



THE ROLE OF COMPUTATIONAL MECHANICS TODAY

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Abstract

With the proliferation of computers, mechanical problems arising in mechanical engineering practice can be effectively solved. Among the many numerical methods, the present article reports on the development of the finite element method, its development directions, its application possibilities, the importance of error estimation, the requirements supported by the programs and the author's experiences, presenting some numerical examples.

Keywords: *finite element method, approximation, accuracy, variational methods.*

1. Introduction

The science of mechanics has a history spanning several centuries and continues to be important today. Moreover, with the expansion of computational capabilities, it is possible to better describe and model real-world conditions.

The way in which the mechanical model defined by the engineer is solved significantly influences the success and the validity of the result obtained. This article draws attention to the 70-year history of the finite element method, one of the most significant approximate calculation methods, the possibilities of approximation methods, the importance of error estimation, and illustrates the versatility of this calculation method through the presentation of author's own experiences and the analysis of practical problems.

1.1. The beginning of mechanics

The movement of the planets has for a long time fascinated mankind, what orbits they move in, why we see them in different positions depending on the season, etc.

Significant breakthroughs were achieved by the findings of Nicolaus Copernicus (1473-1543) (the Earth is not the center of the world), Johannes Kepler (1571-1630) with the laws of planetary motion, and Galileo Galilei's (1561-1642) experiments with free fall and movement on a slope, Isaac Newton's (1643-1727) understanding of gravity, the creation of the laws of motion, and Gottfried Wilhelm Leibnitz's (1646-1716) creation of differ-

ential calculus, which provided the mathematical basis for the scientific description of motion.

After these, many of the "giants" of the Early New Age, such as Daniel Bernoulli (1700-1782), Leonhard Euler (1707-1783), Jean Le Rond d'Alembert (1717-1783), Joseph-Louis Lagrange (1736-1813), Johann Carl Friedrich Gauss (1777-1855), Simeon-Denis Poisson (1781-1841), Claude Louis Marie Henri Navier (1785-1836), Augustin Louis Cauchy (1789-1857), Sir George Biddell Airy (1801-1892), William Rowan Hamilton (1805-1865), Gustav Robert Kirchhoff (1824-1887), and Lord John William Strutt Rayleigh (1842-1919), contributed significantly to the development of what is called classical mechanics.

1.2. Mechanics from the 1900s

With the development of industry and the design of modern devices, the ability to perform reliable strength and dynamic calculations became an important requirement. The exact solutions developed for simpler shapes had to be replaced by some approximate method. First, Walter Ritz (1878-1909), then Boris Grigorievich Galerkin (1871-1945) succeeded in treating the differential equation system as an algebraic equation system using the variational method. Eric Reissner (1913-1996) approximated the stress field in addition to the displacement field, thus initiating the research of what is called multi-field variational principles.

In the mechanics of solids, we operate with the following basic quantities: displacement field $\mathbf{u}=\mathbf{u}(x, y, z, t)$, strain tensor field $\mathbf{A}=\mathbf{A}(x, y, z, t)$, stress tensor field $\mathbf{T}=\mathbf{T}(x, y, z, t)$, which represent 3+6+6=15 unknown scalar fields at any point of the body. According to linear theory, they can be defined as the geometric equation $\mathbf{A}=0.5(\mathbf{u}\cdot\nabla+\nabla\cdot\mathbf{u})$ material law $\mathbf{T}=\mathbf{T}(\mathbf{A}(\mathbf{u}))$ and equation of motion $\mathbf{T}\cdot\nabla+\mathbf{b}=\rho\ddot{\mathbf{u}}$ (in the case of time-independent $\nabla\cdot\mathbf{T}+\mathbf{b}=0$ equilibrium equation). These equations are supplemented by the dynamic $\mathbf{T}\cdot\mathbf{n}=\mathbf{t}$, $\mathbf{r}\in S_t$ and geometric $\mathbf{r}\in S_u$ $\mathbf{u}=\mathbf{u}_u(x, y, z, t)=\mathbf{u}_u(\mathbf{r}, t)$, boundary conditions, furthermore, in the dynamic case, the following initial conditions: $\mathbf{u}=\mathbf{u}_0(x, y, z, t=0)$, $\dot{\mathbf{u}}=\dot{\mathbf{u}}_0(x, y, z, t=0)$. Here are given $\mathbf{u}_u, \mathbf{t}, \mathbf{u}_0(t=0), \dot{\mathbf{u}}_0(t=0)$ is the ∇ nabla diff. operator, „ \cdot ” is the dyadic, „ \cdot ” is the sign of scalar multiplication, \mathbf{n} is the surface normal vector, S_p, S_u are the signs of the surface subdomains, \mathbf{r} is the location vector. These quantities are illustrated in **Figures 1–3**.

To determine the 15 unknown fields, a system of partial differential equations can be derived for the displacement field and the stress field. In general, their exact solution is only given for a few geometries and special boundary conditions.

It is known that in the case of special geometry, the three-dimensional 3D problem can be reduced to 1D so-called rod, 2D plate or shell problems. For rods, the Jacob Bernoulli and Stephen P. Timoshenko (1878-1972) models were widespread, while for plates, the Raymond David Mindlin (1906-1987) model, which also takes into account shear deformation, and for shells, the Paul M. Naghdi (1924-1994) model. The classical plate and shell models are based on the Gustav Robert Kirchhoff (1824-1887) principle (the normal remains perpendicular to the curved mid-surface during deformation).

These models have successfully solved many problems in engineering practice, and their teaching is still valid in today's higher technical education.

In practical life, an important aspect of design is the prediction of the essential properties of the object to be designed before its implementation, in order to achieve an optimal design. This also requires reliable mechanical calculations, for which a model must be prepared. **Figure 4** shows the possibility of taking various aspects into account.

In the following we outline the history of the finite element method [1, 2, 3] related to the approximate solution.

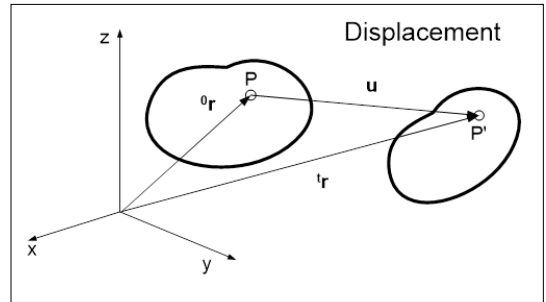


Fig. 1. Displacement vector.

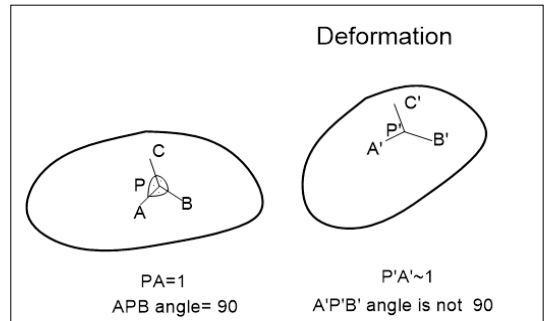


Fig. 2. Deformation detection on the accompanying trider at point P.

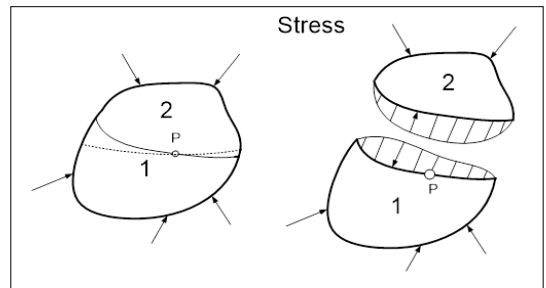


Fig. 3. The internal surface distributed load that ensures equilibrium is the stress.

2. Finite element method

The essence of the method can be easily explained through the example of a stretched rod outlined in **Figure 5**. The rod is divided into elements, the points at the boundary of the elements are called nodes. The displacement at these points is $u_i, i=1, \dots, 6$. Let us approximate the displacement within the elements with a linear function (dashed line). Obviously, if we include more elements, the dashed linear function within the elements becomes closer to the actual one. Since the solution is not known initially, its approximate function can be obtained from the principle of minimum potential energy, i.e. we will calculate

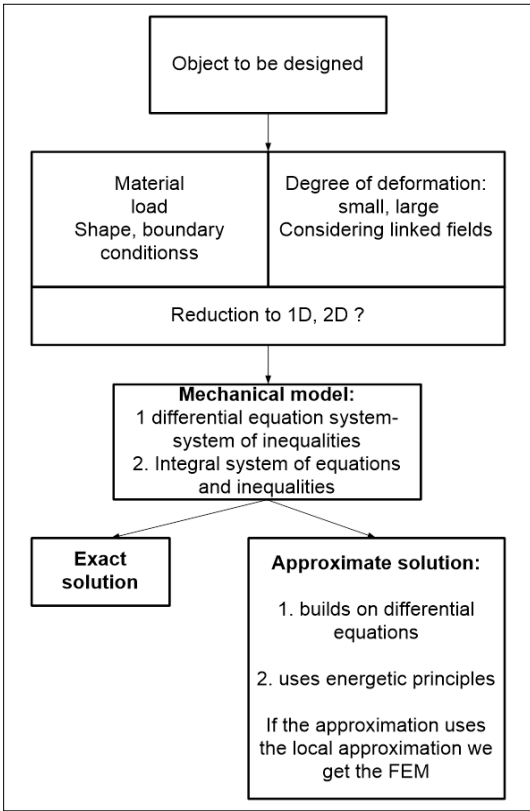


Fig. 4. Main aspects and direction of modeling.

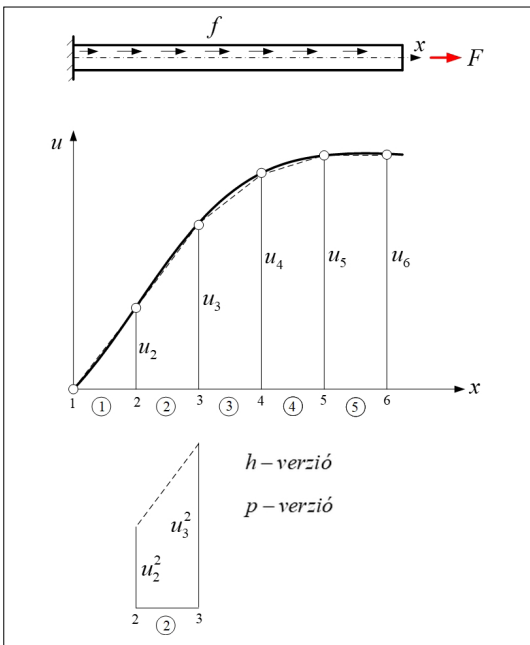


Fig. 5. Displacement field approximation using the finite element method.

the displacements at the nodes from an algebraic system of equations. We feel that with more elements we will get a more accurate solution.

Since the length of the elements has been reduced, this approximate solution is called the h -version calculation.

Another solution method is also possible. We do not increase the number of elements, but the degree of the approximating polynomial within the element. This was first proposed in the world in 1973 by Barna Szabó (currently an external member of the Hungarian Academy of Sciences, professor emeritus at Washington University in St. Louis). At his suggestion, mathematicians began to deal with this p -version approximation method, examining the degree of convergence of the solution. In the division techniques where the size of the elements in the vicinity of stress singularities is also reduced along with the increase in degree, the procedure is called the hp -version calculation. The log value of the error of deformation energies varies significantly in the log value of the total number of unknowns, according to different functions, which is illustrated in Figure 6. The h -version technique has the slowest convergence [1, 2, 3]. Therefore, it is worth using programs suitable for p, hp version calculations when solving specific problems.

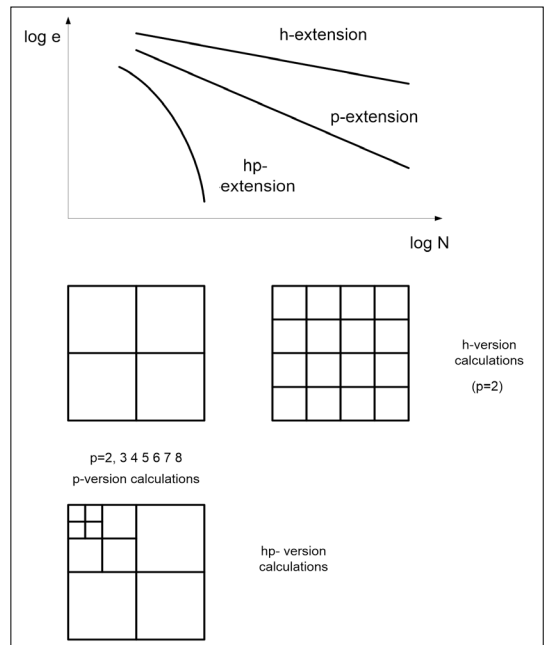


Fig. 6. Convergence of the solution.

It has been proven [1] that the error can be calculated in terms of $\|e\| \leq kN^{-\lambda}$ where N is the number of unknowns in the problem to be solved.

Since the energy of the exact solution in e is not known, nor are the parameters k and λ a minimum of 3 calculations with different orders of approximation must be performed to estimate the error.

Regarding the history of the development of the method, we mention that the article written by M.J. Turner and his colleagues published in 1956 is considered to be the first, and then in 1960 the name finite element became associated with the name of R.W. Clough. Within a short time, several large program systems were developed (1964: ASKA, 1969: SAP, NONSAP, 1969: MARC, ANSYS, NASTRAN).

In 1967, the first book on the finite element method was published, written by O.C. Zienkiewicz and Y.K. Cheung. Further software systems were created: ABAQUS, ADINA in 1972, DYNA in 1976, SYSTUS, COSAR, LSDYNA, PROBE, StressCheck, RASNA, I-DEAS, COSMOS/M, FEAP, FLUENT, SYSWELD, ProCAST, and DEFORM in the 1980s.

On a theoretical level, the convergence of the p version calculation was proven in 1981, and so was the h version in 1984. In 1991, hierarchical modeling was formulated, in 2006, ASME standardized the conditions for the correct use of the finite element program, and in 2012, the formulation of the reliability of the calculation began.

Conditions formulated against finite element programs:

Expectations of finite element programs:

- convenient description of geometry (using one's own system program or relying on data taken from another system).
- wide range of elements (low and high-degree elements).
- possibility of automatic element division.
- simple definition of constraints and other external effects.
- possibility of special modeling constraints (e.g. substructure technology).
- wide range of material laws.
- possibility of using linear and nonlinear theories.
- wide range of loads.
- fast calculation (multiprocessor algorithm, efficient equation system solvers: with direct solver or iteration).
- error analysis.
- possibility of precision of the calculation.

–combining and graphically illustrating the results.

In the early days, finite element program systems were installed on large central computers, which were accessible via terminals. Starting in the 80s, after the appearance of personal computers, the finite element method has become closer to the user, and even the graphical possibilities have made the method user-friendly. Considering the importance of the topic, special courses were organized in higher education institutions to introduce the method and to access it directly by means of computer rooms. Research has also increased significantly, focusing on the creation of new elements and procedures and their industrial dissemination. Several conferences have been organized for those working with finite elements, to build fruitful relationships.

Several programs were developed at our department to overcome the disadvantages of the embargo at that time: among others, TESZGA was developed for the calculation of shells and tanks, AUTOBUSZVAZ for the calculation of bus frames (also using the support of AUTÓKUT), BANYAVAG for KBFI, LAPÁT for GANZ-MÁVAG, ROBOT and SIK for VIDEOTON, ABRONCS 1, 2, 3 commissioned by TAURUS, and FEM3D, a finite element program system for solving strength and dynamic problems of complex flexible structures.

It is worth comparing the shape functions of rectangular elements used in the h -version calculations operating with classical Lagrange shape functions with the Legendre functions used for p -version calculations, and also analyzing the computational advantages and disadvantages that can be deduced from these.

On the left side of [Figure 7](#) the shape functions applied to the first side of the element in the p version can be seen, similarly, on the right side, the h version element using Lagrange functions [3].

We see that with the p version, as p increases, the top two figures belong to $p=1$ at $p=2$ a new quadratic function enters, and then so does the cubic function below in sub-[Figure 7](#). So if the approximation is $p=2$ only the functions belonging to $p=1$ and $p=2$ work on the element, if $p=3$, then new functions enter, without spoiling the previous ones. This is no longer the case in the h -version case outlined on the right. In the quadratic approximation, in addition to the corner points, the side midpoints appear, i.e., a total of eight points, while in the cubic approximation, we have another 1-1 on the sides, i.e., a total of 12 nodes, but the assumed shape functions are no

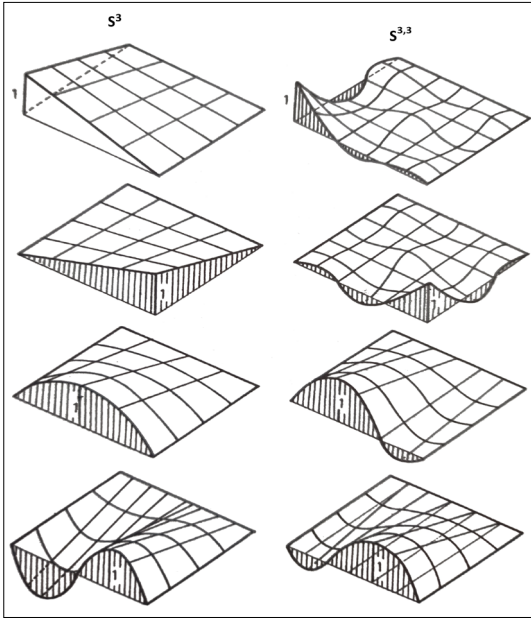


Fig. 7. Shape functions for p and h version elements.

longer the same as the previous ones, since the Lagrange functions must be zero at the nodes except for one point. This means that the stiffness matrix calculated in the lower-order approximation will no longer be usable, while in the p -version the previous one will correspond to a part of the new one. The requirement to perform a minimum of 3 calculations is very well programmable in the p -version (see e.g. StressCheck program [4]).

Formally, after discretization

$$\mathbf{M}\ddot{\mathbf{q}} + \mathbf{C}\dot{\mathbf{q}} + \mathbf{K}\mathbf{q} = \mathbf{f} \tag{1}$$

we obtain a differential equation, where \mathbf{M} is the mass, \mathbf{C} is the damping matrix, \mathbf{K} is the stiffness matrix, \mathbf{f} is the load vector, \mathbf{q} is the nodal displacement vector (unknown).

In the nonlinear (static) case, the equation for the increments can be written as

$$\mathbf{K}_T \Delta \mathbf{q} = \mathbf{f}_n - \mathbf{f}_{\sigma_n} \tag{2}$$

Due to the location of the nodes, the above matrices have a band structure. Reducing the band width reduces the calculation time, which is why sophisticated finite element software systems also include band reduction procedures.

Automatic re-meshing strategies have also been developed to reduce the errors that occur during the calculation.

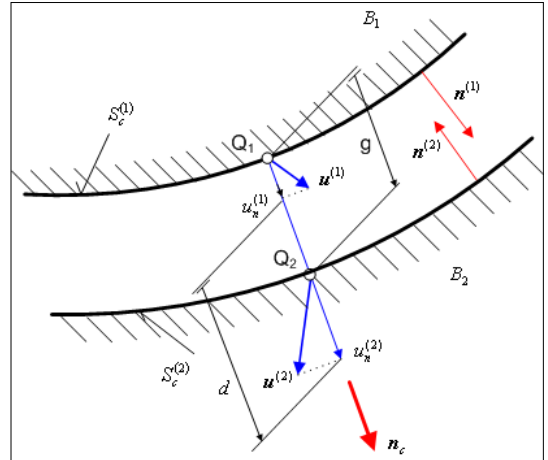


Fig. 8. Contact displacements and the initial gap g .

A separate class of nonlinear problems is the clarification of the contact relations between bodies. In this case, inequalities can be written for on the contact surfaces [5]. So, a system of inequalities must be solved!

To solve these, mathematical programming procedures or special iteration methods can be used to create contact finite elements.

The gap between the bodies after deformation in the relevant contact region: (see Figure 8):

$$d = u_n^{(2)} - u_n^{(1)} + g \quad \mathbf{r} \in S_c = S_o \cup S_p \tag{3}$$

If

$$\begin{aligned} d \geq 0, p_n = 0 \quad \mathbf{r} \in S_o \quad \text{gap,} \\ d = 0 \quad p_n \geq 0 \quad \mathbf{r} \in S_p \quad \text{contact,} \end{aligned} \tag{4}$$

but $d p_{n=0} \quad \mathbf{r} \in S_c$

3. Examples

Here we present only two simpler tasks.

3.1. Contact of cylindrical bodies

Let us examine the contact problem of two cylindrical bodies made of the same material. The radius of the inner holes is $r_i = 20$ mm, the outer ones are $r_e^{(1)} = 120$ mm, $r_e^{(2)} = 140$ mm, respectively, while their heights are $b = b^{(1)} = b^{(2)} = 100$ mm (Figure 9). The lower meridian of the upper cylinder B_1 is characterized by the following function:

$$g = 0,0035 (r - 20)^2, \tag{5}$$

which also corresponds to the initial gap between the bodies. The Young's modulus of the material of the bodies is $E = 2 \cdot 10^5$ MPa and the Poisson's ra-

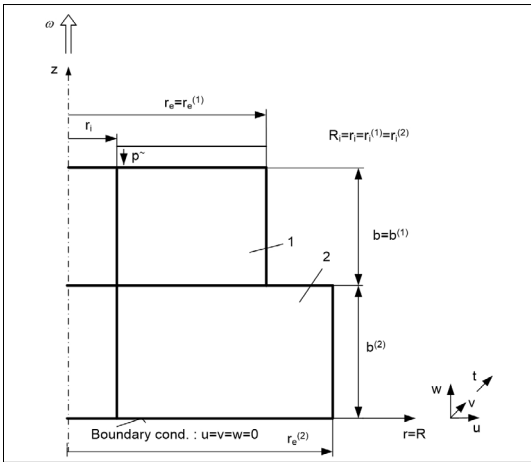


Fig. 9. KContact of two cylindrical bodies.

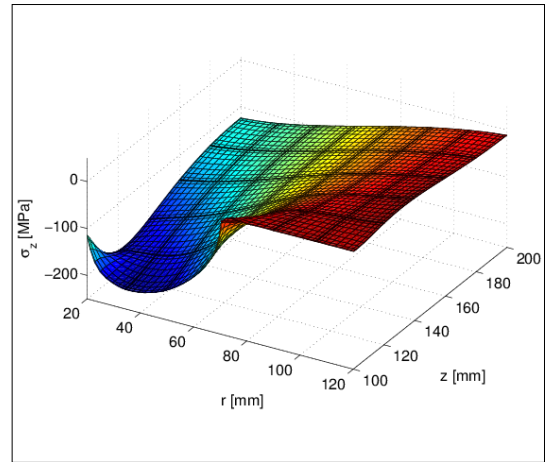


Fig. 12. σ_z normal stress distribution.

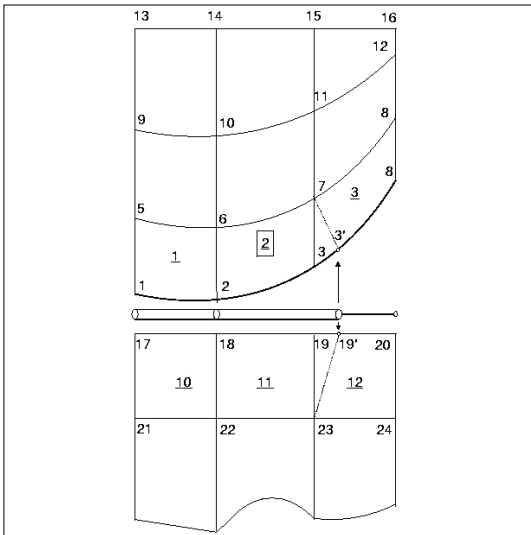


Fig. 10. Moving the nodes 3→3', 19→19'.

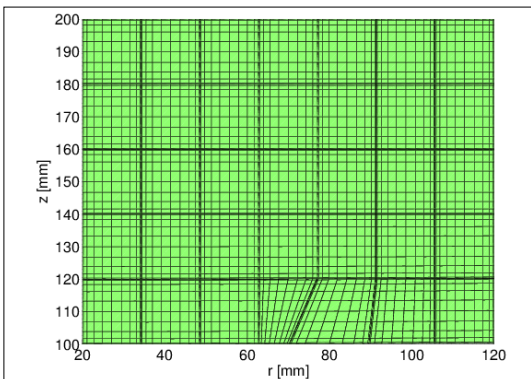


Fig. 11. Modified 7×5 mesh, radius of the edge of the contact area $r_p = 70.37$ mm.

tio is $\nu = 0.3$. The vertical displacement of the upper edge of the body B_1 is given by: $w_0 = 0.15$ mm.

The solution is modeled with finite elements of degree $p = 8$. The initial mesh is modified using the so-called positioning technique in such a way that the node of the finite element coincides with the boundary of the contact area [6]. This allows us to achieve freedom from oscillation of the stresses. In Figure 10, the end of the contact region does not coincide with the right-hand ends of elements 2 and 11, so the nodes 3 and 19 must be moved. Figure 11 shows this for a 7×5 mesh, and the distribution of the normal stress σ_z can be seen in Figure 12.

For further refinement solution techniques, see the paper under [7].

3.2. Roller rounding

We now examine a type of problem completely different from those described above. The rounding of a cylindrical roller must be determined in such a way that the pressure in the meridian plane follows a course specified by us. This can be achieved by partially controlling the contact pressure [5, 8].

The basic idea of the method is clearly shown in Figure 13.

We have a “roof” under which the pressure must be located. Since the highest value occurs in the meridian plane, here the pressure touches the roof, and in other places it is below it. The course of the pressure in the meridian plane can be prescribed using spline (Hermite) functions.

The figure shows half of the roller. Depending on the length of the constant section, we will get pressures of different heights and slopes. The de-

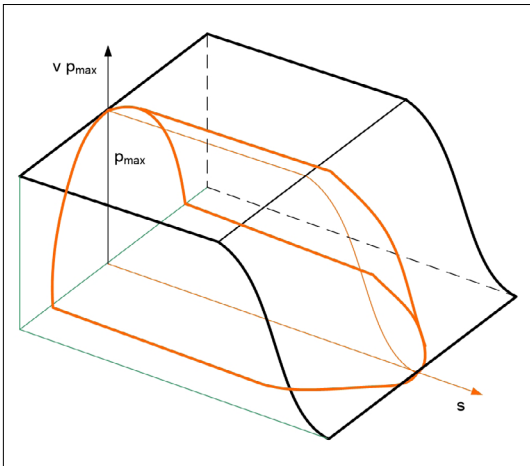


Fig. 13. Function prescribing partial pressure control.

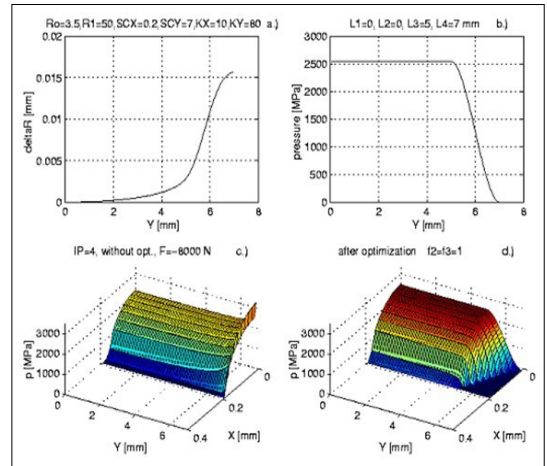


Fig. 14. Result of the roller rounding.

signer has to decide which variant he considers the best. For example, the fatigue limit specification criterion for roller bearings can be decisive.

Figure 14 shows the change in radius obtained from the solution of the optimization problem, the pressure falling into the meridian, the pressure rising high at the end obtained for the original constant radius roller, and the pressure of the optimal rounded roller, looking at a quarter of the solution.

The control function ensures that the pressure at the end of the roller and its derivative with respect to s are also zero. When setting up the problem, the elastic bodies were taken into account by means of the influence function obtained from the Boussinesq solution for the elastic half-space [5]. The number of surface elements taken is 10×40 . The load is 8000 N, the roller radius is 3.5 mm, and the contact area tested is 0.2×7 mm.

4. The future of the finite element method

It is a fact that while in the 1950s, measurement represented 95% of linear stress analysis, in the 1970s it represented 50%, and in the 1990s it represented only 1%, compared to 99% computer simulation. The possibilities of numerical mechanics are increasing in everyday practice, and their practical benefits are clear.

We also experience that the domestic and international conferences organized focus on numerical research in many cases, presenting the diversity of design engineering tasks. For example, at the USACM (USA Congress on Computational Me-

chanics) held in San Francisco in 1997, 450 presentations were given in 96 sections.

In Europe, ECCOMAS organizes similar conferences every 4 years.

The large number of participants also indicates the importance of the research, and the fact that serious forces are at work, conquering newer and newer areas in the field of mechanics.

The large number of journals dealing with finite elements, and the quality of the books published underline the importance of this scientific field, its will to live, and the urgent need to educate the next generation.

5. Conclusions

We can state that a new profession has emerged, which deals with

- the analysis and optimization of mechanical problems,
- the development of specialist software,
- the identification of models,
- the training and installation of computer systems,
- the development and teaching of mechanical numerical methods.

In connection with all this, we can also state that the role of mechanics is being valued and strengthened, since

- more precise models can be built,
- the static and dynamic examinations of complex systems (solid-body-liquid/gas systems, intelligent structures containing piezoelectric elements) become feasible possible.
- When designing new machines and structures, and examining machine manufacturing pro-

cesses, the problems to be solved have become conveniently manageable with numerical techniques based on the high-degree-of-freedom finite element/boundary element method for simulating mechanical states.

- Error estimation and automatic refinement of results are services expected and required from the program.
- Complex design systems are available, to which computer programs suitable for analyzing mechanical models can be easily connected.
- In the fierce market competition related to the development of new products and technologies, modern strength, dynamics, and multidisciplinary knowledge are of strategic importance, which greatly increases the value of engineers in possession of this knowledge.

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