1, Y_2$ .

- (iii) if  $X_1 = \emptyset$  then  $V = S_2(x') \cup S_2(y)$ ,
- (iv) if  $Y_1 = \emptyset$  then  $V = S_2(x') \cup S_2(x)$ ,
- (v) if  $X_2 = \emptyset$  then  $V = S_1(x') \cup S_1(x)$ ,
- (vi) if  $Y_2 = \emptyset$  then  $V = S_1(x') \cup S_1(y')$ .

From (iii)-(vi) we may assume that the partitions  $X_1, X_2$  and  $Y_1, Y_2$  are not degenerate. From these assumptions the coloring of  $G^c$  is restricted enough to conclude the proof with a covering of  $V$  by the vertices of two monochromatic  $C_5^+$ -s (note that this is the only place in the proof of Lemma 5 where we are using  $C_5^+$ -s instead of stars).

Indeed, consider the cyclic order  $x, y', X_1, x', Y_1, x$  of five disjoint subsets of  $V$ . Observe that between any two cyclically consecutive sets we have a complete bipartite graph in color 1. Thus the union of these complete bipartite graphs in  $G_1^c$  is a member of  $C_5^+$ , consequently has diameter two in color 1. Similarly, consecutive sets in the cyclic order  $x, y, Y_2, x', X_2, x$  of five disjoint subsets form complete bipartite graphs in color 2 and again, the union of these complete bipartite graphs in  $G_2^c$  is a member of  $C_5^+$ , consequently has diameter two in color 2. Observing that the sets involved cover  $V$ , the proof of Lemma 5 and Theorem 1 are concluded. □

### 3 Concluding Remarks

For  $r = 2$  Conjecture 1 is equivalent to König’s theorem [10] and the  $r = 3$  case was proved by Aharoni [1]. For  $\alpha(G) = 1$  (i.e. for complete graphs) the conjecture holds for  $r = 4$  ([7], [13]) and for  $r = 5$  ([14]).

Milićević [12] conjectured that for  $\alpha = 1$  there exists a smallest  $f = f(r)$  bounding the diameter of the monochromatic components in Conjecture 1. It is known that  $f(2) = 3$  (folklore), the bound  $f(3) \leq 8$  in [11] is improved to  $f(3) \leq 4$  in [3]. This conjecture is extended for every  $\alpha$  in [3] defining  $f = f(r, \alpha)$  as the smallest bound for the diameter of the monochromatic components in Conjecture 1. It is known that  $f(2, \alpha) \leq \alpha^2 + 12\alpha + 4$  [2]. The bound  $f(2, 2) \leq 6$  in [3] is improved to 4 in [9] but it is possible that the best bound is 3.

For certain graphs with  $\alpha = 2$ , including odd antiholes, there is a covering by two monochromatic subgraphs of diameter at most three in every 2-colorings [9]. However, it does not seem easy to decide whether the same is true for graphs defined by vertex disjoint odd antiholes with all possible edges between them.

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